# A MONOIDAL VIEW ON FIXPOINT CHECKS 

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#### Abstract

Fixpoints are ubiquitous in computer science as they play a central role in providing a meaning to recursive and cyclic definitions. Bisimilarity, behavioural metrics, termination probabilities for Markov chains and stochastic games are defined in terms of least or greatest fixpoints. Here we show that our recent work which proposes a technique for checking whether the fixpoint of a function is the least (or the largest) admits a natural categorical interpretation in terms of gs-monoidal categories. The technique is based on a construction that maps a function to a suitable approximation. The compositionality properties of this mapping are naturally interpreted as a gs-monoidal functor. This guides the development of a tool, called UDEfix that allows us to build functions (and their approximations) like a circuit out of basic building blocks and subsequently perform the fixpoints checks. We also show that a slight generalisation of the theory allows one to treat a new relevant case study: coalgebraic behavioural metrics based on Wasserstein liftings.


## 1. Introduction

Graph compositionality has always been an important part of the theory of graph rewriting. For instance, one way to explain the double-pushout approach [19] is to view the graph to be rewritten as a composition of a left-hand side and a context, then compose the context with the right-hand side to obtain the result of the rewriting step.

Several algebras have been proposed for a compositional view on graphs, see for instance [9, 23, 11]. For the compositional modelling of graphs and graph-like structures it

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has in particular proven useful to use the notion of monoidal categories [24], i.e., categories equipped with a tensor product. There are several variants of such categories, such as gsmonoidal categories, that have been shown to be suitable for specifying term graph rewriting (see e.g. [21, 22]). In essence gs-monoidal categories describe graph-like structures with dedicated input and output interfaces, operators for disjoint union (tensor), duplication and termination of wires, quotiented by the axioms satisfied by these operators. Particularly useful are gs-monoidal functors that preserve such operators and hence naturally describe compositional operations.

We show that gs-monoidal categories and the composition concepts that come with them can be fruitfully used in a scenario that - at first sight - might seem quite unrelated: methods for fixpoints checks. In particular, we build upon $[8,6]$ where a theory is proposed for checking whether a fixpoint of a given function is the least (greatest) fixpoint. The theory applies to a variety of fairly diverse application scenarios, such as bisimilarity [28], behavioural metrics [17, 31, 12, 4], termination probabilities for Markov chains [3] and simple stochastic games [13]. Such theory applies to non-expansive functions of the kind $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Y}$, where $\mathbb{M}$ is a set of values and $Y$ is a finite set. More precisely, $\mathbb{M}$ is an MV-chain, i.e., a totally ordered complete lattice endowed with suitable operations of sum and complement, a prototypical example being the interval $[0,1]$ with truncated sum.

The rough idea consists in mapping such functions to corresponding approximations, whose fixpoints can be computed effectively and give information on the fixpoints of the original function.

Here we show that the approximation framework and its compositionality properties can be naturally interpreted in categorical terms. This is done by introducing two gs-monoidal categories in which the concrete functions respectively their approximations live as arrows, together with a gs-monoidal functor, called \#, mapping one to the other. Besides shedding further light on the theoretical approximation framework of [6], this view guided the realisation of a tool, called UDEfix that allows to build functions (and their approximations) like a circuit out of basic building blocks and subsequently perform the fixpoints checks.

The function characterising the system to be checked is viewed as a (hyper)graph whose edges represent some basic building blocks. On the one hand, the graphical interpretation naturally combines with the underlying theory which builds the approximation of a complex function compositionally out of the approximation of the basic components. On the other hand, it leads to an intuitive visual representation which inspires the development of the tool UDEfix.

We also show that the functor \# can be extended to deal with functions $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Y}$, where $Y$ is not necessarily finite, becoming a union of lax functors. We prove some properties of this functor that enable us to give a recipe for finding approximations for a special type of functions: predicate liftings that have been introduced for coalgebraic modal logic [26, 29]. This recipe allows us to include a new case study in the machinery for fixpoint checking: coalgebraic behavioural metrics, based on Wasserstein liftings.

The paper is organized as follows: In Section 2 we give some high-level motivation, while in Section 3 we review the theory from [6]. Subsequently in Section 4 we introduce two (gs-monoidal) categories $\mathbb{C}, \mathbb{A}$ (of concrete and abstract functions), investigate the properties of the approximation map \# between these categories. We then show how to handle predicate liftings (Section 5) and behavioural metrics (Section 6) in that setting. Next, for the finitary case, we show that the categories $\mathbb{C}, \mathbb{A}$ are indeed gs-monoidal and \#


Figure 1. Two probabilistic transition systems.
is gs-monoidal functor (Section 7) and lastly discuss the tool UDEfix in Section 8. We end by drawing some conclusions (Section 9).

This article is an extended version of the paper [5] presented at ICGT '23. With respect to the conference version, the present paper contains additional explanations and examples and full proofs of the results.

## 2. Motivation

We start with some motivations for our theory and the tool UDEfix, which is based on it, via a case study on behavioural metrics. We consider probabilistic transition systems (Markov chains) with labelled states. These are given by a finite set of states $X$, a function $\eta: X \rightarrow \mathcal{D}(X)$ mapping each state $x \in X$ to a probability distribution on $X$ and a labelling function $\ell: X \rightarrow \Lambda$, where $\Lambda$ is a fixed set of labels (for examples see Figure 1). Our aim is to determine the behavioural distance of two states, whose definition is based on the socalled Kantorovich or Wasserstein lifting [32] that measures the distance of two probability distributions on $X$, based on a distance $d: X \times X \rightarrow[0,1]$. In more detail: given $d$, we define $d^{\mathcal{D}}: \mathcal{D}(X) \times \mathcal{D}(X) \rightarrow[0,1]$ as

$$
d^{\mathcal{D}}\left(p_{1}, p_{2}\right)=\inf \left\{\sum_{x_{1}, x_{2} \in X} d\left(x_{1}, x_{2}\right) \cdot t\left(x_{1}, x_{2}\right) \mid t \in \Gamma\left(p_{1}, p_{2}\right)\right\}
$$

where $\Gamma\left(p_{1}, p_{2}\right)$ is the set of couplings of $p_{1}, p_{2}$ (i.e., distributions $t: X \times X \rightarrow[0,1]$ such that $\sum_{x_{2} \in X} t\left(x_{1}, x_{2}\right)=p_{1}\left(x_{1}\right)$ and $\left.\sum_{x_{1} \in X} t\left(x_{1}, x_{2}\right)=p_{2}\left(x_{2}\right)\right)$. The Wasserstein lifting gives in fact the solution of a transport problem, where we interpret $p_{1}, p_{2}$ as the supply respectively demand at each point $x \in X$. Transporting a unit from $x_{1}$ to $x_{2} \operatorname{costs} d\left(x_{1}, x_{2}\right)$ and $t$ is a transport plan ( $=$ coupling) whose marginals are $p_{1}, p_{2}$. In other words $d^{\mathcal{D}}\left(p_{1}, p_{2}\right)$ can be seen as the cost of the optimal transport plan, moving the supply $p_{1}$ to the demand $p_{2}$.

The behavioural metric is then defined as the least fixpoint of the function $\mathcal{W}:[0,1]^{X \times X} \rightarrow$ $[0,1]^{X \times X}$ where $\mathcal{W}(d)\left(x_{1}, x_{2}\right)=1$ if $\ell\left(x_{1}\right) \neq \ell\left(x_{2}\right)$ and $\mathcal{W}(d)\left(x_{1}, x_{2}\right)=d^{\mathcal{D}}\left(\eta\left(x_{1}\right), \eta\left(x_{2}\right)\right)$ otherwise. For instance, the best transport plan for the system on the left-hand side of Figure 1 and the distributions $\eta(1), \eta(2)$ is $t$ with $t(3,3)=1 / 3, t(3,4)=1 / 6, t(4,4)=1 / 2$ and 0 otherwise.

One can observe that the function $\mathcal{W}$ can be decomposed as

$$
\mathcal{W}=\max _{\rho} \circ\left(c_{k} \otimes(\eta \times \eta)^{*} \circ \min _{u} \circ \tilde{\mathcal{D}}\right),
$$

where $\otimes$ stands for disjoint union (on functions) and we use the functions given in Table 1. Note that the operators can be generalised to the case in which the underlying sets are infinite, by replacing min and max with inf and sup. More concretely, the types of the components and the parameters $k, u, \rho$ are given as follows, where $Y=X \times X$ :

| Function | $c_{k}$ | $g^{*}$ | $\min _{u}$ | $\max _{u}$ | $\operatorname{av}_{D}=\tilde{\mathcal{D}}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $k: Z \rightarrow \mathbb{M}$ | $g: Z \rightarrow Y$ | $u: Y \rightarrow Z$ | $u: Y \rightarrow Z$ | $\mathbb{M}=[0,1], Z=\mathcal{D}(Y)$ |
| Name | constant | reindexing | minimum | maximum | expectation |
| $a \mapsto \ldots$ | $k$ | $a \circ g$ | $\lambda z \cdot \min _{u(y)=z} a(y)$ | $\lambda z \cdot \max _{u(y)=z} a(y)$ | $\lambda z \cdot \sum_{y \in Y} z(y) \cdot a(y)$ |

TABLE 1. Basic functions of type $\mathbb{M}^{Y} \rightarrow \mathbb{M}^{Z}, a: Y \rightarrow \mathbb{M}$.


Figure 2. Decomposition of the fixpoint function $\mathcal{W}$ for computing behavioural metrics.

- $c_{k}:[0,1]^{\emptyset} \rightarrow[0,1]^{Y}$ where $k\left(x, x^{\prime}\right)=1$ if $\ell(x) \neq \ell\left(x^{\prime}\right)$ and 0 otherwise.
- $\tilde{\mathcal{D}}:[0,1]^{Y} \rightarrow[0,1]^{\mathcal{D}(Y)}$.
- $\min _{u}:[0,1]^{\mathcal{D}(Y)} \rightarrow[0,1]^{\mathcal{D}(X) \times \mathcal{D}(X)}$ where $u: \mathcal{D}(Y) \rightarrow \mathcal{D}(X) \times \mathcal{D}(X), u(t)=(p, q)$ with $p(x)=\sum_{x^{\prime} \in X} t\left(x, x^{\prime}\right), q(x)=\sum_{x^{\prime} \in X} t\left(x^{\prime}, x\right)$.
- $(\eta \times \eta)^{*}:[0,1]^{\mathcal{D}}(X) \times \mathcal{D}(X) \rightarrow[0,1]^{Y}$.
- $\max _{\rho}:[0,1]^{Y+Y} \rightarrow[0,1]^{Y}$ where $\rho: Y+Y \rightarrow Y$ is the obvious map from the coproduct (disjoint union of sets) $Y+Y$ to $Y$.
In fact, this decomposition can be depicted diagrammatically, as in Figure 2.
The function $\mathcal{W}$ is a monotone function on a complete lattice, hence it has a least fixpoint by Knaster-Tarski's fixpoint theorem [30], which is the behavioural metric. By giving a transport plan as above, it is possible to provide an upper bound for the Wasserstein lifting and hence there are strategy iteration algorithms that can approach a fixpoint from above. The problem with these algorithms is that they might get stuck at a fixpoint that is not the least. Hence, it is essential to be able to determine whether a given fixpoint is indeed the smallest one (cf. [2]).

Consider for instance the transition system in Figure 1 on the right. It contains two states 1,2 on a cycle. In fact these two states should be indistinguishable and hence, if $d=\mu \mathcal{W}$ is the least fixpoint of $\mathcal{W}$, then $d(1,2)=d(2,1)=0$. However, the metric $a$ with $a(1,2)=a(2,1)=1$ ( 0 otherwise) is also a fixpoint and the question is how to determine that it is not the least.

For this, we use the techniques developed in [6] that require, in particular, that $\mathcal{W}$ is nonexpansive (i.e., given two metrics $d_{1}, d_{2}$, the sup-distance of $\mathcal{W}\left(d_{1}\right), \mathcal{W}\left(d_{2}\right)$ is smaller or equal than the sup-distance of $\left.d_{1}, d_{2}\right)$. In this case we can associate $\mathcal{W}$ with an approximation $\mathcal{W}_{\#}^{a}$ on subsets of $X \times X$ such that, given $Y^{\prime} \subseteq X \times X$, the set $\mathcal{W}_{\#}^{a}\left(Y^{\prime}\right)$ intuitively contains all pairs $\left(x_{1}, x_{2}\right)$ such that, decreasing function $a$ by some value $\delta$ over $Y^{\prime}$, resulting in a function $b$ (defined as $b\left(x_{1}, x_{2}\right)=a\left(x_{1}, x_{2}\right)-\delta$ if $\left(x_{1}, x_{2}\right) \in Y^{\prime}$ and $b\left(x_{1}, x_{2}\right)=a\left(x_{1}, x_{2}\right)$ otherwise) and applying $\mathcal{W}$, we obtain a function $\mathcal{W}(b)$, where the same decrease takes place
at $\left(x_{1}, x_{2}\right)$ (i.e., $\left.\mathcal{W}(b)\left(x_{1}, x_{2}\right)=\mathcal{W}(a)\left(x_{1}, x_{2}\right)-\delta\right)$. Concretely, here $\mathcal{W}_{\#}^{a}(\{(1,2)\})=\{(2,1)\}$, since a decrease at $(1,2)$ will cause a decrease at $(2,1)$ in the next iteration. In fact the greatest fixpoint of $\mathcal{W}_{\#}^{a}$, which here is $\{(1,2),(2,1)\}$, gives us those elements that have a potential for decrease (intuitively there is "slack" or "wiggle room") and form a vicious cycle as above. It holds that $a$ is the least fixpoint of $\mathcal{W}$ iff the the greatest fixpoint of $\mathcal{W}_{\#}^{a}$ is the empty set, a non-trivial result proved in [6].

The importance of the decomposition stems from the fact that the approximation is compositional, that is $\mathcal{W}_{\#}^{a}$ can be built out of the approximations of $\max _{\rho}, c_{k},(\delta \times \delta)^{*}$, $\min _{u}, \tilde{\mathcal{D}}=\operatorname{av}_{D}$, which can be easily determined (see Table 2).

The function $\tilde{\mathcal{D}}$ can be obtained as the predicate lifting of the distribution functor $\mathcal{D}$ (see Section 5). But for general functors, beyond $\mathcal{D}$, the characterization is still missing and will be provided in this paper.

We anticipate that in our tool UDEfix we can draw a diagram as in Figure 2, from which the approximation and its greatest fixpoint is automatically computed in a compositional way, allowing us to perform such fixpoint checks.

## 3. Preliminaries

This section reviews some background used throughout the paper. This includes the basics of lattices and MV-algebras, where the functions of interest take values. We also recap some results from [6] useful for detecting if a fixpoint of a given function is the least (or greatest).

We will also need some standard notions from category theory, in particular categories, functors and natural transformations. The definition of (strict) gs-monoidal categories is spelled out in detail later in Definition 7.1.

For sets $X, Y$, we denote by $\mathcal{P}(X)$ the powerset of $X$ and $\mathcal{P}_{f}(X)$ the set of finite subsets of $X$. The set of functions from $X$ to $Y$ is denoted by $Y^{X}$.

A partially ordered set $(P, \sqsubseteq)$ is often denoted simply as $P$, omitting the order relation. For $x, y \in P$, we write $x \sqsubset y$ when $x \sqsubseteq y$ and $x \neq y$. For a function $f: X \rightarrow P$, we will write $\arg \min _{x \in X} f(x)$ to denote the set of elements where $f$ reaches the minimum, i.e., $\{x \in X \mid \forall y \in X . f(x) \sqsubseteq f(y)\}$ and, abusing the notation, we will write $z=\arg \min _{x \in X} f(x)$ instead of $z \in \arg \min _{x \in X} f(x)$. The join and the meet of a subset $X \subseteq P$ (if they exist) are denoted $\bigsqcup X$ and $\Pi X$.

A complete lattice is a partially ordered set $(\mathbb{L}, \sqsubseteq)$ such that each subset $X \subseteq \mathbb{L}$ admits a join $\bigsqcup X$ and a meet $\Pi X$. A complete lattice $(\mathbb{L}, \sqsubseteq)$ always has a least element $\perp=\Pi \mathbb{L}$ and a greatest element $T=\bigsqcup \mathbb{L}$.

A function $f: \mathbb{L} \rightarrow \mathbb{L}$ is monotone if for all $l, l^{\prime} \in \mathbb{L}$, if $l \sqsubseteq l^{\prime}$ then $f(l) \sqsubseteq f\left(l^{\prime}\right)$. By Knaster-Tarski's theorem [30, Theorem 1], any monotone function on a complete lattice has a least fixpoint $\mu f$ and a greatest fixpoint $\nu f$, characterised as the meet of all pre-fixpoints $\mu f=\Pi\{l \mid f(l) \sqsubseteq l\}$ and, dually, a greatest fixpoint $\nu f=\bigsqcup\{l \mid l \sqsubseteq f(l)\}$, characterised as the join of all post-fixpoints. We denote by Fix $(f)$ the set of all fixpoints of $f$.

For a set $Y$ and a complete lattice $\mathbb{L}$, the set of functions $\mathbb{L}^{Y}=\{f \mid f: Y \rightarrow \mathbb{L}\}$ with pointwise order (for $a, b \in \mathbb{L}^{Y}, a \sqsubseteq b$ if $a(y) \sqsubseteq b(y)$ for all $y \in Y$ ), is a complete lattice.

We are also interested in the set of finitely supported probability distributions $\mathcal{D}(Y) \subseteq$ $[0,1]^{Y}$, i.e., functions $\beta: Y \rightarrow[0,1]$ with finite support such that $\sum_{y \in Y} \beta(y)=1$.

Definition 3.1 (MV-algebra). An $M V$-algebra [25] is a tuple $\mathbb{M}=(M, \oplus, 0,(\cdot))$ where $(M, \oplus, 0)$ is a commutative monoid and $\overline{(\cdot)}: M \rightarrow M$ maps each element to its complement, such that, if we define $1=\overline{0}$ and subtraction $x \ominus y=\overline{\bar{x} \oplus y}$, then for all $x, y \in M$ it holds that
(1) $\overline{\bar{x}}=x$;
(2) $x \oplus 1=1$;
(3) $(x \ominus y) \oplus y=(y \ominus x) \oplus x$.

MV-algebras are endowed with a partial order, the so-called natural order, defined for $x, y \in M$, by $x \sqsubseteq y$ if $x \oplus z=y$ for some $z \in M$. When $\sqsubseteq$ is total, $\mathbb{M}$ is called an $M V$-chain. We will often write $\mathbb{M}$ instead of $M$.

The natural order gives an MV-algebra a lattice structure where $\perp=0, \top=1, x \sqcup y=$ $(x \ominus y) \oplus y$ and $x \sqcap y=\overline{\bar{x}} \sqcup \bar{y}=x \ominus(x \ominus y)$. We call the MV-algebra complete if it is a complete lattice, which is not true in general, e.g., $([0,1] \cap \mathbb{Q}, \leq)$.

Example 3.2. A prototypical MV-algebra is $([0,1], \oplus, 0, \overline{(\cdot)})$ where $x \oplus y=\min \{x+y, 1\}$, $\bar{x}=1-x$ and $x \ominus y=\max \{0, x-y\}$ for $x, y \in[0,1]$. The natural order is $\leq$ (less or equal) on the reals. Another example is $K=(\{0, \ldots, k\}, \oplus, 0, \overline{(\cdot)})$ where $n \oplus m=\min \{n+m, k\}$, $\bar{n}=k-n$ and $n \ominus m=\max \{n-m, 0\}$ for $n, m \in\{0, \ldots, k\}$. Both MV-algebras are complete and MV-chains.

We next briefly recap the theory from [6] which will be helpful in the paper for checking whether a fixpoint is the least or the greatest fixpoint of some underlying endo-function. For the purposes of the present paper we actually need a generalisation of the theory which provides the approximation also for functions with an infinite domain (while the theory in [6] was restricted to finite sets). Hence in the following, sets $Y$ and $Z$ are possibly infinite.

Given $a \in \mathbb{M}^{Y}$ we define its norm as $\|a\|=\sup \{a(y) \mid y \in Y\}$. A function $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Z}$ is non-expansive if for all $a, b \in \mathbb{M}^{Y}$ it holds $\|f(b) \ominus f(a)\| \sqsubseteq\|b \ominus a\|$. It can be seen that non-expansive functions are monotone. A number of standard operators are non-expansive (e.g., constants, reindexing, max and min over a relation, average in Table 1), and nonexpansiveness is preserved by composition and disjoint union (see [6, Theorem 5.2]). Given $Y^{\prime} \subseteq Y$ and $\delta \in \mathbb{M}$, we write $\delta_{Y^{\prime}}$ for the function defined by $\delta_{Y^{\prime}}(y)=\delta$ if $y \in Y^{\prime}$ and $\delta_{Y^{\prime}}(y)=0$, otherwise.

Let $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Y}, a \in \mathbb{M}^{Y}$ and $0 \sqsubset \delta \in \mathbb{M}$. For a non-expansive endo-function $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Y}$ and $a \in \mathbb{M}^{Y}$, the theory in [6] provides a so-called $a$-approximation $f_{\#}^{a}$ of $f$, which is an endo-function over a suitable subset of $\mathcal{P}(Y)$. More precisely, for $a, b \in \mathbb{M}^{Y}$, let $[a, b]=\left\{c \in \mathbb{M}^{Y} \mid a \sqsubseteq c \sqsubseteq b\right\}$ and let $[Y]^{a}=\{y \in Y \mid a(y) \neq 0\}$. Consider the functions $\alpha^{a, \delta}: \mathcal{P}\left([Y]^{a}\right) \rightarrow[a \ominus \delta, a]$ and $\gamma^{a, \delta}:[a \ominus \delta, a] \rightarrow \mathcal{P}\left([Y]^{a}\right)$, defined, for $Y^{\prime} \in \mathcal{P}\left([Y]^{a}\right)$ and $b \in[a \ominus \delta, a]$, by

$$
\alpha^{a, \delta}\left(Y^{\prime}\right)=a \ominus \delta_{Y^{\prime}} \quad \gamma^{a, \delta}(b)=\left\{y \in[Y]^{a} \mid a(y) \ominus b(y) \sqsupseteq \delta\right\} .
$$

For a non-expansive function $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Z}$ and $\delta \in \mathbb{M}$, define $f_{\#}^{a, \delta}: \mathcal{P}\left([Y]^{a}\right) \rightarrow$ $\mathcal{P}\left([Z]^{f(a)}\right)$ as $f_{\#}^{a, \delta}=\gamma^{f(a), \delta} \circ f \circ \alpha^{a, \delta}$. The function $f_{\#}^{a, \delta}$ is antitone in the parameter $\delta$ and we define the $a$-approximation of $f$ as

$$
f_{\#}^{a}=\bigcup_{\delta \sqsupset 0} f_{\#}^{a, \delta} .
$$

For finite sets $Y$ and $Z$ there exists a suitable value $\iota_{f}^{a} \sqsupset 0$, such that all functions $f_{\#}^{a, \delta}$ for $0 \sqsubset \delta \sqsubseteq \iota_{f}^{a}$ are equal. Here, the $a$-approximation is given by $f_{\#}^{a}=f_{\#}^{a, \delta}$ for $\delta=\iota_{f}^{a}$.

Let $\delta_{a}=\min \left\{a(y) \mid y \in[Y]^{a}\right\}$ be the least non-zero value assumed by a function $a$. When $\delta \sqsubseteq \delta_{a}$, the pair $\left\langle\alpha^{a, \delta}, \gamma^{a, \delta}\right\rangle$ is a (contra-variant) Galois connection, a notion at the heart of abstract interpretation [15, 16], and $f_{\#}^{a, \delta}=\gamma^{f(a), \delta} \circ f \circ \alpha^{a, \delta}$ is the best correct approximation of $f$.

Intuitively, given some $Y^{\prime}$, the set $f_{\#}^{a}\left(Y^{\prime}\right)$ contains the points where a decrease of the values of $a$ on the points in $Y^{\prime}$ "propagates" through the function $f$. The greatest fixpoint of $f_{\#}^{a}$ gives us the subset of $Y$ where such a decrease is propagated in a cycle (a so-called "vicious cycle"). Whenever $\nu f_{\#}^{a}$ is non-empty, one can argue that $a$ cannot be the least fixpoint of $f$ since we can decrease the value in all elements of $\nu f_{\#}^{a}$, obtaining a smaller prefixpoint. Interestingly, for non-expansive functions, also the converse holds, i.e., emptiness of the greatest fixpoint of $f_{\#}^{a}$ implies that $a$ is the least fixpoint.

Theorem 3.3 (soundness and completeness for fixpoints). Let $\mathbb{M}$ be a complete MV-chain, $Y$ a finite set and $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Y}$ be a non-expansive function. Let $a \in \mathbb{M}^{Y}$ be a fixpoint of $f$. Then $\nu f_{\#}^{a}=\emptyset$ if and only if $a=\mu f$.

If $a$ is not the least fixpoint and thus $\nu f_{\#}^{a} \neq \emptyset$ then there is $0 \sqsubset \delta \in \mathbb{M}$ such that $a \ominus \delta_{\nu f_{\#}^{a}}$ is a pre-fixpoint of $f$.

Using the above theorem we can check whether some fixpoint $a$ of $f$ is the least fixpoint. Whenever $a$ is a fixpoint, but not yet the least fixpoint of $f$, it can be decreased by a fixed value in $\mathbb{M}$ (see [6, Proposition 4.5] for the details) on the points in $\nu f_{\#}^{a}$ to obtain a smaller pre-fixpoint. In this way we obtain $a^{\prime} \sqsubset a$ such that $f\left(a^{\prime}\right) \sqsubseteq a^{\prime}$.

This results in the following complete proof rule, where "complete" means that it can be used in both directions.

$$
\frac{a=f(a) \quad \nu f_{\#}^{a}=\emptyset}{a=\mu f}
$$

Whenever $a \in \mathbb{M}^{Y}$ is not a fixpoint, but a post-fixpoint of $f$ (i.e., $a \sqsubseteq f(a)$ ), one can still consider a restriction of the $a$-approximation of $f$ and obtain a sound, but not complete rule.
Lemma 3.4 (soundness for post-fixpoints). Let $\mathbb{M}$ be a complete MV-chain, $Y$ a finite set and $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Y}$ a non-expansive function, $a \in \mathbb{M}^{Y}$ such that $a \sqsubseteq f(a)$. Define $[Y]^{a=f(a)}=\left\{y \in[Y]^{a} \mid a(y)=f(a)(y)\right\}$ and restrict the abstraction $f_{\#}^{a}: \mathcal{P}\left([Y]^{a}\right) \rightarrow$ $\mathcal{P}\left([Y]^{f(a)}\right)$ to an endo-function $f_{*}^{a}:[Y]^{a=f(a)} \rightarrow[Y]^{a=f(a)}$ by letting $f_{*}^{a}\left(Y^{\prime}\right)=f_{\#}^{a}\left(Y^{\prime}\right) \cap$ $[Y]^{a=f(a)}$. If $\nu f_{*}^{a}=\emptyset$ then $a \sqsubseteq \mu f$.

Written more compactly, we obtain the following (incomplete) proof rule:

$$
\frac{a \sqsubseteq f(a) \quad \nu f_{*}^{a}=\emptyset}{a \sqsubseteq \mu f}
$$

The above theory can be easily be dualised to checking greatest fixpoints.
Some basic functions which can be used as basic building blocks for constructing nonexpansive functions have already been given in Table 1. In Table 2 these basic non-expansive functions are listed together with their approximations. Note that for min and max we
slightly generalize and allow relations instead of functions as parameters. Note, in particular, that the approximation of reindexing is given by the inverse image, a fact that will be often used in the paper.

| function $f$ | definition of $f$ | $f_{\#}^{a}\left(Y^{\prime}\right)$ |
| :--- | :--- | :--- |
| $c_{k}$ <br> $\left(k \in \mathbb{M}^{Z}\right)$ | $f(a)=k$ | $\emptyset$ |
| $u^{*}$ <br> $(u: Z \rightarrow Y)$ | $f(a)=a \circ u$ | $u^{-1}\left(Y^{\prime}\right)$ |
| $\min _{\mathcal{R}}$ <br> $(\mathcal{R} \subseteq Y \times Z)$ | $f(a)(z)=\min _{y \mathcal{R} z} a(y)$ | $\left\{z \in[Z]^{f(a)} \mid \arg \min _{y \in \mathcal{R}^{-1}(z)} a(y) \cap Y^{\prime} \neq \emptyset\right\}$ |
| $\max _{\mathcal{R}}$ <br> $(\mathcal{R} \subseteq Y \times Z)$ | $f(a)(z)=\max _{y \mathcal{R} z} a(y)$ | $\left\{z \in[Z]^{f(a)} \mid \arg \max _{y \in \mathcal{R}^{-1}(z)} a(y) \subseteq Y^{\prime}\right\}$ |
| $\tilde{\mathcal{D}} \quad(\mathbb{M}=[0,1]$, | $f(a)(p)=\sum_{y \in Y} p(y) \cdot a(y)$ | $\left\{p \in[D]^{f(a)} \mid \operatorname{supp}(p) \subseteq Y^{\prime}\right\}$ |
| $Z=D \subseteq \mathcal{D}(Y))$ |  |  |$\quad$|  |
| :--- |

TABLE 2. Basic functions $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Z}$ (constant, reindexing, minimum, maximum, average) and the corresponding approximations $f_{\#}^{a}: \mathcal{P}\left([Y]^{a}\right) \rightarrow$ $\mathcal{P}\left([Z]^{f(a)}\right)$.
Notation: $\mathcal{R}^{-1}(z)=\{y \in Y \mid y \mathcal{R} z\} ; \operatorname{supp}(p)=\{y \in Y \mid p(y)>0\}$ for $p \in \mathcal{D}(Y) ; \arg \min _{y \in Y^{\prime}} a(y)$, resp. $\arg \max _{y \in Y^{\prime}} a(y)$ are the sets of elements where $\left.a\right|_{Y^{\prime}}$ reaches the minimum, resp. the maximum, for $Y^{\prime} \subseteq Y$ and $a \in$ $\mathbb{M}^{Y}$.

In Section 2 we already gave a motivating example for fixpoint checks. Here we consider another example instantiating the framework.

Example 3.5. We consider (unlabelled) Markov chains given by $(S, T, \eta)$, where $S$ is a set of states, $T \subseteq S$ is a set of terminal states and $\eta: S \backslash T \rightarrow \mathcal{D}(S)$ assigns to each non-terminal state a probability distribution over its successors. We are interested in the termination probability of a given state of the Markov chain, which can be computed by taking the least fixpoint of a function $\mathcal{T}:[0,1]^{S} \rightarrow[0,1]^{S}$ :

$$
\begin{aligned}
& \mathcal{T}:[0,1]^{S} \rightarrow[0,1]^{S} \\
& \mathcal{T}(t)(s)= \begin{cases}1 & \text { if } s \in T \\
\sum_{s^{\prime} \in S} \eta(s)\left(s^{\prime}\right) \cdot t\left(s^{\prime}\right) & \text { otherwise }\end{cases}
\end{aligned}
$$

As an example we consider the Markov chain in Figure 3 with state set $S=\{x, y, u, z\}$, where $u$ is the only terminal state. Let $p_{s}=\eta(s)$ be the probability distribution associated with state $s$. The least fixpoint $\mu \mathcal{T}$ of $\mathcal{T}$ is given in Figure 3 in green (left) and the greatest fixpoint $\nu \mathcal{T}$ in red (right). These are two of infinitely many fixpoints of $\mathcal{T}$. The cycle on the left (including $y, z$ ) can be seen as some kind of vicious cycle, where $y, z$ convince each other that they terminate with a probability that might be too high.

Now let $a=\nu \mathcal{T}$ be the greatest fixpoint of $\mathcal{T}$, i.e., the function $S \rightarrow[0,1]$ that maps every state to 1 . All states still have "wiggle room" and can be decreased, hence $[S]^{a}=S$.


Figure 3. A Markov chain with two fixpoints of $\mathcal{T}$ (right)
Given $S^{\prime} \subseteq S$, the approximation $\mathcal{T}_{\#}^{a}$ is as follows:

$$
\mathcal{T}_{\#}^{a}\left(S^{\prime}\right)=\left\{s \in[S]^{\mathcal{T}(a)} \mid s \notin T \wedge \operatorname{supp}(\eta(s)) \subseteq S^{\prime}\right\}
$$

where $\operatorname{supp}(p)$ denoted the support of a probability distribution $p: S \rightarrow[0,1]$, i.e., the set of states $v$ for which $p(v)>0$.

Intuitively, if we decide to decrease the $a$-values of all states in $S^{\prime}$ by some fixed amount $\delta$, the states in $\mathcal{T}_{\#}^{a}\left(S^{\prime}\right)$ will also decrease their $a$-values by $\delta$. This is true for all states that are not terminal and whose successors are all included in $S$.

In the example $\mathcal{T}_{\#}^{a}(\{y, z\})=\{y, z\}$ and $\{y, z\}$ is indeed the greatest fixpoint of the approximation. Since it is non-empty, we can deduce that $a$ is not the least fixpoint. Furthermore we could now subtract a small value (for details on how to obtain this value see [6, Proposition 4.5]) from $a(y), a(z)$ to obtain a smaller pre-fixpoint, from where one can continue to iterate to the least fixpoint (see also [7]).

## 4. A Categorical View of the Approximation Framework

The framework from [6], summarized in the previous section, is not based on category theory, but - as we shall see - it can be naturally reformulated in a categorical setting. In particular, casting the compositionality results into a monoidal structure (see Section 7) is a valuable basis for our tool. The starting point for this section is the observation that the operation of mapping a function $f$ to its approximation $f_{\#}^{a}$ (given a fixpoint $a$ of $f$ ) is compositional: it respects (under some constraints) function composition and disjoint union, making it a monoidal functor.

We however have to take care to choose adequate categories and fix an appropriate setting, to obtain the desired results. First, we will show how the operation \# of taking the $a$-approximation of a function can be seen as a union of lax functors between two categories: a concrete category $\mathbb{C}$ whose arrows are the non-expansive functions for which we seek the least (or greatest) fixpoint and an abstract category $\mathbb{A}$ whose arrows are the corresponding approximations.

More precisely, recall that, as discussed in the previous section, given a non-expansive function $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Z}$, the approximation of $f$ is relative to a fixed map $a \in \mathbb{M}^{Y}$. Hence we let objects in $\mathbb{C}$ be elements $a \in \mathbb{M}^{Y}$ and an arrow from $a \in \mathbb{M}^{Y}$ to $b \in \mathbb{M}^{Z}$ is a nonexpansive function $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Z}$ required to map $a$ into $b$. The approximations instead live in a different category $\mathbb{A}$. Recall that the approximation $f_{\#}^{a}$ is of type $\mathcal{P}\left([Y]^{a}\right) \rightarrow \mathcal{P}\left([Z]^{b}\right)$. Since the domain and codomain are again dependent on maps $a$ and $b$, we still employ elements of $\mathbb{M}^{Y}$ as objects, but arrows are functions between powersets.

Definition 4.1 (concrete and abstract categories). We define two categories, the concrete category $\mathbb{C}$ and the abstract category $\mathbb{A}$ and a functor $\#: \mathbb{C} \rightarrow \mathbb{A}$. The concrete category $\mathbb{C}$
has as objects maps $a \in \mathbb{M}^{Y}$ where $Y$ is a (possibly infinite) set. Given $a \in \mathbb{M}^{Y}, b \in \mathbb{M}^{Z}$ an arrow $f: a \rightarrow b$ is a non-expansive function $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Z}$, such that $f(a)=b$.

The abstract category $\mathbb{A}$ has again maps $a \in \mathbb{M}^{Y}$ as objects. Given $a \in \mathbb{M}^{Y}, b \in \mathbb{M}^{Z}$ an arrow $f: a \rightarrow b$ is a monotone (wrt. inclusion) function $f: \mathcal{P}\left([Y]^{a}\right) \rightarrow \mathcal{P}\left([Z]^{b}\right)$. Arrow composition and identities are the obvious ones.

The approximation maps $\#: \mathbb{C} \rightarrow \mathbb{A}$ and $\#^{\delta}: \mathbb{C} \rightarrow \mathbb{A}($ for $\delta \sqsupset 0)$ are defined as follows: for an object $a \in \mathbb{M}^{Y}$, we let $\#(a)=\#^{\delta}(a)=a$ and, given an arrow $f: a \rightarrow b$, we let $\#^{\delta}(f)=f_{\#}^{a, \delta}$ and $\#(f)=\bigcup_{\delta \sqsupset 0} \#^{\delta}(f)=f_{\#}^{\delta}$.

Note that formal arrows are dashed $(--\rightarrow)$, while the underlying functions are represented by standard arrows $(\rightarrow)$.
Lemma 4.2 (well-definedness). The categories $\mathbb{C}$ and $\mathbb{A}$ are well-defined and the $\#^{\delta}$ are lax functors, i.e., identities are preserved and $\#^{\delta}(f) \circ \#^{\delta}(g) \subseteq \#^{\delta}(f \circ g)$ for composable arrows $f, g$ in $\mathbb{C}$.
Proof.
(1) $\mathbb{C}$ is a well-defined category: Given arrows $f: a \rightarrow b$ and $g: b \rightarrow c$ then $g \circ f$ is nonexpansive (since non-expansiveness is preserved by composition) and $(g \circ f)(a)=$ $g(b)=c$, thus $g \circ f: a \longrightarrow c$. Associativity holds and the identities are the units of composition as for standard function composition.
(2) $\mathbb{A}$ is a well-defined category: Given arrows $f: a \rightarrow b$ and $g: b \rightarrow c$ then $g \circ f$ is monotone (since monotonicity is preserved by composition) and hence $g \circ f: a \rightarrow c$.

Again associativity and the fact that the identities are units is immediate.
(3) $\#^{\delta}: \mathbb{C} \rightarrow \mathbb{A}$ is a lax functor: we first check that identities are preserved. Let $U \subseteq[Y]^{a}$, then

$$
\begin{aligned}
\#^{\delta}\left(i d_{a}\right)(U) & =\left(i d_{a}\right)_{\#}^{a, \delta}(U) \\
& =\left\{y \in[Y]^{i d_{a}(a)} \mid i d_{a}(a)(y) \ominus i d_{a}\left(a \ominus \delta_{U}\right)(y) \sqsupseteq \delta\right\} \\
& =\left\{y \in[Y]^{a} \mid a(y) \ominus\left(a \ominus \delta_{U}\right)(y) \sqsupseteq \delta\right\} \\
& =U=i d_{a}(U)=i d_{\#^{\delta}(a)}(U) .
\end{aligned}
$$

where in the second last line we use the fact that $U \subseteq[Y]^{a}$.
Let $a \in \mathbb{M}^{Y}, b \in \mathbb{M}^{Z}, c \in \mathbb{M}^{V}, f: a \rightarrow b, g: b \longrightarrow c$ be arrows in $\mathbb{C}$ and $Y^{\prime} \subseteq[Y]^{a}$. Then

$$
\begin{aligned}
\left(\#^{\delta}(g) \circ \#^{\delta}(f)\right)\left(Y^{\prime}\right) & =g_{\#}^{b, \delta}\left(f_{\#}^{a, \delta}\left(Y^{\prime}\right)\right) \\
& =\left(\gamma^{c, \delta} \circ g \circ \alpha^{b, \delta} \circ \gamma^{b, \delta} \circ f \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right) \\
& \subseteq\left(\gamma^{g(f(a)), \delta} \circ g \circ f \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right) \\
& =(g \circ f)_{\#}^{a, \delta}\left(Y^{\prime}\right) \\
& =\#^{\delta}(g \circ f)\left(Y^{\prime}\right)
\end{aligned}
$$

The inequality holds since for $c \in \mathbb{M}^{Z}$ :

$$
\alpha^{b, \delta}\left(\gamma^{b, \delta}(c)\right)=\alpha^{b, \delta}(\{y \in Y \mid b(y) \ominus c(y) \sqsupseteq \delta\})=b \ominus \delta_{\{y \in Y \mid b(y) \ominus c(y) \sqsupseteq \delta\}} \sqsupseteq c .
$$

Then the inequality follows from the antitonicity of $\gamma^{c, \delta}$. (Remember that we are working with a contra-variant Galois connection.)

Note that while \# clearly also preserves identities, the question whether it is a lax functor (or even a proper functor) is currently open. It is however the union of lax functors.

Hence in the following it will be convenient to restrict to the subcategory of $\mathbb{C}$ where arrows are reindexings and to subcategories of $\mathbb{C}, \mathbb{A}$ with maps on finite sets.

Definition 4.3 (reindexing subcategory). We denote by $\mathbb{C}^{*}$ the lluf ${ }^{1}$ sub-category of $\mathbb{C}$ where arrows are reindexing, i.e., given objects $a \in \mathbb{M}^{Y}, b \in \mathbb{M}^{Z}$ we consider only arrows $f: a \rightarrow b$ such that $f=g^{*}$ for some $g: Z \rightarrow Y$ (hence, in particular, $b=g^{*}(a)=a \circ g$ ).

We prove the following auxiliary lemma that basically shows that reindexings are preserved by $\alpha, \gamma$ :
Lemma 4.4. Given $a \in \mathbb{M}^{Y}, g: Z \rightarrow Y$ and $0 \sqsubset \delta \in \mathbb{M}$, then we have
(1) $\alpha^{a \circ g, \delta} \circ g^{-1}=g^{*} \circ \alpha^{a, \delta}$
(2) $\gamma^{a \circ g, \delta} \circ g^{*}=g^{-1} \circ \gamma^{a, \delta}$

This implies that for two $\mathbb{C}$-arrows $f: a \rightarrow b, h: b \rightarrow c$, it holds that $\#(h \circ f)=\#(h) \circ \#(f)$ whenever $f$ or $h$ is a reindexing, i.e., is contained in $\mathbb{C}^{*}$. Hence $\#: \mathbb{C}^{*} \rightarrow \mathbb{A}$ is a functor. Proof.
(1) Let $Y^{\prime} \subseteq[Y]^{a}$. Then

$$
\begin{aligned}
g^{*}\left(\alpha^{a, \delta}\left(Y^{\prime}\right)\right) & =g^{*}\left(a \ominus \delta_{Y^{\prime}}\right)=\left(a \ominus \delta_{Y^{\prime}}\right) \circ g=a \circ g \ominus \delta_{Y^{\prime}} \circ g \\
& =a \circ g \ominus \delta_{g^{-1}\left(Y^{\prime}\right)}=\alpha^{a \circ g, \delta}\left(g^{-1}\left(Y^{\prime}\right)\right)
\end{aligned}
$$

where we use that $\left(\delta_{Y^{\prime}} \circ g\right)(z)=\delta$ if $g(z) \in Y^{\prime}$, equivalent to $z \in g^{-1}\left(Y^{\prime}\right)$, and 0 otherwise. Hence $\delta_{Y^{\prime}} \circ g=\delta_{g^{-1}\left(Y^{\prime}\right)}$.
(2) Let $b \in \mathbb{M}^{Y}$ with $a \ominus \delta \sqsubseteq b \sqsubseteq a$. Then

$$
\begin{aligned}
\gamma^{a \circ g, \delta} \circ g^{*}(b) & =\{z \in Z \mid a(g(z)) \ominus b(g(z)) \sqsupseteq \delta\}=\left\{z \in Z \mid g(z) \in \gamma^{a, \delta}(b)\right\} \\
& =g^{-1}\left(\gamma^{a, \delta}(b)\right)
\end{aligned}
$$

It is left to show that $\#(h \circ f)=\#(h) \circ \#(f)$ whenever $f$ or $h$ is a reindexing. Recall that on reindexings it holds that $\#\left(g^{*}\right)=g^{-1}$.

Let $a \in \mathbb{M}^{Y}, b \in \mathbb{M}^{Z}, c \in \mathbb{M}^{W}$ and assume first that $f$ is a reindexing, i.e., $f=g^{*}$ for some $g: Z \rightarrow Y$. Let $Y^{\prime} \subseteq[Y]^{a}$, then

$$
\begin{align*}
\#(h \circ f) & =(h \circ f)_{\#}^{a}=\bigcup_{\delta \sqsupset 0}\left(\gamma^{h(f(a)), \delta} \circ h \circ f \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right) \\
& =\bigcup_{\delta \sqsupset 0}\left(\gamma^{h(f(a)), \delta} \circ h \circ g^{*} \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right) \\
& =\bigcup_{\delta \sqsupset 0}\left(\gamma^{h(f(a)), \delta} \circ h \circ \alpha^{a \circ g, \delta}\right)\left(g^{-1}\left(Y^{\prime}\right)\right)  \tag{1}\\
& =\bigcup_{\delta \sqsupset 0}\left(\gamma^{h(f(a)), \delta} \circ h \circ \alpha^{f(a), \delta}\right)\left(\#\left(g^{*}\right)\left(Y^{\prime}\right)\right) \\
& =(\#(h) \circ \#(f))\left(Y^{\prime}\right)
\end{align*}
$$

[^0]Now we assume that $h$ is a reindexing, i.e., $h=g^{*}$ for some $g: W \rightarrow Z$. Let again $Y^{\prime} \subseteq[Y]^{a}$, then:

$$
\begin{align*}
\#(h \circ f) & =(h \circ f)_{\#}^{a}=\bigcup_{\delta \sqsupset 0}\left(\gamma^{h(f(a)), \delta} \circ h \circ f \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right) \\
& =\bigcup_{\delta \sqsupset 0}\left(\gamma^{f(a) \circ g, \delta} \circ g^{*} \circ f \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right) \\
& =\bigcup_{\delta \sqsupset 0} g^{-1}\left(\left(\gamma^{f(a), \delta} \circ f \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right)\right)  \tag{2}\\
& =g^{-1}\left(\bigcup_{\delta \sqsupset 0}\left(\gamma^{f(a), \delta} \circ f \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right)\right) \\
& =\#\left(g^{*}\right)\left(\bigcup_{\delta \sqsupset 0}\left(\gamma^{f(a), \delta} \circ f \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right)\right) \\
& =(\#(h) \circ \#(f))\left(Y^{\prime}\right)
\end{align*}
$$

$$
=g^{-1}\left(\bigcup_{\delta \sqsupset 0}\left(\gamma^{f(a), \delta} \circ f \circ \alpha^{a, \delta}\right)\left(Y^{\prime}\right)\right) \quad \text { [preimage preserves union] }
$$

Definition 4.5 (finite subcategories). We denote by $\mathbb{C}_{f}, \mathbb{A}_{f}$ the full sub-categories of $\mathbb{C}, \mathbb{A}$ where objects are of the kind $a \in \mathbb{M}^{Y}$ for a finite set $Y$.

Lemma 4.6. The approximation map $\#: \mathbb{C} \rightarrow \mathbb{A}$ restricts to $\#: \mathbb{C}_{f} \rightarrow \mathbb{A}_{f}$, which is a (proper) functor.

Proof. Clearly the restriction to categories based on finite sets is well-defined.
We show that $\#$ is a (proper) functor. Let $a \in \mathbb{M}^{Y}, b \in \mathbb{M}^{Z}, c \in \mathbb{M}^{V}, f: a \rightarrow b$, $g: b \rightarrow c$ and $Y^{\prime} \subseteq[Y]^{a}$. Then

$$
\#(g \circ f)=(g \circ f)_{\#}^{a}=g_{\#}^{f(a)} \circ f_{\#}^{a}=g_{\#}^{b} \circ f_{\#}^{a}=\#(g) \circ \#(f),
$$

The second inequality above is a consequence of the compositionality result in [6, Proposition D.3]. This requires finiteness of the sets $Y, Z, V$.

The rest follows from Lemma 4.2.

We will later show in Theorem 7.4 that \# also respects the monoidal operation $\otimes$ (disjoint union of functions). Using this knowledge, we can now apply the framework to an example.

Example 4.7. We revisit Example 3.5 and observe that the function $\mathcal{T}$, whose least fixpoint is termination probability, can be be decomposed as follows:

$$
\mathcal{T}=\left(\eta^{*} \circ \tilde{\mathcal{D}}\right) \otimes c_{k}
$$

For the Markov chain in Figure 3 these parameters instantiate as follows (cf. Table 1):

- $c_{k}:[0,1]^{\emptyset} \rightarrow[0,1]^{T}, k: T \rightarrow[0,1]$ with $k(u)=1$
- $\tilde{\mathcal{D}}:[0,1]^{S} \rightarrow[0,1]^{D}$, where $D$ is a suitable finite subset of $\mathcal{D}(S)$ (to ensure compositionality).
- $\eta^{*}:[0,1]^{D} \rightarrow[0,1]^{S \backslash T}$ where $\eta: S \backslash T \rightarrow D$ with $\eta(s)=p_{s}$, where $p_{x}(y)=p_{x}(u)=$ $1 / 2, p_{y}(z)=p_{z}(y)=1$ and all other values are 0 . The set $D$ has to contain at least $p_{x}, p_{y}, p_{z}$.

Now given the greatest fixpoint $a=\nu \mathcal{T}$, we can compute $\mathcal{T}_{\#}^{a}$ compositionally. We instantiate the results of this section and consider $\mathcal{T}: a \rightarrow a$ as an arrow living in $\mathbb{C}_{f}$. The subfunctions of $\mathcal{T}$ given above can also be considered as arrows in $\mathbb{C}_{f}$ for appropriate domains and codomains, which we refrain from spelling out explicitly.

Now:

$$
\mathcal{T}_{\#}^{a}=\# \mathcal{T}=\#\left(\left(\eta^{*} \circ \tilde{\mathcal{D}}\right) \otimes c_{k}\right)=\#\left(\eta^{*}\right) \circ \#(\tilde{\mathcal{D}}) \otimes \#\left(c_{k}\right)
$$

This knowledge enables us to obtain an approximation $\mathcal{T}_{\#}^{a}$ compositionally out of the approximations of the subfunctions.

## 5. Predicate Liftings

In this section we discuss how predicate liftings [26, 29] can be integrated into our theory. In this context the idea is to view a map in $\mathbb{M}^{Y}$ as a predicate over $Y$ with values in $\mathbb{M}$ (e.g., if $\mathbb{M}=\{0,1\}$ we obtain Boolean predicates). Then, given a functor $F$, a predicate lifting transforms a predicate over $Y$ (a map in $\mathbb{M}^{Y}$ ), to a predicate over $F Y$ (a map in $\mathbb{M}^{F Y}$ ). Predicate liftings have been studied for arbitrary quantales, for instance, in [10] and one can show that every complete MV-algebra is a quantale with respect to $\oplus$ and the inverse of the natural order. The result can be easily derived from [18]. Below we provide an explicit proof.
5.1. Quantales. Recall that a quantale is a complete lattice with an associative operator that distributes over arbitrary joins. It is called unital if its associative operator has a neutral element (unit).

Lemma 5.1 (complete MV -algebras are quantales). Let $\mathbb{M}$ be a complete MV-algebra. Then $(\mathbb{M}, \oplus, \supseteq)$ is a unital commutative quantale.

Proof. We know $\mathbb{M}$ is a complete lattice. Binary meets are given by

$$
\begin{equation*}
x \sqcap y=\overline{\bar{x} \oplus \bar{y} \oplus \bar{y}} . \tag{5.1}
\end{equation*}
$$

Moreover $\oplus$ is associative and commutative, with 0 as neutral element.
It remains to show that $\oplus$ distributes with respect to $\square$ (note that $\square$ is the join for the reverse order), i.e., that for all $X \subseteq \mathbb{M}$ and $a \in \mathbb{M}$, it holds

$$
a \oplus \bigcap X=\bigcap\{a \oplus x \mid x \in X\}
$$

Clearly, since $\Pi X \leq x$ for all $x \in X$ and $\oplus$ is monotone, we have $a \oplus\rceil X \sqsubseteq \Pi\{a \oplus x \mid$ $x \in X\}$. In order to show that $a \oplus \Pi X$ is the greatest lower bound, let $z$ be another lower bound for $\{a \oplus x \mid x \in X\}$, i.e., $z \sqsubseteq a \oplus x$ for all $x \in X$. Then observe that for $x \in X$, using (5.1), we get

$$
x \sqsupseteq x \sqcap \bar{a}=\overline{\overline{(x \oplus a)} \oplus a} \sqsupseteq \overline{\bar{z} \oplus a}=z \ominus a
$$

Therefore $\Pi X \sqsupseteq z \ominus a$ and thus

$$
a \oplus\rceil X \sqsupseteq a \oplus(z \ominus a) \sqsupseteq z
$$

as desired.
5.2. Predicate Liftings and their Properties. We now define predicate liftings and then first characterise which predicate liftings are non-expansive and second, derive their approximations. We will address both these issues in this section and then use predicate liftings to define behavioural metrics in Section 6.

The fact that there are some functors $F$, for which $F Y$ is infinite, even if $Y$ is finite, is the reason why the categories $\mathbb{C}$ and $\mathbb{A}$ also include infinite sets. However note that the resulting fixpoint function will be always defined for finite sets, although intermediate functions might not conform to this.
Definition 5.2 (predicate lifting). Given a functor $F:$ Set $\rightarrow$ Set, a predicate lifting is a family of functions $\tilde{F}_{Y}: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{F Y}$ (where $Y$ is a set), such that for all $g: Z \rightarrow Y$, $a: Y \rightarrow \mathbb{M}$ it holds that $(F g)^{*}\left(\tilde{F}_{Y}(a)\right)=\tilde{F}_{Z}\left(g^{*}(a)\right)$.

That is, predicate liftings must commute with reindexings. The index $Y$ will be omitted if clear from the context. Such predicate liftings are in one-to-one correspondence to so called evaluation maps ev: $F \mathbb{M} \rightarrow \mathbb{M}$. (This follows from the Yoneda lemma, see e.g. [24].) Given $e v$, we define the corresponding lifting to be $\tilde{F}(a)=e v \circ F a: F Y \rightarrow \mathbb{M}$, where $a: Y \rightarrow \mathbb{M}$.

In the sequel we will only consider well-behaved liftings [4, 10], i.e., we require that (i) $\tilde{F}$ is monotone; (ii) $\tilde{F}\left(0_{Y}\right)=0_{F Y}$ where 0 is the constant 0 -function; (iii) $\tilde{F}(a \oplus b) \sqsubseteq \tilde{F}(a) \oplus \tilde{F}(b)$ for $a, b: Y \rightarrow \mathbb{M}$; (iv) $F$ preserves weak pullbacks.

We aim to have not only monotone, but non-expansive liftings.
Lemma 5.3 (non-expansive predicate lifting). Let $e v: F \mathbb{M} \rightarrow \mathbb{M}$ be an evaluation map and assume that its corresponding lifting $\tilde{F}: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{F Y}$ is well-behaved. Then $\tilde{F}$ is non-expansive iff for all $\delta \in \mathbb{M}$ it holds that $\tilde{F} \delta_{Y} \sqsubseteq \delta_{F Y}$.
Proof. The proof is inspired by [33, Lemma 3.9] and uses the fact that a monotone function $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Z}$ is non-expansive iff $f(a \oplus \delta) \sqsubseteq f(a) \oplus \delta$ for all $a, \delta$.
$(\Rightarrow)$ Fix a set $Y$ and assume that $\tilde{F}: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{F Y}$ is non-expansive. Then

$$
\tilde{F}(\delta)=\tilde{F}(0 \oplus \delta) \sqsubseteq \tilde{F}(0) \oplus \delta \sqsubseteq 0 \oplus \delta=\delta
$$

$(\Leftarrow)$ Now assume that $\tilde{F}(\delta) \sqsubseteq \delta$. Then, using the lemma referenced above,

$$
\tilde{F}(a \oplus \delta) \sqsubseteq \tilde{F}(a) \oplus \tilde{F}(\delta) \sqsubseteq \tilde{F}(a) \oplus \delta
$$

Above we write $\delta$ for both $\delta_{Y}, \delta_{F Y}$ and both deductions rely on the fact that $\tilde{F}$ is well-behaved.

Example 5.4 (Finitely supported distributions). We consider the (finitely supported) distribution functor $\mathcal{D}$ that maps a set $X$ to all maps $p: X \rightarrow[0,1]$ that have finite support and satisfy $\sum_{x \in X} p(x)=1$. (Here $\mathbb{M}=[0,1]$.) A possible evaluation map is ev: $\mathcal{D}[0,1] \rightarrow[0,1]$ defined by $e v(p)=\sum_{r \in[0,1]} r \cdot p(r)$, where $p$ is a distribution on $[0,1]$ (expectation). It is easy to see that $\tilde{\mathcal{D}}$ is well-behaved and non-expansive. The latter follows from $\tilde{\mathcal{D}}\left(\delta_{Y}\right)=\delta_{\mathcal{D} Y}$.
Example 5.5 (Finite powerset). Another example is given by the finite powerset functor $\mathcal{P}_{f}$. We are given the evaluation map $e v: \mathcal{P}_{f} \mathbb{M} \rightarrow \mathbb{M}$, defined for finite $S \subseteq \mathbb{M}$ as $e v(S)=$ $\max S$, where $\max \emptyset=0$. The lifting $\tilde{\mathcal{P}}_{f}$ is well-behaved (see [4]) and non-expansive. To show the latter, observe that $\tilde{\mathcal{P}}_{f}\left(\delta_{Y}\right)=\delta_{\mathcal{P}_{f}(Y) \backslash\{\emptyset\}} \sqsubseteq \delta_{\mathcal{P}_{f}(Y)}$.

Non-expansive predicate liftings can be seen as functors $\tilde{F}: \mathbb{C}^{*} \rightarrow \mathbb{C}^{*}$. To be more precise, $\tilde{F}$ maps an object $a \in \mathbb{M}^{Y}$ to $\tilde{F}(a) \in \mathbb{M}^{F Y}$ and an arrow $g^{*}: a \rightarrow a \circ g$, where $g: Z \rightarrow Y$, to $(F g)^{*}: \tilde{F} a \rightarrow \tilde{F}(a \circ g)$.
5.3. Approximations of Predicate Liftings. We now study approximations of predicate liftings, where we first investigate an auxiliary natural transformation. It involves the approximation functor $\#: \mathbb{C}^{*} \rightarrow \mathbb{A}$ (restricted to $\left.\mathbb{C}^{*}\right)$ and the predicate lifting $\tilde{F}: \mathbb{C}^{*} \rightarrow \mathbb{C}^{*}$ introduced before.
Proposition 5.6. Let $\tilde{F}$ be a (non-expansive) predicate lifting. There is a natural transformation $\beta: \# \Rightarrow \# \tilde{F}$ between functors $\#, \# \tilde{F}: \mathbb{C}^{*} \rightarrow \mathbb{A}$, whose components, for $a \in \mathbb{M}^{Y}$, are $\beta_{a}: a \rightarrow \tilde{F}(a)$ in $\mathbb{A}$, defined by $\beta_{a}(U)=\tilde{F}_{\#}^{a}(U)$ for $U \subseteq[Y]^{a}$.

That is, the following diagrams commute for every $g: Z \rightarrow Y$ (the diagram on the left indicates the formal arrows, while the one on the right reports the underlying functions).

$$
\begin{array}{ccc}
\#(a) & \#\left(g^{*}\right) & \\
\beta_{a} & & \#(a \circ g) \\
\vdots(\tilde{F} a) & \#\left(\tilde{F}\left(g^{*}\right)\right) & \vdots \\
\hdashline-\cdots & \#(\tilde{F}(a \circ g))
\end{array}
$$

$$
\begin{array}{ccc}
\mathcal{P}\left([Y]^{a}\right) & g^{-1} & \mathcal{P}\left([Z]^{a \circ g}\right) \\
\left.\tilde{F}_{\#}^{a} \downarrow^{( }\right) & & \|_{\#}^{\tilde{F}_{\#}^{a \circ g}} \\
\mathcal{P}\left([F Y]^{\tilde{F}(a)}\right) & \xrightarrow{(F g)^{-1}} & \mathcal{P}\left([F Z]^{\tilde{F}(a \circ g)}\right)
\end{array}
$$

Proof. We first define a natural transformation $\eta: \operatorname{Id}_{\mathbb{C}^{*}} \Rightarrow \tilde{F}$ (between the identity functor and $\tilde{F}$ ) with components $\eta_{a}: a \rightarrow \tilde{F}(a)$ (for $a \in \mathbb{M}^{Y}$ ) by defining $\eta_{a}(b)=\tilde{F}(b)$ for $b \in \mathbb{M}^{Y}$. The $\eta_{a}$ are non-expansive by assumption. In addition, $\eta$ is natural due to the definition of a predicate lifting, i.e., $(F g)^{*} \circ \tilde{F}=\tilde{F} \circ g^{*}$ for $g: Z \rightarrow Y$.

Now we apply \# and use the fact that \# is functorial, even for the full category $\mathbb{C}$, whenever one of the two arrows to which \# is applied is a reindexing (see Lemma 4.4). Furthermore we observe that $\beta=\#(\eta)$. This immediately gives commutativity of the diagram on the left. (The diagram on the right just displays the underlying functions.)

For using the above result we next characterize $\tilde{F}_{\#}^{d}\left(Y^{\prime}\right)$. We rely on the fact that $d$ can be decomposed into $d=\pi_{1} \circ \bar{d}$, where the projection $\pi_{1}$ is independent of $d$ and $\bar{d}$ is dependent on $Y^{\prime}$, and exploit the natural transformation in Proposition 5.6.

Proposition 5.7. We fix $Y^{\prime} \subseteq Y=X \times X$ and $d: Y \rightarrow \mathbb{M}$. Let $\pi_{1}: \mathbb{M} \times\{0,1\} \rightarrow \mathbb{M}$ be the projection to the first component and $\bar{d}: Y \rightarrow \mathbb{M} \times\{0,1\}$ with $\bar{d}(y)=\left(d(y), \chi_{Y^{\prime}}(y)\right)$ where $\chi_{Y^{\prime}}: Y \rightarrow\{0,1\}$ is the characteristic function of $Y^{\prime}$. Then $\tilde{F}_{\#}^{d}\left(Y^{\prime}\right)=(F \bar{d})^{-1}\left(\tilde{F}_{\#}^{\pi_{1}}((\mathbb{M} \backslash\{0\}) \times\right.$ \{1\})).
Proof. Let $d \in \mathbb{M}^{Y}$ and $Y^{\prime} \subseteq[Y]^{d}$. Note that $\bar{d}^{-1}((\mathbb{M} \backslash\{0\}) \times\{1\})=Y^{\prime}$ and $d=\pi_{1} \circ \bar{d}$, thus by Proposition 5.6:

$$
\begin{aligned}
\tilde{F}_{\#}^{d}\left(Y^{\prime}\right) & =\tilde{F}_{\#}^{\pi_{1} \circ \bar{d}}\left(\bar{d}^{-1}((\mathbb{M} \backslash\{0\}) \times\{1\})\right) \\
& =(F \bar{d})^{-1}\left(\tilde{F}_{\#}^{\pi_{1}}((\mathbb{M} \backslash\{0\}) \times\{1\})\right)
\end{aligned}
$$

Here $\tilde{F}_{\#}^{\pi_{1}}((\mathbb{M} \backslash\{0\}) \times\{1\}) \subseteq F(\mathbb{M} \times\{0,1\})$ is independent of $d$ and has to be determined only once for every predicate lifting $\tilde{F}$. We will show how this set looks like for our example functors. We first consider the distribution functor.
Lemma 5.8. Consider the lifting of the distribution functor presented in Example 5.4 and let $Z=[0,1] \times\{0,1\}$. Then we have

$$
\tilde{\mathcal{D}}_{\#}^{\pi_{1}}((0,1] \times\{1\})=\{p \in \mathcal{D} Z \mid \operatorname{supp}(p) \in(0,1] \times\{1\}\} .
$$

Proof. Let $\delta>0$. We define

$$
\tilde{\pi}_{1}^{\delta}:=\alpha^{\pi_{1}, \delta}((0,1] \times\{1\})
$$

where $\tilde{\pi}_{1}^{\delta}(x, 0)=x, \tilde{\pi}_{1}^{\delta}(x, 1)=x \ominus \delta$ for $x \in[0,1]$. Note that $[\mathcal{D} Z]^{\tilde{\mathcal{D}} \pi_{1}}=\{p \in D Z \mid \exists(x, b) \in$ $\operatorname{supp}(p)$ with $x \geq 0\}$. Now

$$
\begin{aligned}
\tilde{\mathcal{D}}_{\#}^{\pi_{1}, \delta}((0,1] \times\{1\})= & \left\{p \in[\mathcal{D} Z]^{\tilde{\mathcal{D}} \pi_{1}} \mid \tilde{\mathcal{D}} \pi_{1}(p) \ominus \tilde{\mathcal{D}}\left(\tilde{\pi}_{1}^{\delta}\right)(p) \geq \delta\right\} \\
= & \left\{p \in[\mathcal{D} Z]^{\tilde{\mathcal{D}} \pi_{1}} \mid\left(\sum_{x \in[0,1]} x \cdot p(x, 0) \oplus \sum_{x \in[0,1]} x \cdot p(x, 1)\right)\right. \\
& \left.\ominus\left(\sum_{x \in[0,1]} x \cdot p(x, 0) \oplus \sum_{x \in[0,1]}(x \ominus \delta) \cdot p(x, 1)\right) \geq \delta\right\} \\
= & \left\{p \in[\mathcal{D} Z]^{\tilde{\mathcal{D}} \pi_{1}} \mid \sum_{x \in[0, \delta)} x \cdot p(x, 1)+\sum_{x \in[\delta, 1]} \delta \cdot p(x, 1) \geq \delta\right\} \\
= & \left\{p \in[\mathcal{D} Z]^{\tilde{\mathcal{D}} \pi_{1}} \mid \operatorname{supp}(p) \in[\delta, 1] \times\{1\}\right\} .
\end{aligned}
$$

Where the second last equality uses the fact that $x \ominus(x \ominus \delta)=\delta$ if $x \geq \delta$ and $x$ otherwise.
Now, we obtain

$$
\tilde{\mathcal{D}}_{\#}^{\pi_{1}}((0,1] \times\{1\})=\bigcup_{\delta \sqsupset 0} \tilde{\mathcal{D}}_{\#}^{\pi_{1}, \delta}((0,1] \times\{1\})=\{p \in \mathcal{D} Z \mid \operatorname{supp}(p) \in(0,1] \times\{1\}\} .
$$

This means intuitively that a decrease or "slack" can exactly be propagated for elements whose probabilities are strictly larger than 0 .

We now turn to the powerset functor.
Lemma 5.9. Consider the lifting of the finite powerset functor from Example 5.5 and let $Z=\mathbb{M} \times\{0,1\}$. Then we have

$$
\left(\tilde{\mathcal{P}}_{f}\right)_{\#}^{\pi_{1}}((\mathbb{M} \backslash\{0\}) \times\{1\})=\left\{S \in\left[\mathcal{P}_{f} Z\right]^{\tilde{\mathcal{P}}_{f} \pi_{1}} \mid \exists(s, 1) \in S \forall\left(s^{\prime}, 0\right) \in S: s \sqsupset s^{\prime}\right\} .
$$

Proof. Let $\delta \sqsupset 0$ and define $\tilde{\pi}_{1}^{\delta}$ as in the proof of Lemma 5.8. Then

$$
\begin{aligned}
\left(\tilde{\mathcal{P}}_{f}\right)_{\#}^{\pi_{1}, \delta}((\mathbb{M} \backslash\{0\}) \times\{1\}) & =\left\{S \in\left[\mathcal{P}_{f} Z\right]^{\tilde{\mathcal{P}}_{f} \pi_{1}} \mid \tilde{\mathcal{P}}_{f} \pi_{1}(S) \ominus \tilde{\mathcal{P}}_{f}\left(\tilde{\pi}_{1}^{\delta}\right)(S) \sqsupseteq \delta\right\} \\
& =\left\{S \in\left[\mathcal{P}_{f} Z\right]^{\tilde{\mathcal{P}}_{f} \pi_{1}} \mid \max _{(s, b) \in S} s \ominus\left(\max _{(s, b) \in S} s \ominus b \cdot \delta\right) \sqsupseteq \delta\right\} \\
& =\left\{S \in\left[\mathcal{P}_{f} Z\right]^{\tilde{\mathcal{P}}_{f} \pi_{1}} \mid \exists(s, 1) \in S \forall\left(s^{\prime}, 0\right) \in S: s \ominus \delta \sqsupseteq s^{\prime}\right\}
\end{aligned}
$$

For the last step we note that this condition ensures that the second maximum equates to $\max _{(s, b) \in S} s \ominus \delta$ which is required for the inequality to hold. Now, we obtain

$$
\begin{aligned}
\left(\tilde{\mathcal{P}}_{f}\right)_{\#}^{\pi_{1}}((\mathbb{M} \backslash\{0\}) \times\{1\}) & =\bigcup_{\delta \sqsupset 0} \tilde{F}_{\#}^{\pi_{1}, \delta}((\mathbb{M} \backslash\{0\}) \times\{1\}) \\
& =\left\{S \in\left[\mathcal{P}_{f} Z\right]^{\tilde{\mathcal{P}}_{f} \pi_{1}} \mid \exists(s, 1) \in S \forall\left(s^{\prime}, 0\right) \in S: s \sqsupset s^{\prime}\right\} .
\end{aligned}
$$

The idea is that the maximum of a set $S$ decreases if we decrease at least one its values and all values which are not decreased are strictly smaller.
Remark 5.10. Note that \# is a functor on the subcategory $\mathbb{C}_{f}$ (see Lemma 4.6), while some liftings (e.g., the one for the distribution functor) involve infinite sets, for which we would lose compositionality. In this case, given a finite set $Y$, we will actually focus on a finite subset $D \subseteq F Y$. (This is possible since we consider coalgebras with finite state space and impose some extra conditions, the details are given below.) Then we consider $\tilde{F}_{Y}: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{F Y}$ and $e: D \hookrightarrow F Y$ (the embedding of $D$ into $F Y$ ). We set $f=e^{*} \circ \tilde{F}_{Y}: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{D}$. Given $a: Y \rightarrow \mathbb{M}$, we view $f$ as an arrow $a \rightarrow \tilde{F}_{Y}(a) \circ e$ in $\mathbb{C}$. The approximation adapts to the "reduced" lifting, which can be seen as follows (cf. Lemma 4.4, which shows that \# preserves composition if one of the arrows is a reindexing):

$$
f_{\#}^{a}=\#(f)=\#\left(e^{*} \circ \tilde{F}_{Y}\right)=\#\left(e^{*}\right) \circ \#\left(\tilde{F}_{Y}\right)=e^{-1} \circ \#\left(\tilde{F}_{Y}\right)=\#\left(\tilde{F}_{Y}\right) \cap D .
$$

## 6. Wasserstein Lifting and Behavioural Metrics

In this section we show how the framework for fixpoint checking described before can be used to deal with coalgebraic behavioural metrics.

We build on [4], where an approach is proposed for canonically defining a behavioural pseudo-metric for coalgebras of a functor $F:$ Set $\rightarrow$ Set, that is, for functions of the form $\xi: X \rightarrow F X$ where $X$ is a set. Intuitively $\xi$ specifies a transition system whose branching type is given by $F$. Given such a coalgebra $\xi$, the idea is to endow $X$ with a pseudo-metric $d_{\xi}: X \times X \rightarrow \mathbb{M}$ defined as the least fixpoint of the map $d \mapsto d^{F} \circ(\xi \times \xi)$ where ${ }_{-}^{F}$ lifts a metric $d: X \times X \rightarrow \mathbb{M}$ to a metric $d^{F}: F X \times F X \rightarrow \mathbb{M}$. Here we focus on the so-called Wasserstein lifting and show how approximations of the functions involved in the definition of the pseudo-metric can be determined.
6.1. Wasserstein Lifting. Hereafter, $F$ denotes a fixed endo-functor on Set and $\xi: X \rightarrow$ $F X$ is a coalgebra over a finite set $X$. We also fix a well-behaved non-expansive predicate lifting $\tilde{F}$.

In order to define a Wasserstein lifting, a first ingredient is that of a coupling. Given $t_{1}, t_{2} \in F X$ a coupling of $t_{1}$ and $t_{2}$ is an element $t \in F(X \times X)$, such that $F \pi_{i}(t)=t_{i}$ for $i=1,2$, where $\pi_{i}: X \times X \rightarrow X$ are the projections. We write $\Gamma\left(t_{1}, t_{2}\right)$ for the set of all such couplings.

Definition 6.1 (Wasserstein lifting). The Wasserstein lifting ${ }_{-}^{F}: \mathbb{M}^{X \times X} \rightarrow \mathbb{M}^{F X \times F X}$ is defined for $d: X \times X \rightarrow \mathbb{M}$ and $t_{1}, t_{2} \in F X$ as

$$
d^{F}\left(t_{1}, t_{2}\right)=\inf _{t \in \Gamma\left(t_{1}, t_{2}\right)} \tilde{F} d(t)
$$

For more intuition on the Wasserstein lifting see the motivations in Section 2. Note that a coupling correspond to a transport plan. It can be shown that for well-behaved $\tilde{F}$, the lifting preserves pseudo-metrics (see [4, 10]).

In order to make the theory for fixpoint checks effective we will need to restrict to a subclass of liftings.
Definition 6.2 (finitely coupled lifting). We call a lifting $\tilde{F}$ finitely coupled if for all $X$ and $t_{1}, t_{2} \in F X$ there exists a finite $\Gamma^{\prime}\left(t_{1}, t_{2}\right) \subseteq \Gamma\left(t_{1}, t_{2}\right)$, which can be computed given $t_{1}, t_{2}$, such that $\inf _{t \in \Gamma\left(t_{1}, t_{2}\right)} \tilde{F} d(t)=\min _{t \in \Gamma^{\prime}\left(t_{1}, t_{2}\right)} \tilde{F} d(t)$ for all $d$.

The lifting in Example 5.5 (for the finite powerset functor) is obviously finitely coupled. For the lifting $\tilde{\mathcal{D}}$ from Example 5.4 we note that the set of couplings $t \in \Gamma\left(t_{1}, t_{2}\right)$ forms a polytope with a finite number of vertices, which can be effectively computed and $\Gamma^{\prime}\left(t_{1}, t_{2}\right)$ consists of these vertices. The infimum (minimum) is obtained at one of these vertices [1, Remark 4.5].
6.2. Decomposing the Behavioural Metrics Function. As mentioned above, for a coalgebra $\xi: X \rightarrow F X$ the behavioural pseudo-metric $d: X \times X \rightarrow \mathbb{M}$ is the least fixpoint of the behavioural metrics function $\mathcal{W}=\left({ }_{-}^{F}\right) \circ(\xi \times \xi)$ where $\left({ }_{-}^{F}\right)$ is the Wasserstein lifting.

The Wasserstein lifting can be decomposed as ${ }_{-}^{F}=\min _{u} \circ \tilde{F}$ where $\tilde{F}: \mathbb{M}^{X \times X} \rightarrow$ $\mathbb{M}^{F(X \times X)}$ is the predicate lifting - which we require to be non-expansive (cf. Lemma 5.3) - and $\min _{u}$ is the minimum over the coupling function $u: F(X \times X) \rightarrow F X \times F X$ defined as $u(t)=\left(F \pi_{1}(t), F \pi_{2}(t)\right)$, which means that $\min _{u}: \mathbb{M}^{F(X \times X)} \rightarrow \mathbb{M}^{F X \times F X}$ (see Table 1).

Therefore the behavioural metrics function can be expressed as

$$
\mathcal{W}=(\xi \times \xi)^{*} \circ \min _{u} \circ \tilde{F}
$$

Explicitly, for $d \in[0,1]^{X \times X}$ and $x, y \in X$,

$$
\begin{aligned}
\mathcal{W}(d)(x, y) & =\min _{u} \circ \tilde{F}(d)(\xi(x), \xi(y))=\min _{u(t)=(\xi(x), \xi(y))} \tilde{F}(d)(t)= \\
& =\min _{t \in \Gamma(\xi(x), \xi(y))} \tilde{F} d(t)=d^{F}(\xi(x), \xi(y))
\end{aligned}
$$

Note that the fixpoint equation for behavioural metrics is sometimes equipped with a discount factor that reduces the effect of deviations in the (far) future and ensures that the fixpoint is unique by contractivity of the function. Here we focus on the undiscounted case where the fixpoint equation may have several solutions.

Example 6.3. Note that we can recover the motivating example from Section 2 by setting $\mathbb{M}=[0,1]$ and using the functor $F X=\Lambda \times \mathcal{D}(X)$, where $\Lambda$ is a fixed set of labels. We observe that couplings of $\left(a_{1}, p_{1}\right),\left(a_{2}, p_{2}\right) \in F X$ only exist if $a_{1}=a_{2}$ and - if they do not exist - the Wasserstein distance is the infimum of an empty set, hence 1. If $a_{1}=a_{2}$, couplings correspond to the usual Wasserstein couplings of $p_{1}, p_{2}$ and the least fixpoint of $\mathcal{W}$ equals the behavioural metrics, as explained in Section 2.
6.3. Approximation of the Behavioural Metrics Function. The decomposition of the behavioural metrics function $\mathcal{W}$ above, can be used to derive its $d$-approximation.

Proposition 6.4. Assume that $\tilde{F}$ is non-expansive and finitely coupled and fix a coalgebra $\xi: X \rightarrow F X$. Let $Y=X \times X$, where $X$ is finite. For $d \in \mathbb{M}^{Y}$ and $Y^{\prime} \subseteq[Y]^{d}$ we have

$$
\mathcal{W}_{\#}^{d}\left(Y^{\prime}\right)=\left\{(x, y) \in[Y]^{d} \mid \exists t \in \tilde{F}_{\#}^{d}\left(Y^{\prime}\right), u(t)=(\xi(x), \xi(y)), \tilde{F} d(t)=\mathcal{W}(d)(x, y)\right\}
$$

Proof. We first remark that since $X$ is finite and $\tilde{F}$ is finitely coupled it is sufficient to restrict to finite subsets of $F(X \times X)$ and $F X \times F X$ (cf. Remark 5.10). In other words $\mathcal{W}$ can be obtained as composition of functions living in $\mathbb{C}_{f}$, hence $\#$ is a proper functor and approximations can obtained compositionally. We exploit this fact in the following.

More concretely, we restrict $u$ to $u: V \rightarrow W$, where $V \subseteq F(X \times X), W \subseteq F X \times F X$. We require that $W$ contains all pairs $(\xi(x), \xi(y))$ for $x, y \in X$ and $V=\bigcup_{\left(t_{1}, t_{2}\right) \in W} \Gamma^{\prime}\left(t_{1}, t_{2}\right)$. Hence both $V, W$ are finite.

The function $\tilde{F}$ is restricted accordingly to a map $\mathbb{M}^{Y} \rightarrow \mathbb{M}^{V}$ as explained in Remark 5.10.

For $d \in \mathbb{M}^{Y}$ and $Y^{\prime} \subseteq[Y]^{d}$ we have, using the fact that $\mathcal{W}=(\xi \times \xi)^{*} \circ \min _{u} \circ \tilde{F}$, compositionality and the approximations listed in Table 2:

$$
\begin{aligned}
\mathcal{W}_{\#}^{d}\left(Y^{\prime}\right)= & \left\{(x, y) \in[Y]^{d} \mid(\xi(x), \xi(y)) \in\left(\min _{u}\right)_{\#}^{\tilde{F}(d)}\left(\tilde{F}_{\#}^{d}\left(Y^{\prime}\right) \cap V\right)\right\} \\
= & \left\{(x, y) \in[Y]^{d} \mid \arg \min _{t \in u^{-1}(\xi(x), \xi(y))} \tilde{F}(d)(t) \cap \tilde{F}_{\#}^{d}\left(Y^{\prime}\right) \cap V \neq \emptyset\right\} \\
= & \left\{(x, y) \in[Y]^{d} \mid \exists t \in \tilde{F}_{\#}^{d}\left(Y^{\prime}\right) \cap V, u(t)=(\xi(x), \xi(y)),\right. \\
& \left.\tilde{F} d(t)=\min _{t^{\prime} \in V} \tilde{F} d\left(t^{\prime}\right)\right\} \\
= & \left\{(x, y) \in[Y]^{d} \mid \exists t \in \tilde{F}_{\#}^{d}\left(Y^{\prime}\right) \cap V, u(t)=(\xi(x), \xi(y)),\right. \\
& \left.\tilde{F} d(t)=\min _{t^{\prime} \in \Gamma^{\prime}(\xi(x), \xi(y))} \tilde{F} d\left(t^{\prime}\right)\right\} \\
= & \left\{(x, y) \in[Y]^{d} \mid \exists t \in \tilde{F}_{\#}^{d}\left(Y^{\prime}\right), u(t)=(\xi(x), \xi(y)), \tilde{F} d(t)=\min _{t^{\prime} \in \Gamma^{\prime}(\xi(x), \xi(y))} \tilde{F} d\left(t^{\prime}\right)\right\} \\
= & \left\{(x, y) \in[Y]^{d} \mid \exists t \in \tilde{F}_{\#}^{d}\left(Y^{\prime}\right), u(t)=(\xi(x), \xi(y)), \tilde{F} d(t)=\mathcal{W}(d)(x, y)\right\}
\end{aligned}
$$

The first equality is based on Remark 5.10 and uses the fact that the approximation for the restricted $\tilde{F}$ maps $Y^{\prime}$ to $\tilde{F}_{\#}^{d}\left(Y^{\prime}\right) \cap V$.

The second-last inequality also needs explanation, in particular, we have to show that the set on the second-last line is included in the one on the previous line, although we omitted the intersection with $V$.

Hence let $(x, y) \in[Y]^{d}$ such that there exists $s \in \tilde{F}_{\#}^{d}\left(Y^{\prime}\right), u(s)=(\xi(x), \xi(y))$ and $\tilde{F} d(s)=\min _{t^{\prime} \in \Gamma^{\prime}(\xi(x), \xi(y))} \tilde{F} d\left(t^{\prime}\right)$. We have to show that there exists a $t$ with the same properties that is also included in $V$.

The fact that $s \in \tilde{F}_{\#}^{d}\left(Y^{\prime}\right)$ implies that $\tilde{F}(d)(s) \ominus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(s) \sqsupseteq \delta$ for small enough $\delta$, using the fact that $\tilde{F}_{\#}^{d}=\gamma^{\tilde{F}(d), \delta} \circ \tilde{F} \circ \alpha^{d, \delta}$ (for an appropriate value $\delta$ ).

Since the minimum of the Wasserstein lifting is always reached in $\Gamma^{\prime}(\xi(x), \xi(y))$, independently of the argument, there exists $t \in \Gamma^{\prime}(\xi(x), \xi(y)) \subseteq V$ (hence $u(t)=(\xi(x), \xi(y))$ ), such that

$$
\tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(t)=\min _{t^{\prime} \in \Gamma(\xi(x), \xi(y))} \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)\left(t^{\prime}\right)
$$

This implies that $\tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(t) \sqsubseteq \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(s)$ (since $s \in \Gamma(\xi(x), \xi(y))$ ). From the assumption $\tilde{F} d(s)=\min _{t^{\prime} \in \Gamma^{\prime}(\xi(x), \xi(y))} \tilde{F} d\left(t^{\prime}\right)$ we obtain $\tilde{F} d(s) \sqsubseteq \tilde{F} d(t)$. Hence, using the fact that
$\ominus$ is monotone in the first and antitone in the second argument, we have:

$$
\delta \sqsubseteq \tilde{F}(d)(s) \ominus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(s) \sqsubseteq \tilde{F}(d)(t) \ominus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(t) \sqsubseteq \delta .
$$

The last inequality follows from non-expansiveness. Hence

$$
\tilde{F}(d)(s) \ominus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(s)=\tilde{F}(d)(t) \ominus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(t)=\delta,
$$

which in particular implies that $t \in \tilde{F}_{\#}^{d}\left(Y^{\prime}\right)$.
In order to conclude we have to show that $\tilde{F} d(t)=\min _{t^{\prime} \in \Gamma^{\prime}(\xi(x), \xi(y))} \tilde{F} d\left(t^{\prime}\right)$. We first observe that in an MV-chain $\mathbb{M}$, whenever $x \sqsubseteq y$ (for $x, y \in \mathbb{M}$ ) we can infer that ( $y \ominus$ $x) \oplus x=y$ (this follows for instance from Lemma 2.4(6) in [6] and duality). The inequality $\tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right) \sqsubseteq \tilde{F}(d)$ holds by monotonicity and we can conclude that

$$
\begin{aligned}
\tilde{F} d(t) & \left.=\left(\tilde{F}(d)(t) \ominus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(t)\right) \oplus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(t)\right) \\
& \left.=\left(\tilde{F}(d)(s) \ominus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(s)\right) \oplus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(t)\right) \\
& \left.\sqsubseteq\left(\tilde{F}(d)(s) \ominus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(s)\right) \oplus \tilde{F}\left(d \ominus \delta_{Y^{\prime}}\right)(s)\right)=\tilde{F}(d)(s) .
\end{aligned}
$$

The other inequality $\tilde{F} d(s) \sqsubseteq \tilde{F} d(t)$ holds anyway and hence $\tilde{F} d(t)=\tilde{F} d(s)$. This finally implies, as desired, that

$$
\tilde{F} d(t)=\tilde{F} d(s)=\min _{t^{\prime} \in \Gamma^{\prime}(\xi(x), \xi(y))} \tilde{F} d\left(t^{\prime}\right)
$$

Intuitively the statement of Proposition 6.4 means that the optimal coupling which reaches the minimum in the Wasserstein lifting must be contained in $\tilde{F}_{\#}^{d}\left(Y^{\prime}\right)$. Note that $\tilde{F}_{\#}^{d}$ has already been characterized earlier in Section 5.3.
6.4. Examples on Behavioural Metrics. We now consider two examples involving coalgebras over the distribution and powerset functor.
Example 6.5. We start with an example involving probabilistic transition systems. Let $X=\{x, y, z\}$. We define a coalgebra $\xi: X \rightarrow \Lambda \times \mathcal{D} X$ via $\xi(x)(x)=1, \xi(y)(y)=\xi(y)(z)=$ $1 / 2$ and $\xi(z)(z)=1$. All other distances are 0 . Assume that all states have the same label (not indicated).


Since all states have the same label, they are in fact probabilistically bisimilar and hence have behavioural distance 0 , given by the least fixpoint of $\mathcal{W}$. Now consider the pseudo-metric $d: X \times X \rightarrow[0,1]$ with $d(x, y)=d(x, z)=d(y, z)=1$ and 0 for the reflexive pairs. This is a also a fixpoint of $\mathcal{W}(d=\mathcal{W}(d))$, but it clearly over-estimates the true behavioural metric.

We can detect this by computing the greatest fixpoint of $\mathcal{W}_{\#}^{d}$, which is

$$
Y^{\prime}=\{(x, y),(y, x),(x, z),(z, x),(y, z),(z, y)\} \neq \emptyset,
$$

containing the pairs that still have slack in $d$ and whose distances can be reduced. We explain why $Y^{\prime}=\mathcal{W}_{\#}^{d}\left(Y^{\prime}\right)$ by focussing on the example pair $(x, y)$ and check that $(x, y) \in \mathcal{W}_{\#}^{d}\left(Y^{\prime}\right)$. For this we use the definition of $\mathcal{W}_{\#}^{d}$ given in Proposition 6.4.

A valid coupling $t \in \mathcal{D}(X \times X)$ of $\xi(x), \xi(y)$ is given by $t(x, y)=t(x, z)=1 / 2$. It satisfies $u(t)=(\xi(x), \xi(y))$ and is optimal since it is the only one. We obtain the Wasserstein lifting

$$
\mathcal{W}(d)(x, y)=d^{\mathcal{D}}(\xi(x), \xi(y))=\min _{t^{\prime} \in \Gamma(\xi(x), \xi(y))} \tilde{\mathcal{D}} d\left(t^{\prime}\right)=\tilde{\mathcal{D}} d(t)=1 / 2 \cdot 1+1 / 2 \cdot 1=1 .
$$

It is left to show that $t \in \tilde{\mathcal{D}}_{\#}^{d}\left(Y^{\prime}\right)$, for which we use the characterization in Proposition 5.7. We have $\bar{d}(x, y)=\bar{d}(x, z)=(1,1)$ where

$$
\mathcal{D} \bar{d}(t)=p \in \mathcal{D} Z \text { with } p(1,1)=1 / 2+1 / 2=1 .
$$

From Lemma 5.8 we obtain $p \in \tilde{\mathcal{D}}_{\#}^{\pi_{1}}((0,1] \times\{1\})$ and by definition $t \in(\mathcal{D} \bar{d})^{-1}(p)$, i.e. $t \in \tilde{\mathcal{D}}_{\#}^{d}\left(Y^{\prime}\right)$. So we can conclude that $(x, y) \in \mathcal{W}_{\#}^{d}\left(Y^{\prime}\right)$.
Example 6.6. We now consider an example in the non-deterministic setting. Let $X=$ $\{x, y\}$ and a coalgebra $\xi: X \rightarrow \mathcal{P} X$ be given by $\xi(x)=\{x, y\}, \xi(y)=\{x\}$.


Since all states have successors, they are in fact bisimilar and hence have behavioural distance 0 , given by the least fixpoint of $\mathcal{W}$. Now consider the pseudo-metric $d: X \times X \rightarrow$ $[0,1]$ with $d(x, y)=d(y, x)=1 / 2$ and 0 for the reflexive pairs. This is a also a fixpoint of $\mathcal{W}(d=\mathcal{W}(d))$, but it clearly over-estimates the true behavioural metric.

We can detect this by computing the greatest fixpoint of $\mathcal{W}_{\#}^{d}$, which is

$$
Y^{\prime}=\{(x, y),(y, x)\} \neq \emptyset
$$

containing the pairs that still have slack in $d$ and whose distances can be reduced. We explain why $Y^{\prime}=\mathcal{W}_{\#}^{d}\left(Y^{\prime}\right)$ by focussing on the example pair $(x, y)$ and check that $(x, y) \in \mathcal{W}_{\#}^{d}\left(Y^{\prime}\right)$. For this we use the definition of $\mathcal{W}_{\#}^{d}$ given in Proposition 6.4.

A valid (and optimal) coupling $t \in \mathcal{P}(X \times X)$ of $\xi(x), \xi(y)$ is given by $t=\{(x, x),(y, x)\}$. It satisfies $u(t)=(\xi(x), \xi(y))$. We obtain the Wasserstein lifting

$$
\mathcal{W}(d)(x, y)=d^{\mathcal{P}}(\xi(x), \xi(y))=\min _{t \in \Gamma(\xi(x), \xi(y))} \tilde{\mathcal{P}} d(t)=\tilde{\mathcal{P}} d(t)=\max \{0,1 / 2\}=1 / 2
$$

It is left to show that $t \in \tilde{\mathcal{P}}_{\#}^{d}\left(Y^{\prime}\right)$, for which we use the characterization in Proposition 5.7. We have $\bar{d}(x, x)=\bar{d}(y, y)=(0,0), \bar{d}(x, y)=\bar{d}(y, x)=(1 / 2,1)$ and

$$
\mathcal{P} \bar{d}(t)=S=\{(0,0),(1 / 2,1)\}
$$

From Lemma 5.9 we obtain $S \in \tilde{\mathcal{P}}_{\#}^{\pi_{1}}((\mathbb{M} \backslash\{0\}) \times\{1\})$ and $t \in(\mathcal{P} \bar{d})^{-1}(S)$, i.e. $t \in \tilde{\mathcal{P}}_{\#}^{d}\left(Y^{\prime}\right)$. So we can conclude that $(x, y) \in \mathcal{W}_{\#}^{d}\left(Y^{\prime}\right)$.

## 7. GS-Monoidality

We will now show that the categories $\mathbb{C}_{f}$ and $\mathbb{A}_{f}$ can be turned into gs-monoidal categories, making \# a gs-monoidal functor. This will give us a way to assemble functions and their approximations compositionally and this will form the basis for the tool. We first define gs-monoidal categories in detail (cf. [22, Definition 7] and [20]).

Definition 7.1 (gs-monoidal categories). A strict gs-monoidal category is a strict symmetric monoidal category, where $\otimes$ denotes the tensor and $e$ its unit and symmetries are given by $\rho_{a, b}: a \otimes b \rightarrow b \otimes a$. For every object $a$ there exist morphisms $\nabla_{a}: a \rightarrow a \times a$ (duplicator) and $!_{a}: a \rightarrow e$ (discharger) satisfying the axioms given below. (See also their visualizations as string diagrams in Figure 4.)
(1) functoriality of tensor:

- $\left(g \otimes g^{\prime}\right) \circ\left(f \otimes f^{\prime}\right)=(g \circ f) \otimes\left(g^{\prime} \circ f^{\prime}\right)$
- $i d_{a \otimes b}=i d_{a} \otimes i d_{b}$
(2) monoidality:
- $(f \otimes g) \otimes h=f \otimes(g \otimes h)$
- $f \otimes i d_{e}=f=i d_{e} \otimes f$
(3) naturality:
- $\left(f^{\prime} \otimes f\right) \circ \rho_{a, a^{\prime}}=\rho_{b, b^{\prime}} \circ\left(f \otimes f^{\prime}\right)$
(4) symmetry:
- $\rho_{e, e}=i d_{e}$
- $\rho_{b, a} \circ \rho_{a, b}=i d_{a \otimes b}$
- $\rho_{a \otimes b, c}=\left(\rho_{a, c} \otimes i d_{b}\right) \circ\left(i d_{a} \otimes \rho_{b, c}\right)$
(5) gs-monoidality:
- $!_{e}=\nabla_{e}=i d_{e}$
- coherence axioms:

$$
\begin{aligned}
& -\left(i d_{a} \otimes \nabla_{a}\right) \circ \nabla_{a}=\left(\nabla_{a} \otimes i d_{a}\right) \circ \nabla_{a} \\
& -i d_{a}=\left(i d_{a} \otimes!_{a}\right) \circ \nabla_{a} \\
& -\rho_{a, a} \circ \nabla_{a}=\nabla_{a}
\end{aligned}
$$

- monoidality axioms:

$$
\begin{aligned}
& -!_{a \otimes b}=!_{a} \otimes!b \\
& -\left(i d_{a} \otimes \rho_{a, b} \otimes i d_{b}\right) \circ\left(\nabla_{a} \otimes \nabla_{b}\right)=\nabla_{a \otimes b} \\
& \left.\quad \text { (or, equivalently, } \nabla_{a} \otimes \nabla_{b}=\left(i d_{a} \otimes \rho_{b, a} \otimes i d_{b}\right) \circ \nabla_{a \otimes b}\right)
\end{aligned}
$$

A functor $F: \mathbb{C} \rightarrow \mathbb{D}$ is gs-monoidal if the following holds:
(1) $\mathbb{C}$ and $\mathbb{D}$ are gs-monoidal categories
(2) monoidality:

- $F(e)=e^{\prime}$
- $F(a \otimes b)=F(a) \otimes^{\prime} F(b)$
(3) symmetry:
- $F\left(\rho_{a, b}\right)=\rho_{F(a), F(b)}^{\prime}$
(4) gs-monoidality:
- $F\left(!_{a}\right)=!_{F(a)}^{\prime}$
- $F\left(\nabla_{a}\right)=\nabla_{F(a)}^{\prime}$
where the primed operators are from category $\mathbb{D}$, the others from $\mathbb{C}$.
In fact, in order to obtain strict gs-monoidal categories with disjoint union, we will work with the skeleton categories where every finite set $Y$ is represented by an isomorphic copy $\{1, \ldots,|Y|\}$. This enables us to make disjoint union strict, i.e., associativity holds on the nose and not just up to isomorphism. In particular for finite sets $Y, Z$, we define disjoint union as $Y+Z=\{1, \ldots,|Y|,|Y|+1, \ldots,|Y|+|Z|\}$.

$$
\stackrel{1}{f} e|=\stackrel{\downarrow}{f}=| e \begin{gathered}
\stackrel{1}{f} \\
\underset{T}{f}
\end{gathered}
$$



Figure 4. String diagrams for the axioms of gs-monoidal categories.
Theorem 7.2 ( $\mathbb{C}_{f}$ is gs-monoidal). The category $\mathbb{C}_{f}$ with the following operators is gsmonoidal:
(1) The tensor $\otimes$ on objects $a \in \mathbb{M}^{Y}$ and $b \in \mathbb{M}^{Z}$ is defined as

$$
a \otimes b=a+b \in \mathbb{M}^{Y+Z}
$$

where for $k \in Y+Z$ we have $(a+b)(k)=a(k)$ if $k \leq|Y|$ and $(a+b)(k)=b(k-|Y|)$ if $|Y|<k \leq|Y|+|Z|$.

On arrows $f: a \longrightarrow b$ and $g: a^{\prime} \longrightarrow b^{\prime}$ (with $a^{\prime} \in \mathbb{M}^{Y^{\prime}}, b^{\prime} \in \mathbb{M}^{Z^{\prime}}$ ) tensor is given by

$$
f \otimes g: \mathbb{M}^{Y+Y^{\prime}} \rightarrow \mathbb{M}^{Z+Z^{\prime}}, \quad(f \otimes g)(u)=f\left(\overleftarrow{u}_{Y}\right)+g\left(\vec{u}_{Y}\right)
$$

for $u \in \mathbb{M}^{Y+Y^{\prime}}$ where $\bar{u}_{Y} \in \mathbb{M}^{Y}$ and $\vec{u}_{Y} \in \mathbb{M}^{Y^{\prime}}$, defined as $\bar{u}_{Y}(k)=u(k)(1 \leq k \leq$ $|Y|)$ and $\vec{u}_{Y}(k)=u(|Y|+k)\left(1 \leq k \leq\left|Y^{\prime}\right|\right)$.
(2) The symmetry $\rho_{a, b}: a \otimes b \rightarrow b \otimes a$ for $a \in \mathbb{M}^{Y}, b \in \mathbb{M}^{Z}$ is defined for $u \in \mathbb{M}^{Y+Z}$ as

$$
\rho_{a, b}(u)=\vec{u}_{Y}+\grave{u}_{Y} .
$$

(3) The unit $e$ is the unique mapping $e: \emptyset \rightarrow \mathbb{M}$.
(4) The duplicator $\nabla_{a}: a \rightarrow a \otimes a$ for $a \in \mathbb{M}^{Y}$ is defined for $u \in \mathbb{M}^{Y}$ as

$$
\nabla_{a}(u)=u+u .
$$

(5) The discharger $!_{a}: a \rightarrow e$ for $a \in \mathbb{M}^{Y}$ is defined for $u \in \mathbb{M}^{Y}$ as $!_{a}(u)=e$.

Proof. In the following let $a \in \mathbb{M}^{Y}, a^{\prime} \in \mathbb{M}^{Y^{\prime}}, b \in \mathbb{M}^{Z}, b^{\prime} \in \mathbb{M}^{Z^{\prime}}, c \in \mathbb{M}^{W}, c^{\prime} \in \mathbb{M}^{W^{\prime}}$ be objects in $\mathbb{C}_{f}$.

We know that $\mathbb{C}_{f}$ is a well-defined category from Lemma 4.2. We also note that disjoint unions of non-expansive functions are non-expansive. Moreover, given $f: a \rightarrow b$ and $g: a^{\prime} \rightarrow b^{\prime}$, that

$$
\begin{aligned}
& (f \otimes g)\left(a \otimes a^{\prime}\right)=(f \otimes g)\left(a+a^{\prime}\right) \\
& =f\left(\overleftarrow{\left.\left(a+a^{\prime}\right)_{Y}\right)+g\left({\overrightarrow{\left(a+a^{\prime}\right.}}_{Y}\right)=f(a)+g\left(a^{\prime}\right)}\right. \\
& =b+b^{\prime}=b \otimes b^{\prime}
\end{aligned}
$$

Thus $f \otimes g$ is a well-defined arrow $a \otimes a^{\prime} \rightarrow b \otimes b^{\prime}$.
We next verify all the axioms of gs-monoidal categories given in Definition 7.1. In general the calculations are straightforward, but they are provided here for completeness.

In the sequel we will often use the fact that $\bar{u}_{Y}+\vec{u}_{Y}=u$ whenever $Y$ is a subset of the domain of $u$.
(1) functoriality of tensor:

- $i d_{a \otimes b}=i d_{a} \otimes i d_{b}$

Let $u \in \mathbb{M}^{Y+Z}$. Then

$$
\left(i d_{a} \otimes i d_{b}\right)(u)=i d_{a}\left(\overleftarrow{u}_{Y}\right)+i d_{b}\left(\vec{u}_{Y}\right)=\overleftarrow{u}_{Y}+\vec{u}_{Y}=u=i d_{a \otimes b}(u)
$$

- $\left(g \otimes g^{\prime}\right) \circ\left(f \otimes f^{\prime}\right)=(g \circ f) \otimes\left(g^{\prime} \circ f^{\prime}\right)$

This is required to hold when both sides are defined. Hence let $f: a \rightarrow b$, $g: b \longrightarrow c, f^{\prime}: a^{\prime} \longrightarrow b^{\prime}, g^{\prime}: b^{\prime} \rightarrow c^{\prime}$ and $u \in \mathbb{M}^{Y+Y^{\prime}}$. We obtain:

$$
\begin{aligned}
& \left(g \otimes g^{\prime}\right) \circ\left(f \otimes f^{\prime}\right)(u)=\left(g \otimes g^{\prime}\right)\left(f\left(\bar{u}_{Y}\right)+f^{\prime}\left(\vec{u}_{Y}\right)\right) \\
& =g\left(f\left(\bar{u}_{Y}\right)\right)+g^{\prime}\left(f^{\prime}\left(\vec{u}_{Y}\right)\right)=\left((g \circ f) \otimes\left(g^{\prime} \circ f^{\prime}\right)\right)\left(\bar{u}_{Y}+\vec{u}_{Y}\right) \\
& =\left((g \circ f) \otimes\left(g^{\prime} \circ f^{\prime}\right)\right)(u)
\end{aligned}
$$

(2) monoidality:

- $f \otimes i d_{e}=f=i d_{e} \otimes f$

Let $f: a \longrightarrow b$ and $u \in \mathbb{M}^{Y}$. It holds that:

$$
\begin{aligned}
& \left(f \otimes i d_{e}\right)(u)=\left(f \otimes i d_{e}\right)(u+e)=f(u)+i d_{e}(e) \\
& =f(u)+e=f(u)=e+f(u) \\
& =i d_{e}(e)+f(u)=\left(i d_{e} \otimes f\right)(e+u)=\left(i d_{e} \otimes f\right)(u)
\end{aligned}
$$

- $(f \otimes g) \otimes h=f \otimes(g \otimes h)$

Let $f: a \longrightarrow a^{\prime}, g: b \longrightarrow b^{\prime}$ and $h: c \longrightarrow c^{\prime}$ and $u \in \mathbb{M}^{Y+Z+W}$, then

$$
\begin{aligned}
& ((f \otimes g) \otimes h)(u)=(f \otimes g)\left(\bar{u}_{Y+Z}\right)+h\left(\vec{u}_{Y+Z}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =f\left(\overleftarrow{u}_{Y}\right)+\left(g\left({\left.\overleftarrow{\vec{u}_{Y}}\right)_{Z}}\right)+h\left({\left.\left.\left.\overrightarrow{\left(\overrightarrow{u_{Y}}\right.}\right)_{Z}\right)\right)}\right.\right. \\
& =f\left(\bar{u}_{Y}\right)+(g \otimes h)\left(\vec{u}_{Y}\right)=(f \otimes(g \otimes h))(u)
\end{aligned}
$$

where we use the fact that ${\left.\overrightarrow{\left(\bar{u}_{Y+Z}\right.}\right)_{Y}}={\left.\overleftarrow{\left(\vec{u}_{Y}\right.}\right)}_{Z}$.
(3) naturality:

- $\left(f^{\prime} \otimes f\right) \circ \rho_{a, a^{\prime}}=\rho_{b, b^{\prime}} \circ\left(f \otimes f^{\prime}\right)$

Let $f: a \longrightarrow b$ and $f^{\prime}: a^{\prime} \longrightarrow b^{\prime}$. Then for $u \in \mathbb{M}^{Y+Y^{\prime}}$

$$
\begin{aligned}
& \left(\rho_{b, b^{\prime}} \circ\left(f \otimes f^{\prime}\right)\right)(u)=\rho_{b, b^{\prime}}\left(f\left(\overleftarrow{u}_{Y}\right)+f^{\prime}\left(\vec{u}_{Y}\right)\right) \\
& =f^{\prime}\left(\vec{u}_{Y}\right)+f\left(\overleftarrow{u}_{Y}\right)=\left(f^{\prime} \otimes f\right)\left(\vec{u}_{Y}+\bar{u}_{Y}\right)=\left(\left(f^{\prime} \otimes f\right) \circ \rho_{a, a^{\prime}}\right)(u)
\end{aligned}
$$

(4) symmetry:

- $\rho_{e, e}=i d_{e}$

We note that $e$ is the unique function from $\emptyset$ to $\mathbb{M}$ and furthermore $e \otimes e=$ $e+e=e$. Then

$$
\rho_{e, e}(e)=\rho_{e, e}(e+e)=e+e=e=i d_{e}(e)
$$

- $\rho_{b, a} \circ \rho_{a, b}=i d_{a \otimes b}$

Let $u \in \mathbb{M}^{Y+Z}$, then:

$$
\left(\rho_{b, a} \circ \rho_{a, b}\right)(u)=\rho_{b, a}\left(\vec{u}_{Y}+\overleftarrow{u}_{Y}\right)=\bar{u}_{Y}+\vec{u}_{Y}=u=i d_{a \otimes b}(u)
$$

- $\rho_{a \otimes b, c}=\left(\rho_{a, c} \otimes i d_{b}\right) \circ\left(i d_{a} \otimes \rho_{b, c}\right)$

Let $u \in \mathbb{M}^{Y+Z+W}$, then:

$$
\begin{aligned}
& \left(\left(\rho_{a, c} \otimes i d_{b}\right) \circ\left(i d_{a} \otimes \rho_{b, c}\right)\right)(u) \\
& =\left(\rho_{a, c} \otimes i d_{b}\right)\left(i d_{a}\left(\overleftarrow{u}_{Y}\right)+\rho_{b, c}\left(\vec{u}_{Y}\right)\right) \\
& =\left(\rho_{a, c} \otimes i d_{b}\right)\left(\overleftarrow{u}_{Y}+{\left.\overrightarrow{\left(\vec{u}_{Y}\right.}\right)}_{Z}+{\left.\overleftarrow{\left(\vec{u}_{Y}\right.}\right)}_{Z}\right) \\
& \left.=\rho_{a, c}\left(\overleftarrow{u}_{Y}+\vec{u}_{Y+Z}\right)+i d_{b}\left(\overleftarrow{\overleftarrow{u}_{Y}}\right) Z\right)=\vec{u}_{Y+Z}+\bar{u}_{Y}+{\overrightarrow{\left(\overleftarrow{u}_{Y+Z}\right)_{Y}}} \\
& =\vec{u}_{Y+Z}+\overleftarrow{u}_{Y+Z}=\rho_{a \otimes b, c}(u)
\end{aligned}
$$

where we use the fact that ${\overrightarrow{(\vec{u}})^{\prime}}_{Z}=\vec{u}_{Y+Z}$ and $\overleftarrow{\left(\vec{u}_{Y}\right)}{ }_{Z}={\left.\overline{\left(\bar{u}_{Y+Z}\right.}\right)_{Y}}$.
(5) gs-monoidality:

- $!_{e}=\nabla_{e}=i d_{e}$

Since $e$ is the unique function of type $\emptyset \rightarrow \mathbb{M}$ and $e+e=e$, we obtain:

$$
!_{e}(e)=e=i d_{e}(e)=e=e+e=\nabla_{e}(e)
$$

- coherence axioms:

For $u \in \mathbb{M}^{Y}$, we note that $\overleftarrow{(u+u)}_{Y}={\overrightarrow{(u+u)_{Y}}}_{Y}=u$.
$-\left(i d_{a} \otimes \nabla_{a}\right) \circ \nabla_{a}=\left(\nabla_{a} \otimes i d_{a}\right) \circ \nabla_{a}$
Let $u \in \mathbb{M}^{Y}$, then:

$$
\begin{aligned}
&\left(\left(i d_{a} \otimes \nabla_{a}\right) \circ \nabla_{a}\right)(u)=\left(i d_{a} \otimes \nabla_{a}\right)(u+u) \\
&=i d_{a}(u)+\nabla_{a}(u)=u+u+u=\nabla_{a}(u)+i d_{a}(u) \\
&=\left(\nabla_{a} \otimes i d_{a}\right)(u+u)=\left(\nabla_{a} \otimes i d_{a}\right)\left(\nabla_{a}(u)\right) \\
&-i d_{a}=\left(i d_{a} \otimes!a\right) \circ \nabla_{a}
\end{aligned}
$$

Let $u \in \mathbb{M}^{Y}$, then:

$$
\begin{aligned}
& \left(\left(i d_{a} \otimes!a\right) \circ \nabla_{a}\right)(u)=\left(i d_{a} \otimes!_{a}\right)(u+u) \\
& =i d_{a}(u)+!_{a}(u)=i d_{a}(u)+e=i d_{a}(u)
\end{aligned}
$$

$$
-\rho_{a, a} \circ \nabla_{a}=\nabla_{a}
$$ Let $u \in \mathbb{M}^{Y}$, then:

$$
\left(\rho_{a, a} \circ \nabla_{a}\right)(u)=\rho_{a, a}(u+u)=u+u=\nabla_{a}(u)
$$

- monoidality axioms:
$-!_{a \otimes b}=!_{a} \otimes!_{b}$
Let $u \in \mathbb{M}^{Y+Z}$, then:

$$
!_{a \otimes b}(u)=e=e+e=!_{a}\left(\overleftarrow{u}_{Y}\right)+!_{b}\left(\vec{u}_{Y}\right)=\left(!_{a} \otimes!_{b}\right)(u)
$$

$-\nabla_{a} \otimes \nabla_{b}=\left(i d_{a} \otimes \rho_{b, a} \otimes i d_{b}\right) \circ \nabla_{a \otimes b}$
Let $u \in \mathbb{M}^{Y+Z}$, then:

$$
\begin{aligned}
& \left(i d_{a} \otimes \rho_{b, a} \otimes i d_{b}\right)\left(\nabla_{a \otimes b}(u)\right)=\left(i d_{a} \otimes \rho_{b, a} \otimes i d_{b}\right)(u+u) \\
& =\left(i d_{a} \otimes \rho_{b, a} \otimes i d_{b}\right)\left(\overleftarrow{u}_{Y}+\vec{u}_{Y}+\bar{u}_{Y}+\vec{u}_{Y}\right) \\
& =\overleftarrow{u}_{Y}+\overleftarrow{u}_{Y}+\vec{u}_{Y}+\vec{u}_{Y}=\nabla_{a}\left(\overleftarrow{u}_{Y}\right)+\nabla_{b}\left(\vec{u}_{Y}\right) \\
& =\left(\nabla_{a} \otimes \nabla_{b}\right)\left(\overleftarrow{u}_{Y}+\vec{u}_{Y}\right)=\left(\nabla_{a} \otimes \nabla_{b}\right)(u)
\end{aligned}
$$

We now turn to the abstract category $\mathbb{A}_{f}$. Note that here functions have as parameters sets of the form $U \subseteq[Y]^{a} \subseteq Y$. Hence, (the cardinality of) $Y$ cannot be determined directly from $U$ and we need extra care with the tensor.

Theorem 7.3 ( $\mathbb{A}_{f}$ is gs-monoidal). The category $\mathbb{A}_{f}$ with the following operators is gsmonoidal:
(1) The tensor $\otimes$ on objects $a \in \mathbb{M}^{Y}$ and $b \in \mathbb{M}^{Z}$ is again defined as $a \otimes b=a+b$.

On arrows $f: a \rightarrow b$ and $g: a^{\prime} \longrightarrow b^{\prime}\left(\right.$ where $a^{\prime} \in \mathbb{M}^{Y^{\prime}}, b^{\prime} \in \mathbb{M}^{Z^{\prime}}$ and $f: \mathcal{P}\left([Y]^{a}\right) \rightarrow$ $\mathcal{P}\left([Z]^{b^{\prime}}\right), g: \mathcal{P}\left(\left[Y^{\prime}\right]^{a^{\prime}}\right) \rightarrow \mathcal{P}\left(\left[Z^{\prime}\right]^{b^{\prime}}\right)$ are the underlying functions), the tensor is given by
$f \otimes g: \mathcal{P}\left(\left[Y+Y^{\prime}\right]^{a+a^{\prime}}\right) \rightarrow \mathcal{P}\left(\left[Z+Z^{\prime}\right]^{b+b^{\prime}}\right), \quad(f \otimes g)(U)=f\left(\grave{U}_{Y}\right) \cup_{Z} g\left(\vec{U}_{Y}\right)$
where $\overleftarrow{U}_{Y}=U \cap\{1, \ldots,|Y|\}$ and $\vec{U}_{Y}=\{k| | Y \mid+k \in U\}$. Furthermore:

$$
U \cup_{Y} V=U \cup\{|Y|+k \mid k \in V\} \quad(\text { where } U \subseteq Y)
$$

(2) The symmetry $\rho_{a, b}: a \otimes b \rightarrow b \otimes a$ for $a \in \mathbb{M}^{Y}, b \in \mathbb{M}^{Z}$ is defined for $U \subseteq[Y+Z]^{a+b}$ as

$$
\rho_{a, b}(U)=\vec{U}_{Y} \cup_{Z} \dot{U}_{Y} \subseteq[Z+Y]^{b+a}
$$

(3) The unit $e$ is again the unique mapping $e: \emptyset \rightarrow \mathbb{M}$.
(4) The duplicator $\nabla_{a}: a \rightarrow a \otimes a$ for $a \in \mathbb{M}^{Y}$ is defined for $U \subseteq[Y]^{a}$ as

$$
\nabla_{a}(U)=U \cup_{Y} U \subseteq[Y+Y]^{a+a} .
$$

(5) The discharger $!_{a}: a \rightarrow e$ for $a \in \mathbb{M}^{Y}$ is defined for $U \subseteq[Y]^{a}$ as $!_{a}(U)=\emptyset$.

Proof. Let $a \in \mathbb{M}^{Y}, a^{\prime} \in \mathbb{M}^{Y^{\prime}}, b \in \mathbb{M}^{Z}, b^{\prime} \in \mathbb{M}^{Z^{\prime}}, c \in \mathbb{M}^{W}, c^{\prime} \in \mathbb{M}^{W^{\prime}}$ be objects in $\mathbb{A}_{f}$.
We know that $\mathbb{A}_{f}$ is a well-defined category from Lemma 4.2. We note that, disjoint unions of monotone functions are monotone, making the tensor well-defined.

We now verify the axioms of gs-monoidal categories (see Definition 7.1). The calculations are mostly straightforward.

In the following we will often use the fact that $\overleftarrow{U}_{Y} \cup_{Y} \vec{U}_{Y}=U$ whenever $U \in \mathcal{P}\left([Z]^{b}\right)$ and $Y \subseteq Z$.
(1) functoriality of tensor:

- $i d_{a \otimes b}=i d_{a} \otimes i d_{b}$

Let $U \subseteq[Y+Z]^{a+b}$, then:

$$
\begin{aligned}
& \left(i d_{a} \otimes i d_{b}\right)(U)=\left(i d_{a} \otimes i d_{b}\right)\left(\overleftarrow{U}_{Y} \cup_{Y} \vec{U}_{Y}\right) \\
& =i d_{a}\left(\overleftarrow{U}_{Y}\right) \cup_{Y} i d_{b}\left(\vec{U}_{Y}\right)=\overleftarrow{U}_{Y} \cup_{Y} \vec{U}_{Y}=U=i d_{a \otimes b}(U)
\end{aligned}
$$

- $\left(g \otimes g^{\prime}\right) \circ\left(f \otimes f^{\prime}\right)=(g \circ f) \otimes\left(g^{\prime} \circ f^{\prime}\right)$

Let $f: a \rightarrow b, g: b \rightarrow c, f^{\prime}: a^{\prime} \rightarrow b^{\prime}, g^{\prime}: b^{\prime} \rightarrow c^{\prime}$ and $u \in \mathbb{M}^{Y+Y^{\prime}}$. We obtain:

$$
\begin{aligned}
& \left(\left(g \otimes g^{\prime}\right) \circ\left(f \otimes f^{\prime}\right)\right)(U)=\left(g \otimes g^{\prime}\right)\left(f\left(\overleftarrow{U}_{Y}\right) \cup_{Z} f^{\prime}\left(\vec{U}_{Y}\right)\right) \\
& =g\left(f\left(\overleftarrow{U}_{Y}\right)\right) \cup_{W} g^{\prime}\left(f^{\prime}\left(\vec{U}_{Y}\right)\right)=\left((g \circ f) \otimes\left(g^{\prime} \circ f^{\prime}\right)\right)\left(\overleftarrow{U}_{Y} \cup_{Y} \vec{U}_{Y}\right) \\
& =\left((g \circ f) \otimes\left(g^{\prime} \circ f^{\prime}\right)\right)(U)
\end{aligned}
$$

(2) monoidality:

- $f \otimes i d_{e}=f=i d_{e} \otimes f$

Let $f: a \rightarrow b$ and $U \subseteq[Y]^{a}$. It holds that:

$$
\begin{aligned}
& \left(f \otimes i d_{e}\right)(U)=f\left(\stackrel{U}{U}_{Y}\right) \cup_{Z} i d_{e}\left(\vec{U}_{Y}\right)=f(U) \cup_{Z} i d_{e}(\emptyset)=f(U) \cup_{Z} \emptyset \\
& =f(U)=\emptyset \cup_{\emptyset} f(U)=i d_{e}(\emptyset) \cup_{\emptyset} f(U)=i d_{e}\left(\overleftarrow{U}_{\emptyset}\right) \cup_{\emptyset} f\left(\vec{U}_{\emptyset}\right) \\
& =\left(i d_{e} \otimes f\right)(U)
\end{aligned}
$$

where we use $\overleftarrow{U}_{Y}=U$ and $\vec{U}_{Y}=\emptyset$, since $U \subseteq Y$, as well as $\overleftarrow{U}_{\emptyset}=\emptyset$ and $\vec{U}_{\emptyset}=U$.

- $(f \otimes g) \otimes h=f \otimes(g \otimes h)$

Let $f: a \longrightarrow a^{\prime}, g \in: b \longrightarrow b^{\prime}$ and $h: c \longrightarrow c^{\prime}$ and $U \subseteq[Y+Z+W]^{a+b+c}$. Then:

$$
\begin{aligned}
& ((f \otimes g) \otimes h)(U)=(f \otimes g)(\stackrel{\overleftarrow{U}}{Y+Z}) \cup_{Y^{\prime}+Z^{\prime}} h\left(\vec{U}_{Y+Z}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =\left(f\left(\overleftarrow{U}_{Y}\right) \cup_{Y^{\prime}} g\left(\overleftarrow{\left(\vec{U}_{Y}\right)_{Z}}\right)\right) \cup_{Y^{\prime}+Z^{\prime}} h\left(\vec{U}_{Y+Z}\right) \\
& =f\left(\overleftarrow{U}_{Y}\right) \cup_{Y^{\prime}}\left(g\left(\overleftarrow{\left(\vec{U}_{Y}\right)_{Z}}\right) \cup_{Z^{\prime}} h\left({\left.\overrightarrow{\left(\vec{U}_{Y}\right.}\right)_{Z}}\right)\right) \\
& =f\left(\stackrel{\leftarrow}{U}_{Y}\right) \cup_{Y^{\prime}}(g \otimes h)\left(\vec{U}_{Y}\right)=(f \otimes(g \otimes h))(U)
\end{aligned}
$$

where we use ${\overleftarrow{\left(\overleftarrow{U}_{Y+Z}\right)_{Y}}}_{Y}=\overleftarrow{U}_{Y},{\overrightarrow{\left(\overleftarrow{U}_{Y+Z}\right)_{Y}}}_{Y}={\overleftarrow{\left(\vec{U}_{Y}\right)}}_{Z}$ and $\vec{U}_{Y+Z}={\overrightarrow{(\vec{U}})_{Y}}_{Z}$
(3) naturality:

- $\left(f^{\prime} \otimes f\right) \circ \rho_{a, a^{\prime}}=\rho_{b, b^{\prime}} \circ\left(f \otimes f^{\prime}\right)$

Let $f: a \rightarrow b$ and $f^{\prime}: a^{\prime} \rightarrow b^{\prime}$. Then for $U \subseteq\left[Y+Y^{\prime}\right]^{a+a^{\prime}}$ it holds that:

$$
\begin{aligned}
& \left(\rho_{b, b^{\prime}} \circ\left(f \otimes f^{\prime}\right)\right)(U) \\
& =\rho_{b, b^{\prime}}\left(f\left(\stackrel{U}{U}_{Y}\right) \cup_{Z} f^{\prime}\left(\vec{U}_{Y}\right)\right) \\
& ={\overrightarrow{\left(f\left(\overleftarrow{U}_{Y}\right) \cup_{Z} f^{\prime}\left(\vec{U}_{Y}\right)\right)_{Z}} \cup_{Z^{\prime}}}^{\left(f\left(\overleftarrow{U}_{Y}\right) \cup_{Z} f^{\prime}\left(\vec{U}_{Y}\right)\right)_{Z}}
\end{aligned}
$$

$$
\begin{aligned}
& =f^{\prime}\left(\vec{U}_{Y}\right) \cup_{Z^{\prime}} f\left(\overleftarrow{U}_{Y}\right) \\
& =f^{\prime}\left(\left({\stackrel{\left(\vec{U}_{Y}\right.}{ } \cup_{Y^{\prime}} \stackrel{U}{U}_{Y}}_{Y_{Y^{\prime}}}\right) \cup_{Z^{\prime}} f\left(\left(\vec{U}_{Y} \cup_{Y^{\prime}} \stackrel{U}{U}_{Y}\right)_{Y^{\prime}}\right)\right. \\
& =\left(f^{\prime} \otimes f\right)\left(\vec{U}_{Y} \cup_{Y^{\prime}} \overleftarrow{U}_{Y}\right)=\left(f^{\prime} \otimes f\right)\left(\rho_{a, a^{\prime}}(U)\right)
\end{aligned}
$$

where we use ${\overleftarrow{\left(U \cup_{Y} V\right)}}_{Y}=U$ and ${\overrightarrow{\left(U \cup_{Y} V\right)_{Y}}}_{Y}=V$.
(4) symmetry:

- $\rho_{e, e}=i d_{e}$

Note that the only possibly argument is $\emptyset$ and hence:
$\rho_{e, e}(\emptyset)=\vec{\emptyset}_{\emptyset} \cup_{\emptyset} \overleftarrow{\emptyset}_{\emptyset}=\emptyset \cup_{\emptyset} \emptyset=\emptyset=i d_{e}(\emptyset)$

- $\rho_{b, a} \circ \rho_{a, b}=i d_{a \otimes b}$
- $\rho_{b, a} \circ \rho_{a, b}=i d_{a \otimes b}$

Let $U \subseteq[Y+Z]^{a+b}$, then:

$$
\begin{aligned}
& \left(\rho_{b, a} \circ \rho_{a, b}\right)(U)=\rho_{b, a}\left(\vec{U}_{Y} \cup_{Z} \stackrel{\leftarrow}{U}_{Y}\right) \\
& ={\overrightarrow{\left(\vec{U}_{Y} \cup_{Z} \stackrel{\leftarrow}{U}_{Y}\right)}}_{Z} \cup_{Y}{\overleftarrow{\left(\vec{U}_{Y} \cup_{Z} \stackrel{\leftarrow}{U}_{Y}\right)_{Z}}=\stackrel{\leftarrow}{U}_{Y} \cup_{Y} \vec{U}_{Y}=U} \\
& =i d_{a \otimes b}(U)
\end{aligned}
$$

- $\rho_{a \otimes b, c}=\left(\rho_{a, c} \otimes i d_{b}\right) \circ\left(i d_{a} \otimes \rho_{b, c}\right)$

Let $U \subseteq[Y+Z+W]^{a+b+c}$, then:

$$
\begin{aligned}
& \left(\left(\rho_{a, c} \otimes i d_{b}\right) \circ\left(i d_{a} \otimes \rho_{b, c}\right)\right)(U) \\
& =\left(\rho_{a, c} \otimes i d_{b}\right)\left(i d_{a}\left(\overleftarrow{U}_{Y}\right) \cup_{Y} \rho_{b, c}\left(\vec{U}_{Y}\right)\right) \\
& =\left(\rho_{a, c} \otimes i d_{b}\right)\left(\overleftarrow{U}_{Y} \cup_{Y}\left(\underset{\left(\vec{U}_{Y}\right)_{Z}}{ } \cup_{W} \overleftarrow{\left(\vec{U}_{Y}\right)_{Z}}\right)\right) \\
& =\left(\rho_{a, c} \otimes i d_{b}\right)\left(\overleftarrow{U}_{Y} \cup_{Y}\left(\vec{U}_{Y+Z} \cup_{W} \overleftarrow{\left(\vec{U}_{Y}\right)} Z\right)\right) \\
& =\left(\rho_{a, c} \otimes i d_{b}\right)\left(\left(\overleftarrow{U}_{Y} \cup_{Y} \vec{U}_{Y+Z}\right) \cup_{Y+W} \overleftarrow{\left.\left(\vec{U}_{Y}\right)_{Z}\right)}\right) \\
& =\rho_{a, c}\left(\overleftarrow{U}_{Y} \cup_{Y} \vec{U}_{Y+Z}\right) \cup_{W+Y} i d_{b}\left(\overleftarrow{\left(\vec{U}_{Y}\right)_{Z}}\right) \\
& =\left(\vec{U}_{Y+Z} \cup_{W} \overleftarrow{U}_{Y}\right) \cup_{W+Y} \overleftarrow{\left(\vec{U}_{Y}\right)_{Z}} \\
& =\vec{U}_{Y+Z} \cup_{W}\left(\overleftarrow{U}_{Y} \cup_{Y} \overleftarrow{\left.\left(\vec{U}_{Y}\right)_{Z}\right)}\right. \\
& =\vec{U}_{Y+Z} \cup_{W}\left({\overleftarrow{\left(\overleftarrow{U}_{Y+Z}\right)}}_{Y} \cup_{Y} \overline{\left(\overleftarrow{U}_{Y+Z}\right)_{Y}}\right) \\
& =\vec{U}_{Y+Z} \cup_{W} \overleftarrow{U}_{Y+Z}=\rho_{a \otimes b, c}(U)
\end{aligned}
$$

where we use ${\overrightarrow{(\vec{U}})_{Z}}_{Z}=\vec{U}_{Y+Z}, \dot{U}_{Y}={\overleftarrow{\left(\dot{U}_{Y+Z}\right)}}_{Y}$ and ${\overleftarrow{\left(\vec{U}_{Y}\right)_{Z}}}_{Z}={\overrightarrow{\left(\dot{U}_{Y+Z}\right)}}_{Y}$.
(5) gs-monoidality:

- $!_{e}=\nabla_{e}=i d_{e}$

In this case $\emptyset$ is the only possible argument and we have:

$$
!e(\emptyset)=\emptyset=i d_{e}(\emptyset)=\emptyset=\emptyset \cup_{\emptyset} \emptyset=\nabla_{e}(\emptyset)
$$

- coherence axioms:

For $U \subseteq[Y]^{a}$, we note that ${\overleftarrow{\left(U \cup_{Y} U\right)_{Y}}}_{Y}={\overline{\left(U \cup_{Y} U\right)_{Y}}}_{Y}=U$.
$-\left(i d_{a} \otimes \nabla_{a}\right) \circ \nabla_{a}=\left(\nabla_{a} \otimes i d_{a}\right) \circ \nabla_{a}$
Let $U \subseteq[Y]^{a}$, then:

$$
\begin{aligned}
& \left(\left(i d_{a} \otimes \nabla_{a}\right) \circ \nabla_{a}\right)(U)=\left(i d_{a} \otimes \nabla_{a}\right)\left(U \cup_{Y} U\right) \\
& =i d_{a}(U) \cup_{Y} \nabla_{a}(U)=U \cup_{Y}\left(U \cup_{Y} U\right) \\
& =\left(U \cup_{Y} U\right) \cup_{Y+Y} U=\nabla_{a}(U) \cup_{Y+Y} i d_{a}(U) \\
& =\left(\nabla_{a} \otimes i d_{a}\right)\left(U \cup_{Y} U\right)=\left(\nabla_{a} \otimes i d_{a}\right)\left(\nabla_{a}(U)\right)
\end{aligned}
$$

$$
-i d_{a}=\left(i d_{a} \otimes!a\right) \circ \nabla_{a}
$$

$$
\text { Let } U \subseteq[Y]^{a} \text {, then: }
$$

$$
\begin{aligned}
& \quad\left(\left(i d_{a} \otimes!_{a}\right) \circ \nabla_{a}\right)(U)=\left(i d_{a} \otimes!_{a}\right)\left(U \cup_{Y} U\right) \\
& =i d_{a}(U) \cup_{Y}!_{a}(U)=i d_{a}(U) \cup_{Y} \emptyset \\
& =i d_{a}(U) \\
& -\rho_{a, a} \circ \nabla_{a}=\nabla_{a}
\end{aligned}
$$

Let $U \subseteq[Y]^{a}$, then:

$$
\left(\rho_{a, a} \circ \nabla_{a}\right)(U)=\rho_{a, a}\left(U \cup_{Y} U\right)=U \cup_{Y} U=\nabla_{a}(U)
$$

- monoidality axioms:

$$
-!_{a \otimes b}=!_{a} \otimes!_{b}
$$

Let $U \subseteq[Y+Z]^{a+b}$, then:

$$
\begin{aligned}
!_{a \otimes b}(U) & =\emptyset=\emptyset \cup_{\emptyset} \emptyset=!_{a}\left(\overleftarrow{U}_{Y}\right) \cup_{\emptyset}!_{b}\left(\vec{U}_{Y}\right) \\
& =\left(!_{a} \otimes!_{b}\right)\left(\overleftarrow{U}_{Y} \cup_{Y} \vec{U}_{Y}\right)=\left(!_{a} \otimes!_{b}\right)(U)
\end{aligned}
$$

$-\nabla_{a} \otimes \nabla_{b}=\left(i d_{a} \otimes \rho_{b, a} \otimes i d_{b}\right) \circ \nabla_{a \otimes b}$
Let $U \subseteq[Y+Z]^{a+b}$, then:

$$
\begin{aligned}
& \left(i d_{a} \otimes \rho_{b, a} \otimes i d_{b}\right)\left(\nabla_{a \otimes b}(U)\right) \\
& =\left(i d_{a} \otimes\left(\rho_{b, a} \otimes i d_{b}\right)\right)\left(U \cup_{Y+Z} U\right) \\
& =i d_{a}\left({\left.\overleftarrow{\left(U \cup_{Y+Z} U\right.}\right)}_{Y}\right) \cup_{Y}\left(\rho_{b, a} \otimes i d_{b}\right)\left(\overrightarrow{\left(U \cup_{Y+Z} U\right)_{Y}}\right) \\
& =i d_{a}\left(\overleftarrow{U}_{Y}\right) \cup_{Y}\left(\rho_{b, a} \otimes i d_{b}\right)\left(\left(\overline{\left.\left(U \cup_{Y+Z} U\right)_{Y}\right)}\right.\right. \\
& =\overleftarrow{U}_{Y} \cup_{Y}\left(\rho_{b, a} \otimes i d_{b}\right)\left(\overrightarrow{\left.\left(U \cup_{Y+Z} U\right)_{Y}\right)}\right. \\
& =\overleftarrow{U}_{Y} \cup_{Y}\left(\rho_{b, a} \otimes i d_{b}\right)\left(\left(\vec{U}_{Y} \cup_{Z} \overleftarrow{U}_{Y}\right) \cup_{Z+Y} \vec{U}_{Y}\right) \\
& =\overleftarrow{U}_{Y} \cup_{Y}\left(\left(\overleftarrow{U}_{Y} \cup_{Y} \vec{U}_{Y}\right) \cup_{Y+Z} \vec{U}_{Y}\right) \\
& =\left(\overleftarrow{U}_{Y} \cup_{Y} \overleftarrow{U}_{Y}\right) \cup_{Y+Y}\left(\vec{U}_{Y} \cup_{Z} \vec{U}_{Y}\right) \\
& =\nabla_{a}\left(\overleftarrow{U}_{Y}\right) \cup_{Y+Y} \nabla_{b}\left(\vec{U}_{Y}\right) \\
& =\left(\nabla_{a} \otimes \nabla_{b}\right)\left(\overleftarrow{U}_{Y} \cup_{Y} \vec{U}_{Y}\right)=\left(\nabla_{a} \otimes \nabla_{b}\right)(U)
\end{aligned}
$$

where we use the fact that ${\overleftarrow{\left(U \cup_{Y+Z} U\right)_{Y}}}_{Y}=\dot{U}_{Y}$ and ${\overline{\left(U \cup_{Y+Z} U\right)_{Y}}}_{Y}=$ $\left(\vec{U}_{Y} \cup_{Z} \overleftarrow{U}_{Y}\right) \cup_{Z+Y} \vec{U}_{Y}$.

Finally, the approximation \# is indeed gs-monoidal, i.e., it preserves all the additional structure (tensor, symmetry, unit, duplicator and discharger).

Theorem 7.4 (\# is gs-monoidal). $\#: \mathbb{C}_{f} \rightarrow \mathbb{A}_{f}$ is a gs-monoidal functor.
Proof. We write $e^{\prime}, \otimes^{\prime},!^{\prime}, \nabla^{\prime}, \rho^{\prime}$ for the corresponding operators in category $\mathbb{A}_{f}$. Note that by definition $e=e^{\prime}$ and $\otimes, \otimes^{\prime}$ agree on objects.

First, categories $\mathbb{C}_{f}$ and $\mathbb{A}_{f}$ are gs-monoidal by Theorem 7.2 and 7.3.
Furthermore we verify that:
(1) monoidality:

- $\#(e)=e^{\prime}$

We have $\#(e)=e=e^{\prime}$

- $\#(a \otimes b)=\#(a) \otimes^{\prime} \#(b)$

We have:

$$
\#(a \otimes b)=a \otimes b=\#(a) \otimes^{\prime} \#(b)
$$

(2) symmetry:

- $\#\left(\rho_{a, b}\right)=\rho_{\#(a), \#(b)}^{\prime}$

Let $U \subseteq[Y+Z]^{a+b}$, then for sufficiently small $\delta \sqsupset 0$ (note that such $\delta$ exists due to finiteness):

$$
\begin{aligned}
& \#\left(\rho_{a, b}\right)(U) \\
& =\left(\rho_{a, b}\right)_{\#}^{a+b, \delta}(U) \\
& =\left\{w \in[Z+Y]^{b+a} \mid \rho_{a, b}(a+b)(w) \ominus \rho_{a, b}\left((a+b) \ominus \delta_{U}\right)(w) \sqsupseteq \delta\right\} \\
& =\left\{w \in[Z+Y]^{b+a} \mid(b+a)(w) \ominus\left((b+a) \ominus \delta_{\rho_{a, b}^{\prime}(U)}\right)(w) \sqsupseteq \delta\right\} \\
& =\rho_{a, b}^{\prime}(U)=\rho_{\#(a), \#(b)}^{\prime}(U)
\end{aligned}
$$

since $\rho_{a, b}$ distributes over componentwise subtraction and $\rho_{a, b}\left(\delta_{U}\right)=\delta_{\rho_{a, b}^{\prime}(U)}$.
The second-last equality holds since for all $w$ in the set we have $(b+a)(w) \sqsupset 0$.
(3) gs-monoidality:

- $\#\left(!_{a}\right)=!^{\prime} \#(a)$

Let $U \subseteq[Y]^{a}$, then for some $\delta$ :

$$
\#\left(!_{a}\right)(U)=\left(!_{a}\right)_{\#}^{a, \delta}(U)=\emptyset=!_{a}^{\prime}(U)=!_{\#(a)}^{\prime}(U)
$$

since the codomain of $\left(!a_{\#}^{a, \delta}(U)\right.$ is $\mathcal{P}(\emptyset)$ and hence the only possible value for $(!a)_{\#}^{a, \delta}(U)$ is $\emptyset$.

- $\#\left(\nabla_{a}\right)=\nabla_{\#(a)}^{\prime}$

Let $U \subseteq[Y]^{a}$, then for sufficiently small $\delta \sqsupset 0$ :

$$
\begin{aligned}
& \#\left(\nabla_{a}\right)(U) \\
& =\left(\nabla_{a}\right)_{\#}^{a, \delta}(U) \\
& =\left\{w \in[Y+Y]^{a+a} \mid \nabla_{a}(a)(w) \ominus \nabla_{a}\left(a \ominus \delta_{U}\right)(w) \sqsupseteq \delta\right\} \\
& =\left\{w \in[Y+Y]^{a+a} \mid(a+a)(w) \ominus\left((a+a) \ominus \delta_{\nabla_{a}^{\prime}(U)}\right)(w) \sqsupseteq \delta\right\} \\
& =\nabla_{a}^{\prime}(U)=\nabla_{\#(a)}^{\prime}(U)
\end{aligned}
$$

| Basic Function | $c_{k}$ | $u^{*}$ | $\min _{\mathcal{R}} / \max _{\mathcal{R}}$ | $a d d_{w}$ | $\operatorname{sub}_{w}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| Req. Parameter | $k \in \mathbb{M}^{Z}$ | $u: Z \rightarrow Y$ | $R \subseteq Y \times Z$ | $w \in \mathbb{M}^{Y}$ | $w \in \mathbb{M}^{Y}$ |

Table 3. Additional parameters for the basic functions from Table 2.
since $\nabla_{a}$ distributes over componentwise subtraction and $\nabla_{a}\left(\delta_{U}\right)=\delta_{\nabla_{a}^{\prime}(U)}$. The second-last equality holds since for all $w$ in the set we have $(a+a)(w) \sqsupset 0$.

## 8. UDEfix: A Tool for Fixpoints Checks

8.1. Overview. We present a tool, called UDEfix, which exploits gs-monoidality as discussed before and allows the user to construct functions $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Y}$ as a sort of circuit. As basic components, UDEfix can handle all functions presented in Section 4 (Table 2) and addition and subtraction by a fixed constant $w\left(a d d_{w}, s u b_{w}\right.$, both are non-expansive functions). Since the approximation functor \# is gs-monoidal, this circuit can be then transformed automatically, in a compositional way, into the corresponding abstraction $f_{\#}^{a}$, for some given $a \in \mathbb{M}^{Y}$. By computing the greatest fixpoint of $f_{\#}^{a}$ and checking for emptiness, UDEfix can check whether $a=\mu f$. In addition, it is possible to check whether a given post-fixpoint $a$ is below the least fixpoint $\mu f$ (recall that in this case the check is sound but not complete). The dual checks (for greatest fixpoint and pre-fixpoints) are implemented as well.

The tool is shipped with pre-defined functions implementing examples concerning case studies on termination probability, bisimilarity, simple stochastic games, energy games, behavioural metrics and Rabin automata.

UDEfix is a Windows-Tool created in Python, which can be obtained from https:// github.com/TimoMatt/UDEfix.

Building the desired function $f: \mathbb{M}^{Y} \rightarrow \mathbb{M}^{Y}$ requires three steps:

- Choosing the MV-algebra $\mathbb{M}$ of interest.
- Creating the required basic functions by specifying their parameters.
- Assembling $f$ from these basic functions.
8.2. Tool Areas. Concretely, the GUI of UDEfix is separated into three areas: the Content area, Building area and Basic-Functions area. Under File-Settings the user can set the MValgebra. Currently the MV-chains $[0, k]$ (algebra 1) and $\{0, \ldots, k\}$ (algebra 2) for arbitrary $k$ are supported (see Example 3.2)

Basic-Functions Area: The Basic-Functions area (see Figure 5) contains the basic functions, encompassing those listed in Table 2 in Section 2 as well as addition and subtraction by a constant $w$. Via drag-and-drop (or right-click) these basic functions can be added to the Building area to create a Function box. Each such box requires three (in the case of $\tilde{\mathcal{D}}$ two) Contents: The Input set, the Output set and an additional required parameter (see Table 3). These Contents can be created in the Content area.

Additionally the Basic-Functions area offers functionalities for composing functions via disjoint union (more concretely, this is handled by the the auxiliary Higher-Order Function) and the Testing functionality for fixpoint checks which we will discuss in the next paragraph.


Figure 5. List of basic functions


Figure 6. Assembling the function $\mathcal{W}$ from Section 2.
Building Area: The user can connect the created Function boxes to construct the function $f$ of interest. Composing functions is as simple as connecting two Function boxes in the correct order by mouse-click. Disjoint union is achieved by connecting two boxes to the same box. Note that Input and Output sets of connected Function boxes need to match. As an example, in Figure 6 we show how the function $\mathcal{W}$, discussed in Section 2, for computing the behavioural distance of a labeled Markov chain can be assembled. The construction exactly follows the structure of the diagram in Figure 2. Here, the parameters are instantiated for the labeled Markov chain displayed in Figure 1 (left-hand side).

The special box Testing is always required to appear at the end. Here, the user can enter some mapping $a: Y \rightarrow \mathbb{M}$, test if $a$ is a fixpoint of the function $f$ of interest and then verify if $a=\mu f$. As explained before, this is realised by computing the greatest fixpoint of the approximation $\nu f_{\#}^{a}$. In case this is not empty and thus $a \neq \mu f$, the tool can produce a suitable value which can be used for decreasing $a$, needed for iterating to the least fixpoint
from above (respectively increasing $a$ for iterating to the greatest fixpoint from below). There is also support for comparison with pre- and post-fixpoints.

Example 8.1. As an example, consider the left-hand system in Figure 1 and consider the function $\mathcal{W}$ from Section 2 whose least fixpoint corresponds to the behavioural distance for a labeled Markov chain.

We now define a fixpoint of $\mathcal{W}$, which is not the least, namely $d: Y \rightarrow[0,1]$ with $d(3,3)=0, d(1,1)=1 / 2, d(1,2)=d(2,1)=d(2,2)=2 / 3$ and 1 for all other pairs. Note that $d$ is not a pseudo-metric, but is a fixpoint of $\mathcal{W}$. We give a hint why $d$ is a fixpoint, by exemplarily considering the pair $(1,2)$. Due to the labels, an optimal coupling for the successors of 1,2 assigns $(3,3) \mapsto 1 / 3,(4,4) \mapsto 1 / 2,(3,4) \mapsto 1 / 6$. Hence, by the fixpoint equation, we have

$$
d(1,2)=1 / 3 \cdot d(3,3)+1 / 2 \cdot d(4,4)+1 / 6 \cdot d(3,4)=1 / 3 \cdot 0+1 / 2 \cdot 1+1 / 6 \cdot 1=1 / 2+1 / 6=2 / 3 .
$$

A similar argument can be made for the remaining pairs.
By clicking Compute in the Testing-box, UDEfix displays that $d$ is a fixpoint and tells us that $d$ is in fact not the least and not the greatest fixpoint. It also computes the greatest fixpoints of the approximations step by step (via Kleene iteration) and displays the results to the user. In this case $\nu f_{\#}^{a}=\{(4,4)\}$, indicating that the distance $d(4,4)=1$ overestimates the true value and can be decreased. The fact that also other pairs over-estimate their value (but to a lesser degree), will be detected in later steps in the iteration to the least fixpoint from above.

Content Area: Here the user can create sets, mappings and relations which are used to specify the basic functions. The user can create a variety of different types of sets, such as $X=\{1,2,3,4\}$, which is a basic set of numbers, and the set $D=\left\{p_{1}, p_{2}, p_{3}, p_{4}\right\}$ which is a set of mappings representing probability distributions. These objects are called Content.

Once Input and Output sets are created we can specify the required parameters (cf. Table 3) for a function. Here, the created sets can be chosen as domain and co-domain. Relations can be handled in a similar fashion: Given the two sets one wants to relate, creating a relation can be easily achieved by checking some boxes. Some useful in-built relations like"is-element-of"-relation and projections to the $i$-th component are pre-defined.

By clicking on the icon " + " in a Function box, a new function with the chosen Input and Output sets is created. The additional parameters (cf. Table 3) have domains and co-domains which need to be created by the user or are provided by the chosen MV-algebra.

The Testing function $a$ (i.e., the candidate (pre/post-)fixpoint) is a mapping as well and can be created as all other functions.

In Figure 7 we give examples of how contents can be created: we show the creation of a set $(Y=X \times X)$, a distance function $(d)$ and a relation $(\rho)$.
8.3. Tutorial. We now provide a small tutorial intended to clarify the use of the tool. It deals with a simple example: a function whose least fixpoint provides the termination probability of a Markov chain.

We continue Examples 3.5 and 4.7 and consider the Markov chain $(S, T, \eta)$ in Figure 3.
Remember that the function $\mathcal{T}$, whose least fixpoint is the termination probability, can be decomposed as follows:

$$
\mathcal{T}=\left(\eta^{*} \circ \tilde{\mathcal{D}}\right) \otimes c_{k}
$$



Figure 7. Contents: Set $Y$, Mapping $d$, Relation $\rho$.


Figure 8. Creation of set $S$ and probability distribution $p_{x}$ for the example.
In order to start, one first chooses the correct MV-algebra under Settings and creates the sets $S, T, S \backslash T$ (Input and Output sets). The creation of set $S$ is exemplified in Figure 8 (left-hand side). The tool supports several types of sets and operators on sets (such as complement, which makes it easy to create $S \backslash T$. Next, we create the set $D$ of probability distributions, consisting of the mappings $p_{x}, p_{y}, p_{z}$. In Figure 8 (right-hand side) we show the creation of $p_{x}$.


Figure 9. Assembling the function $\mathcal{T}$.


Figure 10. Creation of the parameters $k$ and $\eta^{*}$ and the greatest fixpoint $a_{1}=\mu \mathcal{T}$.

Now we create the basic function boxes and connect them in the correct way (see Figure 9). The additional parameters according to Table 3 - in this case the map $c_{k}$ and the reindexing $\eta^{*}$ based on the successor map - can be created by clicking the icon " + " in the corresponding box (see Figure 10) (left-hand side for $k$ and middle for $\eta$ ).

We can also assemble several test functions (e.g., possible candidate fixpoints), among them the greatest fixpoint $a_{1}=\nu \mathcal{T}$ (see Figure 10, right-hand side).


Figure 11. Checking the candidate fixpoint $a_{1}$.
When testing $a_{1}$ we obtain the results depicted in Figure 11. In fact $\nu \mathcal{T}_{\#}^{a_{1}}=\{y, z\} \neq \emptyset$, which tells us that $a_{1}$ is not the least fixpoint.

Similarly, one can test $a_{2}=\mu \mathcal{T}$, the least fixpoint, obtaining $\nu \mathcal{T}_{\#}^{a_{2}}=\emptyset$. This allows the user to deduce that $a_{2}$ is indeed the least fixpoint. As mentioned before, one can also test whether a pre-fixpoint is below the greatest fixpoint or a post-fixpoint above the least fixpoint, although such tests are sound but not complete.

## 9. Conclusion, Related and Future Work

We have shown that a framework originally introduced in $[8,6]$ for analysing fixpoint of non-expansive functions over MV-algebras can be naturally cast into a gs-monoidal setting. The non-expansive functions of interest live in a so-called concrete gs-monoidal category, their approximations in a gs-monoidal abstract category and they are related by a gsmonoidal functor \#. We also developed a general theory for constructing approximations of predicate liftings, which find natural application in the definition of behavioural metrics over coalgebras.

The compositionality properties arising from the gs-monoidal view of the theory are at the basis of the development of a prototype tool UDEfix. The tool allows one to build the concrete function of interest out of some basic components. Then the approximation of the function can be obtained compositionally from the approximations of the components and one can check whether some fixpoint is the least or greatest fixpoint of the function of interest. Additionally, one use the tool to show that some pre-fixpoint is above the greatest fixpoint or some post-fixpoint is below the least fixpoint.

Related work: This paper is based on fixpoint theory, coalgebras, as well as on the theory of monoidal categories. Monoidal categories [24] are categories equipped with a tensor. It has long been realized that monoidal categories can have additional structure such as braiding or symmetries. Here we base our work on so called gs-monoidal categories [14, 22], called s-monoidal in [21]. These are symmetric monoidal categories, equipped with a discharger and a duplicator. Note that "gs" originally stood for "graph substitution" as such categories were first used for modelling term graph rewriting.

We view gs-monoidal categories as a means to compositionally build monotone nonexpansive functions on complete lattices, for which we are interested in the (least or greatest) fixpoint. Such fixpoints are ubiquitous in computer science, here we are in particular interested in applications in concurrency theory and games, such as bisimilarity [28], behavioural metrics $[17,31,12,4]$ and simple stochastic games [13]. In recent work we have considered strategy iteration procedures inspired by games for solving fixpoint equations [7].

Fixpoint equations also arise in the context of coalgebra [27], a general framework for investigating behavioural equivalences for systems that are parameterized - via a functor over their branching type (labelled, non-deterministic, probabilistic, etc.). Here in particular we are concerned with coalgebraic behavioural metrics [4], based on a generalization of the Wasserstein or Kantorovich lifting [32]. Such liftings require the notion of predicate liftings, well-known in coalgebraic modal logics [29], lifted to a quantitative setting [10].

Future work: One important question is still open: we defined an approximation \#, relating the concrete category $\mathbb{C}$ of functions of type $\mathbb{M}^{Y} \rightarrow \mathbb{M}^{Z}$ - where $Y, Z$ might be infinite - to their approximations, living in $\mathbb{A}$. It is unclear whether $\#$ is a lax or even proper functor, i.e., whether it (laxly) preserves composition. For finite sets functoriality derives from a non-trivial result in [6] and it is unclear whether it can be extended to the infinite case. If so, this would be a valuable step to extend the theory to infinite sets.

In this paper we illustrated the approximation for predicate liftings via the powerset and the distribution functor. It would be interesting to study more functors and hence broaden the applicability to other types of transition systems.

Concerning UDEfix, we plan to extend the tool to compute fixpoints, either via Kleene iteration or strategy iteration (strategy iteration from above and below), as detailed in [7]. Furthermore for convenience it would be useful to have support for generating fixpoint functions directly from a given coalgebra respectively transition system.
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[^0]:    ${ }^{1}$ A lluf sub-category is a sub-category that contains all objects.

