# Tail recursion transformation for invertible functions 

Joachim Tilsted Kristensen ${ }^{1}$, Robin Kaarsgaard ${ }^{2}$, and Michael Kirkedal Thomsen ${ }^{1,3}$<br>${ }^{1}$ University of Oslo, Norway<br>${ }^{2}$ University of Edinburgh, UK<br>${ }^{3}$ University of Copenhagen, Denmark


#### Abstract

Tail recursive functions allow for a wider range of optimisations than general recursive functions. For this reason, much research has gone into the transformation and optimisation of this family of functions, in particular those written in continuation passing style (CPS).

Though the CPS transformation, capable of transforming any recursive function to an equivalent tail recursive one, is deeply problematic in the context of reversible programming (as it relies on troublesome features such as higher-order functions), we argue that relaxing (local) reversibility to (global) invertibility drastically improves the situation. On this basis, we present an algorithm for tail recursion conversion specifically for invertible functions. The key insight is that functions introduced by program transformations that preserve invertibility, need only be invertible in the context in which the functions subject of transformation calls them. We show how a bespoke data type, corresponding to such a context, can be used to transform invertible recursive functions into a pair of tail recursive function acting on this context, in a way where calls are highlighted, and from which a tail recursive inverse can be straightforwardly extracted.


Keywords: tail recursion, CPS transformation, program transformation, program inversion

## 1 Introduction

When a function calls itself, either directly or indirectly, we say that the function is recursive. Furthermore, when the last operation of all branches in the definition of a recursive function is the recursive call, we say that the function is tail recursive. Unlike generally recursive functions, tail recursive functions can be easily compiled into loops in imperative languages (in particular assembly languages) doing away with the overhead of function calls entirely. This makes tail recursion a desirable programming style.

Recall that a program is reversible when it is written such that it only consists of invertible combinations of invertible atomic operations; this is the idea of reversibility as local phenomenon. While every reversible program is also
invertible (in the sense that it has an inverse), the converse is not the case, as an invertible program may consist of a number of non-invertible functions that simply happen to interact in a way as to make the program invertible. As such, invertibility is a global phenomenon.

While recursion has been employed in both imperative and functional reversible programming languages $[12,24]$ for many years, tail recursion has been more cumbersome to handle. Here, we argue that relaxing (local) reversibility to (global) invertibility can drastically simplify the handling of tail recursion and even make it possible to use (adaptations of) conventional CPS transformation methods for transforming general recursive to tail recursive functions. To see this, consider the list reversal and list append functions

```
reverse1 [ ] = []
reverse1 (x : xs) =
    let ys = reverse1 xs in
    let (zs:_x) = snoc1 (ys,x) in
    (zs:_x)
```

```
snoc1 ([ ], x) = x : []
snoc1 (y : ys, x) =
    let (zs:_x) = snoc1 (ys, x) in
    (y : zs:_
```

The careful reader will have already realised that reverse1 is its own inverse. Here, we will refrain from clever realisations and focus on purely mechanical ways of providing inverse functions. For instance, the inverses

```
unsnoc1 (y : zs:_x) =
    let (ys, x) = unsnoc1 (zs:_x) in
    (y : ys, x)
unsnoc1 (x : [ ]) = ([ ], x)
unreverse1 (zs:_x)=
```

are produced by rewriting "let $\mathrm{y}=\mathrm{f} \mathrm{x}$ in t " to "let $\mathrm{x}=\mathrm{unf} \mathrm{y}$ in t ", and then swapping the order bindings in the remaining program t , starting from the last line and ending with the first, much in the style of Romanenko [19]. To transform these recursive functions into tail recursive functions, the standard technique is to introduce an iterator that passes around an explicit argument for accumulating the deferred part of the computation, e.g.,

```
reverse2 xs = reverse2_iter (xs, [])
reverse2_iter ([ ], accum) = accum
reverse2_iter (x : xs, accum) = reverse2_iter (xs, x : accum)
```

Implementing list reversal in this style makes it tail recursive, but it also loses an important property, namely branching symmetry. This is crucial, since branching symmetry was the entire reason why we could mechanically invert the implementations of snoc1 and reverse1 so easily: because the leaves of their cases are syntactically orthogonal. For instance, in reverse1, when the input is an empty list, the result is also an empty list, and when the input is nonempty, the result is also nonempty.

As a consequence of this loss of symmetry, the iterator function reverse2_iter it is not considered well-formed for inversion as defined by Glück \& Kawabe [6]. Consequently, it cannot be implemented in a reversible functional programming language such as RFun [20,24] or CoreFun [8], as it breaks the symmetric first
match policy; the base case returning accum will also return the same value from the iterative case. Even worse, reverse2 iter cannot be inverted to a deterministic function using known methods [5, 16, 18]. Of course, this is because reverse 2 _iter is not injective, so the outputs of a particular input is not unique.

It does not take much effort to show that reverse1 and reverse2 are semantically equivalent. Thus, since the latter does nothing but call reverse2 iter it is surprising that we cannot invert it. A brief analysis of the problem concludes that reverse2 restricts itself to a subset of the domain of reverse2_iter, and since reverse2 is clearly injective, reverse2_iter as restricted to this smaller domain must be injective as well. By further analysis, we realise that the second component of the arguments to reverse2_iter, as called by reverse2, is static and can be ignored. In this context reverse2_iter is in one of three configurations: accepting the restricted input, iterating, or returning an output. By introducing a data type, we can explicitly restrict reverse2_iter to this smaller domain:

```
data Configuration a = Input a | Iteration (a, a)| Output a
reverse3 xs = let (Output ys) = iterate (Input xs) in ys
iterate (Input xs) = iterate (Iteration (xs, [ ]))
iterate (Iteration (x : xs, ys)) = iterate (Iteration (xs, x : ys))
iterate (Iteration ([ ], ys)) = (Output ys)
```

Even further, just like reverse1 this definition can be mechanically inverted:

```
uniterate (Output ys) = uniterate (Iteration ([ ], ys))
uniterate (Iteration (xs, x : ys)) = uniterate (Iteration (x : xs, ys))
uniterate (Iteration (xs, [ ])) = (Input xs)
unreverse3 ys = let (Input xs) = uniterate (Output ys) in xs
```

Moreover, these four function definitions are all tail recursive, which was what we wanted.

Structure: In this article we will show an algorithm that can perform this transformation. First, in Section 2, we illustrate the program transformation by example, before describing it formally in Section 3 and prove its correctness. Afterwards, we discuss a couple of known limitations (Section 4) of our approach, and show how the resulting constructs can be compiled to a flow-chart language (Section 5). Finally, we discuss related in Section 6 and end in Section 7 with some concluding remarks.

## 2 Tail recursion transformation, by example

The transformation we propose assumes a functional programming language with first order functions, algebraic datatypes, and recursion, as these are the features commonly found in reversible functional programming languages [8, 9 , $20,24]$. Moreover, as the subject of the transformation, we only consider functions that are well-formed for inversion [6] as usual, meaning that the patterns of case-expressions are orthogonal, either syntactically, or by guard statements as suggested in Mogensen's semi-inversion for guarded-equations [15]. Furthermore, we require that expressions and patterns are linear (any variable binding is used exactly once), and (for simplicity) that a variable cannot be redefined in expressions that nest binders (such as let and case).

Such programming languages usually introduce the notion of tail recursion by introducing an imperative style language feature. For instance, Mogensen's language for guarded equations [15] features a loop construct that allows it to call a partial function until it fails (by pattern matching not exhaustive), as illustrated by the function reverse 4 , defined by:

```
reverse4(xs) = let ([], acc) = loop revStep (xs, []) in acc
    where revStep(x:xs, acc) = (xs, x : acc);
```

Likewise, the Theseus programming language [9] provides a trace operation encoded via so-called iteration labels, as demonstrated in reverse5 below.

```
iso reverse5 :: [a] <-> [a]
|ss}=\mathrm{ iterate $ inL xs, []
iterate inL $ (x:xs) ys = iterate $ inL xs, (x : ys)
| iterate inL $ [], ys = iterate $ inR ys
| iterate $ inR ys = ys
    where
        iterate :: ([a] * [a]) + [a]
```

We do not introduce a new language feature, but instead relax the requirement that all functions must be well-formed for inversion. Instead we require only that the subject of the transformation must be well-formed for inversion. For instance, recall that the function snoc 1 from Section 1 is well-formed for inversion, and consider Nishida \& Vidal's CPS transformation of first-order functions [17]

```
data Continuation a = Id | Fa (Continuation a)
snoc2 p = snoc2_iter (p, Id)
    where
        snoc2_iter (([ ], x), g) = snoc2_call (g, x : [])
        snoc2_iter ((y : ys, x), g) = snoc2_iter ((ys, x), F y g)
        snoc2_call (ld , zs:_x) = zs:_x
        snoc2_call (F y g, zs:_x) = snoc2_call (g, y : zs:_x)
```

Here, the computation has been split into two parts; one that computes a structure corresponding to the closure of the usual continuation function, and another that corresponds to evaluating the call to said continuation. Now, just as with reverse2_iter, snoc2_iter and snoc2_call are not injective functions, but can be restricted to such when recognizing that one or more of its arguments are static (Id and [] respectively). Consequently, we can introduce a datatype that does away with these, and invert snoc2 as

```
data Configuration' input acc arg output =
    Input' input
    Iterate (input, Continuation acc)
    | Call (Continuation acc, arg)
    | Output' output
snoc6 (ys, x) =
    let (Output' (zs:_x)) = snoc6_call (snoc6_iter (Input' (ys, x))) in (zs:_x)
snoc6_iter (Input' (ys, x)) = snoc6_iter (Iterate ((ys, x), Id ))
snoc6_iter (Iterate ((y : ys, x), g)) = snoc6_iter (Iterate ((ys, x), F y g))
snoc6_iter (Iterate (([ ], x), g)) = (Call (g, [x]))
```

```
snoc6 call (Call (g, [x])) = snoc6_call (Iterate ([x], g))
snoc6_call (Iterate ((zs:_x), F y g)) = snoc6_call (Iterate (y : (zs:_x), g))
snoc6_call (Iterate ((zs:_x), Id )) = (Output' ((zs:_x)))
unsnoc6_call (Output' ((zs:_x))) = unsnoc6_call (Iterate ((zs:_x), Id ))
unsnoc6_call (Iterate (y : (zs:_x), g)) = unsnoc6_call (Iterate ((zs:_x), F y g))
unsnoc6_-call (Iterate ([x], g))}=(\mathrm{ Call (g, [x]))
unsnoc6_iter (Call (g, [x])) = unsnoc6_iter (Iterate (([ ], x), g))
unsnoc6_iter (Iterate ((ys, x), F y g)) = unsnoc6_iter (Iterate ((y : ys, x), g))
unsnoc6_iter (Iterate ((ys, x), Id )) = (Input' (ys, x))
unsnoc6 (zs: x) =
    let (Input' (ys, x)) = unsnoc6_iter (unsnoc6_call (Output' (zs:_x))) in (ys, x)
```

Moreover, because the iterator does not use Output' and the call simulation does not use Input', we can introduce two separate datatypes, and a couple of gluing functions to improve composition.

```
data Configuration2 input acc arg
    = Input2 input
    | Iterate2 (input, Continuation acc)
    | Output2 (Continuation acc, arg)
data Configuration3 acc arg output
    = Input3 (Continuation acc, arg)
    | Iterate3 (Continuation acc, output)
    | Output3 output
input a = (Input2 a)
uninput (Input2 a) =a
glue (Output2 a) = (Input3 a)
unglue (Input3 a) = (Output2 a)
output a = (Output3 a)
unoutput (Output3 a) = a
```

Now, because reverse1 was also well-formed for inversion, we can apply the usual CPS transformation, and obtain a tail-recursive inverse program by the exact same procedure:

```
reverse6 = unoutput . call6 . glue . iterate6 . input
iterate6 (Input2 xs) = iterate6 (Iterate2 (xs, Id ))
iterate6 (Iterate2 (x: xs, g)) = iterate6 (Iterate2 (xs, F x g))
iterate6 (Iterate2 ([], g)) = (Output2 (g, []))
call6 (Input3 (g, [])) = call6 (Iterate3 (g, []))
call6 (Iterate3 (F x g, ys)) = call6 (Iterate3 (g, zs:_x))
    where zs:_x = snoc6 (ys, x)
call6 (Iterate3 (Id , ys)) = (Output3 ys)
uncall6 (Output3 ys) = uncall6 (Iterate3 (Id , ys))
uncall6 (Iterate3 (g, zs:_x)) = uncall6 (Iterate3 (F x g, ys))
    where (ys, x) = unsnoc6 (zs:_x)
uncall6 (Iterate3 (g, [])) - = (Input3 (g, []))
uniterate6 (Output2 (g, [])) = uniterate6 (Iterate2 ([], g))
uniterate6 (Iterate2 (xs, F x g)) = uniterate6 (Iterate2 (x : xs, g))
uniterate6 (Iterate2 (xs, Id)) = (Input2 xs)
```

$$
\begin{array}{rlr}
p & :=c\left(p_{i}\right) & \text { (Constructor). } \\
& \mid x & \text { (Variable). } \\
t: & \text { (Terms.) } \\
\Delta & ::=f p|f p| \text { case } t \text { of } p_{i} \rightarrow t_{i} & \text { (where } \left.p_{i}=t_{i}\right] . \Delta \mid \text { data } \tau=c_{j}\left(\tau_{i}\right) . \Delta \mid \epsilon \\
\text { (Programs). }
\end{array}
$$

Figure 1: The syntax for a first order functional programming language.

$$
\begin{aligned}
& \llbracket f t \rrbracket::=\text { case } \llbracket t \rrbracket \text { of } p \rightarrow f p \\
& \llbracket c\left(t_{i}\right) \rrbracket::=\text { case } \llbracket t_{i} \rrbracket \text { of } p_{i} \rightarrow c\left(p_{i}\right) \\
& \llbracket \text { let } p=t_{1} \text { in } t_{2} \rrbracket:=\text { case } \llbracket t_{1} \rrbracket \text { of } p \rightarrow \llbracket t_{2} \rrbracket \\
& \llbracket f p_{i}=t_{i} \rrbracket::=f x=\text { case } x \text { of } p_{i} \rightarrow \llbracket t_{i} \rrbracket \\
& \llbracket f p=t \text { where } p_{i}=t_{i} \rrbracket:=f p=\text { case } \llbracket t_{i} \rrbracket \text { of } p_{i} \rightarrow \llbracket t^{\prime} \rrbracket
\end{aligned}
$$

Figure 2: Disambiguation of syntactic sugar.
${ }^{17}$
unreverse6 $=$ uninput . uniterate6 . unglue . uncall6 . output
It is important to note that even though reverse6 seems a bit more complicated than reverse3, we did not start with a tail recursive function, and the transformation process was entirely mechanical. First we converted into tail recursive form using continuation passing style. Then, we restricted the functions introduced by the transformation, to the domain on which they are called by the function you are inverting. Finally, we inverted all the operations performed by reverse1, and this was also entirely mechanical (since reverse1 was well-formed for inversion), and produced the inverse program by swapping the input and output arguments (keeping the recursive call in front of the Iterate data structure).

## 3 Tail recursion transformation, formally

In the interest of simplicity we will show how the transformation works on a small, idealised subset of the Haskell programming language as shown in Figure 1, restricted to first order function application and conditionals. A term is a pattern, a function applied to a pattern, or a case-expression, though in program examples, we might use a where-clause or a let-statement when the syntactic disambiguation is obvious. Functions applied to terms, patterns that consist of terms, and let-statements are disambiguated as shown in Figure 2.

First, we give the definition of the requirements for the transformation to work.

Definition 1. A term $t$ is closed under a pattern $p$ precisely if all of the variables that occur in $p$ appear in $t$ exactly once.

Definition 2. A function $f$, as defined by the equation $f p=t$, is well-formed for inversion, if $t$ is closed under $p$. Moreover,

- If $t$ is an application, then $t \equiv g p_{0}$, where $g$ is well-formed for inversion as well.
- If t is a case-expression, then $t \equiv$ case $t_{0}$ of $p_{i} \rightarrow t_{i}$, where then $t_{0}$ is wellformed for inversion, each $t_{i}$ is closed under the corresponding pattern $p_{i}$, and for all indices $j$ and $k$, if $j<k$ then $p_{j}$ is syntactically distinguishable from $p_{k}$ and the leaf terms of $t_{j}$ are all syntactically distinguishable from the corresponding leaf terms of $t_{k}$.

When a function is well-formed for inversion in this way, we know how to invert it using existing methods, even though such methods may require some expensive search. However, functions that do not contain a case-expression are all trivially and efficiently invertible, and we can focus on the hard part, namely conditionals.

Functions that are well-formed for inversion will be implemented with function clauses of the following two forms

$$
\begin{aligned}
f p_{k} & =t_{k} \\
f p_{i} & =g_{i}\left(t_{i 0}, t_{i 1}\right)
\end{aligned}
$$

Each term $t_{k}$ is well-formed for inversion and do not contain recursive calls to $f, k$ is less than $i$, and $g_{i}$ is well-formed for inversion. Furthermore, $t_{i 0}$ may contain recursive calls to $f$ but $t_{i 1}$ is free of such calls. Moreover, the result of calling $g_{i}$ with these arguments yield patterns that are distinguishable from the results of calling $g_{j}$ on $\left(t_{j 0}, t_{j 1}\right)$ whenever $i<j$.

The first order CPS transformation proposed by Nishida \& Vidal essentially defers the call to $g_{i}$ by storing the unused parts of $p_{i}$ and $t_{i 1}$ in a data structure, yielding the program transforms

$$
\begin{aligned}
\text { data } \tau & =\mathrm{Id} \mid \mathrm{G}_{i}\left(\tau_{t_{i 1}}, \tau\right) \\
f_{0} x & =f_{1}(\mathrm{Id}, x) \\
f_{j}\left(g, p_{k}\right) & =f_{n}\left(g, t_{k}\right) \\
f_{j}\left(g, p_{i}\right) & =f_{l}\left(\mathrm{G}_{i}\left(t_{i 1}, g\right), t_{i 0}\right) \\
f_{n}(\mathrm{Id}, y) & =y \\
f_{n}\left(\mathrm{G}_{i}\left(p_{i 1}, g\right), p_{i 0}\right) & =f_{n}\left(g, g_{i}\left(p_{i 0}, p_{i 1}\right)\right),
\end{aligned}
$$

where $1 \leq j \leq n$ and $1 \leq l \leq n$. This transformation clearly preserves semantics (in the sense that $f$ is semantically equivalent with $f_{0}$ ) since $f_{j}$ essentially builds up a stack of calls to respective $g_{i}$ 's, while $f_{n}$ performs these calls in the expected order.

The only problem is that each $f_{j}$ may not be well-formed for inversion, and $f_{n}$ is certainly not well-formed; the variable pattern in the first case is a catch-all that later cases cannot be syntactically orthogonal to. Consequently, we cannot use existing methods to invert these functions. Instead, we realize that their origins are well-formed for inversion, so we should have been able to invert them in a way that is "well formed enough". The idea is to represent
each intermediate function with a datatype, and use the fact that each $g_{i}$ is well-formed for inversion to construct the invertible program as

$$
\begin{aligned}
\text { data } \tau & =\operatorname{Id} \mid \mathrm{G}_{i}\left(\tau_{t_{i 1}}, \tau\right) . \\
\text { data } \tau^{\prime} & =\operatorname{In}\left(\tau_{x}\right)\left|\mathrm{F}_{j 0}\left(\tau, \tau_{i 0}\right)\right| \mathrm{H}_{k 1}\left(\tau, \tau_{k}\right) . \\
\text { data } \tau^{\prime \prime} & =\mathrm{H}_{k 2}\left(\tau, \tau_{k}\right)\left|\operatorname{Eval}\left(\tau, \tau_{k}\right)\right| \operatorname{Out}\left(\tau_{y}\right) . \\
f_{0}^{\prime} & =f_{2}^{\prime} \circ h \circ f_{1}^{\prime} . \\
h\left(\mathrm{H}_{k 1}(f, x)\right) & =\mathrm{H}_{k 2}(f, x) . \\
f_{1}^{\prime}\left(\operatorname{In}\left(p_{l}\right)\right) & =f_{1}^{\prime}\left(\mathrm{F}_{l}\left(\mathrm{Id}, p_{l}\right)\right) \\
f_{1}^{\prime}\left(\mathrm{F}_{j}\left(g, p_{j}\right)\right) & =f_{1}^{\prime} \mathrm{F}_{j 0}\left(\mathrm{G}_{j}\left(t_{i 1}, g\right), t_{i 0}\right) . \\
f_{1}^{\prime}\left(\mathrm{F}_{j}\left(g, p_{k}\right)\right) & =f_{1}^{\prime} \mathrm{H}_{k 1}\left(g, p_{k}\right) . \\
f_{2}^{\prime}\left(\mathrm{H}_{k 2}\left(g, p_{k}\right)\right) & =f_{2}^{\prime} \operatorname{Eval}\left(g, p_{k}\right) . \\
f_{2}^{\prime}\left(\operatorname{Eval}\left(\mathrm{F}_{i 0}\left(\mathrm{G}_{i}\left(p_{i 1}, g\right)\right), p_{i 0}\right)\right) & =f_{2}^{\prime}\left(g, y_{i}\right) \text { where } y_{i}=g_{i}\left(p_{i 0}, p_{i 1}\right) . \\
f_{2}^{\prime}(\operatorname{Eval}(\operatorname{Id}, y)) & =\operatorname{Out}(y) .
\end{aligned}
$$

Now, just as with the CPS transformation, $f_{0}^{\prime}$ is semantically equivalent to $f$ because $f_{1}^{\prime}$ collects calls $\mathrm{G}_{j}$ and $f_{2}^{\prime}$ evaluates them. As such, the only difference is that the input is wrapped in In and Out. However, this time we can derive an inverse program as

$$
\begin{align*}
& f_{0}^{\prime-1}=f_{1}^{\prime-1} \circ h^{-1} \circ f_{2}^{\prime-1}  \tag{1}\\
& h^{-1}\left(\mathrm{H}_{k 2}(f, x)\right)=\mathrm{H}_{k 1}(f, x)  \tag{2}\\
& f_{2}^{\prime-1}(\operatorname{Out}(y))=f_{2}^{\prime-1}(\operatorname{Eval}(\operatorname{Id}, y))  \tag{3}\\
& f_{2}^{\prime-1}\left(g, y_{i}\right)=f_{2}^{\prime-1}\left(\operatorname{Eval}\left(\mathrm{~F}_{i 0}\left(\mathrm{G}_{i}\left(p_{i 1}, g\right)\right), p_{i 0}\right)\right)  \tag{4}\\
& \text { where }\left(p_{i 0}, p_{i 1}\right)=g_{i}^{-1}\left(y_{i}\right)  \tag{5}\\
& f_{2}^{\prime-1}\left(\operatorname{Eval}\left(g, p_{k}\right)\right)=\mathrm{H}_{k 2}\left(g, p_{k}\right)  \tag{6}\\
& f_{1}^{\prime-1}\left(\mathrm{H}_{k 1}\left(g, p_{k}\right)\right)=f_{1}^{\prime-1}\left(\mathrm{~F}_{j}\left(g, p_{k}\right)\right)  \tag{7}\\
& f_{1}^{\prime-1} \mathrm{~F}_{j 0}\left(\mathrm{G}_{j}\left(p_{i 1}, g\right), p_{i 0}\right)=f_{1}^{\prime-1}\left(\mathrm{~F}_{j}\left(g, p_{j}\right)\right)  \tag{8}\\
& f_{1}^{\prime-1}\left(\mathrm{~F}_{l}\left(\operatorname{Id}, p_{l}\right)\right)=\left(\operatorname{In}\left(p_{l}\right)\right) \tag{9}
\end{align*}
$$

The correctness of this technique can be shown as follows.
Theorem 1. The function $f_{0}^{\prime-1}$ is inverse to $f_{0}$.
Proof. We remark the following for each step of the transformation:

- (1) $f_{0}^{\prime-1}$ is inverse to $f_{0}^{\prime}$ (by definition of function composition) precisely if $h^{-1}, f_{1}^{\prime-1}$, and $f_{2}^{\prime-1}$ are the inverse to $h, f_{1}^{\prime}$, and $f_{2}^{\prime}$ respectively.
- (2) $h^{-1}$ is trivially inverse to $h$ since it does not contain application or case-expressions.
- (3) There is only one way of constructing the arguements to $f_{2}^{\prime-1}$, namely using the constructor Out on the output of $f_{2}^{\prime}$.
- (4-5) Since $g_{i}$ was well-formed for inversion, the output $y_{i}$ is syntactically orthogonal to outputs of $g_{j}$ when $i \neq j$. The patterns it takes as arguments
$\left(p_{i 0}, p_{i 1}\right)$ are syntactically orthogonal to all other such patterns, so the choice of constructors $\mathrm{F}_{i 0}$ and $\mathrm{G}_{i}$ has to be unique as well.
- (6) $p_{k}$ is trivially recognized as one of the syntactically orthogonal parts of the left-hand side of $f$, which was well-formed for inversion.
- (7) There is only one way of constructing $\mathrm{H}_{k 1}$, namely using $h^{-1}$.
- (8) These are exactly the arguments of $g_{j}$, Since $f$ was well-formed for inversion, they must be closed under $p_{j}$ (Definition 2), which we may now reconstruct by copying.
- (9) Finally, the first argument of $\mathrm{F}_{l}$ could only have been Id in one program point, and the result has to be constructed using In, and we are done.

By equations (7)-(9), $f_{1}^{\prime-1}$ is inverse to $f_{1}^{\prime}$, and by equations (3)-(6), $f_{2}^{\prime-1}$ is inverse to $f_{2}^{\prime}$. Since, by equation (2), $h^{-1}$ is inverse to $h$, it follows by equation (1) that $f_{0}^{\prime-1}$ is inverse to $f_{0}^{\prime}$. Now, since $f_{0}^{\prime}$ was semantically equivalent to $f_{0}$, $f_{0}^{\prime-1}$ must be inverse to $f_{0}$ as well, and we are done.

## 4 Known limitations

In Definition 1 we required linearity, which is slightly stronger than it needs to be. The reason why we chose this restriction is because it commonly occurs in reversible programming [20,24], and makes it easy to reject programs that are trivially non-invertible. However, the linearity restriction could be relaxed to relevance (i.e., that variables must occur at least once rather than exactly once) as in [8]. Moreover, we might even want to relax this restriction even further to say that all values that were available to a particular function of interest must be used at least once on every execution path. We do not believe that it can relaxed further than that, as an invertible program cannot lose information when it is not redundant.

Additionally, one may want to relax the constraints of local invertibility to be operations for which an inverse is symbolically derivable. For instance, consider extending the syntax for patterns with integer literals, and terms with addition and subtraction. Hence, the following formulation of the Fibonacci-pair function is possible.

```
fib \(\quad(a, b)=(a+b, a)\)
dec \(n=n-1\)
fib_pair \(0=(1,1)\)
fib_pair \(n=\) fib (fib_pair \((\operatorname{dec} n)\) )
```

While this program is invertible, it requires a bit of inference to derive the inverse. For instance, that one of the arguments of fib is preserved in its output, which is needed to infer unfib. Likewise for dec and undec, the compiler must infer that subtracting a constant can be automatically inverted.

Additionally, while the algebraic data-representation of natural number constants is syntactically distinguishable, with integer constants and variables the compiler has to insert guards, as in

```
fib_pair = unoutput . call7 . glue . iterate7 . input
iterate7 (Input2 n) = iterate7 (Iterate2 (n, Id))
iterate7 (Iterate2 (n, f)) | n ==0 = iterate7 (Iterate2 (n', F () f))
    where n' = dec n
iterate7 (Iterate2 (0, f)) = (Output2 (f, (1,1)))
call7 (Input3 (f, (1, 1))) = call7 (Iterate3 (f, (1, 1)))
call7 (Iterate3 (F () f, pair)) = call7 (Iterate3 (f, y))
    where y = fib pair
call7 (Iterate3 (Id , x)) = (Output3 x)
uncall7 (Output3 x) = uncall7 (Iterate3 (Id, x))
uncall7 (Iterate3 (f, y)) | y /= (1, 1) = uncall7 (Iterate3 (F () f, pair))
    where pair = unfib y
uncall7 (Iterate3 (f, (1, 1))) = (Input3 (f, (1, 1)))
uniterate7 (Output2 (f, (1, 1))) = uniterate7 (Iterate2 (0, f))
uniterate7 (Iterate2 (n', F () f)) = uniterate7 (Iterate2 (n, f))
    where n = undec n'
uniterate7 (Iterate2 (n, Id )) = (Input2 n)
unfib_pair = uninput . uniterate7 . unglue . uncall7 . output
unfib (ab, a) = (a, ab - a)
undec n = n +1
```

However, the necessary guards are essentially predicates stating that future clauses do not match (so, they can all be formulated using the $\neq$-operator). Moreover, the additional meta theory needed for this kind of support is fairly simple. In this case, that adding a constant can be inverted by subtracting it, and that one of the arguments of addition must be an available expression in the term returned by the call.

## 5 Translation to flowchart languages

One reason for putting recursive functions on a tail recursive form is for efficiency, as tail recursive programs can be easily compiled to iterative loopconstructs, eliminating the overhead of function calls. We sketch here how the transformed programs can be translated to a reversible loop-construct from flowchart languages [23] (see also [3,4]), which can be implemented in Janus [12, 25] and later be compiled [2] to reversible abstract machines such as PISA [22] or BobISA [21].

We remind the reader that the reversible loop has the following structure:


The entry assertion must only be true on entry to the loop, while the exit condition will only be true in the final iterations. For completeness there are two statements in the loop: the upper (called pre/post statement) we can use to transform between the input/output state and the iterative state, while the lower (called iterative statement) is the most widely used as this has similar semantics to the normal while-loop.

We will show the translation based on the reverse3 example from before.

```
data Configuration a = Input a | Iteration (a, a)| Output a
reverse3 xs = let (Output ys) = interpret (Input xs) in ys
interpret (Input xs) = interpret (Iteration (xs, [ ]))
interpret (Iteration (x:xs, ys)) = interpret (Iteration (xs, x : ys))
interpret (Iteration ([ ], ys)) = (Output ys)
```

The first step is to apply our transformation to yield a tail recursive function; here, this has already been done. Next, we must translate the functional abstract data types to imperative values. The Configuration type will be translated into an enumeration type, with the values Input, Iteration, and Output encoded at integers (e.g. 1, 2, and 3). We would also need to encoded the function data (here the two lists), which could be done with an arrays and a given length. We will, however, not dwell on the data encoding, as our focus is the translation of code that our translation generates.

We can now construct the reverse3 procedure that will contain the loop. This will be given the encoded list and return the reversed encoded list. Here a full compiler (again outside our scope) should also be aware that e.g. Janus restricts to call-by-reference, making it needed to compile the function to inline data handling. Though, this is not a restriction in reversible assembly languages. In the beginning of reverse3 we will create a local variable configuration that is initialised to Input. After the loop, this variable will be delocalised with the value Output. At the entry to the loop, the available variables will, thus, be configuration and the function data (i.e. the encoding of the two lists).

The reversible loop will implement the interpret function. We assume that there exist a translation of the data handling, meaning that we have the two procedures
empty that checks if the encoded list is empty, and
move that move the first element of an encode list to the other.
With this, we mechanically derive the four components of the loop as
Entry assertion: configuration = Input. We have defined that is the only valid value at entrance. Afterwards it will not be used.

Exit condition: configuration = Output. Similar to before, this value is only used on exit from the function.

Pre/post statement: Line 1 and 3. These two lines can be implemented as two conditions in sequence, similar to

$$
\begin{aligned}
& \text { if } \quad(\text { configuration }=\text { Input }) \\
& \text { then configuration++ // Update from enum Input to Iteration }
\end{aligned}
$$

```
fi \(\quad(\) configuration \(=\) Iteration and empty(ys))
if ( configuration \(=\) Iteration and empty(xs))
then configuration \(++/ /\) Update from enum Iteration to Output
\(\mathrm{fi} \quad(\) configuration \(=\) Output \()\)
```

Here, the first condition transforms the Input value to an Iteration value with the assertion that the resulting list is empty, while the second condition transforms a Iteration value with an empty list to an Output value with an assertion that we now have an output value.

Iterative statement: Line 2. This performs the iterative computation, generating code similar to

```
8 if ( configuration = Iteration and (not empty(xs)))
9 then move(xs,ys) // Update from enum Input to Iteration
10 fi (configuration = Iteration and (not empty(ys)))
```

For completeness we check and assert that configuration $=$ Iteration, though this is clear from the translation. We also assure correct data handling, by checking that the relevant lists are non-empty (matching the pattern matching of the function) and implement the relevant data handling (the move function).

The generated program could be more efficient, but it clearly demonstrates how the datatype Configuration translates to a reversible loop. The hard work is in the encoding of the data. This approach also applies to functions that have more one function clause with Input and Output cases, and more iterative clauses.

## 6 Discussion and related work

While it is possible to invert all injective functions [1, 14], inverse programs constructed this way are often not very efficient. In spite of this, specific inversion methods tend to have well-defined subsets of programs for which they can produce efficient inverses.

Precisely classifying the problems which can be efficiently inverted is hard, so the problem is usually approached from a program-specific perspective. One approach is to restricting programs to be formulated in a way that is particularly conducive to inversion. Another approach is grammar-based-inversion, which works by classifying how hard it is to invert a function, based on the properties of a grammar derived from the function body that decides whether or not a given value is in its range [7,13].

An alternative perspective on finding efficient inverse programs is to acknowledge the huge body of knowledge that has been produced in order to optimized programs running in the forward direction for time complexity, and see if we can bring those optimizations into the realm of reversible computing. In doing so we have not found a need to invent new class of programs to invert. Instead, we enable existing techniques for optimizing CPS transformed programs to be leveraged on programs which do not naturally allow for CPS transformation.

The technique we use for transforming programs into tail recursive form is essentially Nishida \& Vidal's method for continuation passing style for first
order programs [17]. In doing so, we introduce an extra function that evaluates a data type that represents a continuation.

In related work on grammar/syntax based inversion techniques [6, 18], wellformed with respect to inversion means that the function is linear in its arguments (and so does not throw anything away), and that cases are syntactically orthogonal. Programs that are well-formed in this sense allow inversion by applying known inversion methods to the iteration function, which then becomes a non-deterministic inverse program (since it need not be injective). However, existing methods for non-determinism elimination can be applied to solve this problem since the original program was well-formed.

## 7 Conclusion

In this work we have shown that invertible programs admit a tail recursion transformation, provided that they are syntactically well-formed. This was achieved using a version of the first order CPS transformation tailored to invertible programs. Alternatives that do not have tail recursion optimisation must instead rely on search, which can be prohibitively expensive. Instead of searching, we can enforce determinism by pattern matching. That is, transformations where the non-injective part is introduced by the compiler, we can use a "new datatype trick". Finally, we have shown correctness of our transformation and how the transformed programs can be efficiently compiled to the reversible loops found in reversible flowchart languages, which in turn may serve as a basis for efficient implementations in reversible abstract machines.

### 7.1 Future work

Currently, the transformation is implemented for at subset of Haskell. Future work will be to integrate this into a invertible functional programming languages such as Jeopardy [10, 11].

This work avoids the need for a symbolic and relational intermediate representation. Perhaps future iterations on such an approach will enable a relaxation of the existing methods' very strict requirements (such as linearity), and thus a less restrictive notion of well-formedness, but also a less syntactic notion of the complexity of function invertibility.

A major improvement to the complexity of function invertibility would also be to eschew classifying programs that are hard to invert in favor of classifying problems. One approach could be to see if the grammar-based approach from [13] can be relaxed to grammars that recognize the output of the function, rather than grammars generated by the syntactic structure of the output of a program.

An example of such a relaxation would to allow existential variables. That is, to split the mechanism of introducing a variable symbol from the mechanism that associates it with a value (its binder). This is customary in logic programming languages such as Prolog, where programs express logical relationships that are solved for all possible solutions based on backtracking that redefines variable bindings. In a functional language, such a mechanism could try to postpone the need to use a free variable until as late as possible, allowing partially invertible functions that accept and return partial data structures (containing logical variables) that may be combined to complete ones (free of logical variables) when
composed in certain ways. We are currently exploring this concept further in related work on the Jeopardy programming language [11].

The use of existential variables could further enable the relaxation of the linearity constraint beyond relevance, such that an iterator function may reconstruct a partial term (containing free varaibles) which is then unified with the available knowledge about its origin, if it is possible to unify it to a complete term (not containing free variables). We have developed an analysis to infer per-program-point sets of such information [10], which may be combined with control flow analysis to decide on a suitable program point in which to unify.

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