# Frank number and nowhere-zero flows on graphs\*

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**Abstract.** An edge e of a graph G is called *deletable* for some orientation o if the restriction of o to G-e is a strong orientation. Inspired by a problem of Frank, in 2021 Hörsch and Szigeti proposed a new parameter for 3-edge-connected graphs, called the Frank number, which refines k-edge-connectivity. The Frank number is defined as the minimum number of orientations of G for which every edge of G is deletable in at least one of them. They showed that every 3-edge-connected graph has Frank number at most 7 and that in case these graphs are also 3-edge-colourable the parameter is at most 3. Here we strengthen both results by showing that every 3-edge-connected graph has Frank number at most 4 and that every graph which is 3-edge-connected and 3-edge-colourable has Frank number 2. The latter also confirms a conjecture by Barát and Blázsik. Furthermore, we prove two sufficient conditions for cubic graphs to have Frank number 2 and use them in an algorithm to computationally show that the Petersen graph is the only cyclically 4-edge-connected cubic graph up to 36 vertices having Frank number greater than 2.

**Keywords:** Frank number, Connectivity, Orientation, Snark, Nowhere-zero flows

## 1 Introduction

An orientation (G, o) of a graph G is a directed graph with vertices V(G) such that each edge  $uv \in E(G)$  is oriented either from u to v or from v to u by the function o. An orientation is called strong if, for every pair of distinct vertices u and v, there exists an oriented uv-path, i.e. an oriented path starting at vertex u and ending at vertex v. It is not difficult to see that an orientation is strong if and only if each edge cut contains edges oriented in both directions.

An edge e is deletable in an orientation (G, o) if the restriction of o to  $E(G) - \{e\}$  yields a strong orientation of G - e. A graph in which the removal of fewer than k edges cannot separate the graph into two components which both contain a cycle, is called cyclically k-edge-connected. The cyclic edge connectivity of a graph G is the largest k for which G is cyclically k-edge-connected.

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Inspired by a problem of Frank, in 2021, Hörsch and Szigeti [10] proposed a new parameter for 3-edge-connected graphs called the Frank number. This parameter can be seen as a generalisation of a theorem by Nash-Williams [13] stating that a graph has a k-arc-connected orientation if and only if it is 2k-edge-connected.

**Definition 1.** For a 3-edge-connected graph G, the Frank number – denoted by fn(G) – is the minimum number k for which G admits k orientations such that every edge  $e \in E(G)$  is deletable in at least one of them.

Note that Definition 1 does not make sense for graphs which are not 3-edge-connected as such a graph G has at least one edge which is not deletable in any orientation of G.

A first general upper bound for the Frank number was established by Hörsch and Szigeti in [10]. They proved that  $fn(G) \leq 7$  for every 3-edge-connected graph G. Moreover, in the same paper it is shown that the Berge-Fulkerson conjecture [14] implies that  $fn(G) \leq 5$ . We improve the former result by showing the following upper bound.

**Theorem 1.** Every 3-edge-connected graph G has  $fn(G) \leq 4$ .

We would also like to note that in [2] Barát and Blázsik very recently independently proved that  $fn(G) \leq 5$  using methods similar to ours based on our Lemma 1 from this paper.

In [10], Hörsch and Szigeti also conjectured that every 3-edge-connected graph G has  $fn(G) \leq 3$  and showed that the Petersen graph has Frank number equal to 3. In this paper we conjecture a stronger statement:

Conjecture 1. The Petersen graph is the only cyclically 4-edge-connected graph with Frank number greater than 2.

Barát and Blázsik showed in [1] that for any 3-edge-connected graph G, there exists a 3-edge-connected cubic graph H with  $fn(H) \geq fn(G)$ . Similarly, it is not difficult to see that if G is a cyclically 4-edge-connected 3-edge-connected graph, then there exists a cubic cyclically 4-edge-connected graph H such that  $fn(H) \geq fn(G)$ . It is obtained by inflating each vertex of degree greater than 3 to a circuit with the correct cyclic ordering. Hence, it is enough to prove the conjecture for cubic graphs and in the remainder, we will mainly focus on them. Note that since cubic graphs cannot be 4-edge-connected, their Frank number is at least 2.

Hörsch and Szigeti proved in [10] that every 3-edge-connected 3-edge-colourable graph has Frank number at most 3. We remark that such graphs are always cubic. We strengthen this result by showing that these graphs have Frank number equal to 2. In fact, we prove the following more general theorem.

**Theorem 2.** If G is a 3-edge-connected graph admitting a nowhere-zero 4-flow, then  $fn(G) \leq 2$ . In particular, fn(G) = 2 for every 3-edge-connected 3-edge-colourable graph G.

It is also verified in [1] that several well-known infinite families of 3-edge-connected graphs have Frank number 2. This includes wheel graphs, Möbius ladders, prisms, flower snarks and an infinite subset of the generalised Petersen graphs. Note that except for the wheel graphs and flower snarks, these families all consist of 3-edge-colourable graphs. In the same paper it is also conjectured that every 3-edge-connected hamiltonian cubic graph has Frank number 2. Since every hamiltonian cubic graph is 3-edge-colourable, Theorem 2 also proves this conjecture.

The main tool in the proofs of the two mentioned results make use of nowhere-zero integer flows. We give a sufficient condition for an edge to be deletable in an orientation

which is the underlying orientation of some all-positive nowhere-zero k-flow and construct two specific nowhere-zero 4-flows that show that the Frank number is 2.

Moreover, we also give two sufficient conditions for cyclically 4-edge-connected cubic graphs to have Frank number 2. We propose a heuristic algorithm and an exact algorithm for determining whether the Frank number of a 3-edge-connected cubic graph is 2. The heuristic algorithm makes use of the sufficient conditions mentioned earlier. Using our implementation of these algorithms we show that the Petersen graph is the only cyclically 4-edge-connected cubic graph up to 36 vertices with Frank number greater than 2. This implies a positive answer for Conjecture 1 up to this order in the family of cubic graphs.

After the introduction and preliminaries, our paper is divided into two main sections: Section 2 which is devoted to theoretical results and Section 3 which focuses on the algorithmic aspects of this problem. More precisely, in Section 2 we first prove our key Lemma 1 and use it to prove Theorems 1 and 2. Here, we also provide sufficient conditions for a cubic graph to have Frank number 2. In Section 3 we describe the algorithms and use them to check Conjecture 1 for nontrivial non-3-edge-colourable cubic graphs up to 36 vertices. Together with our theoretical results this proves that there is no cubic counterexample to Conjecture 1 up to 36 vertices.

#### 1.1 Preliminaries

Let H be an abelian group. An H-flow (o, f) on a graph G consists of an orientation (G, o) and a valuation  $f: E(G) \to H$  assigning elements of H to the edges of G in such a way that for every vertex v of G the sum of the values on the incoming edges is the same as the sum of the values on the outgoing edges from v. A  $\mathbb{Z}$ -flow is called a k-flow if the function f only takes values in  $\{0, \pm 1, \pm 2 \dots, \pm (k-1)\}$ . An H-flow (o, f) (or a k-flow) is said to be nowhere-zero if the value of f is not the identity element  $0 \in H$   $(0 \in \mathbb{Z})$  for any edge of E(G).

A nowhere-zero k-flow on G is said to be *all-positive* if the value f(e) is positive for every edge e of G. Every nowhere-zero k-flow can be transformed to an all-positive nowhere-zero k-flow by changing the orientation of the edges with negative f(e) and changing negative values of f(e) to -f(e).

Let (G, o) be an orientation of a graph G. Let H be a subgraph of G. If the context is clear we write (H, o) to be the orientation of H where o is restricted to H. We define the set  $D(G, o) \subseteq E(G)$  to be the set of all edges of G which are deletable in (G, o). Let  $u, v \in V(G)$ , if the edge uv is oriented from u to v, we write  $u \to v$ .

In the following proofs we will combine two flows into a new one as follows. Let  $(o_1, f_1)$  and  $(o_2, f_2)$  be k-flows on subgraphs  $G_1$  and  $G_2$  of a graph G, respectively. For  $i \in \{1, 2\}$  we extend the flow  $(o_i, f_i)$  to flows on G, still called  $(o_i, f_i)$ , by setting the value of  $f_i$  to be 0 and setting the orientation  $o_i$  arbitrarily for the edges not in  $G_i$ . For those edges e of G where  $o_1(e) \neq o_2(e)$ , we change both the orientation of  $o_2(e)$  and the value  $f_2(e)$  to  $-f_2(e)$  thereby transforming  $(o_2, f_2)$  to a flow  $(o_1, f'_2)$ . The combination of flows  $(o_1, f_1)$  and  $(o_2, f_2)$  is the flow  $(o_1, f_1 + f'_2)$  on G. Transforming this obtained flow to an all-positive flow, we get a flow (o, f) on G, which we call the positive combination of flows  $(o_1, f_1)$  and  $(o_2, f_2)$ .

A *smooth orientation* of a set of circuits is an orientation such that at each vertex, one edge is incoming and one edge is outgoing.

Let  $G_1$  and  $G_2$  be two subgraphs of G and let  $(G_1, o_1)$  and  $(G_2, o_2)$  be two orientations. Let  $Z \subseteq E(G_1) \cap E(G_2)$ . We say that  $(G_1, o_1)$  and  $(G_2, o_2)$  are consistent on Z if  $o_1$  and  $o_2$  agree on all the edges from Z.

## 2 Theoretical results

Let (o, f) be an all-positive nowhere-zero k-flow on a cubic graph G. An edge e with f(e) = 2 is called a *strong* 2-edge if G has no 3-edge-cut containing the edge e such that the remaining edges of the cut have value 1 in f.

**Lemma 1.** Let G be a 3-edge-connected graph and let (o, f) be an all-positive nowhere-zero k-flow on G for some integer k. Then all edges of G which receive value 1 and all strong 2-edges in (o, f) are deletable in o.

Proof. Let G be a 3-edge-connected graph. Let (o, f) be an all-positive nowhere-zero k-flow on G for some integer k. Assume further that e is an edge with f(e) = 1. Suppose that there exist two vertices of G, say u and v, such that there is no oriented uv-path in (G - e, o). Let W be the set of vertices of G to which there exists an oriented path from u in (G - e, o). Obviously, we have  $u \in W$  and  $v \notin W$ . Let W' = V(G) - W. Let us look at the edge-cut G between G and G and G and G are well as G between G and G are well as G in G must be oriented from G to this vertex. Moreover, as G is 3-edge-connected, we have  $|G| \ge 2$ .

Now consider the edge-cut  $S^*$  between W and W' in G; either  $S^* = S$  or  $S^* = S \cup \{e\}$ . Recall that (o, f) is an all-positive flow. Since (o, f) is a flow, it holds that on any edge-cut the sum of the values on the edges oriented in one direction equals the sum of the values on the edges oriented in the other direction. Since all the edges of S are oriented in the same direction, it cannot happen that  $S^* = S$ . So it must be the case that  $S^* = S \cup \{e\}$  and all the edges of S are oriented in the same direction and e is oriented in the opposite orientation. But f(e) = 1 and since  $|S| \ge 2$  and all the values of f on the edges of S are positive, this cannot happen either. Therefore we conclude that for any two vertices e and e there exists an oriented e e0 and so e1 is deletable.

Assume now that e is a strong 2-edge and suppose that e is not deletable. We define the cuts S in G - e and  $S^*$  in G similarly as above. Since f is all-positive and G is 3-edge-connected,  $S^*$  has to contain exactly three edges, e and two edges in S oriented oppositely from e and valuated 1. But this is impossible since e is a strong 2-edge. Therefore e is deletable.

## 2.1 The Frank number of graphs with a nowhere-zero 4-flow

In this section, we prove Theorem 2. In the proof we utilize Lemma 1 and carefully apply the fact that every nowhere-zero 4-flow can be expressed as a combination of two 2-flows.

Proof of Theorem 2. Since G admits a nowhere-zero 4-flow, it also admits a nowhere-zero  $(\mathbb{Z}_2 \times \mathbb{Z}_2)$ -flow by a famous result of Tutte [16]. Let us denote by A the set of edges with value (0,1), by B the set of edges with value (1,0) and by C the set of edges with value (1,1) in this flow. It is not difficult to check that at any vertex, the parity of the number of A-edges is the same as the parity of the number of B-edges and that the parity of the number of A- and B-edges is the same as the parity of the number of C-edges.

Consider the subgraph  $G_1$  of G induced by all A- and B-edges. Clearly, the degree of every vertex of  $G_1$  is even, therefore  $G_1$  consists of edge-disjoint circuits. Note that a vertex can belong to more than one circuit. Similarly, the subgraph  $G_2$  induced by all B- and C-edges has every vertex of even degree and so consists of edge-disjoint circuits.

Now fix an orientation  $(G_1, o_1)$  of the circuits in  $G_1$  and an orientation  $(G_2, o_2)$  of the circuits in  $G_2$ . Set the value  $f_i$  to be i for the edges lying in  $G_i$ . Denote by (o, f) the positive combination of the flows  $(o_1, f_1)$  and  $(o_2, f_2)$ . The value 1 in (o, f) is on all the edges of A and on those edges of B that have different orientation in  $(G_1, o_1)$  and  $(G_2, o_2)$ .

Now we construct a complementary all-positive nowhere-zero 4-flow on G in a sense that this flow will have 1 exactly on those edges where (o, f) had not. For  $i \in \{1, 2\}$  we set  $o'_i = o_i$  on  $G_i$ . We set  $f'_1(e) = 2$  if  $e \in G_1$  and  $f'_2(e) = -1$  if  $e \in G_2$ . We create a flow (o', f') of G as the positive combination of the flows  $(o'_1, f'_1)$  and  $(o'_2, f'_2)$ .

Summing up, we have constructed two all-positive nowhere-zero 4-flows on G flows (o, f) and (o', f'). The edges of A are valuated 1 in (o, f). The edges of C are valuated 1 in (o', f'). The edges e of B where  $o_1(e) \neq o_2(e)$  are valuated 1 in (o, f). The edges e of B where  $o_1(e) = o_2(e)$  are valuated 1 in (o', f'). Therefore, each edge has value 1 either in (o, f) or in (o', f') and by Lemma 1 we have that fn(G) = 2.

It is known that a cubic graph is 3-edge-colourable if and only if it admits a nowhere-zero 4-flow. Therefore the second part of the theorem follows.  $\Box$ 

Since every hamiltonian cubic graph is 3-edge-colourable, we have also shown the following conjecture by Barát and Blázsik [1].

Corollary 1. If G is a 3-edge-connected cubic graph admitting a hamiltonian cycle, then fn(G) = 2.

## 2.2 A general upper bound for the Frank number

In this section we prove Theorem 1 and thereby improve the previous general upper bound by Hörsch and Szigeti from 7 to 4. As a main tool we again use Lemma 1.

*Proof of Theorem 1.* By a result of Barát and Blázsik it is sufficient to prove the theorem for 3-edge-connected cubic graphs.

Let G be a 3-edge-connected cubic graph. By Seymour's 6-flow theorem [15], G has a nowhere-zero ( $\mathbb{Z}_2 \times \mathbb{Z}_3$ )-flow (o, f). The edges  $e \in E(G)$  for which f(e) is non-zero in the first coordinate induce a subgraph D. As every vertex in G can either have none or two of such edges, D is a set of circuits. The edges  $e \in E(G)$  for which f(e) is non-zero in the second coordinate induce a subgraph H'. As H' admits a nowhere-zero 3-flow there are no vertices of degree 1 in H', hence H' consists of a set of circuits and a subdivision of a cubic graph H. The graph H is bipartite since it is cubic and has a nowhere-zero 3-flow.

We will create four all-positive nowhere-zero k-flows on G using the subgraphs D and H' such that each edge is valuated 1 in at least one of the flows. Then Lemma 1 will imply the result.

Since H is cubic and bipartite, by Kőnig's line colouring theorem [12], it is 3-edge-colourable. Hence, it admits a proper edge-colouring with colours a, b, c. Fixing such a colouring  $\varphi$ , we find a (not-necessarily proper) edge-colouring  $\varphi'$  on B' as follows. If an edge e in H corresponds to a path P in H', we colour all the edges of P in H' by  $\varphi(e)$ . All the edges lying on circuits in H' will receive the colour a in  $\varphi'$ . Denote the set of edges with colour a, b, or c in  $\varphi'$  in H' by A, B, or C, respectively.

Now fix an orientation  $(D, o_D)$  of the circuits in D and an orientation of H by directing all edges from one partite set to the other. We can find an orientation  $(H', o_{H'})$  of H' as follows. Each oriented edge in H will correspond to an oriented path in H', oriented in the same direction. We take any orientation of the circuits of H'. This fixes an orientation  $(H', o_{H'})$  of H'.

We now partition the edges of G based on the orientations and colors of D and H'. Denote by  $D_0$  the set of edges which lie only in D. Denote by  $A_0$  the set of edges which lie only in A, by  $A_+$  the set of edges e lying in A and D such that  $o_{H'}$  and  $o_D$  have the same direction for e and by  $A_-$  the set of edges e in A and D such that  $o_{H'}$  and  $o_D$  direct e oppositely. Similarly, we define  $B_0$ ,  $B_+$ ,  $B_-$  and  $C_0$ ,  $C_+$  and  $C_-$ . It is easy to check that every edge belongs to exactly one of  $D_0$ ,  $A_0$ ,  $A_+$ ,  $A_-$ ,  $B_0$ ,  $B_+$ ,  $B_-$ ,  $C_0$ ,  $C_+$ , and  $C_-$ .

Flow	$g_{i,D}$	$g_{i,A}$	$g_{i,B}$	$g_{i,C}$	$h_i(e) = \pm 1$
$(o_1, h_1)$	1	2	2	-4	$e \in D_0 \cup A \cup B$
$(o_2, h_2)$	3	1	1	-2	$e \in A_0 \cup B_0 \cup C_+$
$(o_3, h_3)$	2	3	-4	1	$e \in C_0 \cup A \cup C$
$(o_4,h_4)$	2	-3	-1	4	$e \in A_+ \cup B_+ \cup B_0$

Table 1: Four nowhere-zero k-flows defined by the procedure described in the proof of Theorem 1.

Each of the four nowhere-zero flows  $(o_i, h_i)$  for  $i \in \{1, 2, 3, 4\}$  will be the positive combination of a flow on D and a flow on H'. In each of the four cases cases we proceed as follows.

We define flows  $(o_{i,1}, g_{i,1})$  on D for  $i \in \{1, 2, 3, 4\}$ . For edges  $e \in E(D)$ , let  $o_{i,1}(e) = o_D(e)$  and  $g_{i,1}(e) = g_{i,D}$  where  $g_{i,D}$  is the value according to Table 1.

We define flows  $(o_{i,2}, g_{i,2})$  on H' for  $i \in \{1, 2, 3, 4\}$ . For edges  $e \in E(H')$ , let  $o_{i,2}(e) = o_{H'}(e)$  and let  $g_{i,2}(e)$  equal  $g_{i,A}$ ,  $g_{i,B}$  or  $g_{i,C}$  if e is in A, B or C, respectively, where  $g_{i,A}$ ,  $g_{i,B}$ , and  $g_{i,C}$  are three values that sum to 0, according to Table 1.

The flow  $(o_i, h_i)$  will be the positive combination of  $(o_{i,1}, g_{i,1})$  and  $(o_{i,2}, g_{i,2})$ . For each  $i \in \{1, 2, 3, 4\}$  the flow values are given in Table 1.

Since for each  $i \in \{1, 2, 3, 4\}$ , the sum of  $g_{i,A}$ ,  $g_{i,B}$  and  $g_{i,C}$  is zero, it is easy to see that  $(o_i, h_i)$  is a k-flow. Moreover, as  $g_{i,D}$ ,  $g_{i,A}$ ,  $g_{i,B}$  and  $g_{i,C}$  are non-zero and  $|g_{i,D}|$  differs from  $|g_{i,A}|$ ,  $|g_{i,B}|$  or  $|g_{i,C}|$ , we see that each  $(o_i, h_i)$  is nowhere-zero.

Finally, one can see that for every edge e,  $h_i(e)$  is 1 for at least one i. The result follows.

## 2.3 Sufficient conditions for Frank number 2

The following lemmas and theorems give two sufficient conditions for a cyclically 4-edge-connected cubic graph to have Frank number 2. These will be used in the algorithm in Section 3.

In the proof of the next lemma we will use the following proposition which can be found in [1] as Proposition 2.2.

**Proposition 1.** Let (G, o) be a strong orientation of a graph G. Assume that an edge e = uv is oriented from u to v in (G, o). The edge e is deletable in (G, o) if and only if there exists an oriented uv-path in (G - e, o).

**Lemma 2.** Let (G, o) be a strong orientation of a cubic graph G. Let  $e_1 = u_1v_1$  and  $e_2 = u_2v_2$  be two nonadjacent edges in G such that (G, o) contains  $u_1 \to v_1$  and  $u_2 \to v_2$ . Assume that both  $e_1$  and  $e_2$  are deletable in (G, o). Create a cubic graph G' from G by subdividing the edges  $e_1$  and  $e_2$  with vertices  $e_1$  and  $e_2$ , respectively, and adding a new edge between  $e_1$  and  $e_2$ . Let  $e_1$  be the orientation of  $e_1$  containing  $e_1$  and  $e_2$  are deletable in  $e_1$  be the orientation of  $e_2$  containing  $e_1$  and  $e_2$  and such that  $e_2$  be the remaining edges of  $e_1$ . Then

$$D(G',o') \supseteq (D(G,o) - \{e_1,e_2\}) \cup \{x_1v_1,x_1x_2,u_2x_2\}.$$

*Proof.* First of all, define (G', o') as in the statement of this Lemma. Then it is a strong orientation. Indeed, since (G, o) is a strong orientation of G, any edge-cut in G contains edges in both directions in (G, o). Therefore, any edge-cut in G' contains edges in both directions in (G', o').

Now we are going to show that

$$D(G',o') \supseteq D(G,o) - \{e_1,e_2\} \cup \{x_1v_1,x_1x_2,u_2x_2\}.$$

Let  $e = pr \in D(G, o) - \{e_1, e_2\}$  and let  $p \to r$  be an oriented edge in (G, o). By Proposition 1 we have to show that there is an oriented pr-path in (G' - e, o'). If neither of p and r belongs to  $\{x_1, x_2\}$ , that is p and r belong to V(G), then, since (G, o) is a strong orientation of G, there exists an oriented pr-path R in (G, o). Then R with possible subdivisions by  $x_1$  and  $x_2$  if  $e_1 \in R$  or  $e_2 \in R$  is the required pr-path in (G' - e, o').

If  $pr = x_1v_1$ , then we find a  $v_2v_1$ -path  $R_1$  in  $(G-e_1, o)$ , which exists since  $e_1$  is deletable in o. The path  $x_1x_2v_2R_1$  is an oriented  $x_1v_1$ -path in  $(G'-x_1v_1, o')$ .

If  $pr = u_2x_2$ , then we find a  $u_2u_1$ -path  $R_2$  in  $(G - e_2, o)$ , which exists since  $e_2$  is deletable in o. The path  $R_2u_1x_1x_2$  is an oriented  $u_2x_2$ -path in  $(G' - u_2x_2, o')$ .

Finally, if  $pr = x_1x_2$ , we find a  $v_1u_2$ -path  $R_3$  in (G, o). The path  $x_1v_1R_3u_2x_2$  is an oriented  $x_1x_2$ -path in  $(G' - x_1x_2, o')$ .

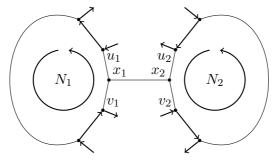


Figure 1: A smooth orientation of the circuits in  $F - \{x_1x_2\} \cup M$  and of those in C consistent on the edges of  $N_i$  at distance 1 from  $x_i$ , for  $i \in \{1, 2\}$ .

**Theorem 3.** Let G be a cyclically 4-edge-connected cubic graph. Let C be a 2-factor of G with exactly two odd circuits, say  $N_1$  and  $N_2$  (and possibly some even circuits). Let  $e = x_1x_2$  be an edge of G such that  $x_1 \in V(N_1)$  and  $x_2 \in V(N_2)$ . Let F = G - C and let G be a maximum matching in G and G if there exists a smooth orientation of the circuits in G in G in G and a smooth orientation of G which are consistent on the edges of G is at distance 1 from G for both G in G see G in G

*Proof.* For  $i \in \{1,2\}$  denote by  $u_i$  and  $v_i$  the vertices of  $N_i$  which are adjacent to  $x_i$  and let the rest of notation be as in the statement of the theorem. Consider the graph  $G \sim e$ , that is the graph created by deleting e and smoothing the two 2-valent vertices. First, we observe that  $G \sim e$  is cyclically 3-edge-connected. This can be easily seen, because if  $G \sim e$  had a cycle-separating k-edge-cut S for some k < 3, then  $S \cup \{e\}$  would be a set of k+1 edges separating two components containing circuits, so the cyclic connectivity of G would be at most k+1 < 4, a contradiction.

Let  $(F - \{e\} \cup M, o_2)$  be an orientation of the circuits in  $F - \{e\} \cup M$  consistent on the edges of  $N_i$  which are at distance 1 from  $x_i$  for both  $i \in \{1, 2\}$ . By our assumption such an orientation exists. Let  $(C, o_1)$  be such an orientation of the circuits in C that the edges of  $N_i \cap M$  incident with  $u_i$  and  $v_i$  on  $N_i$  are oriented equally in  $(C, o_1)$  and  $(F - \{e\} \cup M, o_2)$ . With slight abuse of notation, we will also consider C and  $F - \{e\} \cup M$  to be subgraphs of  $C \sim e$  and consider C = e and consider C = e orientations of these subgraphs.

We define two nowhere-zero 4-flows (o', f') and (o'', f'') on  $G \sim e$  such that  $D(G \sim e, o') \cup D(G \sim e, o'') = E(G \sim e)$  and  $\{u_1v_1, u_2v_2\} \subset D(G \sim e, o') \cap D(G \sim e, o'')$ . This will by Lemma 2 imply that fn(G) = 2.

We define flows  $(o_1, f_1')$  on C and  $(o_2, f_2')$  on  $F - \{e\} \cup M$ . For edges  $d \in E(C)$ , let  $f_1'(d) = 1$ . For edges  $d \in E(F - \{e\} \cup M)$ , let  $f_2'(d) = -2$ . The flow (o', f') on  $G \sim e$  will be the positive combination of  $(o_1, f_1')$  and  $(o_2, f_2')$ .

Similarly, we define flows  $(o_1, f_1'')$  on C and  $(o_2, f_2'')$  on  $F - \{e\} \cup M$ . For edges  $d \in E(C)$ , let  $f_1''(d) = 2$ . For edges  $d \in E(F - e \cup M)$ , let  $f_2''(d) = 1$ . The flow (o'', f'') on  $G \sim e$  will be the positive combination of  $(o_1, f_1'')$  and  $(o_2, f_2'')$ .

It is easy to see that both (o', f') and (o'', f'') are nowhere-zero. We also see that the edges of C-M are valuated 1 in (o', f'), the edges of  $F-\{e\}$  are valuated 1 in (o'', f'') and the edges of M are valuated 1 in (o', f') if  $o_1$  and  $o_2$  agree and are valuated 1 in (o'', f'') if  $o_1$  and  $o_2$  disagree. Therefore, by Lemma 1, all edges of  $G \sim e$  are deletable in one of these orientations and so  $D(G \sim e, o') \cup D(G \sim e, o'') = E(G \sim e)$ .

It remains to show that the edges  $u_1v_1$  and  $u_2v_2$  are deletable both in  $(G \sim e, o')$  and in  $(G \sim e, o'')$ . By Lemma 1, they are deletable in  $(G \sim e, o')$ . Now we show that  $u_1v_1$  and  $u_2v_2$  are strong 2-edges in (o'', f''). Suppose that, for some  $i \in \{1, 2\}$ , the edge  $u_iv_i$  is not a strong 2-edge in (o'', f''), say  $u_1v_1$  is not a strong 2-edge. Since we have already observed that  $G \sim e$  is cyclically 3-edge-connected, this implies that  $u_1v_1$  belongs to a 3-edge-cut, say R, and the other two edges of this 3-edge-cut have to be valuated 1. The cut R must be cycle-separating. Otherwise, it would separate either  $u_1$  or  $v_1$  from the rest of the graph, contradicting the fact that the edges of M incident with  $u_1$  and  $v_1$  are valuated 3 in (o'', f'').

We will use the symbols  $N_1$  and  $N_2$  to denote also the circuits in  $G \sim e$  corresponding to  $N_1$  and  $N_2$ . Since every edge-cut intersects a circuit an even number of times, R contains two edges from  $N_1$  (one of them is  $u_1v_1$ , let the other be g) and an edge  $f \in E(F) \setminus \{e\}$ . Let  $(V_1, V_2)$  be the partition of V(G) corresponding to the cut R. All the vertices of the circuit  $N_2$  belong either to  $V_1$  or to  $V_2$ . In the former case, the partition  $(V_1 \cup \{x_1, x_2\}, V_2)$  and in the latter case, the partition  $(V_1, V_2 \cup \{x_1, x_2\})$  form a partition corresponding to a 3-edge-cut in G. As the two edges of this cut belonging to  $N_1$  are independent the cut is a cycle-separating 3-edge-cut in G, which is a contradiction.

**Lemma 3.** Let (G, o) be a strong orientation of a cubic graph G. Let  $e_1 = u_1v_1$ ,  $e_2 = u_2v_2$ , and  $f = w_2w_1$  be pairwise independent edges in G such that (G, o) contains  $u_1 \to v_1$ ,  $u_2 \to v_2$ , and  $w_2 \to w_1$  and such that these edges are deletable in (G, o). Let a cubic graph G' be created from G by performing the following steps:

- subdivide the edges  $e_1$  and  $e_2$  with the vertices  $x_1$  and  $x_2$ , respectively,
- subdivide the edge  $w_1w_2$  with the vertices  $y_1$  and  $y_2$  (in this order), and
- add the edges  $x_1y_1$  and  $x_2y_2$ .

Let (G', o') be the orientation of G' containing  $u_1 \to x_1$ ,  $x_1 \to v_1$ ,  $y_1 \to w_1$ ,  $y_2 \to y_1$ ,  $w_2 \to y_2$ ,  $u_2 \to x_2$ ,  $x_2 \to v_2$  and such that o'(e) = o(e) for all the remaining edges of G' except for  $x_1y_1$  and  $x_2y_2$ . Then

- (a) if (G', o') contains  $y_1 \to x_1$  and  $x_2 \to y_2$ , (G', o') is a strong orientation of G' and  $D(G', o') \supseteq D(G, o) \{e_1, e_2, f\} \cup \{u_1x_1, x_1y_1, y_1w_1, y_2w_2, x_2y_2, x_2v_2\}$  (Fig. 2(left));
- (b) if (G', o') contains  $x_1 \to y_1$  and  $y_2 \to x_2$ , (G', o') is a strong orientation of G' and  $D(G', o') \supseteq D(G, o) \{e_1, e_2, f\} \cup \{x_1v_1, y_1y_2, u_2x_2\}$  (Fig. 2(right)).

*Proof.* First of all, if (G', o') is defined either as in (a) or as in (b), then it is a strong orientation. Indeed, since (G, o) is a strong orientation of G, any edge-cut in G contains edges in both directions in (G, o). Therefore, any edge-cut in G' contains edges in both directions in (G', o').

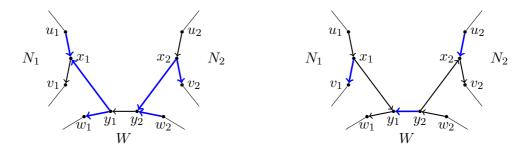


Figure 2: A part of G' and orientation (G', o') as defined in Lemma 3. The left-hand-side corresponds with the orientation of (a) and the right-hand-side corresponds with the orientation of (b). If the conditions of Lemma 3 are met the thick, blue edges will be deletable.

To prove (a) we need to show that every edge from

$$D(G,o) - \{e_1, e_2, f\} \cup \{u_1x_1, x_1y_1, y_1w_1, y_2w_2, x_2y_2, x_2v_2\}$$

is deletable in (G',o'). To do so, it is enough to show that for any edge e=pr in this set oriented as  $p \to r$ , there exists an oriented pr-path in (G'-e,o'). If  $e \in D(G,o)-\{e_1,e_2,f\}$ , then we can use the fact that e is a deletable edge in (G,o) and therefore there is an oriented pr-path in (G-e,o). The corresponding pr-path in (G',o') possibly subdivided by some of the vertices  $x_1, x_2, y_1, y_2$  is a required path.

If  $pr = u_1x_1$ , we take a  $u_1w_2$ -path  $R_1$  in  $G - e_1$  (it exists since  $e_1$  is deletable in (G, o)). Possibly subdivide  $R_1$  to  $R'_1$  in (G', o'). The path  $R'_1w_2y_2y_1x_1$  is an oriented  $u_1x_1$ -path in  $(G' - u_1x_1, o')$ , so  $u_1x_1$  is deletable in (G', o').

We proceed in this way. For every of the five remaining edges we find an oriented path R in (G, o) in which possibly one of the edges  $e_1, e_2, f$  is forbidden (such a path exists as each of these three edges is deletable in (G, o)), possibly subdivide R to R' in (G', o') and finally find the required oriented path T in (G' - e, o'). It is summarised in the following table.

e	R	T
$u_1x_1$	$u_1w_2$ -path in $G - e_1$	$R'w_2y_2y_1x_1$
$y_1x_1$	$w_1u_1$ -path in $G$	$y_1w_1R'u_1x_1$
$y_1w_1$	$v_1w_1$ -path in $G-f$	$y_1x_1v_1R'$
$w_2y_2$	$w_2u_2$ -path in $G-f$	$R'u_2x_2y_2$
$x_2y_2$	$v_2w_2$ -path in $G$	$x_2v_2R'w_2y_2$
$x_2v_2$	$w_1v_2$ -path in $G - e_2$	$x_2y_2y_1w_1R'$

To prove (b) we proceed analogously as in (a). There are only three edges we have to pay special attention to. Each of the edges corresponds to a line of the following table. This completes the proof.

e	R	T
$x_1v_1$	$w_1v_1$ -path in $G-e_1$	$x_1y_1w_1R'$
$y_{2}y_{1}$	$v_2u_1$ -path in $G-f$	$y_2 x_2 v_2 R' u_1 x_1 y_1$
$u_2x_2$	$u_2w_2$ -path in $G - e_2$	$R'w_2y_2x_2$

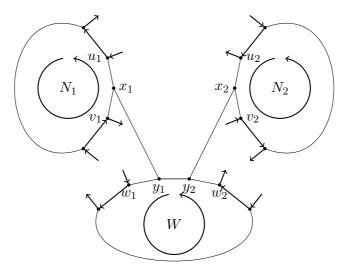


Figure 3: A smooth orientation of the circuits in  $F - \{x_1y_1, y_2x_2\} \cup M$  and of the circuits in C which are consistent on the edges of  $N_i$  at distance 1 from  $x_i$ , for  $i \in \{1, 2\}$ , and on the edges of W at distance 1 from  $y_i$  but not incident with  $y_{3-i}$ , for  $i \in \{1, 2\}$ .

**Theorem 4.** Let G be a cyclically 4-edge-connected cubic graph with a 2-factor C containing precisely two odd circuits  $N_1$  and  $N_2$  and at least one even circuit W. Let  $x_1y_1$ ,  $y_1y_2$  and  $y_2x_2$  be edges of G such that  $x_1 \in V(N_1)$ ,  $x_2 \in V(N_2)$  and  $y_1, y_2 \in V(W)$ . For  $i \in \{1,2\}$  denote by  $u_i$  and  $v_i$  the vertices of  $N_i$  which are adjacent to  $x_i$  and by  $w_i$  the the vertex in  $W - \{y_1, y_2\}$  which is adjacent to  $y_i$  in W. Let F = G - C and let M be a maximum matching in  $C - \{x_1, y_1, y_2, x_2\}$ . If there exists a smooth orientation of the circuits in  $F - \{x_1y_1, x_2y_2\} \cup M$  and a smooth orientation of C which are consistent on the edges of  $N_i$  at distance 1 from  $x_i$  for both  $i \in \{1, 2\}$  and on the edges of W at distance 1 from  $y_i$  but not incident with  $y_{3-i}$ , for  $i \in \{1, 2\}$ , see Fig. 3, and  $G \sim x_1y_1 \sim x_2y_2$  has no cycle-separating set of three edges  $\{e_1, e_2, e_3\}$  with  $e_1 \in \{u_1v_1, u_2v_2, w_1w_2\}$  and  $e_2, e_3 \in E(F - \{x_1y_1, x_2y_2\} \cup M)$ , then  $f_1(G) = 2$ .

Proof. Let  $f_1 = x_1y_1$  and  $f_2 = x_2y_2$ . First we observe that the graph  $G' := G \sim f_1 \sim f_2$  is cyclically 3-edge-connected. Suppose that this is not the case and that we have a cyclic k-edge-cut S, for k < 3. Since G is cyclically 4-edge connected, removing an edge and smoothing the two 2-valent vertices decreases the cyclic connectivity at most by 1. Hence, S must be a cycle separating 2-cut of G'.

It holds that either both edges in the cut belong to  $E(F) \setminus \{f_1, f_2\}$  or both belong to some cycle. Choosing an appropriate third edge such that we get a separating set  $\{e_1, e_2, e_3\}$  as described in the theorem statement, we get a contradiction in the same way as before.

We define nowhere-zero 4-flows (o', f') and (o'', f'') on G' similarly as in the proof of Theorem 3. We can use the arguments from the latter theorem. Since we assume that none of the edges  $u_1v_1, u_2v_2, w_1w_2$  are in a cycle-separating 3-edge-cut with two edges in  $F - \{f_1, f_2\} \cup M$  they are strong 2-edges in (o'', f'') and the result follows from Lemma 1 and Lemma 3.

# 3 Algorithm

We propose two algorithms for computationally verifying whether or not a given 3-edge-connected cubic graph has Frank number 2, i.e. a heuristic and an exact algorithm. Note that the Frank number for 3-edge-connected cubic graphs is always at least 2. Our al-

gorithms are intended for graphs which are not 3-edge-colourable, since 3-edge-connected 3-edge-colourable graphs have Frank number 2 (cf. Theorem 2).

The first algorithm is a heuristic algorithm, which makes use of Theorem 3 and Theorem 4. Hence, it can only be used for cyclically 4-edge-connected cubic graphs. For every 2-factor in the input graph G, we verify if one of the configurations of these theorems is present. If that is the case, the graph has Frank number 2.

More specifically, we look at every 2-factor of G by generating every perfect matching and looking at its complement. We then count how many odd cycles there are in the 2-factor under investigation. If there are precisely two odd cycles, then we check for every edge connecting the two odd cycles whether or not the conditions of Theorem 3 hold. If they hold for one of these edges, we stop the algorithm and return that the graph has Frank number 2. If these conditions do not hold for any of these edges or if there are none, we check for all triples of edges  $x_1y_1, y_1y_2, y_2x_2$ , where  $x_1$  and  $x_2$  lie on different odd cycles and  $y_1$  and  $y_2$  lie on the same even cycle of our 2-factor, whether the conditions of Theorem 4 hold. If they do, then G has Frank number 2 and we stop the algorithm. The pseudocode of this algorithm can be found in Algorithm 1.

Note that in practice, when checking the conditions of Theorem 3 and Theorem 4, we only consider one maximal matching as defined in the statements of the theorems. As we will see, this has no effect on the correctness of the heuristic algorithm and is sufficient for our computations.

The second algorithm is an exact algorithm for determining whether or not a 3-edge-connected cubic graph has Frank number 2. The pseudocode of this algorithm can be found in Algorithm 2. For a graph G, we start by considering each of its strong orientations (G, o) and try to find a complementary orientation (G, o') such that every edge is deletable in either (G, o) or (G, o'). First, we check if there is a vertex in G for which none of its adjacent edges are deletable in (G, o). If this is the case, there exists no complementary orientation as no orientation of a cubic graph can have three deletable edges incident to the same vertex. If (G, o) does not contain such a vertex, we look for a complementary orientation using some tricks to reduce the search space.

More precisely, we first we start with an empty partial orientation, i.e. a directed spanning subgraph of some orientation of G, and fix the orientation of some edge. Note that we do not need to consider the opposite orientation of this edge, since an orientation of a graph in which all arcs are reversed has the same set of deletable edges as the original orientation.

We then recursively orient edges of G that have not yet been oriented. After orienting an edge, the rules of Lemma 4 may enforce the orientation of edges which are not yet oriented. We orient them in this way before proceeding with the next edge. This heavily restricts the number edges which need to be added. As soon as a complementary orientation is found, we can stop the algorithm and return that the graph G has Frank number 2. If for all strong orientations of G no such complementary orientation is found, then the Frank number of G is higher than 2.

Since the heuristic algorithm is much faster than the exact algorithm, we will first apply the heuristic algorithm. After this we will apply the exact algorithm for those graphs for which the heuristic algorithm was unable to decide whether or not the Frank number is 2. In Section 3.1, we give more details on how many graphs pass this heuristic algorithm.

An implementation of these algorithms can be found on GitHub [6]. Our implementation uses bitvectors to store adjacency lists and lists of edges and uses bitoperations to efficiently manipulate these lists.

#### **Algorithm 1** heuristicForFrankNumber2(Graph G)

```
1: for each perfect matching F do
       Store odd cycles of C := G - F in \mathcal{O} = \{N_1, \dots, N_k\}
 2:
        if |\mathcal{O}| is not 2 then
 3:
           Continue with the next perfect matching
 4:
        for all edges x_1x_2 with x_1 \in V(N_1), x_2 \in V(N_2) do
 5:
            // Test if Theorem 3 can be applied
 6:
           Store a maximal matching of C - \{x_1, x_2\} in M
 7:
           Denote the neighbours of x_1 and x_2 in C by u_1, v_1 and u_2, v_2, respectively
 8:
           Denote the set of edges of N_1 and N_2 at distance 1 from x_1 and x_2 by Z
 9:
           Create an empty partial orientation (F - \{x_1, x_2\} \cup M, o)
10:
            for all x \in \{u_1, v_1, u_2, v_2\} do
11:
               if the cycle in F - \{x_1, x_2\} \cup M containing x is not yet oriented then
12:
                   Orient the cycle in F - \{x_1, x_2\} \cup M containing x such that it is smooth
13:
                   and maintaining consistency on Z with some orientation of C if possible.
           if (F - \{x_1, x_2\} \cup M, o) can be extended to a smooth orientation consistent on
14:
            Z with some smooth orientation of C then
               return True // Theorem 3 applies
15:
        for all pairs of edges x_1y_1, x_2y_2 with x_1 \in V(N_1), x_2 \in V(N_2) and y_1, y_2 adjacent
16:
        and on the same even cycle W of C do
            // Test if Theorem 4 can be applied
17:
           Store a maximal matching of C - \{x_1, y_1, y_2, x_2\} in M
18:
           Denote the neighbours of x_1 and x_2 in C by u_1, v_1 and u_2, v_2, respectively
19:
           Denote the neighbour of y_1 in C - y_2 by w_1 and of y_2 in C - y_1 by w_2
20:
           Denote the set of edges of N_i at distance 1 from x_i and of W at distance 1 from
21:
           y_i but not incident with y_{3-i} by Z
22:
           Create an empty partial orientation (F - \{x_1, y_1, y_2, x_2\} \cup M, o)
           for all x \in \{u_1, v_1, u_2, v_2, w_1, w_2\} do
23:
               if the cycle in F - \{x_1, y_1, y_2, x_2\} \cup M with x is not oriented in (G, o) then
24:
                   Orient the cycle in F - \{x_1, y_1, y_2, x_2\} \cup M containing x such that it is
25:
                   smooth and maintaining consistency on Z with some orientation of C
           if (F - \{x_1, y_1, y_2, x_2\} \cup M, o) can be extended to a smooth orientation consistent
26:
           on Z with some orientation of C then
               // Check cycle-separating edge-set condition
27:
               for all pairs of edges e_1, e_2 in F - \{x_1, y_1, y_2, x_2\} \cup M do
28:
                   for all e \in \{u_1x_1, w_1y_1, u_2x_2\} do
29:
                       if \{e, e_1, e_2\} is a cyclic edge-cut in G - x_1y_1 - x_2y_2 then
30:
                           return True // Theorem 4 applies
31:
32: return False
```

**Theorem 5.** Let G be a cyclically 4-edge-connected cubic graph. If Algorithm 1 is applied to G and returns True, G has Frank number 2.

*Proof.* Suppose the algorithm returns True for G. This happens in a specific iteration of the outer for-loop corresponding to a perfect matching F. The complement of F is a 2-factor, say C, and since the algorithm returns True, C has precisely two odd cycles, say  $N_1$  and  $N_2$ , and possibly some even cycles.

Suppose first that the algorithm returns True on Line 15. Then there is an edge  $x_1x_2$  in G with  $x_1 \in V(N_1)$  and  $x_2 \in V(N_2)$ , a maximal matching M of  $C - \{x_1, x_2\}$  and

orientations  $(F - \{x_1x_2\} \cup M, o_1)$  and  $(N_1 \cup N_2, o_2)$  which are consistent on the edges of  $N_i$  at distance 1 from  $x_i$ . Now by Theorem 3 it follows that G has Frank number 2.

Now suppose that the algorithm returns True on Line 31. Then there are edges  $x_1y_1, y_1y_2$  and  $y_2x_2$  such that  $x_1 \in V(N_1), x_2 \in V(N_2)$  and  $y_1, y_2 \in V(W)$  where W is some even cycle in C. Since the algorithm returns True, there is a maximal matching M of  $C - \{x_1, y_1, y_2, x_2\}$  and smooth orientations  $(F - \{x_1x_2\} \cup M, o_1)$  and  $(N_1 \cup N_2 \cup W, o_2)$  which are consistent on the edges of  $N_i$  at distance 1 from  $x_i$  and on the edges of W at distance 1 from  $y_i$  and are not incident with  $y_{3-i}$  for  $i \in \{1, 2\}$ .

Denote the neighbours of  $x_1$  and  $x_2$  in C by  $u_1, v_1$  and  $u_2, v_2$ , respectively and denote the neighbour of  $y_1$  in  $C - y_2$  by  $w_1$  and the neighbour of  $y_2$  in  $C - y_1$  by  $w_2$ . Since no triple  $e, e_1, e_2$ , where  $e \in \{u_1x_1, w_1y_1, u_2x_2\}, e_1, e_2 \in E(F - \{x_1y_1, x_2y_2\} \cup M)$ , is a cycle-separating edge-set of  $G - \{x_1y_1, x_2y_2\}, G \sim x_1y_1 \sim x_2y_2$  has no cycle-separating edge-set  $\{e, e_1, e_2\}$ , where  $e \in \{u_1v_1, u_2v_2, w_1w_2\}$  and  $e_1, e_2 \in E(F - \{x_1y_1, x_2y_2\} \cup M)$ . Now by Theorem 4 it follows that G has Frank number 2.

#### **Algorithm 2** frankNumberIs2(Graph G)

```
1: for all orientations (G, o) of G do
2:
       if (G, o) is not strong then
           Continue with next orientation
3:
       Store deletable edges of (G, o) in a set D
 4:
       for all v \in V(G) do
 5:
 6:
          if no edge incident to v is deletable then
              Continue with next orientation
 7:
       Create empty partial orientation (G, o') of G
8:
       Choose an edge xy in G and fix orientation x \to y in o'
9:
       if not canAddArcsRecursively((G, o'), D, x \rightarrow y) then // Algorithm 4
10:
           Continue loop with next orientation
11:
       if canCompleteOrientation((G, o'), D) then // Algorithm 3
12:
           return True
13:
14: return False
```

## **Algorithm 3** canCompleteOrientation(Partial Orientation (G, o'), Set $\overline{D}$

```
1: if all edges are oriented in (G, o') then
       if D \cup D(G, o') = E(G) then
2:
           return True
3:
       return False
4:
5: //(G,o') still has unoriented edges
6: Store a copy of (G, o') in (G, o'')
 7: Choose an edge uv unoriented in (G, o')
   if canAddArcsRecursively((G, o'), D, u \rightarrow v) then
       if canCompleteOrientation((G, o'), D) then
9:
          return True
10:
11: Reset o' using o''
12: if canAddArcsRecursively((G, o'), D, v \rightarrow u) then
       if canCompleteOrientation((G, o'), D) then
13:
14:
           return True
15: return False
```

We will use the following Lemma for the proof of the exact algorithm's correctness.

**Lemma 4.** Let G be a cubic graph with fn(G) = 2 and let (G, o) and (G, o') be two orientations of G such that every edge  $e \in E(G)$  is deletable in either (G, o) or (G, o'). Then the following hold for (G, o'):

- 1. every vertex has at least one incoming and one outgoing edge in (G, o'),
- 2. let  $uv \notin D(G, o)$ , then u has one incoming and one outgoing edge in (G uv, o'),
- 3. let  $uv, vw \notin D(G, o)$ , then they are oriented either  $u \to v$ ,  $w \to v$  or  $v \to u$ ,  $v \to w$  in (G, o').

*Proof.* We now prove each of the three properties:

- 1. Let u be a vertex such that all its incident edges are either outgoing or incoming in (G, o'). Clearly none of these edges can be deletable in (G, o'). Since there is some edge ux not in D(G, o). We get a contradiction.
- 2. Let  $uv \notin D(G, o)$  and let the remaining edges incident to u be either both outgoing or both incoming in (G, o'). Then uv is not deletable in (G, o') since all oriented paths to (respectively, from) u pass through uv.
- 3. Suppose without loss of generality that we have  $u \to v$  and  $v \to w$  in (G, o'). If the remaining edge incident to v is outgoing, then uv is not deletable in (G, o'). If the remaining edge is incoming, then vw is not deletable in (G, o').

**Theorem 6.** Let G be a cubic graph. Algorithm 2 applied to G returns True if and only if G has Frank number 2.

*Proof.* Suppose that frankNumberIs2(G) returns True. Then there exist two orientations (G, o) and (G, o') for which  $D(G, o) \cup D(G, o') = E(G)$ . Hence, fn(G) = 2.

Conversely, let fn(G) = 2. We will show that Algorithm 2 returns True. Let  $(G, o_1)$  and  $(G, o_2)$  be orientations of G such that every edge of G is deletable in either  $(G, o_1)$  or  $(G, o_2)$ . Every iteration of the loop of Line 1, we consider an orientation of G. If the algorithm returns True before we consider  $(G, o_1)$  in this loop, the proof done. So without loss of generality, suppose we are in the iteration where  $(G, o_1)$  is the orientation under consideration in the loop of Line 1.

Without loss of generality assume that the orientation of xy we fix on Line 9 is in  $(G, o_2)$ . (If not, reverse all edges of  $(G, o_2)$  to get an orientation with the same set of deletable edges.) Let (G, o') be a partial orientation of G and assume that all oriented edges correspond to  $(G, o_2)$ . Let  $u \to v$  be an arc in  $(G, o_2)$ . If  $u \to v$  is present in (G, o'), then canAddArcsRecursively $(G, D(G, o), o', u \to v)$  (Algorithm 4) returns True and no extra edges become oriented in (G, o'). If  $u \to v$  is not present in (G, o'), it gets added on Line 8 of Algorithm 4, since the if-statement on Line 6 of Algorithm 4 will return True by Lemma 4. Note that this is the only place where an arc is added to (G, o') in Algorithm 4. Hence, if we only call Algorithm 4 on arcs present in  $(G, o_2)$ , then all oriented edges e of (G, o') will always be oriented as  $o_2(e)$ . Now we will show that Algorithm 4 indeed only calls itself on arcs in  $(G, o_2)$ .

Again, suppose  $u \to v$  is an arc in  $(G, o_2)$ , that it is not yet oriented in (G, o') and that every oriented edge e of (G, o') has orientation  $o_2(e)$ . The call canAddArcsRecursively $(G, D(G, o), o', u \to v)$  can only call itself on Line 9, i.e. in Algorithm 6. We show that in all cases after orienting uv as  $u \to v$  in (G, o'), the call to Algorithm 4 only happens on arcs oriented as in  $(G, o_2)$ .

Suppose u has two outgoing and no incoming arcs in (G, o'). Let ux be the final unoriented edge incident to u. Then  $(G, o_2)$  must have arc  $x \to u$ , otherwise it has three outgoing arcs from the same vertex. Now suppose v has two incoming and no outgoing arcs in (G, o'). Let vx be the final unoriented edge incident to v. Then  $(G, o_2)$  must have arc  $v \to x$ , otherwise it has three incoming arcs to the same vertex.

Suppose uv is deletable in  $(G, o_1)$ . Let ux also be deletable in  $(G, o_1)$ . Denote the final edge incident to u by uy. Clearly, uy cannot be deletable in  $(G, o_1)$ . Hence, it is deletable in  $(G, o_2)$ . If  $(G, o_2)$  contains  $u \to x$ , then uy is not deletable in  $o_2$ . Hence,  $(G, o_2)$  contains  $x \to u$ . Let vx be a deletable edge of  $(G, o_1)$  and denote the final edge incident to v by vy. Since vy cannot be deletable in  $(G, o_1)$ ,  $(G, o_2)$  must contain arc  $v \to x$ . Suppose that the edges incident with u which are not uv are both not in  $D(G, o_1)$ . Then they must be oriented incoming to u in  $(G, o_2)$ . Similarly, if the edges incident with v which are not uv are both not in  $D(G, o_1)$ , they must both be outgoing from v in  $(G, o_2)$ .

Finally, suppose that uv is not a deletable edge in  $(G, o_1)$ . Suppose that (G, o') still has one unoriented edge incident to u, say ux. If the other incident edges are one incoming and one outgoing from u, then  $(G, o_2)$  contains the arc  $u \to x$ . Otherwise, uv cannot be deletable in  $(G, o_2)$ . Similarly, if (G, o') still has one unoriented edge incident to v, say vx and the remaining incident edges are one incoming and one outgoing, then the arc  $x \to v$  must be present in  $(G, o_2)$ . Otherwise, uv cannot be deletable in  $(G, o_2)$ . If ux is not deletable in  $(G, o_1)$   $x \neq v$ . Then  $(G, o_2)$  contains the arc  $u \to x$ . Otherwise, not both of uv and ux can be deletable in  $(G, o_2)$ . Similarly, if vy is not deletable in  $(G, o_1)$  and  $y \neq u$ , then  $(G, o_2)$  must contain the arc  $y \to v$ . Otherwise, not both of uv and vy can be deletable in  $(G, o_2)$ .

This shows that all calls of canAddArcsRecursively(G, D(G, o), o',  $u \to v$ ) to itself, where all oriented edges of (G, o') and u - > v are oriented as in  $(G, o_2)$ , have as the fourth parameter an arc oriented as in  $(G, o_2)$ . Since, in Algorithm 3, we keep orienting edges until (G, o') is completely oriented and try to orient edge uv as  $v \to u$  if (G, o') cannot be completed with  $u \to v$ , it follows by induction that unless Algorithm 2 returns True in some other case, it will return True when  $(G, o') = (G, o_2)$ .

## **Algorithm 4** canAddArcsRecursively(Partial Orientation (G, o'), Set D, Arc $u \to v$ )

```
1: // Check if u \to v can be added and recursively orient edges for which the orientation is enforced by the rules of Lemma 4
```

- 2: if  $u \to v$  is present in (G, o') then
- 3: **return** True
- 4: if  $v \to u$  is present in (G, o') then
- 5: **return** False
- 6: if adding  $u \to v$  violates rules of Lemma 4 then // Algorithm 5 in Appendix A.1
- 7: **return** False
- 8: Add  $u \to v$  to (G, o')
- 9: **if** the orientation of edges enforced by Lemma 4 yields a contradiction **then** // Algorithm 6 in Appendix A.1
- 10: **return** False
- 11: **return** True

#### 3.1 Results

Since by Theorem 2 all 3-edge-connected 3-edge-colourable (cubic) graphs have Frank number 2, in this section we will focus on *non*-3-edge-colourable cubic graphs, i.e. *snarks*.

In [4] Brinkmann et al. determined all cyclically 4-edge-connected snarks up to order 34 and those of girth at least 5 up to order 36. This was later extended with all cyclically 4-edge-connected snarks on 36 vertices as well [8]. These lists of snarks can be obtained from the House of Graphs [5] at: https://houseofgraphs.org/meta-directory/snarks. Using our implementation of Algorithms 1 and 2, we tested for all cyclically 4-edge-connected snarks up to 36 vertices if they have Frank number 2 or not. This led to the following result.

**Proposition 2.** The Petersen graph is the only cyclically 4-edge-connected snark up to order 36 which has Frank number not equal to 2.

This was done by first running our heuristic Algorithm 1 on these graphs. It turns out that there are few snarks in which neither the configuration of Theorem 3 nor the configuration of Theorem 4 are present. For example: for more than 99.97% of the cyclically 4-edge-connected snarks of order 36, Algorithm 1 is sufficient to determine that their Frank number is 2 (see Table 2 in Appendix A.2 for more details). Thus we only had to run our exact Algorithm 2 (which is significantly slower than the heuristic) on the graphs for which our heuristic algorithm failed. In total about 214 CPU days of computation time was required to prove Proposition 2 using Algorithm 1 and 2 (see Table 3 in Appendix A.2 for more details).

In [11] Jaeger defines a snark G to be a strong snark if for every edge  $e \in E(G)$ ,  $G \sim e$ , i.e. the unique cubic graph such that G - e is a subdivision of  $G \sim e$ , is not 3-edge-colourable. Hence, a strong snark containing a 2-factor which has precisely two odd cycles, has no edge e connecting those two odd cycles, i.e. the configuration of Theorem 3 cannot be present. Therefore, they might be good candidates for having Frank number greater than 2.

In [4] it was determined that there are 7 strong snarks on 34 vertices having girth at least 5, 25 strong snarks on 36 vertices having girth at least 5 and no strong snarks of girth at least 5 of smaller order. By Proposition 2, their Frank number is 2. In [3] it was determined that there are at least 298 strong snarks on 38 vertices having girth at least 5 and the authors of [3] speculate that this is the complete set. We found the following.

**Observation 1.** The 298 strong snarks of order 38 determined in [3] have Frank number 2.

These snarks can be obtained from the House of Graphs [5] by searching for the keywords "strong snark".

The configurations of Theorem 3 and Theorem 4 also cannot occur in snarks of *oddness* 4, i.e. the smallest number of odd cycles in a 2-factor of the graph is 4. Hence, these may also seem to be good candidates for having Frank number greater than 2. In [8, 9] it was determined that the smallest snarks of girth at least 5 with oddness 4 and cyclic edge-connectivity 4 have order 44 and that there are precisely 31 such graphs of this order. We tested each of these and found the following.

**Observation 2.** Let G be a snark of girth at least 5, oddness 4, cyclic edge-connectivity 4 and order 44. Then fn(G) = 2.

These snarks of oddness 4 can be obtained from the House of Graphs [5] at https://houseofgraphs.org/meta-directory/snarks.

## 3.2 Correctness Testing

The correctness of our algorithm was shown in Theorem 5 and Theorem 6. We also performed several tests to verify that our implementations are correct.

Hörsch and Szigeti proved in [10] that the Petersen graph has Frank number 3. In [1] Barát and Blázsik showed that both Blanuša snarks and every flower snark has Frank number 2. We verified that for the Petersen graph both Algorithm 1 and Algorithm 2 give a negative result, confirming that its Frank number is larger than 2. For the Blanuša snarks and the flower snarks up to 40 vertices Algorithm 2 always shows the graph has Frank number 2 and the heuristic Algorithm 1 is able to show this for a subset of these graphs.

We also ran our implementation of Algorithm 2 on the cyclically 4-edge-connected snarks up to 30 vertices without running Algorithm 1 first. Results were in complete agreement with our earlier computation, i.e. Algorithm 2 independently confirmed that each of the snarks which have Frank number 2 according to Algorithm 1 indeed have Frank number 2. For the cyclically 4-edge-connected snarks on 30 vertices Algorithm 2 took approximately 46 hours. Our heuristic Algorithm 1 found the same results for all but 307 graphs in approximately 42 seconds.

During the computation of Algorithm 1 on the strong snarks, we verified it never returned True for the configuration of Theorem 3 and that it returned False for all snarks of oddness 4 mentioned earlier.

We also implemented a method for finding the actual orientations after Algorithm 1 detects one of the configurations. We checked for all cyclically 4-edge-connected snarks up to 32 vertices in which one of these configurations is found, whether the deletable edges for these two orientations form the whole edge set. This was always the case.

Another test was performed using a brute force algorithm which generates all strongly connected orientations of the graph and checks for every pair of these orientations whether the union of the deletable edges of this pair of orientations is the set of all edges. We were able to do this for all cyclically-4-edge-connected snarks up to 26 vertices and obtained the same results as with our other method. Note that this method is a lot slower than Algorithm 1 and 2. For order 26 this took approximately 152 hours, while using Algorithm 1 and 2 this took approximately 1 second.

Our implementation of Algorithm 1 and Algorithm 2 is open source and can be found on GitHub [6] where it can be inspected and used by others.

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## A Appendix

## A.1 Algorithms

```
Algorithm 5 canAddArc(Partial Orientation (G, o'), Set D, Arc u \to v)
 1: // Check if the addition of u \to v will not violate rules of Lemma 4
 2: if u has two outgoing arcs or v has two incoming arcs in (G, o') then
       return False
 4: if uv \in D then
       for all edges e incident to u in G do
 5:
          if e \in D and e is an outgoing arc of u in (G, o') then
 6:
 7.
              return False
       for all edges e incident to v in G do
 8:
          if e \in D and e is an incoming arc of v in (G, o') then
 9:
              return False
10:
11: else
       if in (G, o'), u has two incoming arcs or
12:
       v has two outgoing arcs or
       u has an incoming arc whose corresponding edge is not in D or
       v has an outgoing arc whose corresponding edge is not in D then
          return False
13:
14: return True
```

## **Algorithm 6** canOrientFixedEdges(Partial Orientation (G, o'), Set D, Arc $u \to v$ )

```
1: // Recursively orient edges whose orientation is forced by Lemma 4
 2: if u has two outgoing and no incoming arcs in (G, o') then
 3:
       Let ux be the edge of G which is unoriented in (G, o')
       if not canAddArcsRecursively((G, o'), D, x \rightarrow u) then
 4:
           return False
 5:
 6: if v has two incoming and no outgoing arcs in (G, o') then
       Let vx be the edge of G which is unoriented in (G, o')
 7:
       if not canAddArcsRecursively((G, o'), D, v \rightarrow x) then
 8:
           return False
 9:
10: if uv \in D then
       if not orientDeletable((G, o'), D, u \rightarrow v) then
11:
           return False
12:
13: else
       if not orientNonDeletable((G, o'), D, u \rightarrow v)
14:
        then
           return False
15:
16: return True
```

## **Algorithm 7** orientDeletable(Partial Orientation (G, o'), Set D, Arc $u \to v$ )

```
1: // Recursively orient edges whose orientation is forced by Lemma 4 in the case that
    uv \in D(G,o)
 2: for all edges ux incident to u in G do
       if ux \in D then
 3:
           if not canAddArcsRecursively((G, o'), D, x \rightarrow u) then
 4:
               return False
 5:
   for all edges vx incident to v in G do
 6:
       if vx \in D then
 7:
           if not canAddArcsRecursively((G, o'), D, v \to x) then
 8:
9:
               return False
10: Let ux_1, uy_1 be the two edges incident with u in G such that x_1, y_1 \neq v
11: if \{ux_1, uy_1\} \cap D = \emptyset then
       for z \in \{x_1, y_1\} do
12:
           if not canAddArcsRecursively((G, o'), D, z \rightarrow u) then
13:
14:
               return False
15: Let vx_2, vy_2 be the two edges incident with v in G such that x_2, y_2 \neq u
   if \{vx_2, vy_2\} \cap D = \emptyset then
16:
       for z \in \{x_2, y_2\} do
17:
           if not canAddArcsRecursively((G, o'), D, v \rightarrow z) then
18:
               return False
19:
20: return True
```

## **Algorithm 8** orientNonDeletable(Partial Orientation (G, o'), Set D, Arc $u \to v$ )

```
1: // Recursively orient edges which are forced by Lemma 4 in the case that uv \notin D(G, o)
2: if u has precisely two incident arcs in (G, o') then
       Let ux be the edge of G which is unoriented in (G, o')
3:
       if the arcs incident to u in (G, o') are one incoming and one outgoing then
4:
          if not canAddArcsRecursively((G, o'), D, u \to x) then
 5:
              return False
 6:
 7: if v has precisely two incident arcs in (G, o') then
       Let vx be the edge of G which is unoriented in (G, o')
8:
       if the arcs incident to v in (G,o') are one incoming and one outgoing then
9:
          if not canAddArcsRecursively((G, o'), D, x \to v) then
10:
              return False
12: if there exists an edge ux of G such that x \neq v and ux \notin D then
       if not canAddArcsRecursively((G, o'), D, u \rightarrow x) then
13:
           return False
14:
15: if there exists an edge vy of G such that y \neq u and vy \notin D then
       if not canAddArcsRecursively((G, o'), D, y \rightarrow v) then
16:
           return False
17:
18: return True
```

## A.2 Snarks for which Algorithm 1 is sufficient

Order	Total	Passed
10	1	0
18	2	1
20	6	6
22	31	29
24	155	152
26	1297	1283
28	12517	12472
30	139854	139547
32	1764950	1763302
34	25286953	25273455
36	404899916	404793575

Table 2: Number of cyclically 4-edge-connected snarks for which Algorithm 1 is sufficient to decide that the graph has Frank number 2. In the second column the total number of cyclically 4-edge-connected snarks for the given order can be found. In the third column the number of such snarks in which the configuration of Theorem 3 or Theorem 4 is present is given.

Order	Algorithm 1	Remainder
28	4 s	9 s
30	42 s	$304 \mathrm{\ s}$
32	$585 \mathrm{\ s}$	3 h
34	3 h	106 h
36	54 h	4975  h

Table 3: Runtimes of Algorithm 1 and 2 on the cyclically 4-edge-connected snarks of order 28 to 36. In the second column, the runtime of Algorithm 1 on these graphs can be found for the specific order. In the third column, the runtime of Algorithm 2 **only** on the graphs for which Algorithm 1 failed can be found.