## Robust Classification of High-Dimensional Data using Data-Adaptive Energy Distance

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Abstract. Classification of high-dimensional low sample size (HDLSS) data poses a challenge in a variety of real-world situations, such as gene expression studies, cancer research, and medical imaging. This article presents the development and analysis of some classifiers that are specifically designed for HDLSS data. These classifiers are free of tuning parameters and are robust, in the sense that they are devoid of any moment conditions of the underlying data distributions. It is shown that they yield perfect classification in the HDLSS asymptotic regime, under some fairly general conditions. The comparative performance of the proposed classifiers is also investigated. Our theoretical results are supported by extensive simulation studies and real data analysis, which demonstrate promising advantages of the proposed classification techniques over several widely recognized methods.

**Keywords:** Classification  $\cdot$  Data Mining  $\cdot$  Generalized Energy  $\cdot$  High-Dimensional Asymptotics

#### 1 Introduction

High-dimensional low sample size data is characterized by having a large number of features or variables, but only a few samples or observations. The problem of HDLSS classification has been an important problem in the statistics and machine learning communities. In today's world, high-dimensional low sample size problems are frequently encountered in scientific areas including microarray gene expression studies, medical image analysis, and spectral measurements in chemometrics to name a few.

Traditional classification techniques such as logistic regression, support vector machines, and k-nearest neighbors [9,12] often fail on this type of data [20] when certain regularity conditions on the underlying distributions are not met. In case of k-nearest neighbors, for example, when the dimension of the data is

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far greater than the number of observations, the concept of neighbors becomes loose and ill-defined. Consequently, the k-nearest neighbor classifier exhibits erratic behavior [4]. Due to distance concentration, Euclidean distance (ED)-based classifiers suffer certain limitations in HDLSS situations [2,13]. Some recent work has studied the effect of distance concentration on some widely used classifiers based on Euclidean distances, such as 1-nearest neighbor (1-NN) classifier [15], support vector machines (SVM) [8], etc. They derived conditions under which these classifiers yield perfect classification in the HDLSS setup [14]. Moreover, ED-based classifiers lack robustness to outliers, since ED is sensitive to outliers.

For the HDLSS setup, numerous studies adopt dimension reduction approach as a pre-processing step before performing the classification. These work include modern classifiers and learning techniques centered mainly on feature selection (e.g., correlation-based, information theory-based, feature clustering [27], etc.), projection based on transformation [1,17], regularization (ridge, LASSO, SCAD, and Elastic-net [32]), deep learning (autoencoders [16,30]), etc. However, this is not optimal when the dimension reduction step is conducted independently of the goals of finding reduced features that maximize the separation between classes of signals. In fact, it is inevitable that some information is lost via dimension reduction if a large number of features turn out to be relevant and weakly dependent upon each other. A few studies have conducted classification of HDLSS data without employing dimension reduction (see, e.g., [23,26,31]).

Energy distance was introduced in [3,28] as a statistical measure of distance between two probability distributions on  $\mathbb{R}^d$ . It was primarily designed with a goal of testing for equality of two or more multivariate distributions, and worked particularly well with high-dimensional data. Recently, energy distances have been utilized in the context of classification (see, e.g., [22,23,24]) as well. In this article, we develop classifiers based on a more general version of energy distance that yield asymptotically perfect classification (i.e., zero misclassification rate) under fairly general assumptions in an HDLSS setting, without maneuvering dimension reduction.

Suppose **X** and **Y** are two *d*-variate random vectors following the distribution functions **F** and **G**, respectively. In the context of testing for equality of the two distributions, a constant multiple of the following was introduced in [18] as squared multivariate Cramér-von Mises (CvM) distance between **F** and **G**. It is a special case of the generalized energy distance [25]:

$$\mathcal{W}_{\mathbf{FG}}^* = 2 \int \int \left( \mathbf{F}_{\beta}(t) - \mathbf{G}_{\beta}(t) \right)^2 \, d\mathbf{H}(\boldsymbol{\beta}, t) \,, \tag{1}$$

where  $\boldsymbol{\beta} \in \mathbb{R}^d$ , and  $\mathbf{F}_{\boldsymbol{\beta}}(t) = \mathbb{P}\left[\boldsymbol{\beta}^\top \mathbf{X} \leq t\right]$  and  $\mathbf{G}_{\boldsymbol{\beta}}(t) = \mathbb{P}\left[\boldsymbol{\beta}^\top \mathbf{Y} \leq t\right]$ , for  $t \in \mathbb{R}$ , are the cumulative distribution functions of  $\boldsymbol{\beta}^\top \mathbf{X}$  and  $\boldsymbol{\beta}^\top \mathbf{Y}$  respectively, evaluated at t,  $d\mathbf{H}(\boldsymbol{\beta}, t) = d\mathbf{H}_{\boldsymbol{\beta}}(t) d\lambda(\boldsymbol{\beta})$  with  $\lambda(\boldsymbol{\beta})$  being the uniform probability measure on d-dimensional unit sphere  $\mathbb{S}^{d-1} = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x}^\top \mathbf{x} = 1\}$ , and  $\mathbf{H}_{\boldsymbol{\beta}}(t) = \alpha \mathbf{F}_{\boldsymbol{\beta}}(t) + (1-\alpha) \mathbf{G}_{\boldsymbol{\beta}}(t)$ . Here,  $\alpha$  is a fixed value in (0, 1). In the same context of hypothesis testing, [19] considered a constant multiple of (1), with  $\mathbf{H}(\boldsymbol{\beta}, t)$  as the distribution function of a (d+1)-dimensional normal random vector with mean  $\mathbf{0}_{d+1}$ , the (d+1)-dimensional zero vector, and covariance matrix  $I_{d+1}$ , the identity matrix of order (d + 1). However, considering such a fixed distribution which is not data-dependent may not be useful in general. On the other hand, the weight function  $\mathbf{H}_{\beta}$  considered in [18] is more flexible, since it adapts according to the underlying class distributions. In that sense,  $\mathcal{W}_{\mathbf{FG}}^*$  is referred to as a data-adaptive energy distance between  $\mathbf{F}$  and  $\mathbf{G}$ . It was shown in [18] that  $\mathcal{W}_{\mathbf{FG}}^* = 0$  if and only if  $\mathbf{F} = \mathbf{G}$ . This property of  $\mathcal{W}_{\mathbf{FG}}^*$  says that it has the capability of discriminating between two different distributions. This motivates us to utilize  $\mathcal{W}_{\mathbf{FG}}^*$  in the context of binary classification problems.

#### 1.1 Our contribution

In this article, we start off by developing a classifier based on  $\mathcal{W}^*_{\mathbf{FG}}$ . However, it suffers certain limitations in the HDLSS setting. We investigate and address those issues by modifying  $\mathcal{W}^*_{\mathbf{FG}}$  in different ways, and based on the new measures of distance, we develop classifiers that are robust in the sense that their performance does not depend on the existence of the moments of the underlying class distributions. Moreover, the proposed classifiers are free from tuning parameters and admit strong theoretical guarantees under fairly general assumptions, in an HDLSS setup.

The rest of the paper is organized as follows. In Section 2, we develop a classifier based on  $\mathcal{W}_{\mathbf{FG}}^*$ , discuss its limitations and modify it to obtain three robust classifiers to achieve asymptotically perfect classification under milder conditions. Section 3 provides an analysis of the asymptotic behaviors and a relative comparison of the proposed classifiers. Section 4 demonstrates convincing advantages of the proposed classifiers using numerical simulations and real data analysis. Proofs of the theoretical results are included in Section A of Supplementary Material. Lastly, Section B of the Supplementary Material contains some additional details on the simulation studies and real data analysis. The supplementary material and the relevant R codes for simulation studies and real data analysis are available at: https://github.com/jyotishkarc/sub-834-ecml-2023.

### 2 Methodology

Consider two mutually independent samples

$$\mathbf{X}_1^{(d)}, \mathbf{X}_2^{(d)}, \dots, \mathbf{X}_m^{(d)} \overset{\text{i.i.d.}}{\sim} \mathbf{F}_d \text{ and } \mathbf{Y}_1^{(d)}, \mathbf{Y}_2^{(d)}, \dots, \mathbf{Y}_n^{(d)} \overset{\text{i.i.d.}}{\sim} \mathbf{G}_d$$

where  $\mathbf{X}_{i}^{(d)} = (X_{i1}, X_{i2}, \dots, X_{id})^{\top}$  for  $i = 1, 2, \dots, m$ , and  $\mathbf{Y}_{j}^{(d)} = (Y_{j1}, Y_{j2}, \dots, Y_{jd})^{\top}$  for  $j = 1, \dots, n$  are *d*-dimensional random vectors arising from two different population distributions  $\mathbf{F}_{d}$  and  $\mathbf{G}_{d}$ . For the sake of convenience, we shall drop *d* from notations where dependence on *d* is obvious. As mentioned in Section 1.1, we shall keep the sample sizes *m* and *n* fixed throughout our analysis.

The angular distance between any  $\mathbf{u}, \mathbf{v} \in \mathbb{R}^d$  was defined in [18] as follows:

$$\rho(\mathbf{u}, \mathbf{v}) = \mathbb{E}\left[\rho_0(\mathbf{u}, \mathbf{v}; \mathbf{Q})\right] \text{ with } \mathbf{Q} \sim \alpha \mathbf{F} + (1 - \alpha) \mathbf{G}.$$
(2)

$$\rho_0(\mathbf{u}, \mathbf{v}; \mathbf{w}) = \begin{cases} \frac{1}{\pi} \angle (\mathbf{u} - \mathbf{w}, \mathbf{v} - \mathbf{w}) & \text{if } \mathbf{u} \neq \mathbf{w} \text{ and } \mathbf{v} \neq \mathbf{w}, \\ 0 & \text{otherwise,} \end{cases}$$
(3)

with  $\alpha = \frac{m}{m+n}$ , and  $\angle(\mathbf{a}, \mathbf{b}) = \cos^{-1}\left(\frac{\mathbf{a}^{\top}\mathbf{b}}{\|\mathbf{a}\|_2\|\mathbf{b}\|_2}\right)$  with  $\|\mathbf{v}\|_2$  as the  $l_2$  norm of **v**. Note that  $\rho \in [0, 1]$  since  $\rho_0$  takes values in [0, 1]. It was shown in [18] that  $\mathcal{W}^*_{\mathbf{FG}}$ , defined in (1), has the following closed-form expression:

$$\mathcal{W}_{\mathbf{FG}}^{*} = \mathbb{E}\left[2\rho\left(\mathbf{X}_{1}, \mathbf{Y}_{1}\right) - \rho\left(\mathbf{X}_{1}, \mathbf{X}_{2}\right) - \rho\left(\mathbf{Y}_{1}, \mathbf{Y}_{2}\right)\right].$$
(4)

#### 2.1 A classifier based on $\mathcal{W}_{FG}^*$

Let us consider the unknown expectations  $t_{\mathbf{FF}} = \mathbb{E}[\rho(\mathbf{X}_1, \mathbf{X}_2)]$  and  $t_{\mathbf{GG}} = \mathbb{E}[\rho(\mathbf{Y}_1, \mathbf{Y}_2)]$ . We start off by defining estimators of  $t_{\mathbf{FF}}$  and  $t_{\mathbf{GG}}$  as follows.

$$\hat{t}_{\mathbf{FF}} = \frac{1}{m(m-1)} \sum_{i \neq j} \hat{\rho}\left(\mathbf{X}_i, \mathbf{X}_j\right) \text{ and } \hat{t}_{\mathbf{GG}} = \frac{1}{n(n-1)} \sum_{i \neq j} \hat{\rho}\left(\mathbf{Y}_i, \mathbf{Y}_j\right),$$

where  $\hat{\rho}$  is defined as a sample version of  $\rho$  in the following manner:

$$\hat{\rho}(\mathbf{u}, \mathbf{v}) = \frac{1}{m+n} \Big( \sum_{i=1}^{m} \rho_0(\mathbf{u}, \mathbf{v}, \mathbf{X}_i) + \sum_{j=1}^{n} \rho_0(\mathbf{u}, \mathbf{v}, \mathbf{Y}_j) \Big).$$
(5)

Similarly, for  $\mathbf{z} \in \mathbb{R}^d$ , we define

$$\hat{t}_{\mathbf{F}}(\mathbf{z}) = \frac{1}{m} \sum_{i} \hat{\rho} \left( \mathbf{X}_{i}, \mathbf{z} \right), \ \hat{t}_{\mathbf{G}}(\mathbf{z}) = \frac{1}{n} \sum_{j} \hat{\rho} \left( \mathbf{Y}_{j}, \mathbf{z} \right), \\
l_{\mathbf{F}}(\mathbf{z}) = \hat{t}_{\mathbf{F}}(\mathbf{z}) - \frac{1}{2} \hat{t}_{\mathbf{FF}}, \ l_{\mathbf{G}}(\mathbf{z}) = \hat{t}_{\mathbf{G}}(\mathbf{z}) - \frac{1}{2} \hat{t}_{\mathbf{GG}}.$$
(6)

Finally, the classifier  $\delta_0$  is defined as follows:

$$\delta_0(\mathbf{z}) = \begin{cases} 1 & \text{if } l_{\mathbf{G}}(\mathbf{z}) - l_{\mathbf{F}}(\mathbf{z}) > 0, \\ 2 & \text{otherwise}, \end{cases}$$
(7)

where  $\delta_0(\mathbf{z}) = 1$  or  $\delta_0(\mathbf{z}) = 2$  correspond to assigning  $\mathbf{z}$  to class 1 or class 2 having data distribution  $\mathbf{F}$  or  $\mathbf{G}$ , respectively. Let  $\boldsymbol{\mu}_{\mathbf{F}}$  and  $\boldsymbol{\mu}_{\mathbf{G}}$  denote the mean vectors for  $\mathbf{F}$  and  $\mathbf{G}$ , respectively, and  $\boldsymbol{\Sigma}_{\mathbf{F}}$  and  $\boldsymbol{\Sigma}_{\mathbf{G}}$  denote the covariance matrices for  $\mathbf{F}$  and  $\mathbf{G}$ , respectively. In order to analyze the behavior of the  $\delta_0$  in HDLSS setup, consider the following assumptions:

Assumption 1. There exists a constant  $c < \infty$  such that  $E[|U_k|^4] < c$  for all  $1 \le k \le d$ , where  $\mathbf{U} = (U_1, \cdots, U_d)^\top$  follows either  $\mathbf{F}$  or  $\mathbf{G}$ .

Assumption 2. 
$$\lambda_{\mathbf{FG}} = \lim_{d \to \infty} \left\{ \frac{1}{d} \| \boldsymbol{\mu}_{\mathbf{F}} - \boldsymbol{\mu}_{\mathbf{G}} \|^2 \right\}$$
 and  $\sigma_I^2 = \lim_{d \to \infty} \left\{ \frac{1}{d} \operatorname{trace}(\boldsymbol{\Sigma}_I) \right\}$   
exist for  $I \in \{\mathbf{F}, \mathbf{G}\}$ .

**Assumption 3.** Let  $\mathbf{U}$ ,  $\mathbf{V}$  and  $\mathbf{Z}$  be three independent random vectors such that each of them follows either  $\mathbf{F}$  or  $\mathbf{G}$ . Then,

$$\sum_{i < j} \operatorname{cov}((U_i - Z_i)(V_i - Z_i), (U_j - Z_j)(V_j - Z_j)) = o(d^2).$$

Assumption 1 requires finiteness of the fourth moments of all marginals of  $\mathbf{F}$  and  $\mathbf{G}$ . Assumption 2 demands the existence of the limiting values of the average of the squared mean difference between the marginals of two distributions and the variances of the marginals to exist. Assumption 3 is trivially satisfied when the component variables of the underlying populations are independent. It also holds with certain additional constraints on their dependence structure, e.g., when the sequence  $\{(U_k - Z_k)(V_k - Z_k)\}_{k \geq 1}$  has  $\rho$ -mixing property. In fact, if the sequences  $\{U_k\}_{k \geq 1}, \{V_k\}_{k \geq 1}$  and  $\{Z_k\}_{k \geq 1}$  all have  $\rho$ -mixing property, then the sequence  $\{h(U_k, V_k, Z_k)\}_{k \geq 1}$  also has  $\rho$ -mixing property, for any Borel measurable function h (see [7] for more details).

**Theorem 1.** Suppose assumptions 1 to 3 are satisfied. Then,  $\theta_{\mathbf{FG}}^* = \lim_{d\to\infty} \mathcal{W}_{\mathbf{FG}}^*$  is finite, and for a test observation  $\mathbf{Z}$ ,

(a) if 
$$\mathbf{Z} \sim \mathbf{F}$$
, then  $l_{\mathbf{G}}(\mathbf{Z}) - l_{\mathbf{F}}(\mathbf{Z}) \xrightarrow{\mathbb{P}} \frac{1}{2} \theta_{\mathbf{FG}}^*$  as  $d \to \infty$ ;  
(b) if  $\mathbf{Z} \sim \mathbf{G}$ , then  $l_{\mathbf{G}}(\mathbf{Z}) - l_{\mathbf{F}}(\mathbf{Z}) \xrightarrow{\mathbb{P}} -\frac{1}{2} \theta_{\mathbf{FG}}^*$  as  $d \to \infty$ .

As  $d \to \infty$ ,  $l_{\mathbf{G}}(\mathbf{Z}) - l_{\mathbf{F}}(\mathbf{Z})$  converges in probability to the limit of  $\frac{1}{2}\mathcal{W}_{\mathbf{FG}}^*$  if  $\mathbf{Z} \sim \mathbf{F}$ , and to the negative of it, if  $\mathbf{Z} \sim \mathbf{G}$ . This justifies the construction of the classifier  $\delta_0$  in (7). The probability of misclassification of a classifier  $\delta$  is defined as  $\Delta = \alpha P[\delta(\mathbf{Z}) = 2|\mathbf{Z} \sim \mathbf{F}] + (1 - \alpha)P[\delta(\mathbf{Z}) = 1|\mathbf{Z} \sim \mathbf{G}]$ . Now, we state a result on the convergence of the misclassification probabilities of the classifier  $\delta_0$  (denoted by  $\Delta_0$ ), under the set of assumptions stated above.

**Theorem 2.** Suppose that assumptions 1 to 3 are satisfied, and either  $\lambda_{\mathbf{FG}} \neq 0$  or  $\sigma_{\mathbf{F}}^2 \neq \sigma_{\mathbf{G}}^2$  holds. Then,  $\Delta_0 \to 0$  as  $d \to \infty$ .

It follows from Theorem 2 that if  $\mathbf{F}$  and  $\mathbf{G}$  differ in their locations and/or scales, then  $\Delta_0$  converges to 0 as the dimension grows. Clearly, the asymptotic properties of the classifier  $\delta_0$  are governed by the limiting constants,  $\lambda_{\mathbf{FG}}, \sigma_{\mathbf{F}}$ , and  $\sigma_{\mathbf{G}}$ . Similar issues regarding assumptions on the existence of moments of class distributions were also present in the two-sample test based on  $\mathcal{W}_{\mathbf{FG}}^*$  in [18]. Let us now consider the following two examples:

**Example 1:** 
$$X_{1k} \stackrel{\text{i.i.d.}}{\sim} N(1,1)$$
 and  $Y_{1k} \stackrel{\text{i.i.d.}}{\sim} N(1,2)$   
**Example 2:**  $X_{1k} \stackrel{\text{i.i.d.}}{\sim} N(0,3)$  and  $Y_{1k} \stackrel{\text{i.i.d.}}{\sim} t_3$ 

for  $1 \leq k \leq d$ . Here,  $N(\mu, \sigma^2)$  refers to a Gaussian distribution with mean  $\mu$  and variance  $\sigma^2$ , and  $t_s$  denotes the Student's t-distribution with s degrees of freedom.

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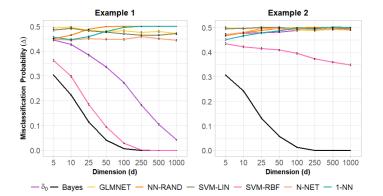


Fig. 1. Average misclassification rates with errorbars for  $\delta_0$ , along with some popular classifiers for increasing dimensions. Bayes classifier is treated as a benchmark.

In **Example 1**,  $\|\boldsymbol{\mu}_{\mathbf{F}} - \boldsymbol{\mu}_{\mathbf{G}}\|^2 = 0$  but  $\sigma_{\mathbf{F}}^2 = 1, \sigma_{\mathbf{G}}^2 = 2$ . It can be observed from Fig. 1 that  $\delta_0$  identifies this difference in scale and its performance improves as d increases, whereas most of the popular classifiers misclassify nearly 45% of the observations.

In **Example 2**,  $\|\boldsymbol{\mu}_{\mathbf{F}} - \boldsymbol{\mu}_{\mathbf{G}}\|^2 = 0$  and  $\sigma_{\mathbf{F}}^2 = \sigma_{\mathbf{G}}^2 = 3$ , i.e., there is no difference between either location parameters or scale parameters. Consequently,  $\delta_0$  (as well as the popular classifiers) fails to classify the test observations correctly since the assumptions in Theorem 2 are not met.

These simulations support Theorem 2 (see Fig. 1) and illustrate the limitations of the classifier  $\delta_0$  when there is no difference in either location parameters or scale parameters. In the next subsection, we refine  $\delta_0$  to develop some classifiers whose asymptotic properties are free of any moment conditions, as well as the limiting constants  $\lambda_{\mathbf{FG}}, \sigma_{\mathbf{F}}^2$ , and  $\sigma_{\mathbf{G}}^2$ , mentioned in assumptions 1 and 2.

#### Refinements of $\delta_0$ $\mathbf{2.2}$

A New Measure of Distance: We modify  $\mathcal{W}^*_{FG}$  by taking the average of the distances between each  $F_k$  and  $G_k$ , the k-th marginals of **F** and **G**, respectively. Define  $\overline{\mathcal{W}}_{\mathbf{FG}}^* = \frac{1}{d} \sum_{k=1}^d \mathcal{W}_{F_k G_k}^*$ . For each  $1 \leq k \leq d$ , it follows from (4) the quantity  $\mathcal{W}_{F_k G_k}^*$  has the following closed-form expression:

$$\mathcal{W}_{F_kG_k}^* = \mathbb{E}\left[2\rho\left(X_{1k}, Y_{1k}\right) - \rho\left(X_{1k}, X_{2k}\right) - \rho\left(Y_{1k}, Y_{2k}\right)\right],\tag{8}$$

where  $\mathbf{X}_1, \mathbf{X}_2 \stackrel{i.i.d.}{\sim} \mathbf{F}$  and  $\mathbf{Y}_1, \mathbf{Y}_2 \stackrel{i.i.d.}{\sim} \mathbf{G}$ . Recall the definition of  $\rho$  in (2). For any two *d*-dimensional random variables  $\mathbf{u} = (u_1, u_2, \dots, u_d)^{\top}$  and  $\mathbf{v} = (v_1, v_2, \dots, v_d)^{\top}$ , we define

$$\bar{\rho}(\mathbf{u}, \mathbf{v}) = \frac{1}{d} \sum_{k=1}^{d} \rho\left(u_k, v_k\right).$$
(9)

We now introduce some notations:

$$T_{\mathbf{FG}} = \mathbb{E}\left[\bar{\rho}\left(\mathbf{X}_{1}, \mathbf{Y}_{1}\right)\right], \ T_{\mathbf{FF}} = \mathbb{E}\left[\bar{\rho}\left(\mathbf{X}_{1}, \mathbf{X}_{2}\right)\right], \text{ and } T_{\mathbf{GG}} = \mathbb{E}\left[\bar{\rho}\left(\mathbf{Y}_{1}, \mathbf{Y}_{2}\right)\right].$$

This implies that  $\overline{\mathcal{W}}_{\mathbf{FG}}^* = 2T_{\mathbf{FG}} - T_{\mathbf{FF}} - T_{\mathbf{GG}}$ . Note that  $\mathcal{W}_{F_kG_k}^* \geq 0$ , and equality holds iff  $F_k = G_k$ . Thus,  $\overline{\mathcal{W}}_{\mathbf{FG}}^* = 0$  iff  $F_k = G_k$  for all  $1 \leq k \leq d$ . This property of  $\overline{\mathcal{W}}_{\mathbf{FG}}^*$  suggests that it can be utilized as a measure of separation between  $\mathbf{F}$  and  $\mathbf{G}$ .

Since  $T_{FF}$ ,  $T_{GG}$ , and  $T_{FG}$  are all unknown quantities, we consider the following estimators based on the sample observations:

$$\begin{split} \hat{T}_{\mathbf{FG}} &= \frac{1}{mn} \sum_{i,j} \hat{\rho} \left( \mathbf{X}_i, \mathbf{Y}_j \right), \\ \hat{T}_{\mathbf{FF}} &= \frac{1}{m(m-1)} \sum_{i \neq j} \hat{\rho} \left( \mathbf{X}_i, \mathbf{X}_j \right), \\ \hat{T}_{\mathbf{GG}} &= \frac{1}{n(n-1)} \sum_{i \neq j} \hat{\rho} \left( \mathbf{Y}_i, \mathbf{Y}_j \right). \end{split}$$

where  $\hat{\bar{\rho}}(\mathbf{u}, \mathbf{v})$  is a natural estimator of  $\bar{\rho}(\mathbf{u}, \mathbf{v})$ , defined as follows:

$$\hat{\bar{\rho}}(\mathbf{u}, \mathbf{v}) = \frac{1}{d} \sum_{k=1}^{d} \hat{\rho}\left(u_k, v_k\right).$$
(10)

This leads to an empirical version of  $\overline{\mathcal{W}}^*_{\mathbf{FG}}$  defined as

$$\hat{\overline{W}}_{\mathbf{FG}}^* = 2\hat{T}_{\mathbf{FG}} - \hat{T}_{\mathbf{FF}} - \hat{T}_{\mathbf{GG}}.$$
(11)

For  $\mathbf{z} = (z_1, z_2, \dots, z_d)^\top \in \mathbb{R}^d$ , we define:

$$\begin{split} \hat{T}_{\mathbf{F}}(\mathbf{z}) &= \frac{1}{m} \sum_{i} \hat{\rho} \left( \mathbf{X}_{i}, z_{i} \right) \ , \ \hat{T}_{\mathbf{G}}(\mathbf{z}) = \frac{1}{n} \sum_{j} \hat{\rho} \left( \mathbf{Y}_{j}, z_{j} \right), \\ L_{\mathbf{F}}(\mathbf{z}) &= \hat{T}_{\mathbf{F}}(\mathbf{z}) - \frac{1}{2} \hat{T}_{\mathbf{FF}} \ , \ L_{\mathbf{G}}(\mathbf{z}) = \hat{T}_{\mathbf{G}}(\mathbf{z}) - \frac{1}{2} \hat{T}_{\mathbf{GG}}, \\ S(\mathbf{z}) &= \hat{T}_{\mathbf{F}}(\mathbf{z}) + \hat{T}_{\mathbf{G}}(\mathbf{z}) - \frac{1}{2} \left( \hat{T}_{\mathbf{FF}} + \hat{T}_{\mathbf{GG}} \right) - \hat{T}_{\mathbf{FG}}. \end{split}$$

**Classifier Based on**  $\overline{\mathcal{W}}_{\mathbf{FG}}^*$ : Define  $\mathscr{D}_1(\mathbf{z}) = L_{\mathbf{G}}(\mathbf{z}) - L_{\mathbf{F}}(\mathbf{z})$ . We prove that  $\mathscr{D}_1(\mathbf{Z})$  converges in probability to  $\frac{1}{2}\overline{\mathcal{W}}_{\mathbf{FG}}^*$ , if  $\mathbf{Z} \sim \mathbf{F}$  and to  $-\frac{1}{2}\overline{\mathcal{W}}_{\mathbf{FG}}^*$ , if  $\mathbf{Z} \sim \mathbf{G}$ , as  $d \to \infty$  (see Theorem 3). This, along with the fact that the average energy distance,  $\overline{\mathcal{W}}_{\mathbf{FG}}^*$  is non-negative, motivates us to consider the following classifier:

$$\delta_1(\mathbf{z}) = \begin{cases} 1 & \text{if } \mathscr{D}_1(\mathbf{z}) > 0, \\ 2 & \text{otherwise.} \end{cases}$$
(12)

Recall that  $(T_{\mathbf{FG}} - T_{\mathbf{FF}})$  and  $(T_{\mathbf{FG}} - T_{\mathbf{GG}})$  sum up to  $\overline{\mathcal{W}}_{\mathbf{FG}}^*$ . So, in case  $T_{\mathbf{FG}}$  lies between  $T_{\mathbf{FF}}$  and  $T_{\mathbf{GG}}$ , adding them up might nearly cancel each other out,

resulting in a very small value of  $\overline{\mathcal{W}}_{\mathbf{FG}}^*$ . Consequently, it may not fully capture the actual dissimilarity between **F** and **G**. A natural way to address this problem is to square the two quantities before adding them. We define

$$\bar{\tau}_{\mathbf{FG}} = \left(T_{\mathbf{FG}} - T_{\mathbf{FF}}\right)^2 + \left(T_{\mathbf{FG}} - T_{\mathbf{GG}}\right)^2$$

It follows from simple calculations that one may write  $\bar{\tau}_{FG}$  in the following form:

$$\bar{\tau}_{\mathbf{FG}} = \frac{1}{2} \overline{\mathcal{W}}_{\mathbf{FG}}^{*2} + \frac{1}{2} \left( T_{\mathbf{FF}} - T_{\mathbf{GG}} \right)^2.$$
(13)

Note that  $\bar{\tau}_{\mathbf{FG}}$  being a convex combination of squares of  $\overline{\mathcal{W}}_{\mathbf{FG}}^* = 2T_{\mathbf{FG}} - T_{\mathbf{FF}} - T_{\mathbf{GG}}$  and  $T_{\mathbf{FF}} - T_{\mathbf{GG}}$ , both of which are measures of disparity between  $\mathbf{F}$  and  $\mathbf{G}$ , can be considered as a new measure of disparity between the two distributions. The modification approach proposed in (13) is similar to what had been suggested in the literature of two-sample hypothesis tests to improve the power of some two-sample tests for HDLSS data in [6].

Classifier Based on  $\bar{\tau}_{\mathbf{FG}}$ : We now develop a classifier that utilizes  $\bar{\tau}_{\mathbf{FG}}$ . Recall the definitions of  $\mathscr{D}_1(\mathbf{z})$  and  $S(\mathbf{z})$ . For  $\mathbf{z} \in \mathbb{R}^d$ , define

$$\mathscr{D}_2(\mathbf{z}) = \frac{1}{2} \hat{\overline{\mathcal{W}}}_{\mathbf{FG}}^* \cdot \mathscr{D}_1(\mathbf{z}) + \frac{1}{2} \left( \hat{T}_{\mathbf{FF}} - \hat{T}_{\mathbf{GG}} \right) \cdot S(\mathbf{z})$$

We show that as  $d \to \infty$ ,  $\mathscr{D}_2(\mathbf{Z})$  converges in probability to  $\bar{\tau}_{\mathbf{FG}}$  (> 0) if  $\mathbf{Z} \sim \mathbf{F}$ , and to  $-\bar{\tau}_{\mathbf{FG}}$  (< 0) if  $\mathbf{Z} \sim \mathbf{G}$  (see Theorem 3 in Section 3 below). This motivates us to consider the following classifier:

$$\delta_2(\mathbf{z}) = \begin{cases} 1 & \text{if } \mathscr{D}_2(\mathbf{z}) > 0, \\ 2 & \text{otherwise.} \end{cases}$$
(14)

We consider another measure of disparity between  $\mathbf{F}$  and  $\mathbf{G}$  (say,  $\bar{\psi}_{\mathbf{FG}}$ ), by simply replacing the squares of  $\overline{\mathcal{W}}_{\mathbf{FG}}^*$  and  $S_{\mathbf{FG}}$  by their absolute values in the expression for  $\bar{\tau}_{\mathbf{FG}}$ . A similar modification has already been considered, in the context of two-sample testing (see, e.g., [29]). Based on this, we define yet another measure of separation:

$$\bar{\psi}_{\mathbf{FG}} = \frac{1}{2} \overline{\mathcal{W}}_{\mathbf{FG}}^* + \frac{1}{2} \left| T_{\mathbf{FF}} - T_{\mathbf{GG}} \right|.$$

Classifier Based on  $\bar{\psi}_{FG}$ : For  $\mathbf{z} \in \mathbb{R}^d$ , we define

$$\mathscr{D}_{3}(\mathbf{z}) = \frac{1}{2} \widehat{\mathcal{W}}_{\mathbf{FG}}^{*} \operatorname{sign}\left(\mathscr{D}_{1}(\mathbf{z})\right) + \frac{1}{2} \left(\hat{T}_{\mathbf{FF}} - \hat{T}_{\mathbf{GG}}\right) \cdot \operatorname{sign}\left(S(\mathbf{z})\right)$$

where sign(·) is defined as sign(x) =  $\frac{x}{|x|}$  for  $x \neq 0$ , and 0 for x = 0.

We prove that as  $d \to \infty$ ,  $\mathscr{D}_3(\mathbf{Z})$  converges in probability to  $\bar{\psi}_{\mathbf{FG}}$ , a positive quantity if  $\mathbf{Z} \sim \mathbf{F}$ , and to  $-\bar{\psi}_{\mathbf{FG}}$ , a negative quantity if  $\mathbf{Z} \sim \mathbf{G}$  (see Theorem 4 in Section 3 below). This motivates us to construct the following classifier:

$$\delta_3(\mathbf{z}) = \begin{cases} 1 & \text{if } \mathscr{D}_3(\mathbf{z}) > 0, \\ 2 & \text{otherwise.} \end{cases}$$
(15)

#### 3 Asymptotics under HDLSS Regime

Suppose  $\mathbf{U} = (U_1, U_2, \dots, U_d)^{\top}$  and  $\mathbf{V} = (V_1, V_2, \dots, V_d)^{\top}$  are drawn independently from  $\mathbf{F}$  or  $\mathbf{G}$ . We assume that the component variables are weakly dependent. In particular, we assume the following.

Assumption 4. For any four d-dimensional random vectors  $\mathbf{U}, \mathbf{V}, \mathbf{Q}, \mathbf{Q}^*$  having distribution  $\mathbf{F}$  or  $\mathbf{G}$ , such that they are mutually independent,

$$i. \sum_{1 \le k_1 < k_2 \le d} \operatorname{cov}(\rho_0(U_{k_1}, V_{k_1}; Q_{k_1}), \rho_0(U_{k_2}, V_{k_2}; Q_{k_2})) = o(d^2);$$
  
$$ii. \sum_{1 < k_1 < k_2 < d} \operatorname{cov}(\rho_0(U_{k_1}, V_{k_1}; Q_{k_1}), \rho_0(U_{k_2}, V_{k_2}; Q_{k_2}^*)) = o(d^2).$$

Assumption 4 is trivially satisfied if the component variables of the underlying distributions are independently distributed, and it continues to hold when the components have  $\rho$ -mixing property.

**Theorem 3.** Suppose assumption 4 is satisfied. For a test observation Z,

(a) if 
$$\mathbf{Z} \sim \mathbf{F}$$
, then  $\left| \mathscr{D}_1(\mathbf{Z}) - \frac{1}{2} \overline{\mathcal{W}}^*_{\mathbf{FG}} \right| \xrightarrow{\mathbb{P}} 0$  and  $\left| \mathscr{D}_2(\mathbf{Z}) - \overline{\tau}_{\mathbf{FG}} \right| \xrightarrow{\mathbb{P}} 0$  as  $d \to \infty$ ;  
(b) if  $\mathbf{Z} \sim \mathbf{G}$ , then  $\left| \mathscr{D}_1(\mathbf{Z}) + \frac{1}{2} \overline{\mathcal{W}}^*_{\mathbf{FG}} \right| \xrightarrow{\mathbb{P}} 0$  and  $\left| \mathscr{D}_2(\mathbf{Z}) + \overline{\tau}_{\mathbf{FG}} \right| \xrightarrow{\mathbb{P}} 0$  as  $d \to \infty$ .

Theorem 3 states that if  $\mathbf{Z} \sim \mathbf{F}$  (respectively,  $\mathbf{Z} \sim \mathbf{G}$ ), the discriminants corresponding to  $\delta_1$  and  $\delta_2$  converge in probability to a positive (respectively, negative) quantity as  $d \to \infty$ . This justifies our construction of the classifiers  $\delta_1$  and  $\delta_2$  in (12) and (14), respectively.

Now, we expect  $\delta_1$  to yield an optimal performance if  $\overline{\mathcal{W}}^*_{\mathbf{FG}}$  does not vanish with increasing dimensions. Hence, it is reasonable to assume the following:

#### Assumption 5.

$$\liminf_{d\to\infty}\overline{\mathcal{W}}_{\mathbf{FG}}^* > 0.$$

Assumption 5 implies that the separation between  $\mathbf{F}$  and  $\mathbf{G}$  is asymptotically non-negligible. Observe that this assumption is satisfied if the component variables of  $\mathbf{F}$  and  $\mathbf{G}$  are identically distributed. We also need the following assumption for the asymptotic analysis of  $\delta_3$ .

#### Assumption 6.

$$\liminf_{d \to \infty} |T_{\mathbf{FF}} - T_{\mathbf{GG}}| > 0.$$

**Theorem 4.** Suppose assumptions 4 to 6 hold true. For a test observation **Z**,

(a) if 
$$\mathbf{Z} \sim \mathbf{F}$$
, then  $\left| \mathscr{D}_{3}(\mathbf{Z}) - \bar{\psi}_{\mathbf{FG}} \right| \stackrel{\mathbb{P}}{\to} 0$  as  $d \to \infty$ ;  
(b) if  $\mathbf{Z} \sim \mathbf{G}$ , then  $\left| \mathscr{D}_{3}(\mathbf{Z}) + \bar{\psi}_{\mathbf{FG}} \right| \stackrel{\mathbb{P}}{\to} 0$  as  $d \to \infty$ 

Theorem 4 states that if  $\mathbf{Z} \sim \mathbf{F}$  (respectively,  $\mathbf{Z} \sim \mathbf{G}$ ), the discriminant  $\mathscr{D}_3$  corresponding to  $\delta_3$  converges in probability to a positive (respectively, negative) quantity as  $d \to \infty$ , which justifies our construction of the classifier  $\delta_3$  in (15).

# 3.1 Misclassification Probabilities of $\delta_1, \delta_2$ , and $\delta_3$ in the HDLSS asymptotic regime

We now show the convergence of the misclassification probabilities of our classifiers  $\delta_i$  (denoted as  $\Delta_i$ ), under some fairly general assumptions for i = 1, 2, 3.

**Theorem 5.** Suppose assumptions 4 and 5 hold. Then,  $\Delta_1 \to 0$  and  $\Delta_2 \to 0$  as  $d \to \infty$ . If, in addition, assumption 6 holds, then  $\Delta_3 \to 0$  as  $d \to \infty$ .

Observe that the asymptotic behaviors of the classifiers  $\delta_1, \delta_2$ , and  $\delta_3$  are no longer governed by the constants  $\lambda_{\mathbf{FG}}, \sigma_{\mathbf{F}}$  and  $\sigma_{\mathbf{G}}$ . In fact, they are robust in terms of moment conditions since their behavior does not depend on the existence of any moments of **F** and **G** altogether.

#### 3.2 Comparison of the classifiers

Although the proposed classifiers yield perfect classification with increasing dimensions, they have some ordering among their misclassification rates under appropriate conditions. The following result describes the same.

**Theorem 6.** Suppose assumptions 4 and 5 hold. Then,

- (a) if  $\liminf_{d\to\infty} (\max\{T_{\mathbf{FF}}, T_{\mathbf{GG}}\} T_{\mathbf{FG}}) > 0$ , there exists  $d'_0 \in \mathbb{N}$  such that  $\Delta_2 \leq \Delta_3 \leq \Delta_1$  for all  $d \geq d'_0$ ,
- (b) if  $\liminf_{d\to\infty} (T_{\mathbf{FG}} \max\{T_{\mathbf{FF}}, T_{\mathbf{GG}}\}) > 0$  and assumption 6 holds, there exists  $d'_0 \in \mathbb{N}$ , such that  $\Delta_2 \geq \Delta_3 \geq \Delta_1$  for all  $d \geq d'_0$ .

**Remark.** If assumption 6 is dropped from Theorem 6(b), it can still be concluded that if  $\liminf_{d\to\infty} (T_{\mathbf{FG}} - \max\{T_{\mathbf{FF}}, T_{\mathbf{GG}}\}) > 0$ , under assumptions 4 and 5, there exists  $d'_0 \in \mathbb{N}$  such that  $\Delta_2 \geq \Delta_1$  for all  $d \geq d'_0$  (see Lemma A.8(b) of the Supplementary Material).

We observe that  $\delta_3$  always works 'moderately' among the proposed classifiers, in the sense that its misclassification probability is neither the largest nor the smallest in both the aforementioned situations. It might be difficult to verify the conditions in Theorem 6 in practice. Under such circumstances, it is more reasonable to use  $\delta_3$  since it is the most 'stable' among the proposed classifiers.

#### 4 Empirical Performance and Results

We examine the performance of our classifiers on a variety of simulated and real datasets, compared to several widely recognized classifiers such as GLMNET [15], Nearest Neighbor Random Projection (NN-RP) [11], Support Vector Machine with Linear (SVM-LIN) as well as Radial Basis Function (SVM-RBF) kernels [8], Neural Networks (N-NET) [5], and k-Nearest Neighbor [12,9] with k = 1 (i.e., 1-NN). Additionally, the Bayes classifier is treated as a benchmark classifier in all the aforementioned simulated examples to assess the performances of the proposed classifiers, since it performs optimally when the true data distributions

are known. All numerical exercises were executed on an Intel Xeon Gold 6140 CPU (2.30GHz, 2295 Mhz) using the R programming language [21]. Details about the packages used to implement the popular classifiers are provided in Section B of Supplementary Material.

#### 4.1 Simulation Studies

We perform our comparative study on five different simulated examples concerning different location problems as well as scale problems. In each example, we consider a binary classification problem with data simulated from two different d-variate distributions. Fixing the training sample size, we increase d to mimic an HDLSS setting. In such situations, our proposed classifiers are expected to achieve perfect classification at higher values of d. We carry out our analysis for eight different values of d, namely, 5, 10, 25, 50, 100, 250, 500, and 1000.

**Examples 1** and **2** were already introduced in Section 2. We consider three more simulated examples as follows:

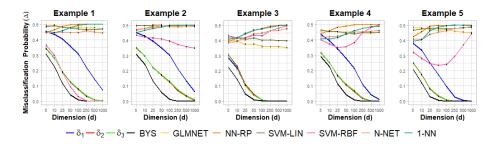
**Example 3:** 
$$X_{1k} \stackrel{\text{i.i.d}}{\sim} C(0,1)$$
 and  $Y_{1k} \stackrel{\text{i.i.d}}{\sim} C(1,1)$ ,  
**Example 4:**  $X_{1k} \stackrel{\text{i.i.d}}{\sim} C(1,1)$  and  $Y_{1k} \stackrel{\text{i.i.d}}{\sim} C(1,2)$ ,  
**Example 5:**  $X_{1k} \stackrel{\text{i.i.d}}{\sim} \frac{9}{10} N(1,1) + \frac{1}{10} C(4,1)$  and  $Y_{1k} \stackrel{\text{i.i.d}}{\sim} \frac{9}{10} N(1,2) + \frac{1}{10} C(4,1)$ ,

for  $1 \le k \le d$ . Here,  $C(\mu, \sigma^2)$  refers to Cauchy distribution with location  $\mu$  and scale  $\sigma$ . **Example 3** and **Example 4** are location and scale problems, respectively. In **Example 5**, we consider the competing distributions to be N(1,1) and N(1,2) but with 10% contamination from a C(4,1) distribution.

For all the examples, a training dataset was formed with a random sample of 20 observations from each class, and a randomly generated test dataset of size 200 (100 from each class) is used. The same process was repeated 100 times independently, and all individual misclassification rates were averaged to estimate the probability of misclassification of  $\delta_1, \delta_2$ , and  $\delta_3$  as well as the popular classifiers, which are reported in Section 2 of the Supplementary Material.

In **Examples 2, 3**, and 4, the competing distributions have identical first and second moments. Consequently,  $\delta_0$  performs poorly in such situations. For this reason, we have dropped  $\delta_0$  from further analysis. Plots of estimated misclassification probabilities of  $\delta_1, \delta_2$ , and  $\delta_3$  along with those of the aforementioned popular classifiers are given in Fig. 2. In each example, since the component variables are i.i.d., assumptions 4 and 5 hold. Consequently,  $\hat{T}_{IJ}$  is a consistent estimator of  $T_{IJ}$  as  $d \to \infty$  (see Lemma A.7(b) of the Supplementary Material) for  $I, J \in {\mathbf{F}, \mathbf{G}}$ . Hence, we estimate  $T_{IJ}$  by  $\hat{T}_{IJ}$  to explain Fig. 2. Fig. 2 shows that  $\Delta_1, \Delta_2$  and  $\Delta_3$  approach zero as d increases for all the examples.

For **Examples 1, 2,** and **4**, we observe  $\max\{\hat{T}_{\mathbf{FF}}, \hat{T}_{\mathbf{GG}}\} > \hat{T}_{\mathbf{FG}}$ . For these three examples, Fig. 2 shows that  $\Delta_2 \leq \Delta_3 \leq \Delta_1$ . For **Example 3**, we observe that  $\max\{\hat{T}_{\mathbf{FF}}, \hat{T}_{\mathbf{GG}}\} < \hat{T}_{\mathbf{FG}}$  (see Table 1, Section B of the Supplementary Material). For this example, Fig. 2 shows that  $\Delta_2 \geq \Delta_3 \geq \Delta_1$ . Thus, the numerical findings are consistent with Theorems 5 and 6 (see Sections 3.1 and 3.2).



**Fig. 2.** Average misclassification rates with errorbars for  $\delta_1$ ,  $\delta_2$ , and  $\delta_3$ , along with some popular classifiers for different dimensions. Bayes classifier is treated as a benchmark.

**Example 5** was specially curated to validate the effectiveness of our classifiers in terms of robustness to outliers. In this example, we have considered two competing mixture distributions with contamination arising from the C(4, 1) distribution. All of  $\delta_1$ ,  $\delta_2$ , and  $\delta_3$  outperform the popular classifiers as d increases. Even in the presence of contamination, all of our proposed classifiers tend to achieve perfect classification as d increases.

#### 4.2 Implementation on Real Data

Alongside the simulation studies, we implement our methods on several HDLSS datasets for a comprehensive performance evaluation. For each dataset, 50% of the observations were selected at random to create a training set, while keeping the proportions of observations from each class consistent with those of all original datasets. The remaining observations were used to create the test set. To obtain stable estimates of the misclassification probabilities, this procedure was repeated 100 times independently, and individual misclassification rates were averaged out to estimate the probability of misclassification of  $\delta_1$ ,  $\delta_2$ , and  $\delta_3$ , as well as the popular classifiers.

Although our methods are primarily designed for binary classification, we implement a majority voting ensemble in the case of *J*-class problems with  $J \ge 3$ . So, for a dataset with *J* different classes, we consider all  $\binom{J}{2} = \frac{J(J-1)}{2}$  many unordered pairs of classes and treat them as separate binary classification problems. For each test observation, we perform classification for all of those  $\binom{J}{2}$  many problems and classify the test observation to the class to which it gets assigned the maximum number of times. Ties are broken at random.

We conduct a case study on six real HDLSS datasets, namely, GSE1577 and GSE89 from the Microarray database<sup>4</sup>, Golub-1999-v2 and Gordon-2002 from the CompCancer database<sup>5</sup>, and Computers and DodgerLoopDay from the UCR Time Series Classification Archive<sup>6</sup> [10]. A brief description follows.

<sup>&</sup>lt;sup>4</sup> Available at https://file.biolab.si/biolab/supp/bi-cancer/projections/.

<sup>&</sup>lt;sup>5</sup> Available at https://schlieplab.org/Static/Supplements/CompCancer/datasets.htm.

<sup>&</sup>lt;sup>6</sup> Available at https://www.cs.ucr.edu/~eamonn/time\_series\_data\_2018/.

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- The **GSE1577** dataset consists of 19 data points and 15434 features. It is divided into 2 classes which are T-cell lymphoblastic lymphoma (T-LL) and T-cell acute lymphoblastic leukemia (T-ALL).
- The **GSE89** dataset consists of 40 data points and 5724 features. It is divided into 3 classes corresponding to three stages of tumor T2+, Ta, and T1.
- The **Golub-1999-v2** dataset consists of 72 data points and 1877 features. It is divided into 3 classes: Acute Myeloid Leukemia (AML), and two types of Acute Lymphoblastic Leukemia - B-cell ALL and T-cell ALL.
- The **Gordon-2002** dataset consists of 181 data points and 1626 features, divided into 2 classes about the pathological distinction between malignant pleural mesothelioma (MPM) and adenocarcinoma (AD) of the lung.
- The **Computers** dataset contains readings on electricity consumption from 500 households in the UK, sampled in two-minute intervals over a month. Each observation has 720 features. The data points are categorized into two classes: 'Desktop' and 'Laptop'.
- The **DodgerLoopDay** dataset consists of 158 data points and 288 features, divided into 7 classes corresponding to the 7 days of a week.

The estimated misclassification probabilities of  $\delta_1, \delta_2$ , and  $\delta_3$ , and the popular classifiers for these datasets are reported in Table 1. The number of classes, data points, and features are denoted by class, N, and d, respectively.

**Table 1.** Estimated misclassification probabilities (in %) with standard errors (in parentheses) for  $\delta_1$ ,  $\delta_2$ , and  $\delta_3$ , and popular classifiers for real datasets. For each dataset, the entries corresponding to the minimum misclassification rates are **boldfaced**.

Dataset	Description			Popular Classifiers						Proposed Classifiers		
	class	Ν	d	GLM- NET	NN- RP	SVM- LIN	SVM- RBF	N-NET	1-NN	$\delta_1$	$\delta_2$	$\delta_3$
GSE1577	2	19	15434	6.51 (0.66)	$11.59 \\ (0.44)$	6.38 (0.35)	30.06 (0.36)	33.87 (0.98)	$11.12 \\ (0.39)$	<b>6.06</b> (0.84)	7.33 (1.04)	<b>6.06</b> (0.84)
GSE89	3	40	5724	25.67 (0.47)	25.15 (0.49)	20.18 (1.05)	$ \begin{array}{c} 41.12 \\ (0.46) \end{array} $	43.03 (1.92)	17.54 (0.28)	$\begin{array}{c} {\bf 15.21} \\ (0.80) \end{array}$	$24.26 \\ (0.97)$	16.53 (0.96)
Golub-1999-v2	3	72	1877	8.78 (0.62)	15.28 (0.44)	10.55 (0.43)	33.05 (0.39)	75.26 (0.67)	8.98 (0.39)	<b>6.89</b> (0.41)	9.40 (0.43)	7.49 (0.44)
Gordon-2002	2	181	1626	$2.58 \\ (0.47)$	4.11 (0.12)	$1.26 \\ (0.09)$	$1.84 \\ (0.09)$	11.28 (2.83)	2.69 (0.10)	<b>0.53</b> (0.01)	$\begin{array}{c} 0.54 \\ (0.01) \end{array}$	<b>0.53</b> (0.01)
Computers	2	500	720	39.99 (0.69)	42.53 (0.62)	47.06 (0.39)	40.76 (0.27)	46.92 (0.41)	41.33 (0.57)	38.65 (0.28)	<b>36.38</b> (0.21)	36.50 (0.22)
DodgerLoopDay	7	158	288	55.45 (0.32)	48.72 (0.79)	$39.42 \\ (0.44)$	47.03 (0.36)	71.32 (1.08)	47.38 (0.68)	<b>37.68</b> (0.39)	44.73 (0.44)	42.23 (0.52)

For the datasets GSE1577, GSE89, Golub-1999-v2, Gordon-2002, and Dodger-LoopDay, the estimated misclassification probabilities of our proposed classifiers

are in the order  $\Delta_1 \leq \Delta_3 \leq \Delta_2$ , i.e., the performance of  $\delta_1$  is the best for these examples and the misclassification probability of  $\delta_3$  lies in between that of  $\delta_1$ and  $\delta_2$ . To understand the relative performance of these classifiers, we computed  $\hat{T}_{\mathbf{FF}}$ ,  $\hat{T}_{\mathbf{FG}}$  and  $\hat{T}_{\mathbf{GG}}$ , and they satisfy  $(\hat{T}_{\mathbf{FG}} - \max\{\hat{T}_{\mathbf{FF}}, \hat{T}_{\mathbf{GG}}\}) > 0$ . As discussed in Theorem 6, this ordering among the empirical versions of  $T_{\mathbf{FF}}$ ,  $T_{\mathbf{FG}}$ and  $T_{\mathbf{GG}}$  is consistent with the relative ordering of the performances of  $\delta_1$ ,  $\delta_2$ and  $\delta_3$ . Furthermore,  $\delta_1$  and  $\delta_3$  performed better than all the popular classifiers. Although  $\delta_2$  performed relatively worse than  $\delta_1$  and  $\delta_3$ , it outperformed NN-RP, SVM-RBF, N-NET, and 1-NN.

For the dataset Computers, the estimated misclassification probabilities of our proposed classifiers are in the order  $\Delta_2 \leq \Delta_3 \leq \Delta_1$ , i.e.,  $\delta_2$  showed the best performance with a misclassification probability close to 36%. The misclassification probability of  $\delta_3$  lies between that of  $\delta_1$  and  $\delta_2$ . It turns out that  $\hat{T}_{\mathbf{FF}}$ ,  $\hat{T}_{\mathbf{FG}}$  and  $\hat{T}_{\mathbf{GG}}$  satisfy (max{ $\hat{T}_{\mathbf{FF}}, \hat{T}_{\mathbf{GG}}$ } -  $\hat{T}_{\mathbf{FG}}$ ) > 0. This ordering among the empirical versions of  $T_{\mathbf{FF}}, T_{\mathbf{FG}}$  and  $T_{\mathbf{GG}}$  is consistent with the relative ordering of the performances of  $\delta_1$ ,  $\delta_2$  and  $\delta_3$  (see Theorem 6 of Section 3.2). All of  $\delta_1$ ,  $\delta_2$ , and  $\delta_3$  performed better than every popular classifier mentioned earlier.

Table 1 shows that  $\delta_1$ ,  $\delta_2$  and  $\delta_3$  outperform widely recognized classifiers in a majority of the reported datasets, which establishes the merit of our proposed methods over the widely recognized ones. In addition, for all the reported datasets, the ordering among  $\hat{T}_{\mathbf{FF}}$ ,  $\hat{T}_{\mathbf{FG}}$  and  $\hat{T}_{\mathbf{GG}}$  were found out to be consistent with the results stated in Theorem 6.

## 5 Concluding Remarks

In this paper, we developed some classification methods that draw good intuition from both classical and recent developments. We proved that under some general conditions, the misclassification probabilities of these classifiers steadily approach 0 in the HDLSS asymptotic regime. The major advantages of our proposed methods are that they are free of tuning parameters, robust in terms of moment conditions, and easy to implement. Theoretical justification and comprehensive empirical studies against other well-established classification methods establish the advantages of our approach.

Nevertheless, when the competing distributions have at most o(d) many different marginals, and the rest are identically distributed, assumptions 5 and 6 will no longer hold. The theoretical guarantees for the optimal performance of the proposed classifiers will break down in such situations. Developing classifiers that avoid these assumptions is a fruitful avenue for further research.

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