# $\alpha-\boldsymbol{\beta}$-Factorization and the Binary Case of Simon's Congruence 

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#### Abstract

In 1991 Hébrard introduced a factorization of words that turned out to be a powerful tool for the investigation of a word's scattered factors (also known as (scattered) subwords or subsequences). Based on this, first Karandikar and Schnoebelen introduced the notion of $k$ richness and later on Barker et al. the notion of $k$-universality. In 2022 Fleischmann et al. presented at DCFS a generalization of the arch factorization by intersecting the arch factorization of a word and its reverse. While the authors merely used this factorization for the investigation of shortest absent scattered factors, in this work we investigate this new $\alpha-\beta$-factorization as such. We characterize the famous Simon congruence of $k$-universal words in terms of 1 -universal words. Moreover, we apply these results to binary words. In this special case, we obtain a full characterization of the classes and calculate the index of the congruence. Lastly, we start investigating the ternary case, present a full list of possibilities for $\alpha \beta \alpha$-factors, and characterize their congruence.


## 1 Introduction

A scattered factor, subsequence, subword or scattered subword of a word $w$ is a word that is obtained by deleting any number of letters from $w$ while preserving the order of the remaining letters. For example, oiaoi and cmbntrcs are both scattered factors of combinatorics. In contrast to a factor, like combinat, a scattered factor is not necessarily contiguous. Note that a scattered factor $v$ can occur in different ways inside a word $w$, for example, ab occurs in aab as aab and aab as marked by the lines below the letters. The relation of $u$ being a scattered factor of $v$ is a partial order on words.

In this paper, we focus on the congruence relation $\sim_{k}$ for $k \in \mathbb{N}_{0}$ which is known as Simon's congruence [22. For two words, we have $u \sim_{k} v$ iff they share all scattered factors up to length $k$. Unions of the congruence classes of this relation are used to form the piecewise testable languages (first studied by Simon [22]), which are a subclass of the regular languages (they are even subregular).

A long-standing open question, posed by Sakarovitch and Simon [21], is the exact structure of the congruence classes of $\sim_{k}$ and the index of the congruence relation itself. Two existing results include a characterization of the congruence
in terms of a special common upper bound of two words [23, Lemma 6], as well as a characterization of the (not unique) shortest elements of the congruence classes [21, Theorem 6.2.9] and 22|4|1. The index of the relation is described asymptotically by Karandikar et al. 12. Currently, no exact formula is known. One approach for studying scattered factors in words is based on the notion of scattered factor universality. A word $w$ is called $\ell$-universal if it contains all words of length $\ell$ over a given alphabet as scattered factors. For instance, the word alfalfa ${ }^{3}$ is 2 -universal since it contains all words of length two over the alphabet $\{\mathrm{a}, \mathrm{l}, \mathrm{f}\}$ as scattered factors. Barker et al. and Fleischmann et al. [15] study the universality of words, as well as how the universality of a word changes when considering repetitions of a word. Fleischmann et al. 6] investigate the classes of Simon's congruence separated by the number of shortest absent scattered factors, characterize the classes for arbitrary alphabets for some fixed numbers of shortest absent scattered factors and give explicit formulas for these subsets. The shortest absent scattered factors of alfalfa are fff, ffl lll, and fll. A main tool in this line of research is a newly introduced factorization, known as the $\alpha$ - $\beta$-factorization [6] which is based on the arch factorization by Hébrard [9. The arch factorization factorizes a word into factors of minimal length containing the complete alphabet. The $\alpha-\beta$-factorization takes also the arch factorization of the reversed word into account. Kosche et al. [16] implicitly used this factorization to determine shortest absent scattered factors in words. In this paper, we study this factorization from a purely combinatorial point of view. The most common algorithmic problems regarding Simon's congruence are SimK (testing whether two words $u, v$ are congruent for a fixed $k$ ) and MaxSimK (the optimization problem of finding the largest $k$ such that they are congruent). The former was approached by finding the (lexicographical least element of the) minimal elements of the congruence classes of $u$ and $v$. Results regarding normal forms and the equation $p w q \sim_{k} r$ for given words $p, q, r$ can be found in 2017 . The computation of the normal form was improved first by Fleischer et al. (4) and later by Barker et al. [1. The latter was approached in the binary case by Hébrard [9], and was solved in linear time using a new approach by Gawrychowski et al. [8]. A new perspective on $\sim_{k}$ was recently given by Sungmin Kim et al. 1415 when investigating the congruence's closure and pattern matching w.r.t. $\sim_{k}$.

Our Contribution. We investigate the $\alpha$ - $\beta$-factorization as an object of independent interest and give necessary and sufficient conditions for the congruence of words in terms of their factors. We characterize $\sim_{k}$ in terms of 1-universal words through their $\alpha \beta \alpha$-factors. We use these results to characterize the congruence classes of binary words and their cardinality, as well as to calculate the index in this special case. Moreover, we give a short and conceptually straightforward algorithm for MAXSimK for binary words. Lastly, we start to transfer the previous results to the ternary alphabet.

Structure of the Work. First, in Section 2 we establish basic definitions and notation. In Section 3, we give our results regarding the $\alpha$ - $\beta$-factorization for

[^0]arbitrary alphabets, including the characterization of the congruence of words w.r.t. $\sim_{k}$ in terms of their $\alpha \beta \alpha$-factors. Second, in Section 4 we present our results regarding binary words. We characterize the congruence classes of binary words in terms of their $\alpha$ - and $\beta$-factors, and apply them to calculate the index of $\sim_{k}$ in this special case. Third, in Section [5] we consider a ternary alphabet and investigate the cases for the $\beta$-factors. Last, in Section 6, we conclude and give ideas for further research.

## 2 Preliminaries

We set $\mathbb{N}:=\{1,2,3, \ldots\}$ and $\mathbb{N}_{0}:=\{0\} \cup \mathbb{N}$ as well as $[m]:=\{1, \ldots, m\}$ and $[m]_{0}:=\{0\} \cup[m]$. We denote disjoint unions by $\sqcup$. If there exists a bijection between two sets $A, B$, then we write $A \cong B$. An alphabet is a finite set $\Sigma$ whose elements are called letters. An alphabet of cardinality $i \in \mathbb{N}$ is abbreviated by $\Sigma_{i}$. A word $w$ is a finite sequence of letters from $\Sigma$ where $w[i]$ denotes the $i^{\text {th }}$ letter of $w$. The set of all words over the alphabet $\Sigma$ is denoted by $\Sigma^{*}$ and the empty word by $\varepsilon$. Set $\Sigma^{+}:=\Sigma^{*} \backslash\{\varepsilon\}$. The length $|w|$ of $w$ is the number of letters in $w$, i.e., $|\varepsilon|=0$. We denote the set of all words of length $k \in \mathbb{N}_{0}$ by $\Sigma^{k}$ and set $\Sigma^{\leq k}:=\left\{w \in \Sigma^{*}| | w \mid \leq k\right\}$. Set alph $(w):=\{w[i] \in \Sigma \mid i \in[|w|]\}$. Set $|w|_{\mathrm{a}}:=|\{i \in[|w|] \mid w[i]=\mathrm{a}\}|$ for all $\mathrm{a} \in \Sigma$. A word $u \in \Sigma^{*}$ is called factor of $w \in \Sigma^{*}$ if there exist $x, y \in \Sigma^{*}$ with $w=x u y$. In the case that $x=\varepsilon, u$ is called prefix of $w$ and suffix if $y=\varepsilon$. The factor of $w$ from its $i^{\text {th }}$ letter to its $j^{\text {th }}$ letter is denoted by $w[i . . j]$ for $1 \leq i \leq j \leq|w|$. For $j<i$ we define $w[i . . j]:=\varepsilon$. If $w=x y$ we write $x^{-1} w$ for $y$ and $w y^{-1}$ for $x$. For $u \in \Sigma^{*}$ we set $u^{0}:=\varepsilon$ and inductively $u^{\ell}:=u u^{\ell-1}$ for all $\ell \in \mathbb{N}$. For $w \in \Sigma^{*}$ define $w^{R}$ as $w[|w|] \cdots w[1]$. For more background information on combinatorics on words see [18.

Now, we introduce the main notion of our work, the scattered factors also known as (scattered) subwords or subsequence (also cf. [21).

Definition 1. A word $u \in \Sigma^{*}$ of length $n \in \mathbb{N}_{0}$ is called a scattered factor of $w \in \Sigma^{*}$ if there exist $v_{0}, \ldots, v_{n} \in \Sigma^{*}$ with $w=v_{0} u[1] v_{1} \cdots v_{n-1} u[n] v_{n}$, denoted by $u \preceq w$. Let $\operatorname{ScatFact}(w):=\left\{v \in \Sigma^{*} \mid v \preceq w\right\}$ as well as $\operatorname{ScatFact}_{k}(w):=$ $\operatorname{ScatFact}(w) \cap \Sigma^{k}$ and $\operatorname{ScatFact} \leq k(w):=\operatorname{ScatFact}(w) \cap \Sigma^{\leq k}$.

For instance, we have and $\preceq$ agenda but nada $\npreceq$ agenda. For comparing words w.r.t. their scattered factors, Simon introduced a congruence relation nowadays known as Simon's congruence [22. Two words are called Simon $k$ congruent, if they have the same set of scattered factors up to length $k$. We refer to this $k$ as the level of the congruence. This set is the full $k$-spectrum of a word, whereas the $k$-spectrum only contains all scattered factors of exactly length $k$.

Definition 2. Let $k \in \mathbb{N}$. Two words $u, v \in \Sigma^{*}$ are called Simon $k$-congruent $\left(u \sim_{k} v\right)$ iff $\operatorname{ScatFact}_{\leq k}(u)=\operatorname{ScatFact}_{\leq k}(v)$. Let $[u]_{\sim_{k}}$ denote the congruence class of $u$ w.r.t. $\sim_{k}$.

For instance, over $\Sigma=\{\mathrm{a}, \mathrm{b}\}$, the words abaaba and baab are Simon 2congruent since both contain each all words up to length 2 as scattered factors.

On the other hand, they are not Simon 3-congruent since we have aaa $\preceq$ abaaba but aaa $\npreceq$ baab.

Starting in 1213 and [11] special $k$-spectra were investigated in the context of piecewise testable languages: the rich resp. $k$-rich words. This work was pursued from the perspective of the universality problem for languages in [13|8|5] with the new notion of $k$-universal words.

Definition 3. A word $w \in \Sigma^{*}$ is called $k$-universal w.r.t. $\Sigma$ if $\operatorname{ScatFact}_{k}(w)=$ $\Sigma^{k}$. The maximal $k$ such that $w$ is $k$-universal is denoted by $\iota_{\Sigma}(w)$ and called $w$ 's universality index.

Remark 4. If we investigate a single word $w \in \Sigma^{*}$, we assume $\Sigma=\operatorname{alph}(w)$ implicitly and omit the $\Sigma$ as index of $\iota$.

In [6] the notion of universality was extended to $m$-nearly $k$-universal words, which are words where exactly $m$ scattered factors of length $k$ are absent, i.e., $\left|\operatorname{ScatFact}_{k}(w)\right|=|\Sigma|^{k}-m$. In the last section of their paper the authors introduce a factorization of words based on the arch factorization (cf. [9]) in order to characterize the 1-nearly $k$-universal words with $\iota(w)=k-1$. This work is closely related to the algorithmic investigation of shortest absent scattered factors [8]. Therefore, we introduce first the arch factorization and based on this the $\alpha-\beta$ factorization from [6]. An arch is a factor of minimal length (when read from left to right) containing the whole alphabet. Consider the word $w=$ abaccaabca. This leads to the arch factorization (abac) • (caab) • ca where the arches are visualized by the brackets.

Definition 5. For a word $w \in \Sigma^{*}$ the arch factorization is given by $w=$ : $\operatorname{ar}_{1}(w) \cdots \operatorname{ar}_{k}(w) \operatorname{re}(w)$ for $k \in \mathbb{N}_{0}$ with $\operatorname{alph}\left(\operatorname{ar}_{i}(w)\right)=\Sigma$ for all $i \in[k]$, the last letter of $\operatorname{ar}_{i}(w)$ occurs exactly once in $\operatorname{ar}_{i}(w)$ for all $i \in[k]$, and $\operatorname{alph}(\operatorname{re}(w)) \subset \Sigma$. The words $\operatorname{ar}_{i}(w)$ are called arches of $w$ and $\operatorname{re}(w)$ is the rest of $w$. Define the modus of $w$ as $\mathrm{m}(w):=\operatorname{ar}_{1}(w)\left[\left|\operatorname{ar}_{1}(w)\right|\right] \cdots \operatorname{ar}_{k}(w)\left[\left|\operatorname{ar}_{k}(w)\right|\right] \in \Sigma^{k}$. For abbreviation let $\operatorname{ar}_{i . . j}(w)$ denote the concatenation from the $i^{\text {th }}$ arch to the $j^{\text {th }}$ arch.

The following remark is a direct consequence of the combination of the $k$ universality and the arch factorization.

Remark 6. Let $w, w^{\prime} \in \Sigma^{*}$ such that $w \sim_{k} w^{\prime}$ for some $k \in \mathbb{N}_{0}$. Then either both $w, w^{\prime}$ have $k$ or more arches or they both have less than $k$ and the same number of arches. Moreover, we have $\iota(w)=k$ iff $w$ has exactly $k$ arches.

A generalization of the arch factorization was introduced in [6 inspired by [16. In this factorization not only the arch factorization of a word $w$ but also the one of $w^{R}$ is taken into consideration. If both arch factorisations, i.e., the one of $w$ and the one of $w^{R}$ are considered simultaneously, we get overlaps of the arches and special parts which start at a modus letter of a reverse arch and end in a modus letter of an arch. For better readability, we use a specific notation for the arch factorisation of $w^{R}$ where we read the parts from left to right: let $\operatorname{arr}_{i}(w):=\left(\operatorname{ar}_{\iota(w)-i+1}\left(w^{R}\right)\right)^{R}$ the $i^{\text {th }}$ reverse $\operatorname{arch}$, let $\overleftarrow{\operatorname{re}}(w):=\left(\operatorname{re}\left(w^{R}\right)\right)^{R}$ the reverse rest, and define the reverse modus $\overline{\mathrm{m}}(w)$ as $\mathrm{m}\left(w^{R}\right)^{R}$.

Definition 7. For $w \in \Sigma^{*}$ define $w$ 's $\alpha$ - $\beta$-factorization (cf. Figure 1) by $w=$ : $\alpha_{0} \beta_{1} \alpha_{1} \cdots \alpha_{\iota(w)-1} \beta_{\iota(w)} \alpha_{\iota(w)}$ with $\operatorname{ar}_{i}(w)=\alpha_{i-1} \beta_{i}$ and $\overline{\operatorname{arr}}_{i}(w)=\beta_{i} \alpha_{i}$ for all $i \in[\iota(w)], \overleftarrow{\mathrm{re}}(w)=\alpha_{0}$, as well as $\mathrm{re}(w)=\alpha_{\iota(w)}$. Define $\operatorname{core}_{i}:=\varepsilon$ if $\left|\beta_{i}\right| \in\{1,2\}$ and $\operatorname{core}_{i}=\beta_{i}\left[2 . .\left|\beta_{i}\right|-1\right]$ otherwise, i.e., as the $\beta_{i}$ without the associated letters of the modus and reverse modus.

For example, consider $w=$ bakebananacake $\in\{\mathrm{a}, \mathrm{b}, \mathrm{c}, \mathrm{k}, \mathrm{e}\}^{*}$. We get $\operatorname{ar}_{1}(w)=$ bakebananac, $\operatorname{re}(w)=$ ake and $\operatorname{ara}_{1}(w)=$ bananacake, $\overleftarrow{\text { re }}(w)=$ bake. Thus, we have $\alpha_{0}=$ bake, $\beta_{1}=$ bananac and $\alpha_{1}=$ ake. Moreover, we have $\mathrm{m}(w)=\mathrm{c}$ and $\overleftarrow{\mathrm{m}}(w)=\mathrm{b}$. This leads to core $_{1}=$ anana.


Fig. 1. $\alpha$ - $\beta$-Factorization of a word $w$ with 4 arches.

Remark 8. In contrast to the arch factorization, the $\alpha$ - $\beta$-factorization is left-right-symmetric. Note that the $i^{\text {th }}$ reverse arch always starts inside the $i^{\text {th }}$ arch since otherwise an arch or the rest would contain at least two reverse arches or a complete arch and thus the arch would contain the complete alphabet more than once or once.

For better readability, we do not parametrize the $\alpha_{i}$ and $\beta_{i}$ by $w$. Instead, we denote the factors according to the word's name, i.e. $\tilde{\alpha}_{i} \tilde{\beta}_{i+1}$ is an arch of $\tilde{w}$.

Remark 9. Since $\left|\operatorname{alph}\left(\alpha_{i}\right)\right| \leq|\Sigma|-1$ we can build the arch factorization of $\alpha_{i}$ w.r.t. some $\Omega$ with $\operatorname{alph}\left(\alpha_{i}\right) \subseteq \Omega \in\binom{\Sigma}{|\Sigma|-1}$. This yields the same factorization for all $\Omega$ because either $\operatorname{alph}\left(\alpha_{i}\right)=\Omega$ or $\operatorname{alph}\left(\alpha_{i}\right) \subset \Omega$ and thus re $\left(\alpha_{i}\right)=\alpha_{i}$.

Last, we recall three lemmata regarding Simon's congruence which we need for our results. The first lemma shows that if we prepend or append a sufficiently universal word to two congruent words each, we obtain congruent words with an increased level of congruence.

Lemma 10 ([12, Lemma 4.1][13, Lemma 3.5]). Let $w, \tilde{w} \in \Sigma^{*}$ such that $w \sim_{k} \tilde{w}$, then for all $u, v \in \Sigma^{*}$ we have $u w v \sim_{\iota(u)+k+\iota(v)} u \tilde{w} v$.

The next lemma characterizes the omittance of suffixes when considering words up to $\sim_{k}$.

Lemma 11 ([23, Lemma 3]). Let $u, v \in \Sigma^{+}$and $\mathrm{x} \in \Sigma$. Then, $u v \sim_{k} u$ iff there exists a factorization $u=u_{1} u_{2} \cdots u_{k}$ such that $\operatorname{alph}\left(u_{1}\right) \supseteq \operatorname{alph}\left(u_{2}\right) \supseteq$ $\ldots \supseteq \operatorname{alph}\left(u_{k}\right) \supseteq \operatorname{alph}(v)$.

The last lemma characterizes letters which can be omitted when we consider words up to $\sim_{k}$. The last two of its conditions follow from the previous lemma.

Lemma 12 ([23, Lemma 4]). Let $u, v \in \Sigma^{*}$ and $\mathrm{x} \in \Sigma$. Then, $u v \sim_{k} u \mathrm{x} v$ iff there exist $p, p^{\prime} \in \mathbb{N}_{0}$ with $p+p^{\prime} \geq k$ and $u \mathrm{x} \sim_{p} u$ and $\mathrm{x} v \sim_{p^{\prime}} v$.

## $3 \alpha-\beta$-Factorization

In this section, we investigate the $\alpha$ - $\beta$-factorization based on results of 12 in the relatively new light of factorizing an arch into an $\alpha$ and a $\beta$ part. The main result states that it suffices to look at 1-universal words in order to gain the information about the congruence classes of $\sim_{k}$.

Remark 13. By the left-right symmetry of the $\alpha$ - $\beta$-factorisation, it suffices to prove most of the claims only for one direction (reading the word from left to right) and the other direction (reading the word from right to left) follows immediately. Thus, these claims are only given for one direction and it is not always mentioned explicitly that the analogous claim holds for the other direction.

Our first lemma shows that cutting of $\ell$ arches from two $k$-congruent words each, leads to ( $k-\ell$ )-congruence. Here, we use the $\alpha$ - $\beta$-factorization's symmetry.

Lemma 14. Let $w, \tilde{w} \in \Sigma^{*}$ with $w \sim_{k} \tilde{w}$ and $\iota(w)=\iota(\tilde{w})<k$. Then we have $\alpha_{i} \beta_{i+1} \alpha_{i+1} \cdots \alpha_{j} \sim_{k-\iota(w)+j-i} \tilde{\alpha}_{i} \tilde{\beta}_{i+1} \tilde{\alpha}_{i+1} \cdots \tilde{\alpha}_{j}$ for all $0 \leq i \leq j \leq \iota(w)$.

Proof. By symmetry, it suffices to show one inclusion of the $(k-1)$-spectra. Let $v \preceq \operatorname{ar}_{1}^{-1}(w) \cdot w=\operatorname{ar}_{2 . . \iota(w)}(w) \operatorname{re}(w)$ with $|v| \leq k-1$ and $\mathrm{m}_{1}:=\mathrm{m}(\tilde{w})[1]$. Then $\mathrm{m}_{1} \cdot v \preceq w$ because $\mathrm{m}_{1} \preceq \operatorname{ar}_{1}(w)$. Since $w \sim_{k} \tilde{w}$, we have $\mathrm{m}_{1} \cdot v \preceq \tilde{w}$ and therefore $v \preceq \operatorname{ar}_{1}^{-1}(\tilde{w}) \cdot \tilde{w}$. This proves the first claim.


The second claim follows by left-right-symmetry and induction by repeatedly cutting $\alpha \beta$-pairs from the left and $\beta \alpha$-pairs form the right.

The following proposition shows that two words having exactly the same $\beta$ factors are $k$-congruent iff the corresponding $\alpha$-factors are congruent at a smaller level. The proof uses a similar idea to the one presented by Karandikar et al. [12, Lemma 4.2].

Proposition 15. For all $w, \tilde{w} \in \Sigma^{*}$ with $m:=\iota(w)=\iota(\tilde{w})<k$ such that $\beta_{i}=\tilde{\beta}_{i}$ for all $i \in[m]$, we have $w \sim_{k} \tilde{w}$ iff $\alpha_{i} \sim_{k-m} \tilde{\alpha}_{i}$ for all $i \in[m]_{0}$.

Proof. By Lemma 14, we directly obtain one direction. Therefore, let $w, \tilde{w} \in \Sigma^{*}$ such that $\beta_{i}=\tilde{\beta}_{i}$ for all $i \in[m]$ and $\alpha_{i} \sim_{k-m} \tilde{\alpha}_{i}$ for all $i \in[m]_{0}$. We obtain by Lemma 10
$\tilde{\alpha}_{0} \beta_{1} \tilde{\alpha}_{1} \cdots \tilde{\alpha}_{i-1} \beta_{i} \alpha_{i} \beta_{i+1} \cdots \beta_{\iota(w)} \alpha_{\iota(w)} \sim_{k} \tilde{\alpha}_{0} \beta_{1} \tilde{\alpha}_{1} \cdots \tilde{\alpha}_{i-1} \beta_{i} \tilde{\alpha}_{i} \beta_{i+1} \cdots \beta_{\iota(w)} \alpha_{\iota(w)}$
for all $i \in[m]_{0}$. Thus, by transitivity of $\sim_{k}$, we have $w \sim_{k} \tilde{w}$.
As an immediate corollary, we obtain the following statement which allows us to normalize the $\alpha$-factors when proving congruence of words.

Corollary 16. Let $w, \tilde{w} \in \Sigma^{*}$ with $m:=\iota(w)=\iota(\tilde{w})<k$, then $w \sim_{k} \tilde{w}$ iff $\alpha_{i} \sim_{k-m} \tilde{\alpha}_{i}$ for all $i \in[m]_{0}$ and for $w^{\prime}:=\alpha_{0} \tilde{\beta}_{1} \alpha_{1} \cdots \tilde{\beta}_{m} \alpha_{m}$ we have $w \sim_{k} w^{\prime}$.

Proof. Note that, $w^{\prime}$ is in $\alpha-\beta$-factorization because the exchanged $\alpha_{i}$ are equivalent, and thus they have the same alphabets. Assume $w \sim_{k} \tilde{w}$. By Lemma 14 we have $\alpha_{i} \sim_{k-m} \tilde{\alpha}_{i}$. Define $w^{\prime}$ as above. We have $w^{\prime} \sim_{k} \tilde{w}$ by Proposition 15 and therefore, by transitivity $w \sim_{k} w^{\prime}$.

Now assume the converse. By $\alpha_{i}^{\prime}=\alpha_{i} \sim_{k-m} \tilde{\alpha}_{i}$ and $\beta_{j}^{\prime}=\tilde{\beta}_{j}$ for all $i \in$ $[m]_{0}, j \in[m]$, we obtain by Proposition 15 that $\tilde{w} \sim_{k} w^{\prime}$. By the assumption and transitivity, we obtain $w \sim_{k} \tilde{w}$.

Next, we show the central result for this section. We can characterize the congruence of words by the congruence of their $\alpha \beta \alpha$-factors. Therefore, it suffices to consider 1-universal words in general. Again, the proof uses Lemma 10 and is inspired by Karandikar et al. [12, Lemma 4.2] and repeatedly exchanges factors up to $k$-Simon congruence.

Theorem 17. Let $w, \tilde{w}_{\tilde{\beta}} \in \Sigma^{*}$ with $m:=\iota(w)=\iota(\tilde{w})<k$. Then, $w \sim_{k} \tilde{w}$ iff $\alpha_{i-1} \beta_{i} \alpha_{i} \sim_{k-m+1} \tilde{\alpha}_{i-1} \tilde{\beta}_{i} \tilde{\alpha}_{i}$ for all $i \in[m]$.

Proof. Assume $w \sim_{k} \tilde{w}$, then the congruences follow directly by Lemma 14 for $i, j \in \mathbb{N}_{0}$ with $|j-i|=1$.

Assume $\alpha_{i-1} \beta_{i} \alpha_{i} \sim_{k-m+1} \tilde{\alpha}_{i-1} \tilde{\beta}_{i} \tilde{\alpha}_{i}$ for all $i \in[m]$. By Lemma 14, we obtain that $\alpha_{i} \sim_{k-m} \tilde{\alpha}_{i}$ for all $i \in[m]_{0}$. By Corollary 16, we have $\alpha_{i-1} \beta_{i} \alpha_{i} \sim_{k-m+1}$ $\alpha_{i-1} \tilde{\beta}_{i} \alpha_{i}$ for all $i \in[m]$, and it suffices to show that $w \sim_{k} \alpha_{0} \tilde{\beta}_{1} \alpha_{1} \cdots \beta_{m} \alpha_{m}$. Now, we have by repeated applications of Lemma 10 that

$$
\alpha_{0} \beta_{1} \alpha_{1} \cdot \beta_{2} \alpha_{2} \cdots \beta_{m} \alpha_{m} \sim_{k} \alpha_{0} \tilde{\beta}_{1} \alpha_{1} \cdot \beta_{2} \alpha_{2} \cdots \beta_{m} \alpha_{m} \sim_{k} \ldots \sim_{k} \tilde{w}
$$

In the light of Theorem 17, in the following, we consider some special cases of these triples w.r.t. the alphabet of both involved $\alpha$. Hence, let $w, \tilde{w} \in \Sigma^{*}$ with $1=\iota(w)=\iota(\tilde{w})$.

Proposition 18. Let $\alpha_{0}=\alpha_{1}=\tilde{\alpha}_{0}=\tilde{\alpha}_{1}=\varepsilon$. Then $w \sim_{k} \tilde{w}$ iff $k=1$ or $k \geq 2$, $\mathrm{m}(w)=\mathrm{m}(\tilde{w}), \overleftarrow{\mathrm{m}}(w)=\overleftarrow{\mathrm{m}}(\tilde{w})$, and $\operatorname{core}_{1} \sim_{k}{\widetilde{\operatorname{core}_{1}}}_{1}$.

Proof. If $k=1$, the claim follows directly. If $k \geq 2, \mathrm{~m}(w)=\mathrm{m}(\tilde{w}), \overleftarrow{\mathrm{m}}(w)=\overleftarrow{\mathrm{m}}(\tilde{w})$ and $w[2 . .|w|-1]=\operatorname{core}_{1} \sim_{k}{\widetilde{\operatorname{core}_{1}}}_{1}=\tilde{w}[2 . .|\tilde{w}|-1]$, then $w \sim_{k} \tilde{w}$ follows directly from the fact that $\sim_{k}$ is a congruence.

Assume $w \sim_{k} \tilde{w}$ and $k \geq 2$. Suppose $\mathrm{m}(w) \neq \mathrm{m}(\tilde{w})$. Then $\mathrm{m}(\tilde{w}) \mathrm{m}(w) \preceq$ $w$ because $\operatorname{alph}(w)=\Sigma$. By $w \sim_{k} \tilde{w}$, we get $\mathrm{m}(\tilde{w}) \mathrm{m}(w) \preceq \tilde{w}=\operatorname{ar}_{1}(\tilde{w})$, a contradiction. Therefore, $\mathrm{m}(w)=\mathrm{m}(\tilde{w})$ and by symmetry $\overline{\mathrm{m}}(w)=\overline{\mathrm{m}}(\tilde{w})$. Now the claim follows because core ${ }_{1}=\pi_{\Omega}(w) \sim_{k} \pi_{\Omega}(\tilde{w})=\widetilde{\text { core }}_{1}$ where $\Omega=\Sigma \backslash$ $\{\mathrm{m}(w), \overleftarrow{\mathrm{m}}(w)\}=\Sigma \backslash\{\mathrm{m}(\tilde{w}), \overleftarrow{\mathrm{m}}(\tilde{w})\}$.
Proposition 19. Let $\operatorname{alph}\left(\alpha_{0}\right)=\operatorname{alph}\left(\alpha_{1}\right)=\operatorname{alph}\left(\tilde{\alpha}_{0}\right)=\operatorname{alph}\left(\tilde{\alpha}_{1}\right) \in\binom{\Sigma}{|\Sigma|-1}$. We have $w \sim_{k} \tilde{w}$ iff $\alpha_{i} \sim_{k-1} \tilde{\alpha}_{i}$ for all $i \in[1]_{0}$.
Proof. In both directions, we have $\beta=\tilde{\beta} \in \Sigma$ by uniqueness of their first and last letter (which are identical). The claim follows from Proposition 15 ,

In the last two propositions, we considered special cases of congruence classes, where all words in such a congruence class have not only the same modus but also the same reverse modus. This is not necessarily always the case witnessed by $w=$ ababeabab $\cdot$ abecd $\cdot \operatorname{cdcdcd} \sim_{4}$ ababeabab $\cdot$ baedc $\cdot \operatorname{cdcdcd}=\tilde{w}$ with $\mathrm{m}(w)=\mathrm{d} \neq \mathrm{c}=\mathrm{m}(\tilde{w})$ and $\widehat{\mathrm{m}}(w)=\mathrm{a} \neq \mathrm{b}=\tilde{\mathrm{m}}(\tilde{w})$. This case occurs if one of the $\alpha$ satisfies $\alpha_{0} \mathrm{x} \sim_{k-1} \alpha_{0}$ and the alphabet of $\alpha_{1}$ factor is missing at least x for all $\mathrm{x} \in\left\{\overleftarrow{\mathrm{m}}(\tilde{w}) \mid \tilde{w} \in[w]_{\sim_{k}}\right\}$. The conditions for $\mathrm{m}(w)$ are analogous. In the last proposition of this section, we show a necessary condition for the $\alpha$-factors of words which are congruent to words with a different modus. The proof uses the same factorization as the proof of Lemma 11(cf. [23, Lemma 3]). By identifying permutable factors, similar ideas also appear when characterizing the shortest elements in congruence classes (cf. [21, Theorem 6.2.9][4, Proposition 6]).
Proposition 20. Let $w \in \Sigma^{*}$ with $\iota(w)=1, k \in \mathbb{N}$, and $\overleftarrow{M}:=\{\overleftarrow{\mathrm{m}}(\tilde{w})[1] \mid \tilde{w} \in$ $\left.[w]_{\sim_{\mathfrak{k}}}\right\}$, i.e., we capture all modus letters of words which are $k$-congruent to $w$. If $|\mathrm{M}| \geq 2$ then there exists a factorization $\alpha_{0}=: u_{1} \cdots u_{k-1}$ with $\operatorname{alph}\left(u_{1}\right) \supseteq$ $\ldots \supseteq \operatorname{alph}\left(u_{k-1}\right) \supseteq \overleftarrow{M}$.
Proof. Let $w, \overleftarrow{M}$ be as above and assume $|\bar{M}| \geq 2$. We define the following factorization inductively. Let $z \in \Sigma^{*}$. If $\bar{M} \nsubseteq \operatorname{alph}(z)$, the factorization of $z$ is $v_{1}:=z$. Otherwise, set $v_{1}$ as the shortest prefix of $z$ containing $\overleftarrow{M}$. Note that the last letter of $v_{1}$ is unique. By induction on the length $z$, there exists a factorization $v_{2} \cdots v_{n}$ of $v_{1}^{-1} z$. The factorisation of $z$ is given by $v_{1} v_{2} \cdots v_{n}$. Note that the last letters of $v_{1}, \ldots, v_{n-1}$ are unique and $\overleftarrow{\mathrm{M}} \nsubseteq \operatorname{alph}\left(v_{n}\right)$.

By factorizing $\alpha_{0}$ this way, we obtain a factorization $v_{1} v_{2} \cdots v_{n}$. Note that we have $\operatorname{alph}\left(v_{i}\right) \supseteq \overline{\mathrm{M}}$ exactly for $i \in[n-1]$. If $n \geq k$, define $u_{i}:=v_{i}$ for $i \in[k-2]$ and $u_{k-1}:=v_{k-1} \cdots v_{n}$, and the claim follows.

Therefore, suppose $n<k$. Define $m:=v_{1}\left[\left|v_{1}\right|\right] \cdots v_{n-1}\left[\left|v_{n-1}\right|\right] \in \Sigma^{\leq k-2}$ as the unique last letters of $v_{1}, \ldots, v_{n-1}$. Choose $\mathrm{x}, \mathrm{y} \in \overleftarrow{\mathrm{M}}$ distinct. Then there exist $w_{\mathrm{x}}, w_{\mathrm{y}} \in[w]_{\sim_{k}}$ with $\widehat{\mathrm{m}}\left(w_{\mathrm{x}}\right)=\mathrm{x}, \overleftarrow{\mathrm{m}}\left(w_{\mathrm{y}}\right)=\mathrm{y}$ and assume by Proposition 15 that they have the same $\alpha$-factors as $w$. By definition, $m \mathrm{xy} \preceq w_{\mathrm{x}}$. Since $w_{\mathrm{x}} \sim_{k} w_{\mathrm{y}}$, we have $m \mathrm{xy} \preceq w_{\mathrm{y}}$ and thus $m \mathrm{x} \preceq \alpha_{0}=v_{1} \cdots v_{n}$ and thus $\mathrm{x} \preceq v_{n}$. Because y , x were chosen arbitrarily, we have $\overline{\mathrm{M}} \subseteq \operatorname{alph}\left(v_{n}\right)$, a contradiction against the construction of $v_{n}$. This implies $n \geq k$.

## 4 The Binary Case of Simon's Congruence

In this section, we apply our previous results to the special case of the binary alphabet. Here, for $x \in \Sigma$, let $\bar{x}$ be the well defined other letter of $\Sigma$. First, we characterize the congruence of binary words in terms of $\alpha$ - and $\beta$-factors. We show that in this scenario in each congruence class of a word $w$ with at most $k$ arches, we have $\left|\left\{\overline{\mathrm{m}}(\tilde{w}) \mid \tilde{w} \in[w]_{\sim_{k}}\right\}\right|=1$ (cf. Proposition (20). We present results such that a full characterization of the structure of the classes in the binary case is given, implying as a byproduct a simple algorithm for MaxSimK in this special case (cf. 9$]$ ). Moreover, we can calculate $\left|\Sigma_{2}^{*} / \sim_{k}\right|$.

Proposition 21. For all $w \in \Sigma_{2}^{*}$, we have for all $i \in[\iota(w)]$

1. $\beta_{i} \in\{\mathrm{a}, \mathrm{b}, \mathrm{ab}, \mathrm{ba}\}$,
2. if $\beta_{i}=\mathrm{x}$, then $\alpha_{i-1}, \alpha_{i} \in \overline{\mathrm{x}}^{+}$with $\mathrm{x} \in \Sigma_{2}$,
3. if $\beta_{i}=\mathrm{x} \overline{\mathrm{x}}$, then $\alpha_{i-1} \in \mathrm{x}^{*}$ and $\alpha_{i} \in \overline{\mathrm{x}}^{*}$ with $\mathrm{x} \in \Sigma_{2}$.

Proof. By definition, the first and last letter of $\beta_{i}$ are unique. Therefore, if $\beta_{i}[1]=$ $\beta_{i}\left[\left|\beta_{i}\right|\right]$ we have $\left|\beta_{i}\right|=1$. Furthermore, if $\beta_{i}[1] \neq \beta_{i}\left[\left|\beta_{i}\right|\right]$ we have $\left|\beta_{i}\right|=2$ because $\left|\Sigma_{2}\right|=2$.

Because alph $\left(\alpha_{i}\right) \subset \Sigma_{2}$, the $\alpha_{i}$ are unary words for all $i \in[m]_{0}$. By symmetry, we only have to show the claim for $\alpha_{i-1}$. The restrictions on the alphabet of $\alpha_{i-1}$ follow directly from $\operatorname{ar}_{i}(w)=\alpha_{i-1} \beta_{i}$ and the uniqueness of the last letter. Furthermore, if $\beta_{i}=\mathrm{x} \in \Sigma_{2}$ then $\alpha_{i-1} \neq \varepsilon$ because $\mathrm{x}, \overline{\mathrm{x}} \preceq \operatorname{ar}_{i}(w)=\alpha_{i-1} \beta_{i}$.

Thus, we get immediately that the $\alpha \beta \alpha$-factors are of the following forms: $\mathrm{a}^{\ell_{1}+1} \mathrm{ba}^{\ell_{2}+1}, \mathrm{~b}^{\ell_{1}+1} \mathrm{ab}^{\ell_{2}+1}, \mathrm{~b}^{\ell_{3}} \mathrm{baa}^{\ell_{4}}$, or $\mathrm{a}^{\ell_{3}} \mathrm{abb}^{\ell_{4}}$ for some $\ell_{1}, \ell_{2}, \ell_{3}, \ell_{4} \in \mathbb{N}_{0}$. The following lemma shows that in the binary case the $k$-congruence of two words with identical universality less than $k$ leads to the same modi and same $\beta$.

Lemma 22. Let $w, w^{\prime} \in \Sigma_{2}^{*}$ with $w \sim_{k} w^{\prime}$ and $m:=\iota(w)=\iota\left(w^{\prime}\right)<k$, then $\mathrm{m}(w)=\mathrm{m}\left(w^{\prime}\right)$ and thus, $\beta_{i}=\beta_{i}^{\prime}$ for all $i \in[m]$.

Proof. By Theorem [17, without loss of generality, we only consider $m=1<k$. Furthermore, it suffices to show that $\mathrm{m}(w)=\mathrm{m}\left(w^{\prime}\right)$ because then symmetry implies $\overline{\mathrm{m}}(w)=\overline{\mathrm{m}}\left(w^{\prime}\right)$, and $\mathrm{m}(w)$ and $\overline{\mathrm{m}}(w)$ fully determine the $\beta$ factors by Proposition 21

We know that the $\alpha$-factors are unary or empty by Proposition 21. Lemma 14 $\operatorname{implies} \operatorname{alph}\left(\alpha_{i}\right)=\operatorname{alph}\left(\alpha_{i}^{\prime}\right)$ for $i \in[1]_{0}$ since we have $w \sim_{k} w^{\prime}$. If $\operatorname{alph}\left(\alpha_{0}\right) \neq \emptyset$, then the arches $\alpha_{0} \beta_{1}$ and $\alpha_{0}^{\prime} \beta_{1}^{\prime}$ start with the same letter and thus $\mathrm{m}(w)=$ $\mathrm{m}\left(w^{\prime}\right)$.

For the second case, assume $\alpha_{0}=\alpha_{0}^{\prime}=\varepsilon$. If $\beta_{1}=\beta_{1}^{\prime}$, then in particular $\mathrm{m}(w)=\mathrm{m}\left(w^{\prime}\right)$, and the claim follows. Thus, suppose $\beta_{1}=\mathrm{y} \overline{\mathrm{y}}$ and $\beta_{1}^{\prime}=\overline{\mathrm{y}} \mathrm{y}$ for some $\mathrm{y} \in \Sigma_{2}$ by Proposition 21 Thus, $\alpha_{1} \in \overline{\mathrm{y}}^{*}$ and $\alpha_{1}^{\prime} \in \mathrm{y}^{*}$. Hence, $\alpha_{1}=\alpha_{1}^{\prime}=\varepsilon$ because y $\notin \operatorname{alph}\left(\alpha_{1}\right)=\operatorname{alph}\left(\alpha_{1}^{\prime}\right) \nexists \overline{\mathrm{y}}$. This contradicts $w \sim_{k} w^{\prime}$ because $w=$ $y \overline{\mathrm{y}} \not \chi_{2} \overline{\mathrm{y}} \mathrm{y}=w^{\prime}$.

Combining the Lemmata 2214 and Proposition 15 yields the following characterization of $\sim_{k}$ for binary words in terms of unary words and factors.

Theorem 23. Let $w, w^{\prime} \in \Sigma_{2}^{*}$ such that $m:=\iota(w)=\iota\left(w^{\prime}\right)<k$, then $w \sim_{k} w^{\prime}$ iff $\beta_{i}=\beta_{i}^{\prime}$ for all $i \in[m]$ and $\alpha_{i} \sim_{k-m} \alpha_{i}^{\prime}$ for all $i \in[m]_{0}$.

Using the characterization, we can also give an $\mathcal{O}(|u|+|v|)$-time algorithm for finding the largest $k$ with $u \sim_{k} v$ for $u, v \in \Sigma_{2}^{*}$. This special case was originally solved by Hébrard [9] just considering arches. Recently, a linear time algorithm for arbitrary alphabets was presented by Gawrychowski et al. 8]. Nonetheless, we give Algorithm [1] as it is a conceptually simple algorithm exploiting that $\alpha_{i}$ factors can be treated similar to $\mathrm{re}(w)$ in the arch factorization.

```
Algorithm 1: MaxSimK for binary words
    Input: \(u, \tilde{u} \in \Sigma_{2}^{*}\)
    Result: if \(u=u^{\prime}\) then \(\infty\) and otherwise the maximum \(k\) such that \(u \sim_{k} \tilde{u}\)
    \(\left(\alpha_{0}, \beta_{1}, \ldots, \alpha_{\iota(u)}\right):=\alpha-\beta-\operatorname{FACT}(u) ; \quad / /\) w.r.t. \(\Sigma_{2}\)
    \(\left(\tilde{\alpha}_{0}, \tilde{\beta}_{1}, \ldots, \tilde{\alpha}_{\iota(\tilde{u})}\right):=\alpha-\beta-\operatorname{FACT}(\tilde{u}) ;\)
    if \(\iota(u) \neq \iota(\tilde{u}) \vee \operatorname{alph}(u) \neq \operatorname{alph}(\tilde{u})\) then // 2nd condition for \(u=\mathrm{x}^{i}, \tilde{u}=\overline{\mathrm{x}}^{j}\)
        return \(\min (\iota(u), \iota(\tilde{u}))\);
    else if \(\beta_{1}=\tilde{\beta}_{1} \wedge \cdots \wedge \beta_{\iota(u)}=\tilde{\beta}_{\iota(\tilde{u})}\) then
        for \(i \in[\iota(u)]_{0}\) do // solve MaxSimK for unary \(\alpha\) pairs
            \(e_{i}:=\) if \(\left|\alpha_{i}\right|=\left|\tilde{\alpha}_{i}\right|\) then \(\infty\) else \(\min \left(\left|\alpha_{i}\right|,\left|\tilde{\alpha}_{i}\right|\right) ;\)
        return \(\iota(u)+\min \left\{e_{i} \mid i \in[\iota(u)]_{0}\right\}\);
    else
        return \(\iota(u)\);
```

We can use Theorem 23 to answer a number of questions regarding the structure of the congruence classes of $\Sigma_{2}^{*} / \sim_{k}$. For instance, for each $w$ with $\left|[w]_{\sim_{k}}\right|=\infty$, we have $\mathrm{x}^{k} \preceq w$ for some $\mathrm{x} \in \Sigma$ by the pigeonhole-principle. The contrary is not true in general witnessed by the word $v=\mathrm{bbabb}$ with respect to $\sim_{4}$. Its scattered factors of length four are bbab, babb and bbbb. Therefore, each word in its class contains exactly one a (aa $\preceq v$ but a $\preceq v$ is), at least two b succeeding and preceding the $\mathrm{a}(\mathrm{bba}, \mathrm{abb} \preceq v)$ but not more than two b (bbba, abbb $\preceq v$ ). Therefore, bbabb is the only word in this class, but it contains $\mathrm{b}^{4}$. By a famous result of Simon [21, Corollary 6.2.8], all congruence classes of $\sim_{k}$ are either infinite or singletons. In the binary case, we can give a straightforward characterization of the finite/singleton and infinite classes.

Theorem 24. Let $w \in \Sigma_{2}^{*}$, then $\left|[w]_{\sim_{k}}\right|<\infty$. In particular, we have $\left|[w]_{\sim_{k}}\right|=1$ iff $\iota(w)<k$ and $\left|\alpha_{i}\right|<k-\iota(w)$ for all $i \in[\iota(w)]_{0}$.

Proof. Let $w \in \sum_{2}^{*}$ with $\iota(w)<k,\left|\alpha_{i}\right| \leq k-\iota(w)$ for all $i \in[\iota(w)]_{0}$, and $\tilde{w} \in[w]_{\sim_{k}}$. By Remark [6, we have $\iota(w)=\iota(\tilde{w})=m$. By Theorem 23, we get $\beta_{i}=\tilde{\beta}_{i}$ for all $i \in[\iota(w)]$ and $\left[\alpha_{i}\right]_{\sim_{k-m}}=\left[\tilde{\alpha}_{i}\right]_{\sim_{k-m}}$ for all $i \in[\iota(w)]_{0}$. Additionally, we have by Proposition 21 and the fact that for $u, v \in \Sigma_{1}^{*}$ we have $u \sim_{\ell} v$, if and only if, $\min (|u|, \ell)=\min (|v|, \ell)$ exactly $\left[\alpha_{i}\right]_{\sim_{k-m}}=\left[\tilde{\alpha}_{i}\right]_{\sim_{k-m}}=\left\{\mathbf{x}^{\left|\alpha_{i}\right|}\right\}$ for all $i \in[\iota(w)]_{0}$. Thus, $\left|[w]_{\sim_{k}}\right|=1$.

Table 1. Index of $\sim_{k}$ restricted to binary words with a fixed number of arches

|  |  | Number of Arches |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | $m$ |
|  | 1 | 3 | 1 |  |  |  |  |  |  |  |
|  | 2 | 5 | 10 | 1 |  |  |  |  |  |  |
|  | 3 | 7 | 26 | 34 | 1 |  |  |  |  |  |
|  | 4 | 9 | 50 | 136 | 116 | 1 |  |  |  |  |
|  | 5 | 11 | 82 | 358 | 712 | 396 | 1 |  |  |  |
|  | 6 | 13 | 122 | 748 | 2564 | 3728 | 1352 | 1 |  |  |
|  | 7 | 15 | 170 | 1354 | 6824 | 18364 | 19520 | 4616 | 1 |  |
| $k \quad 2 k+1$ |  |  |  |  |  |  |  |  |  |  |

Now let $w \in \Sigma_{2}^{*}$ such that $\iota(w) \geq k$ or $\left|\alpha_{i}\right| \geq k-\iota(w)$ for some $i \in[\iota(w)]_{0}$. If $\iota(w) \geq k$, then $\Sigma_{2}^{\leq k}=\operatorname{ScatFact}_{\leq k}(w) \subseteq \operatorname{ScatFact}_{\leq k}(w v)$ and thus $w v \in[w]_{\sim_{k}}$ for all $v \in \Sigma_{2}^{*}$. This implies $\left|[w]_{\sim_{k}}\right| \neq 1$. On the other hand, if $\iota(w)<k$ and there exists $i \in[\iota(w)]_{0}$ with $\left|\alpha_{i}\right| \geq k-\iota(w) \geq 1$, then $\alpha_{i} \sim_{k-\iota(w)} \alpha_{i}^{j}$ for all $j \in \mathbb{N}$. Thus, we have $\alpha_{0} \beta_{1} \alpha_{1} \cdots \beta_{i} \alpha_{i}^{j} \beta_{i+1} \cdots \alpha_{\iota(w)-1} \beta_{\iota(w)} \alpha_{\iota(w)} \sim_{k} w$ for all $j \in \mathbb{N}$ and we have again $\left|[w]_{\sim_{k}}\right| \neq 1$.

In the following, we will use Theorem 23 to derive a formula for the precise value of $\left|\Sigma_{2}^{*} / \sim_{k}\right|$. Note that in the unary case, we have $\left|\Sigma_{1}^{*} / \sim_{k}\right|=k+1$ because the empty word has its own class. By Remark 6, we know that there exists exactly one class w.r.t. $\sim_{k}$ of words with $k$ arches. We can consider the other classes by the common number of arches of their elements. By Theorem [23] we can count classes based on the valid combinations of $\beta$-factors and number of classes for each $\alpha$-factors. Because the $\alpha$ are unary, we already know their number of classes. These valid combinations are exactly given by Proposition 21, The first values for the number of classes separated by the number of arches are given in Table 1

Theorem 25. The number of congruence classes of $\Sigma_{2}^{*} / \sim_{k}$ of words with $m<k$ arches (the entries of Table 1) is given by

$$
\left\|\left(\begin{array}{ccc}
k-m & k-m & k-m \\
1 & 2 & 1 \\
k-m & k-m & k-m
\end{array}\right)^{m} \cdot\left(\begin{array}{c}
k-m \\
1 \\
k-m
\end{array}\right)\right\|_{1}=c_{k}^{m}
$$

where $c_{k}^{-1}:=1, c_{k}^{0}:=2 k+1$, and $c_{k}^{m}:=2 \cdot(k-m+1) \cdot c_{k-1}^{m-1}-2 \cdot(k-m) \cdot c_{k-2}^{m-2}$ where $\|\cdot\|_{1}$ denotes the 1 -norm.

Proof. First, we show that the matrix representation produces the correct values, then we show the characterization as recurrence. Note that $k-m$ is fixed on the diagonals of Table 1 Therefore, increasing both, increases just the exponent of the matrix. We show that the first column is correct and then proceed by induction along the diagonals. Denote the above matrix by $D_{k, m}$.

Let $k \in \mathbb{N}_{0}$ and $w \in \Sigma^{*}$ with $m:=\iota(w)<k$. For $i \in[m]_{0}$, all elements $v \in[w]_{\sim_{k}}$ have $k-m$ congruent $\alpha_{i}$ by Theorem[23. By definition, their alphabets are proper subsets of $\Sigma_{2}$. Therefore, they are either empty or non-empty unary words consisting of just a or b . We separate the choice of $\alpha_{i}$ into these three cases. Let $M_{\varepsilon}^{\ell}:=\left\{[w] \in \Sigma_{2}^{*} / \sim_{(k-m)+\ell} \mid \iota(w)=\ell, \alpha_{0} \sim_{k-m} \varepsilon\right\}$ and $M_{\mathrm{x}}^{\ell}:=$ $\left\{[w] \in \Sigma_{2}^{*} / \sim_{(k-m)+\ell} \mid \iota(w)=\ell, \alpha_{0} \sim_{k-m} \mathrm{x}^{r}, r \in \mathbb{N}\right\}$ for $\mathrm{x} \in \Sigma_{2}$ be sets of $\ell+(k-m)$ congruence classes of words with $\ell$ arches, separated by the alphabet of $\alpha_{0}$. Denote by $e_{k, m}:=\left(\left|M_{\mathrm{a}}^{0}\right|,\left|M_{\varepsilon}^{0}\right|,\left|M_{\mathrm{b}}^{0}\right|\right)^{\top}=(k-m, 1, k-m)^{\top}$ the number of classes for zero arches. We show $\left\|D_{k, m}^{\ell} \cdot e_{k, m}\right\|_{1}=\left(\left|M_{\mathrm{a}}^{\ell}\right|,\left|M_{\varepsilon}^{\ell}\right|,\left|M_{\mathrm{b}}^{\ell}\right|\right)^{\top}$. There are four choices for $\beta_{i}$ which are given by Proposition 21. Each choice of $\beta_{i+1}$ depends on the preceding $\alpha_{i}$ and limits the choices for the succeeding $\alpha_{i+1}$. These are given by Proposition 21 and correspond to the entries of the matrix because for $\ell \geq 1$ we have

$$
\begin{aligned}
M_{\varepsilon}^{\ell}= & \left\{[w]_{\sim_{(k-m)+\ell}} \in M_{\varepsilon}^{\ell} \mid \mathrm{x} \in \Sigma_{2}, \beta_{1}(w)=\overline{\mathrm{x}}, \alpha_{1}(w) \sim_{k-m} \varepsilon\right\} \\
& \sqcup\left\{[w]_{\sim_{(k-m)+\ell}} \in M_{\varepsilon}^{\ell} \mid \mathrm{x} \in \Sigma_{2}, \beta_{1}(w)=\overline{\mathrm{x}}, \alpha_{1}(w) \sim_{k-m} \mathrm{x}^{r}, r \in \mathbb{N}\right\} \\
\cong & \{\mathrm{ab}, \mathrm{ba}\} \times M_{\varepsilon}^{\ell-1} \sqcup\{\mathrm{ab}\} \times M_{\mathrm{b}}^{\ell-1} \sqcup\{\mathrm{ba}\} \times M_{\mathrm{a}}^{\ell-1} \\
M_{\mathrm{x}}^{\ell}= & \left\{[w]_{\sim_{(k-m)+\ell}} \in M_{\mathrm{x}}^{\ell} \mid \beta_{1}(w)=\overline{\mathrm{x}}\right\} \\
& \sqcup\left\{[w]_{\sim_{(k-m)+\ell}} \in M_{\mathrm{x}}^{\ell} \mid \beta_{1}(w)=\mathrm{x} \overline{\mathrm{x}}, \alpha_{1}(w) \sim_{k-m} \overline{\mathrm{x}}^{r}, r \in \mathbb{N}\right\} \\
& \sqcup\left\{[w]_{\sim_{(k-m)+\ell}} \in M_{\mathrm{x}}^{\ell} \mid \beta_{1}(w)=\mathrm{x} \overline{\mathrm{x}}, \alpha_{1}(w) \sim_{k-m} \varepsilon\right\} \\
\cong & {[k-m] \times\left(\{\overline{\mathrm{x}}\} \times M_{\mathrm{x}}^{\ell-1} \sqcup\{\mathrm{x} \overline{\mathrm{x}}\} \times M_{\overline{\mathrm{x}}}^{\ell-1} \sqcup\{\mathrm{x} \overline{\mathrm{x}}\} \times M_{\varepsilon}^{\ell-1}\right) . }
\end{aligned}
$$

Therefore, each multiplication with the matrix increases the number $\ell$ of arches by one. Thus, for $m=\ell$ we have the desired value as $M_{\varepsilon}^{m}$ and $M_{\mathrm{x}}^{m}$ are sets of $k$ congruence classes with $m$ arches. Therefore, $\left\|D_{k, m}^{m} \cdot e_{k, m}\right\|_{1}$ corresponds to the number of classes with respect to $\sim_{k}$ of words with $m$ arches.

The equivalence of the two formulas is left to show. The characteristic polynomial of $D_{k, m}$ is given by $\chi_{D_{k, m}}=\operatorname{det}\left(D_{k, m}-\lambda I\right)=-\lambda^{3}+2 \lambda^{2}+2(k-$ $m) \lambda^{2}-2(k-m) \lambda$. By the Cayley-Hamilton theorem [7], $D_{k, m}$ is a root of its characteristic polynomial and thus satisfies the recurrence

$$
\begin{aligned}
D_{k, m}^{\ell+2} & =2 \cdot D_{k, m}^{\ell+1}+2 \cdot(k-m) \cdot D_{k, m}^{\ell+1}-2 \cdot(k-m) \cdot D_{k, m}^{\ell} \\
& =2 \cdot(k-m+1) \cdot D_{k, m}^{\ell+1}-2 \cdot(k-m) \cdot D_{k, m}^{\ell}
\end{aligned}
$$

for $\ell \in \mathbb{N}$. Note that $e_{k, m}=e_{k+\ell, m+\ell}$ for all $\ell \in \mathbb{N}_{0}$. Now we conclude by induction that

$$
\begin{aligned}
& \left\|D_{k+2, m+2}^{m+2} \cdot e_{k+2, m+2}\right\|_{1}=\left\|D_{k, m}^{m+2} \cdot e_{k, m}\right\|_{1} \\
& =\left\|\left(2 \cdot(k-m+1) \cdot D_{k, m}^{m+1}-2 \cdot(k-m) \cdot D_{k, m}^{m}\right) \cdot e_{k, m}\right\|_{1} \\
& =2 \cdot(k-m+1) \cdot\left\|D_{k, m}^{m+1} \cdot e_{k, m}\right\|_{1}-2 \cdot(k-m) \cdot\left\|D_{k, m}^{m} \cdot e_{k, m}\right\|_{1} \\
& =2 \cdot(k-m+1) \cdot c_{k+1}^{m+1}-2 \cdot(k-m) \cdot c_{k}^{m}=c_{k+2}^{m+2}
\end{aligned}
$$

because $\|u \pm v\|_{1}=\|u\|_{1} \pm\|v\|_{1}$ for all $u=\left(u_{i}\right), v=\left(v_{i}\right) \in \mathbb{R}^{n}$ for which $u_{j} v_{j} \geq 0$ for all $j \in[n]$.

Table 2. Number of classes of perfect universal binary words restricted to a fixed number of arches

|  |  | Number of Arches |  |  |  |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | $m$ |
|  | 2 | 1 | 4 | 1 |  |  |  |  |  |  |  |
|  | 3 | 1 | 6 | 14 | 1 |  |  |  |  |  |  |
|  | 4 | 1 | 8 | 32 | 48 | 1 |  |  |  |  |  |
|  | 5 | 1 | 10 | 58 | 168 | 164 | 1 |  |  |  |  |
|  | 6 | 1 | 12 | 92 | 416 | 880 | 560 | 1 |  |  |  |
|  | 7 | 1 | 14 | 134 | 840 | 2980 | 4608 | 1912 | 1 |  |  |
|  | 8 | 1 | 16 | 184 | 1488 | 7664 | 21344 | 24344 | 6528 | 1 |  |
| $k$ |  | 1 | $2 k$ |  |  |  |  |  |  |  |  |

Remark 26. Note that by setting $\Delta:=k-m$, the family of recurrences depends only on one variable $\Delta$, because $k-m=(k-\ell)-(m-\ell)$ holds for all $\ell \in \mathbb{N}$.

Remark 27. Some sequences in Table 1 are known sequences. The first and second diagonal are A007052 and A018903 resp. in 19. Both sequences are investigated in the work of Janjic [10]. There, the two sequences appear as the number of compositions of $n \in \mathbb{N}$, considering three (resp. five) differently colored 1s. Furthermore, the sequences $c_{k}^{m}$ seem to be equivalent to the family of sequences $\left(s_{n}\right)$ where $s_{0}=1$ and $s_{1}$ is fixed and $s_{n+2}$ is the smallest number such that $\frac{s_{n+2}}{s_{n+1}}>\frac{s_{n+1}}{s_{n}}$. These sequences where studied by Boyd [2].

By Remark 6, we can count the number of classes separated by the universality of words with less than $k$ arches. This leads to the following immediate corollary which allows us to efficently calculate $\left|\Sigma_{2}^{*} / \sim_{k}\right|$.

Corollary 28. Let $k \in \mathbb{N}_{0}$. Over a binary alphabet, the number of congruence classes of $\sim_{k}$ is given by $\left|\Sigma_{2}^{*} / \sim_{k}\right|=1+\sum_{m=0}^{k-1} c_{k}^{m}$.

The first values of the sequence, some of which are already given in [12, are

$$
\begin{aligned}
& 1,4,16,68,312,1560,8528,50864,329248,2298592,17203264,137289920, \\
& 1162805376,10409679744,98146601216,971532333824,10068845515264, \ldots
\end{aligned}
$$

We can use the idea of Theorem 25 to count the number of perfect $k$-universal words, i.e., $k$-universal words with an empty rest (cf. [5]). We can count them by replacing the vector from Theorem [25] with the initial distribution of $\alpha_{i}$ values with $(0,1,0)^{\top}$. Thus, the formula counts words starting or ending with an empty $\alpha$. Because the matrix does not change, we obtain the same recurrence with different initial values. The $k^{\text {th }}$ diagonal, shifted by one, is now given by the Lucas sequence of the first kind $U(2 \cdot k+2,2 \cdot k)$, where $U_{n}(P, Q)$ is given by $U_{0}(P, Q)=0, U_{1}(P, Q)=1, U_{n}(P, Q)=P \cdot U_{n-1}(P, Q)-Q \cdot U_{n-2}(P, Q)$. The first calculated values are given in Table 2. The first three diagonals of the table are the known integer sequences A007070, A084326, and A190978 in [19.

## 5 Towards The Ternary Case of Simon's Congruence

In the following, we will consider cases for the ternary alphabet based on the alphabets of the $\alpha$-factors with the goal of proving similar results to Proposition 21 and Theorem 23 for ternary words, leading to Theorem 30, By Theorem 17, it suffices to consider $\alpha \beta \alpha$-factors for characterizing congruence classes. In Section 3 we already considered some cases for $\alpha \beta \alpha$-factors for arbitrary alphabets. Note that if $\mathrm{m}_{1}(w)=\overleftarrow{\mathrm{m}}_{1}(w)$ then core $=\varepsilon$. Otherwise, if $\mathrm{m}_{1}(w) \neq \overleftarrow{\mathrm{m}}_{1}(w)$, then core $_{1} \in\left(\Sigma \backslash\left\{\mathrm{~m}_{1}(w), \widehat{\mathrm{m}}_{1}(w)\right\}\right)^{*}$. Thus, cores of ternary words are unary, and we denote the well-defined letter of the core by y $\in \Sigma_{3}$.

We use a variant of the Kronecker- $\delta$ for a boolean predicate $P$ as $\delta_{P(x)}=1$ if $P(x)$ is true and 0 otherwise to express a condition on the alphabet of the rest of a binary $\alpha$-factor (cf. Figure 2). If an $\alpha_{0}$ 's rest contains the letter y different from the reverse modus $\mathrm{x}:=\overleftarrow{\mathrm{m}}(w)$, then $\operatorname{re}\left(\alpha_{0}\right) \overleftarrow{\mathrm{m}}(w)$ builds another arch ending before the core (left). This lowers the level of congruence, up to which we can determine the core, by one. If $\mathrm{y} \npreceq \mathrm{re}\left(\alpha_{0}\right)$ the next y is in the core (right).


Fig. 2. Factorization of $\alpha$ in the ternary case assuming core $\in \mathrm{y}^{+}$.

We always assume that $k \geq 2$ because we characterize the congruence of 1-universal words. Moreover, let $w, \tilde{w} \in \Sigma_{3}^{*}$ with $1=\iota(w)=\iota(\tilde{w})$.

First, we prove a useful lemma which characterizes the congruence of two ternary words with the same modus and reverse modus. Together with Proposition 20, this immediately implies several cases.

Lemma 29. Let $\mathrm{m}(w)=\mathrm{m}(\tilde{w})$ and $\overleftarrow{\mathrm{m}}(w)=\overleftarrow{\mathrm{m}}(\tilde{w})$, we have $w \sim_{k} \tilde{w}$ iff $\alpha_{i} \sim_{k-1}$ $\tilde{\alpha}_{i}$ for all $i \in[1]_{0}$ and core $_{1} \sim_{k-c}{\widetilde{\operatorname{core}_{1}}}_{1} \in \mathrm{y}^{*}$ where $c:=\iota\left(\alpha_{0}\right)+\delta_{\mathrm{y} \preceq \mathrm{re}\left(\alpha_{0}\right)}+$ $\iota\left(\alpha_{1}\right)+\delta_{\mathrm{y} \preceq \mathrm{re}}\left(\alpha_{1}\right)$.

Proof. Assume without loss of generality $\beta=01^{\ell} 2$ and $\tilde{\beta}=01^{\tilde{\ell}} 2$ for some $\ell, \tilde{\ell} \in \mathbb{N}_{0}$ and $\ell \leq \tilde{\ell}$.

Further, assume $\alpha_{i} \sim_{k-1} \tilde{\alpha}_{i}$ for all $i \in[1]_{0}$ and $\min (k-c, \ell)=\min (k-c, \tilde{\ell})$ as above. By Corollary [16, we can assume that $\alpha_{i}=\tilde{\alpha}_{i}$ for all $i \in[1]_{0}$. If $\ell=\tilde{\ell}$ or if $\ell<k-c$ and we thus have $\ell=\tilde{\ell}$, we have $w=\tilde{w}$. Hence, assume $k-c \leq \ell<\tilde{\ell}$. We show that $\alpha_{0} \cdot 01^{\ell} 2 \cdot \alpha_{1} \sim_{k} \alpha_{0} \cdot 01^{\ell+1} 2 \cdot \alpha_{1}$ using Lemma 12 , then the claim follows by repeated application of the lemma. Note that $k-\ell \leq$ $c=\iota_{\{0,1\}}\left(\alpha_{0} 0\right)+\iota_{\{1,2\}}\left(2 \alpha_{1}\right)$. Therefore, we can factorize $\alpha_{0} 0$ and $2 \alpha_{1}$ into at least $c$ factors with binary alphabet each. The claim follows with $c+\ell \geq k$.

Because $\alpha_{i} \sim_{k-1} \tilde{\alpha}_{i}$ is a necessary condition for $w \sim_{\tilde{k}} \tilde{w}$ by Lemma 14 , assume this condition holds and $\min (k-c, \ell) \neq \min (k-c, \tilde{\ell})$. Again, by Corollary 16 assume that $\alpha_{i}=\tilde{\alpha}_{i}$ for all $i \in[1]_{0}$. Thus, $\ell<k-c$ and $\ell<\tilde{\ell}$. Define $v:=\mathrm{m}_{\{0,1\}}\left(\alpha_{0} 0\right) \cdot 1^{\min (k-c, \tilde{\ell})} \cdot \widehat{\mathrm{m}}_{\{1,2\}}\left(2 \alpha_{1}\right) \in \Sigma_{3}^{\leq k}$. By construction, we have $v \npreceq w$ because $\ell<\min (k-c, \tilde{\ell})$ and $v \preceq \tilde{w}$ because $\tilde{\ell} \geq \min (k-c, \tilde{\ell})$. Thus, $w \nsim_{k} \tilde{w}$.

Since in the ternary case, there are congruent words having different modi or reverse modi, Lemma 29 does not imply a full characterization. This leads to two cases in the following classification (case 3 and 5 out of the 9 cases in Table 3). These two cases correspond to the first case in the following theorem.
Theorem 30. For $w, \tilde{w} \in \Sigma_{3}^{*}$ we have $w \sim_{k} \tilde{w}$ iff $\alpha_{i} \sim_{k-1} \tilde{\alpha}_{i}$ for all $i \in[1]_{0}$, and one of the following

1. $\left|\operatorname{alph}\left(\alpha_{i}\right)\right|=2, \operatorname{alph}\left(\alpha_{1-i}\right) \cap \operatorname{alph}\left(\alpha_{i}\right)=\emptyset$, and $\iota\left(\alpha_{i}\right) \geq k-1$ for some $i \in[1]_{0}$, 2. $\mathrm{m}(w)=\mathrm{m}(\tilde{w}), \overleftarrow{\mathrm{m}}(w)=\overleftarrow{\mathrm{m}}(\tilde{w})$, core $\sim_{k-c}$ core with $c:=\iota\left(\alpha_{0}\right)+\delta_{\mathrm{y} \preceq \alpha_{0}}+\iota\left(\alpha_{1}\right)+$ $\delta_{\mathrm{y} \in \alpha_{1}}$.
For all possibilities distinguishing the $\beta$-factors, see Table 3 .
Proof. Above, we already considered the cases $\alpha_{0}=\alpha_{1}=\varepsilon$ (Proposition 18) and $\operatorname{alph}\left(\alpha_{0}\right)=\operatorname{alph}\left(\alpha_{1}\right) \in\binom{\Sigma}{|\Sigma|-1}$ (Proposition 19). Next we cover the case of the two alphabets of size two intersecting, finishing the $2=\left|\operatorname{alph}\left(\alpha_{0}\right)\right|=\left|\operatorname{alph}\left(\alpha_{1}\right)\right|$ case. Notice that in the case where both $\alpha$-factors are binary, we only have $\left|\operatorname{alph}\left(\alpha_{0}\right) \cap \operatorname{alph}\left(\alpha_{1}\right)\right|=1$ left to consider.
Claim 1. Let $\left|\operatorname{alph}\left(\alpha_{0}\right)\right|=\left|\operatorname{alph}\left(\alpha_{1}\right)\right|=2$. We have $w \sim_{k} \tilde{w}$ iff $\alpha_{i} \sim_{k-1} \tilde{\alpha}_{i}$ for all $i \in[1]_{0}$ and core $\sim_{k-c} \widetilde{\operatorname{cor}}_{1}$, where $c:=\iota\left(\alpha_{0}\right)+\delta_{\mathrm{y} \preceq \mathrm{re}\left(\alpha_{0}\right)}+\iota\left(\alpha_{1}\right)+\delta_{\mathrm{y} \preceq \mathrm{re}\left(\alpha_{1}\right)}$.
Proof. Because $\left|\operatorname{alph}\left(\alpha_{i}\right)\right|=2$ and in both directions we have $\operatorname{alph}\left(\alpha_{i}\right)=$ $\operatorname{alph}\left(\tilde{\alpha}_{i}\right)$ for all $i \in[1]_{0}$, the claim follows by Lemma 29 ,

The case described in Claim 1 is problematic for the enumeration of classes, because the choices of $\beta$ depend on the universality and the rests' alphabets of the surrounding $\alpha$-factors. This is in contrast to the binary case, where we only had to distinguish between empty and the two types of unary $\alpha$-factors.

Next, we consider the general case $2=\left|\operatorname{alph}\left(\alpha_{0}\right)\right|, 1=\left|\operatorname{alph}\left(\alpha_{1}\right)\right|$. Due to symmetry, it suffices to consider only the cases of the form $\left|\operatorname{alph}\left(\alpha_{0}\right)\right| \geq$ $\left|\operatorname{alph}\left(\alpha_{1}\right)\right|$. The first cases is similar to the one of Claim 1 because the modi are fixed by the structure of the $\alpha$-factors.
Claim 2. Let $\left|\operatorname{alph}\left(\alpha_{0}\right)\right|=2,\left|\operatorname{alph}\left(\alpha_{1}\right)\right|=1$ and $\operatorname{alph}\left(\alpha_{0}\right) \cap \operatorname{alph}\left(\alpha_{1}\right) \neq \emptyset$. Then, $w \sim_{k} \tilde{w}$ iff $k=1$ or $\alpha_{i} \sim_{k-1} \tilde{\alpha}_{i}$ for all $i \in[1]_{0}$ and core $1 \sim_{k-c}{\widetilde{\text { core }_{1}}}_{1}$, where $c:=\iota\left(\alpha_{0}\right)+\delta_{\mathrm{y} \preceq \mathrm{re}\left(\alpha_{0}\right)}+\delta_{\mathrm{y} \preceq \mathrm{re}\left(\alpha_{1}\right)}$.

It is left to consider the case where the two binary alphabets intersect.
Claim 3. Let $\left|\operatorname{alph}\left(\alpha_{0}\right)\right|=2$, $\left|\operatorname{alph}\left(\alpha_{1}\right)\right|=1$ and $\operatorname{alph}\left(\alpha_{0}\right) \cap \operatorname{alph}\left(\alpha_{1}\right)=\emptyset$. Then, $w \sim_{k} \tilde{w}$ iff either $\iota\left(\alpha_{0}\right)=k-1=\iota\left(\tilde{\alpha}_{0}\right), \alpha_{1} \sim_{k-1} \quad \tilde{\alpha}_{1}$, or $\alpha_{i} \sim_{k-1} \tilde{\alpha}_{i}$ for all $i \in[1]_{0}, \iota\left(\alpha_{0}\right)<k-1, \overleftarrow{\mathrm{~m}}(w)=\overleftarrow{\mathrm{m}}(\tilde{w})$, core $_{1} \sim_{k-c}{\widetilde{\operatorname{corer}_{1}}}_{1} \in \mathrm{y}^{+}$where $c:=\iota\left(\alpha_{0}\right)+\delta_{\mathrm{y} \preceq \mathrm{re}}\left(\alpha_{0}\right)$.

Proof. Let $w, \tilde{w} \in \Sigma_{3}^{*}$ as above and assume $w \sim_{k} \tilde{w}$ and without loss of generality that $\alpha_{0} \in\{0,1\}^{*}, \mathrm{~m}(w)=\mathrm{m}(\tilde{w})=2$ and $\alpha_{1} \in 2^{+}$. By Lemma 14, we have $\alpha_{i} \sim_{k-1} \tilde{\alpha}_{i}$ for all $i \in[1]_{0}$. If $\iota\left(\alpha_{1}\right) \geq k-1$, then we are done. Thus, assume $\iota\left(\alpha_{1}\right)<k-1$. We cannot have $\check{\mathrm{m}}(w) \neq \overleftarrow{\mathrm{m}}(\tilde{w})$ by Proposition 20. Thus, we have $\overline{\mathrm{m}}(w)=\overleftarrow{\mathrm{m}}(\tilde{w})$ as well as $\mathrm{m}(w)=\mathrm{m}(\tilde{w})$ and the claim follows by Lemma 29,

Now, for the other direction assume $\alpha_{i} \sim_{k-1} \tilde{\alpha}_{i}$ for all $i \in[1]_{0}$. First, we consider $\iota\left(\alpha_{0}\right)=k-1=\iota\left(\tilde{\alpha}_{0}\right)$. We have $\beta=v 2$ and $\tilde{\beta}=\tilde{v} 2$ for $v, \tilde{v} \in\{0,1\}^{*}$. Because $\alpha_{0} \sim_{k-1} \tilde{\alpha}_{0} \sim_{k-1}(01)^{k-1}$ we have $\alpha_{0} v \sim_{k} \alpha_{0} \tilde{v} \sim_{k} \tilde{\alpha}_{0} \tilde{v}$ by Lemma 10 for $\{0,1\}$. Therefore, we have $w=\alpha_{0} v 2 \alpha_{1} \sim_{k} \tilde{\alpha}_{0} \tilde{v} 2 \tilde{\alpha}_{1}=\tilde{w}$ because $\sim_{k}$ is a congruence. The other case follows again from Lemma 29

The next lemma examines the case $2=\left|\operatorname{alph}\left(\alpha_{0}\right)\right|, 0=\left|\operatorname{alph}\left(\alpha_{1}\right)\right|$. This case is similar to the one above. If $\alpha_{1}$ has at least $k-1$ arches, then we can permute the modi. Otherwise, everything is fixed and the level of congruence of the cores is determined by the structure of the $\alpha$-factors. The proof is analogous to the one of Claim 3 because in the proof we do not use the fact that $\alpha_{0}$ is non-empty.

Claim 4. Let $\left|\operatorname{alph}\left(\alpha_{0}\right)\right|=2$, and $\alpha_{1}=\varepsilon$. Then, $w \sim_{k} \tilde{w}$ iff either $\iota\left(\alpha_{0}\right)=$ $k-1=\iota\left(\tilde{\alpha}_{0}\right)$, or $\alpha_{0} \sim_{k-1} \tilde{\alpha}_{0}, \iota\left(\alpha_{0}\right)<k-1, \overline{\mathrm{~m}}(w)=\overline{\mathrm{m}}(\tilde{w})$, and core ${ }_{1} \sim_{k-c}$ $\widetilde{\operatorname{cor}}_{1} \in \mathrm{y}^{+}$where $c:=\iota\left(\alpha_{0}\right)+\delta_{\mathrm{y} \preceq \text { re }}\left(\alpha_{0}\right)$.

With Claim 4 the case distinction for one of the $\alpha$ containing two letters, is completed. In the following cases we investigate the situation, where both $\alpha$ have maximal 1-letter alphabets.

Claim 5. Let $\left|\operatorname{alph}\left(\alpha_{0}\right)\right|=1$, and $\alpha_{1}=\varepsilon$. Then, $w \sim_{k} \tilde{w}$ iff $\alpha_{1} \sim_{k-1} \tilde{\alpha}_{1}$, $\mathrm{m}(w)=\mathrm{m}(\tilde{w}), \overleftarrow{\mathrm{m}}(w)=\overleftarrow{\mathrm{m}}(\tilde{w})$ and core $\sim_{k-c}{\widetilde{\text { core }_{1}}} \in \mathrm{y}^{+}$where $c:=\delta_{\mathrm{y} \preceq \alpha_{1}}$.

Proof. Assume $w \sim_{k} \tilde{w}$. By Proposition 20, we can assume that the modi are the same. Now the claim, as well as, the other direction follow from Lemma 29 ,

The last two cases follow analogously to Claim 5
Claim 6. Let $\operatorname{alph}\left(\alpha_{1}\right)=\operatorname{alph}\left(\alpha_{0}\right) \in \Sigma_{3}$. Then, $w \sim_{k} \tilde{w}$ iff $\alpha_{1} \sim_{k-1} \tilde{\alpha}_{1}, \mathrm{~m}(w)=$ $\mathrm{m}(\tilde{w}), \overleftarrow{\mathrm{m}}(w)=\overleftarrow{\mathrm{m}}(\tilde{w})$ and $\operatorname{core}_{1} \sim_{k-c}{\widetilde{\text { core }_{1}}}_{1} \in \mathrm{y}^{+}$where $c:=\delta_{\mathrm{y} \preceq \alpha_{0}}+\delta_{\mathrm{y} \in \alpha_{1}}$.
Claim 7. Let $\operatorname{alph}\left(\alpha_{1}\right) \neq \operatorname{alph}\left(\alpha_{0}\right)$ and $\left|\operatorname{alph}\left(\alpha_{1}\right)\right|=\left|\operatorname{alph}\left(\alpha_{0}\right)\right|=1$. Then, $w \sim_{k} \tilde{w}$ iff $\alpha_{1} \sim_{k-1} \tilde{\alpha}_{1}, \mathrm{~m}(w)=\mathrm{m}(\tilde{w}), \overline{\mathrm{m}}(w)=\overline{\mathrm{m}}(\tilde{w})$ and $\operatorname{core}_{1} \sim_{k-c}{\widetilde{\operatorname{corer}_{1}}}_{1} \in \mathrm{y}^{+}$ where $c:=\delta_{\mathrm{y} \preceq \alpha_{0}}+\delta_{\mathrm{y} \in \alpha_{1}}$.

This concludes the proof.

## 6 Conclusion

In 2021, Kosche et al. 16 first implicitly used a new factorization to find absent scattered factors in words algorithmically. Later, in 2022, Fleischmann et al. [6] introduced this factorization as $\alpha$ - $\beta$-factorization and used it to investigate

Table 3. The possibilities for the $\beta$-factor of $w=\alpha_{0} \beta \alpha_{1}$, assuming $\mathbf{a}, \mathrm{b}, \mathrm{c} \in \Sigma_{3}$ different. Note that in the cases $(1,1),(1,0),(0,0)$ the letters not fixed by the $\alpha$-factors can be chosen arbitrarily but differently from $\Sigma_{3}$.

| $\left\|\operatorname{alph}\left(\alpha_{0}\right)\right\|,\left\|\operatorname{alph}\left(\alpha_{1}\right)\right\|$ | $\operatorname{alph}\left(\alpha_{0}\right)$ | $\operatorname{alph}\left(\alpha_{1}\right)$ | $\beta \operatorname{Reg} \operatorname{Exp}$ | Stated In |
| :---: | :--- | :--- | :--- | :--- |
| 2,2 | $\{\mathrm{a}, \mathrm{b}\}$ | $\{\mathrm{a}, \mathrm{c}\}$ | $\mathrm{ba}^{*} \mathrm{c}$ | Prop. [19 |
|  | $\{\mathrm{a}, \mathrm{b}\}$ | $\{\mathrm{a}, \mathrm{b}\}$ | c |  |
| 2,1 | $\{\mathrm{a}, \mathrm{b}\}$ | $\{\mathrm{c}\}$ | $\left(\mathrm{ab}^{+} \mid \mathrm{ba}^{+}\right) \mathrm{c}$ |  |
|  | $\{\mathrm{a}, \mathrm{b}\}$ | $\{\mathrm{a}\}$ | $\mathrm{ba}^{*} \mathrm{c}$ |  |
| 2,0 | $\{\mathrm{a}, \mathrm{b}\}$ | $\emptyset$ | $\left(\mathrm{ab}^{+} \mid \mathrm{b}^{+} \mathrm{a}\right) \mathrm{c}$ |  |
| 1,1 | $\{\mathrm{a}\}$ | $\{\mathrm{b}\}$ | $\mathrm{ab}^{+} \mathrm{c}\left\|\mathrm{ac}^{+} \mathrm{b}\right\| \mathrm{ca}^{+} \mathrm{b}$ |  |
|  | $\{\mathrm{a}\}$ | $\{\mathrm{a}\}$ | $\mathrm{ba}^{*} \mathrm{c}$ |  |
| 0,0 | $\emptyset \mathrm{a}\}$ | $\emptyset$ | $\mathrm{ba}^{*} \mathrm{c} \mid \mathrm{ab}^{+} \mathrm{c}$ |  |
|  | $\emptyset$ | $\emptyset$ | $\mathrm{ab}^{+} \mathrm{c}$ | Prop.[18] |

the classes of Simon's congruence separated by the number of shortest absent scattered factors, to characterize the classes for arbitrary alphabets for some fixed numbers of shortest absent scattered factors and to give explicit formulas for these subsets. In this paper, we investigated the $\alpha$ - $\beta$-factorization as an object of intrinsic interest. This leads to a result characterizing $k$-congruence of $m$-universal words in terms of their 1-universal $\alpha \beta \alpha$-factors. In the case of the binary and ternary alphabet, we fully characterized the congruence of words in terms of their single factors. Moreover, using this characterization, we gave a formula for the number of classes of binary words for each $k$, characterized the finite classes, and gave a conceptually simple linear time algorithm for testing MaxSimK for binary words.

The modus of the layered arch factorizations used in the proof of Proposition 20 and throughout the literature [23|21]4, can be regarded as the optimal word to jump to certain letters in certain parts of the word. The $\alpha$ - $\beta$-factorization encapsulates the first layer (arches w.r.t. $\Sigma$ ) of these factorizations for all indicies. For small alphabets (this paper) and shortest abscent scattered factors (c.f. [6]) this allows the characterization and enumeration of classes. Extending this idea to lower layers (arches w.r.t. some $\Omega \subset \Sigma$ ), is left as future work.

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[^0]:    ${ }^{3}$ Alfalfa (Medicago sativa) is a plant whose name means horse food in Old Persian.

