The Complexity of (P_k, P_ℓ)-Arrowing

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Abstract. For fixed nonnegative integers k and ℓ, the (P_k, P_ℓ)-Arrowing problem asks whether a given graph, G, has a red/blue coloring of $E(G)$ such that there are no red copies of P_k and no blue copies of P_ℓ. The problem is trivial when $\max(k, \ell) \leq 3$, but has been shown to be coNP-complete when $k = \ell = 4$. In this work, we show that the problem remains coNP-complete for all pairs of k and ℓ, except $(3, 4)$, and when $\max(k, \ell) \leq 3$.
Our result is only the second hardness result for (F, H)-Arrowing for an infinite family of graphs and the first for 1-connected graphs. Previous hardness results for (F, H)-Arrowing depended on constructing graphs that avoided the creation of too many copies of F and H, allowing easier analysis of the reduction. This is clearly unavoidable with paths and thus requires a more careful approach. We define and prove the existence of special graphs that we refer to as "transmitters." Using transmitters, we construct gadgets for three distinct cases: 1) $k = 3$ and $\ell \geq 5$, 2) $\ell > k \geq 4$, and 3) $\ell = k \geq 4$. For (P_3, P_4)-Arrowing we show a polynomial-time algorithm by reducing the problem to 2SAT, thus successfully categorizing the complexity of all (P_k, P_ℓ)-Arrowing problems.

Keywords: Graph arrowing · Ramsey theory · Complexity.

1 Introduction and Related Work

Often regarded as the study of how order emerges from randomness, Ramsey theory has played an important role in mathematics and computer science; it has applications in several diverse fields, including, but not limited to, game theory, information theory, and approximation algorithms [16]. A key operator within the field is the arrowing operator: given graphs F, G, and H, we say that $G \to (F, H)$ (read, G *arrows* F, H) if every red/blue edge-coloring of G's edge contains a red F or a blue H. In this work, we categorize the computational complexity of evaluating this operator when F and H are fixed path graphs. The problem is defined formally as follows.

Problem 1 ((F, H)-Arrowing). Let F and H be fixed graphs. Given a graph G, does $G \to (F, H)$?

The problem is clearly in coNP; a red/blue coloring of G with no red F's and no blue H's forms a certificate that can be verified in polynomial time since

F and H are fixed graphs. We refer to such a coloring as an (F, H)-good coloring. The computational complexity of (F, H)-Arrowing has been categorized for a number of pairs (F, H), with a significant amount of work done in the 80s and 90s. Most relevant to our work is a result by Rutenburg, who showed that (P_4, P_4)-Arrowing is coNP-complete [17], where P_n is the path graph on n vertices. Burr showed that (F, H)-Arrowing is in P when F and H are star graphs or when F is a matching [5]. Using "senders"—graphs with restricted (F, H)-good colorings introduced a few years earlier by Burr et al. [6,7], Burr showed that (F, H)-Arrowing is coNP-complete when F and H are members of Γ_3, the family of all 3-connected graphs and K_3. The generalized (F, H)-Arrowing problem, where F and H are also part of the input, was shown to be Π_2^p-complete by Schaefer [18].[1] Aside from categorizing complexity, the primary research avenue concerned with the arrowing operator is focused on finding minimal—with different possible definitions of minimal—graphs for which arrowing holds. The smallest orders of such graphs are referred to as Ramsey numbers. Folkman numbers are defined similarly for graphs with some extra structural constraints. We refer the interested reader to surveys by Radziszowski [15] and Bikov [4] for more information on Ramsey numbers and Folkman numbers, respectively.

Our work provides the first complexity result for (F, H)-Arrowing for an infinite family of graphs since Burr's Γ_3 result from 1990. It is important to note that Burr's construction relies on that fact that contracting less than three vertices between pairs of 3-connected graphs does not create new copies of said graph. Let F be 3-connected and $u, v \in V(F)$. Construct G by taking two copies of F and contracting u across both copies, then contracting v across both copies. Observe that no new copies of F are constructed in this process; if a new F is created then it must be disconnected by the removal of the two contracted vertices, contradicting F's 3-connectivity. This process does not work for paths since contracting two path graphs will always make several new paths across the vertices of both paths. Thus, we require a more careful approach when constructing the gadgets necessary for our reductions. We focus on the problem defined below and prove a dichotomy theorem categorizing the problem to be in P or be coNP-complete. We note that such theorems for other graph problems exist in the literature, e.g., [1,8,10,13].

Problem 2 ((P_k, P_ℓ)-Arrowing). Let k and ℓ be fixed integers such that $2 \le k \le \ell$. Given a graph G, does $G \to (P_k, P_\ell)$?

Theorem 1. *(P_k, P_ℓ)-Arrowing is* coNP*-complete for all k and ℓ unless $k = 2$, $(k, \ell) = (3, 3)$, or $(k, \ell) = (3, 4)$. For these exceptions, the problem is in* P.

Before this, the only known coNP-complete case for paths was when $k = \ell = 4$ [17]. Despite being intuitively likely, generalizing the hardness result to larger paths proved to be an arduous task. Our proof relies on proving the existence of graphs with special colorings—we rely heavily on work by Hook [11], who

[1] $\Pi_2^p = \text{coNP}^{\text{NP}}$, the class of all problems whose complements are solvable by a nondeterministic polynomial-time Turing machine having access to an NP oracle [14].

categorized the (P_k, P_ℓ)-good colorings of the largest complete graphs which do not arrow (P_k, P_ℓ). After showing the existence of these graphs, the reduction is straightforward. The polynomial-time cases are straightforward (Theorem 2) apart from the case where $(k, \ell) = (3, 4)$, wherein we reduce the problem to 2SAT (Theorem 3).

The rest of this paper is organized as follows. We present the necessary preliminaries in Section 2. The proof for Theorem 1 is split into Sections 3 (the polynomial-time cases) and 4 (the coNP-complete cases). We conclude in Section 5.

2 Preliminaries

All graphs discussed in this work are simple and undirected. $V(G)$ and $E(G)$ denote the vertex and edge set of a graph G, respectively. We denote an edge in $E(G)$ between $u, v \in V(G)$ as (u, v). For two disjoint subsets $A, B \subset V(G)$, $E(A, B)$ refers to the edges with one vertex in A and one vertex in B. The neighborhood of a vertex $v \in V(G)$ is denoted as $N(v)$ and $d(v) := |N(v)|$. The path, cycle, and complete graphs on n vertices are denoted as P_n, C_n, and K_n, respectively. The complete graph on n vertices missing an edge is denoted as $K_n - e$. Vertex contraction is the process of replacing two vertices u and v with a new vertex w such that w is adjacent to all remaining neighbors $N(u) \cup N(v)$.

An (F, H)-good coloring of a graph G is a red/blue coloring of $E(G)$ where the red subgraph is F-free, and the blue subgraph is H-free. We say that G is (F, H)-good if it has at least one (F, H)-good coloring. When the context is clear, we will omit (F, H) and refer to the coloring as a good coloring.

Formally, a coloring for G is defined as function $c : E(G) \to \{\text{red}, \text{blue}\}$ that maps edges to the colors red and blue. For an edge (u, v) and coloring c, we denote its color as $c(u, v)$.

3 Polynomial-Time Cases

In this section, we prove the P cases from Theorem 1. Particularly, we describe polynomial-time algorithms for (P_2, P_ℓ)-Arrowing and (P_3, P_3)-Arrowing (Theorem 2) and provide a polynomial-time reduction from (P_3, P_4)-Arrowing to 2SAT (Theorem 3).

Theorem 2. *(P_k, P_ℓ)-Arrowing is in* P *when $k = 2$ and when $k = \ell = 3$.*

Proof. Let G be the input graph. Without loss of generality, assume that G is connected (for disconnected graphs, we run the algorithm on each connected component).
Case 1 ($k = 2$). Coloring any edge in G red will form a red P_2. Thereby, the entire graph must be colored blue. Thus, a blue P_ℓ is avoided if and only if G is P_ℓ-free, which can be checked by brute force, since ℓ is constant.

Case 2 ($k = \ell = 3$). Note that in any (P_3, P_3)-good coloring of G, edges of the same color cannot be adjacent; otherwise, a red or blue P_3 is formed. Thus, we can check if G is (P_3, P_3)-good similarly to how we check if a graph is 2-colorable: arbitrarily color an edge red and color all of its adjacent edges blue. For each blue edge, color its neighbors red and for each red edge, color its neighbors blue. Repeat this process until all edges are colored or a red or blue P_3 is formed. This algorithm is clearly polynomial-time. □

The proof that (P_3, P_4)-Arrowing is in P consists of two parts. A preprocessing step to simplify the graph (using Lemmas 1 and 2), followed by a reduction to 2SAT, which was proven to be in P by Krom in 1967 [12].

Problem 3 (2SAT). Let ϕ be a CNF formula where each clause has at most two literals. Does there exist a satisfying assignment of ϕ?

Lemma 1. *Suppose G is a graph and $v \in V(G)$ is a vertex such that $d(v) = 1$ and v's only neighbor has degree at most two. Then, G is (P_3, P_4)-good if and only if $G - v$ is (P_3, P_4)-good.*

Proof. Let u be the neighbor of v. If $d(u) = 1$, the connected component of v is a K_2 and the statement is trivially true. If $d(u) = 2$, let w be the other neighbor of u, i.e., the neighbor that is not v. Clearly, if G is (P_3, P_4)-good, then $G - v$ is (P_3, P_4)-good. We now prove the other direction. Suppose we have good coloring of $G - v$. It is immediate that we can extend this to a good coloring of G by coloring (v, u) (the only edge adjacent to v) red if (u, w) is colored blue, and blue if (u, w) is colored red. □

Lemma 2. *Suppose G is a graph and there is a P_4 in G with edges $(v_1, v_2), (v_2, v_3)$, and (v_3, v_4) such that $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$. Then, G is (P_3, P_4)-good if and only if $G - v_2$ is (P_3, P_4)-good.*

Proof. If (v_1, v_4) is an edge, then the connected component of v_2 is a C_4 and the statement is trivially true. If not, let $v_0, v_5 \notin \{v_1, v_2, v_3, v_4\}$ be such that (v_0, v_1) and (v_4, v_5) are edges. Note that it is possible that $v_0 = v_5$. Clearly, if G is (P_3, P_4)-good then $G - v_2$ is (P_3, P_4)-good. For the other direction, suppose c is a (P_3, P_4)-good coloring of $G - v_2$. We now construct a coloring c' of G. We color all edges other than $(v_1, v_2), (v_2, v_3)$, and (v_3, v_4) the same as c. The colors of the remaining three edges are determined by the coloring of (v_0, v_1) and (v_4, v_5) as follows.

- If $c(v_0, v_1) = c(v_4, v_5) =$ red, then $c'(v_1, v_2), c'(v_2, v_3), c'(v_3, v_4) =$ blue, red, blue.
- If $c(v_0, v_1) = c(v_4, v_5) =$ blue, then $c'(v_1, v_2), c'(v_2, v_3), c'(v_3, v_4) =$ red, blue, red.
- If $c(v_0, v_1) =$ red and $c(v_4, v_5) =$ blue, then $c'(v_1, v_2), c'(v_2, v_3), c'(v_3, v_4) =$ blue, blue, red.
- If $c(v_0, v_1) =$ blue and $c(v_4, v_5) =$ red, then $c'(v_1, v_2), c'(v_2, v_3), c'(v_3, v_4) =$ red, blue, blue.

Since the cases above are mutually exhaustive, this completes the proof. $\square$

Theorem 3. *(P_3, P_4)-Arrowing is in* P.

Proof. Let G be the input graph. Let G' be the graph obtained by repeatedly removing vertices v described in Lemma 1 and vertices v_2 described in Lemma 2 until no more such vertices exist. As implied by said lemmas, $G' \to (P_3, P_4)$ if and only if $G \to (P_3, P_4)$. Thus, it suffices to construct a 2SAT formula ϕ such that ϕ is satisfiable if and only if G' is (P_3, P_4)-good.

Let r_e be a variable corresponding to the edge $e \in E(G')$, denoting that e is colored red. We construct a formula ϕ, where a solution to ϕ corresponds to a coloring of G'. For each P_3 in G', with edges (v_1, v_2) and (v_2, v_3), add the clause $\left(\overline{r_{(v_1,v_2)}} \vee \overline{r_{(v_2,v_3)}}\right)$. Note that this expresses "no red P_3's." For each P_4 in G', with edges $(v_1, v_2), (v_2, v_3)$, and (v_3, v_4):

1. If $(v_2, v_4) \in E(G')$, add the clause $\left(r_{(v_1,v_2)} \vee r_{(v_3,v_4)}\right)$.
2. If $(v_2, v_4) \notin E(G')$ and $d(v_2) > 2$, then add the clause $\left(r_{(v_2,v_3)} \vee r_{(v_3,v_4)}\right)$.

It is easy to see that the conditions specified above must be satisfied by each good coloring of G', and thus G' being (P_3, P_4)-good implies that ϕ is satisfiable. We now prove the other direction by contradiction. Suppose ϕ is satisfied, but the corresponding coloring c is not (P_3, P_4)-good. It is immediate that red P_3's cannot occur in c, so we assume that there exists a blue P_4, with edges $e = (v_1, v_2), f = (v_2, v_3)$, and $g = (v_3, v_4)$ such that $r_e = r_f = r_g =$ false in the satisfying assignment of ϕ. Without loss of generality, assume that $d(v_2) \geq d(v_3)$.

– If $d(v_2) > 2$, ϕ would contain clause $r_e \vee r_g$ or $r_f \vee r_g$. It follows that $d(v_2) = d(v_3) = 2$.
– If $d(v_1) = 1$, v_1 would have been deleted by applying Lemma 1. It follows that $d(v_1) > 1$. Similarly, $d(v_4) > 1$.
– If $d(v_1) > 2$, then there exists a vertex v_0 such that $(v_0, v_1), (v_1, v_2), (v_2, v_3)$ are a P_4 in G', $d(v_1) > 2$ and $(v_1, v_3) \notin E(G')$ (since $d(v_3) = 2$). This implies that ϕ contains clause $r_e \vee r_f$, which is a contradiction. It follows that $d(v_1) = 2$. Similarly, $d(v_4) = 2$.
– So, we are in the situation that $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 2$. But then v_2 would have been deleted by Lemma 2.

Since the cases above are mutually exhaustive, this completes the proof. $\square$

4 coNP-Complete Cases

In this section, we discuss the coNP-complete cases in Theorem 1. In Section 4.1, we describe how NP-complete SAT variants can be reduced to (P_k, P_ℓ)-Nonarrowing (the complement of (P_k, P_ℓ)-Arrowing: does there exist a (P_k, P_ℓ)-good coloring of G?). The NP-complete SAT variants are defined below.

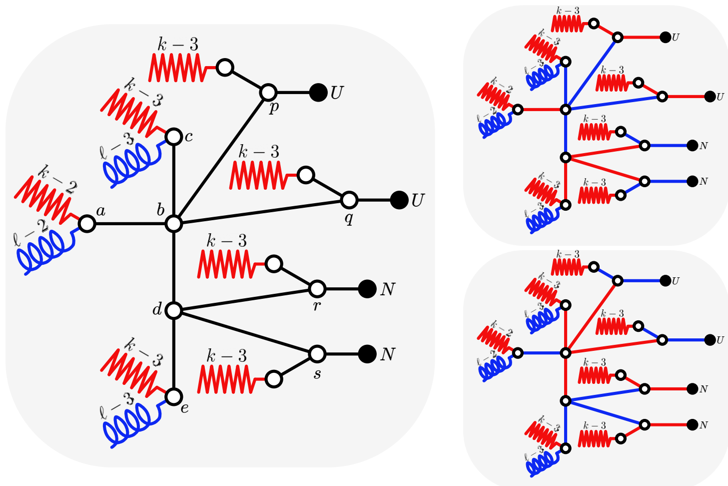

Fig. 1. The variable gadget for (P_k, P_ℓ)-Nonarrowing when $4 \leq k < \ell$ is shown on the left. The output vertices are filled in. Red jagged lines and blue spring lines represent (k, ℓ, x)-red- and (k, ℓ, x)-blue-transmitters, respectively, where the value of x is shown on the top, and the vertex the lines are connected to are the strict endpoints of the monochromatic paths. Observe that when (a, b) is red, other edges adjacent to b must be blue to avoid a red P_k. This, in turn, causes neighbors p and q to have incoming blue $P_{\ell-1}$'s, and vertices marked **U** are now strict endpoints of red P_{k-1}'s. Moreover, edges adjacent to d (except (b, d)) must be red to avoid blue P_ℓ's. Thus, r and s are strict endpoints of red P_{k-1}'s, causing the vertices marked **N** to be strict endpoints of blue P_3's. A similar pattern is observed when (a, b) is blue. Note that for $k \leq 4$, the $(k, \ell, k-3)$-red-transmitter can be ignored. On the right, the two kinds of (P_k, P_ℓ)-good colorings of the gadget are shown.

Problem 4 ((2, 2)-3SAT [3]). Let ϕ be a CNF formula where each clause contains exactly three distinct variables, and each variable appears only four times: twice unnegated and twice negated. Does there exist a satisfying assignment for ϕ?

Problem 5 (Positive NAE E3SAT-4 [2]). Let ϕ be a CNF formula where each clause is an NAE-clause (a clause that is satisfied when its literals are not all true or all false) containing exactly three (not necessarily distinct) variables, and each variable appears at most four times, only unnegated. Does there exist a satisfying assignment for ϕ?

Our proofs depend on the existence of graphs we refer to as "transmitters," defined below. These graphs enforce behavior on special vertices which are *strict endpoints* of red or blue paths. For a graph G and coloring c, we say that v is a strict endpoint of a red (resp., blue) P_k in c if k is the length of the longest red (resp., blue) path that v is the endpoint of. We prove the existence of these graphs in Section 4.2.

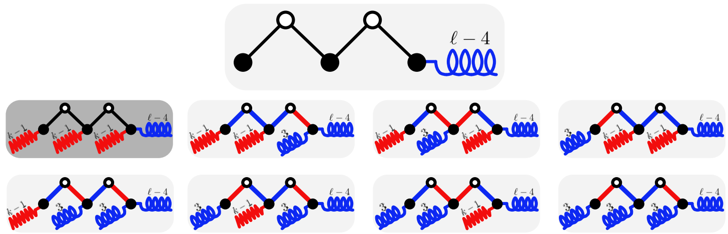

Fig. 2. The clause gadget for (P_k, P_ℓ)-Nonarrowing when $4 \le k < \ell$ is shown on top. The input vertices are filled in. Below it, we show the eight possible combinations of inputs that can be given to the gadget. Observe that a (P_k, P_ℓ)-good coloring is always possible unless the input is three red P_{k-1}'s (top left). As in Figure 1, jagged and spring lines represent transmitters. We use this representation of transmitters to depict the two forms of input to the gadget. For $\ell \le 5$, the $(k, \ell, \ell-4)$-blue-transmitter can be ignored.

Definition 1. *Let $3 \le k < \ell$. For an integer $x \in \{2, 3, \dots, k-1\}$ (resp., $x \in \{2, 3, \dots, \ell-1\}$) a (k, ℓ, x)-red-transmitter (resp., (k, ℓ, x)-blue-transmitter) is a (P_k, P_ℓ)-good graph G with a vertex $v \in V(G)$ such that in every (P_k, P_ℓ)-good coloring of G, v is the strict endpoint of a red (resp., blue) P_x, and is not adjacent to any blue (resp., red) edge.*

Definition 2. *Let $k \ge 3$ and $x \in \{2, 3, \dots, k-1\}$. A (k, x)-transmitter is a (P_k, P_k)-good graph G with a vertex $v \in V(G)$ such that in every (P_k, P_k)-good coloring of G, v is either (1) the strict endpoint of a red P_x and not adjacent to any blue edge, or (2) the strict endpoint of a blue P_x and not adjacent to any red edge.*

4.1 Reductions

We present three theorems that describe gadgets to reduce NP-complete variants of SAT to (P_k, P_ℓ)-Nonarrowing.

Theorem 4. *(P_k, P_ℓ)-Arrowing is* coNP*-complete for all $4 \le k < \ell$.*

Proof. We reduce $(2,2)$-3SAT to (P_k, P_ℓ)-Nonarrowing. Let ϕ be the input to $(2,2)$-3SAT. We construct G_ϕ such that G_ϕ is (P_k, P_ℓ)-good if and only if ϕ is satisfiable. Let VG and CG be the variable and clause gadgets shown in Figures 1 and 2. VG has four output vertices that emulate the role of sending a truth signal from a variable to a clause. We first look at Figure 1. The vertices labeled U (resp., N) correspond to unnegated (resp., negated) signals. Being the strict endpoint of a blue P_3 corresponds to a true signal while being the strict endpoint of a red P_{k-1} corresponds to a false signal. We now look at Figure 2. When three red P_{k-1} signals are sent to the clause gadget, it forces the entire graph to be blue, forming a blue P_ℓ. When at least one blue P_3 is present, a good coloring of CG is possible.

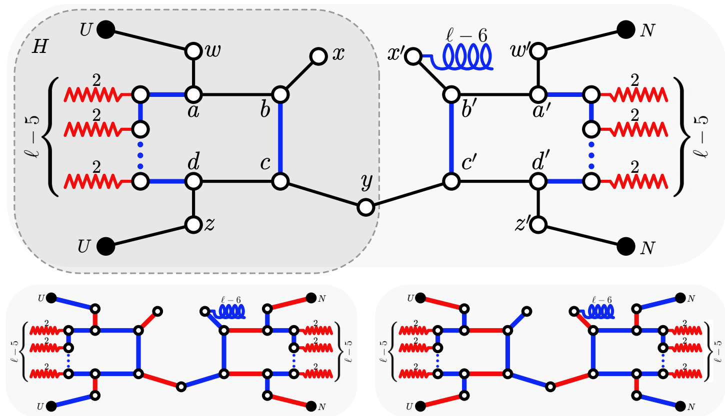

Fig. 3. The variable gadget for (P_3, P_ℓ)-Nonarrowing where $\ell \geq 6$ (top) and its two good colorings (bottom). The variable gadget is a combination of two H's, whose properties we discuss in the proof of Theorem 5. Note that when (a, b) is red in H, then (a', b') is blue in H's copy, and vice versa; if both copies have the same coloring of (a, b), then a red P_3 is formed at y, or a blue P_ℓ is formed from the path from x to x' and the $(3, \ell, \ell - 6)$-blue-transmitter that x' is connected to. When $\ell = 5$, the edge (a, d) is added in H, in lieu of the $\ell - 5$ vertices connected to $(3, \ell, 2)$-red-transmitters. Note that for $\ell \leq 8$, the $(3, \ell, \ell - 6)$-blue-transmitter can be ignored.

We construct G_ϕ like so. For each variable (resp., clause) in ϕ, we add a copy of VG (resp., CG) to G_ϕ. If a variable appears unnegated (resp., negated) in a clause, a U-vertex (resp., N-vertex) from the corresponding VG is contracted with a previously uncontracted input vertex of the CG corresponding to said clause. The correspondence between satisfying assignments of ϕ and good colorings of G_ϕ is easy to see. □

Theorem 5. *(P_3, P_ℓ)-Arrowing is* coNP*-complete for all* $\ell \geq 5$.

Proof. We proceed as in the proof of Theorem 4. The variable gadget is shown in Figure 3. Blue (resp., red) P_2's incident to vertices marked **U** and **N** correspond to true (resp., false) signals. The clause gadget is the same as Theorem 4's, but the good colorings are different since the inputs are red/blue P_2's instead. These colorings are shown in the appendix in Figure 10.

Suppose $\ell \geq 6$. Let H be the graph circled with a dotted line in Figure 3. We first discuss the properties of H. Note that any edge adjacent to a red P_2 must be colored blue to avoid a red P_3. Let $v_1, v_2, \ldots, v_{\ell-5}$ be the vertices connected to $(3, \ell, 2)$-red-transmitters such that v_1 is adjacent to a. Observe that (a, b) and (c, d) must always be the same color; if, without loss of generality, (a, b) is red and (c, d) is blue, a blue P_ℓ is formed via the sequence $a, v_1, \ldots, v_{\ell-5}, d, c, b, x$. In the coloring where (a, b) and (c, d) are blue, the vertices $a, v_1, \ldots, v_{\ell-5}, d, c, b$ form a blue $C_{\ell-1}$, and all edges going out from the cycle must be colored red to

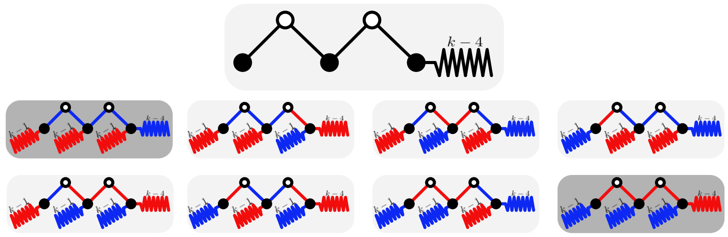

Fig. 4. The clause gadget for (P_k, P_k)-Nonarrowing. The format is similar to Figure 2.

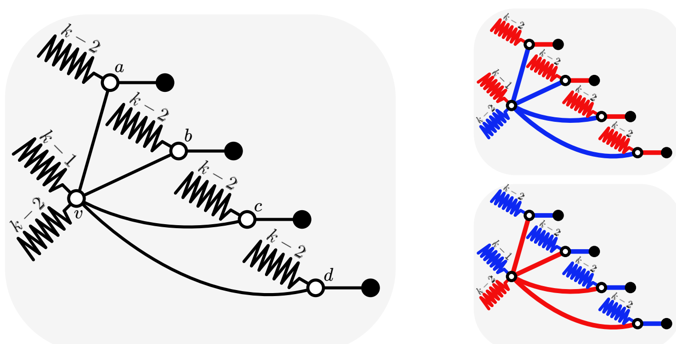

Fig. 5. The variable gadget for (P_k, P_k)-Nonarrowing. Observe that the transmitters connected to v must have different colors; otherwise, a red or blue $P_{k-1+k-2-1}$ is formed, which is forbidden when $k \geq 4$. When the $(k, k-1)$-transmitter is red, v's other neighboring edges must be blue. Thus, vertices a, b, c, and d are strict endpoints of blue P_{k-1}'s, causing the output vertices (filled) to be strict endpoints of red P_{k-1}'s. A similar situation occurs when the $(k, k-1)$-transmitter is blue. Both (P_k, P_k)-good colorings are shown on the right.

avoid blue P_ℓ's. This forces the vertices marked **U** to be strict endpoints of blue P_2's. If (a, b) and (c, d) are red, $w, a, v_1, \ldots, v_{\ell-5}, d, z$ forms a blue $P_{\ell-1}$, forcing the vertices marked **U** to be strict endpoints of red P_2's. Moreover, (x, b) and (y, c) must also be blue.

With these properties of H in mind, the functionality of the variable gadget described in Figure 3's caption is easy to follow. The $\ell = 5$ case uses a slightly different H, also described in the caption. $\square$

Theorem 6. *(P_k, P_k)-Arrowing is* coNP*-complete for all $k \geq 4$.*

Proof. For $k = 4$, Rutenburg showed that the problem is coNP-complete by providing gadgets that reduce from an NAE SAT variant [17]. For $k \geq 5$, we take a similar approach and reduce Positive NAE E3SAT-4 to (P_k, P_k)-Nonarrowing using the clause and variable gadgets described in Figures 4 and 5. The variable gadget has four output vertices, all of which are unnegated. Without loss of

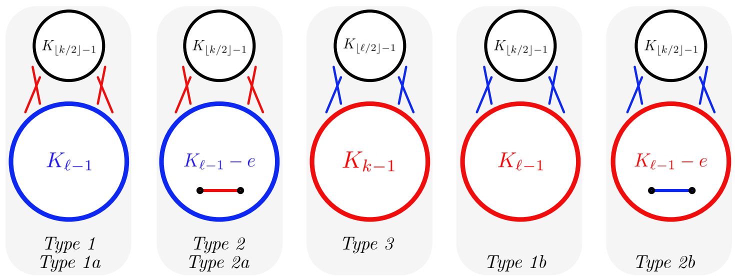

Fig. 6. Illustrations of (P_k, P_ℓ)-good colorings of $K_{R(P_k,P_\ell)-1}$.

generality, we assume that blue P_{k-1}'s correspond to true signals. The graph G_ϕ is constructed as in the proofs of Theorems 4 and 5. Our variable gadget is still valid when $k = 4$, but the clause gadget does not admit a (P_4, P_4)-good coloring for all the required inputs. In Figure 11 in the appendix, we show a different clause gadget that can be used to show the hardness of (P_4, P_4)-Arrowing using our reduction. □

4.2 Existence of Transmitters

Our proofs for the existence of transmitters are corollaries of the following.

Lemma 3. *For integers k, ℓ, where $3 \leq k < \ell$, $(k, \ell, k-1)$-red-transmitters exist.*

Lemma 4. *For all $k \geq 3$, $(k, k-1)$-transmitters exist.*

In the interest of saving space, we only present the proof of one case (when k is even) of Lemma 3 in our main text, and defer the rest to the appendix. We construct these transmitters by carefully combining copies of complete graphs. The Ramsey number $R(P_k, P_\ell)$ is defined as the smallest number n such that $K_n \to (P_k, P_\ell)$. We know that $R(P_k, P_\ell) = \ell + \lfloor k/2 \rfloor - 1$, where $2 \leq k \leq \ell$ [9]. In 2015, Hook characterized the (P_k, P_ℓ)-good colorings of all "critical" complete graphs: $K_{R(P_k,P_\ell)-1}$. We summarize Hook's results below.[2]

Theorem 7 (Hook [11]). *Let $4 \leq k < \ell$ and $r = R(P_k, P_\ell) - 1$. The possible (P_k, P_ℓ)-good colorings of K_r can be categorized into three types. In each case, $V(G)$ is partitioned into sets A and B. The types are defined as follows:*

- *Type 1. Let $|A| = \lfloor k/2 \rfloor - 1$ and $|B| = \ell - 1$. Each edge in $E(B)$ must be blue, and each edge in $E(A, B)$ must be red. Any coloring of $E(A)$ is allowed.*
- *Type 2. Let $|A| = \lfloor k/2 \rfloor - 1$ and $|B| = \ell - 1$, and let $b \in E(B)$. Each edge in $E(B) \setminus \{b\}$ must be blue, and each edge in $E(A, B) \cup \{b\}$ must be red. Any coloring of $E(A)$ is allowed.*

[2] We note that Hook's ordering convention differs from ours, i.e., they look at (P_ℓ, P_k)-good colorings. Moreover, they use m and n in lieu of k and ℓ.

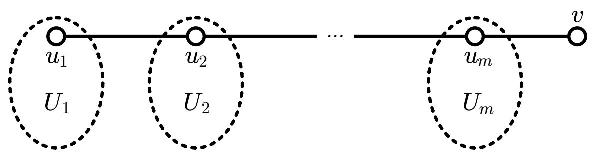

Fig. 7. An (H, u, m)-thread as described in Definition 3.

- *Type 3. Let $|A| = \lfloor \ell/2 \rfloor - 1$ and $|B| = k-1$. Each edge in $E(B)$ must be blue, and each edge in $E(A, B)$ must be red. Any coloring of $E(A)$ is allowed.*

Moreover, the types of colorings allowed vary according to the parity of k. If k is even, then K_r can only have Type 1 colorings. If k is odd and $\ell > k+1$, then K_r can only have Type 1 and 2 colorings. If k is odd and $\ell = k+1$, then K_r can have all types of colorings.

For the case where $k = \ell$, K_r can have Type 1 and 2 colorings as described in the theorem above. Due to symmetry, the colors in these can be swapped and are referred to as Type 1a, 1b, 2a, and 2b colorings. The colorings described have been illustrated in Figure 6. We note the following useful observation.

Observation 1 *Suppose $\ell > k \geq 4$ and $r = R(P_k, P_\ell) - 1$.*

- *In Type 1 (P_k, P_ℓ)-good colorings of K_r: (1) each vertex in B is a strict endpoint of a blue $P_{\ell-1}$, (2) when k is even (resp., odd), each vertex in B is a strict endpoint of a red P_{k-1} (resp., P_{k-2}), and (3) when k is even (resp., odd), each vertex in A is a strict endpoint of a red P_{k-2} (resp., P_{k-3}).*
- *In Type 2 (P_k, P_ℓ)-good colorings of K_r: (1) each vertex in B is a strict endpoint of a blue $P_{\ell-1}$, (2) each vertex in B is a strict endpoint of a red P_{k-1}, and (3) each vertex in A is a strict endpoint of a red P_{k-2}.*
- *In Type 3 (P_k, P_ℓ)-good colorings of K_r: (1) each vertex in B is a strict endpoint of a red P_{k-1}, (2) each vertex in B is a strict endpoint of a blue $P_{\ell-1}$, and (3) each vertex in A is a strict endpoint of a blue $P_{\ell-2}$.*

We justify these claims in the appendix, wherein we also formally define the colorings K_r when $k = \ell$ and justify a similar observation. Finally, we define a special graph that we will use throughout our proofs.

Definition 3 ((H, u, m)-thread). *Let H be a graph, $u \in V(H)$, and $m \geq 1$ be an integer. An (H, u, m)-thread G, is a graph on $m|V(H)|+1$ vertices constructed as follows. Add m copies of H to G. Let $U_i \subset V(G)$ be the vertex set of the i^{th} copy of H, and u_i be the vertex u in H's i^{th} copy. Connect each u_i to u_{i+1} for each $i \in \{1, 2, \ldots, m-1\}$. Finally, add a vertex v to G and connect it to u_m. We refer to v as the thread-end of G. This graph is illustrated in Figure 7.*

Using Theorem 7, Observation 1, and Definition 3 we are ready to prove the existence of $(k, \ell, k-1)$-red-transmitters and $(k, k-1)$-transmitters via construction. Transmitters for various cases are shown in Figures 8 and 9. We present the proof for one case below and the remaining in the appendix.

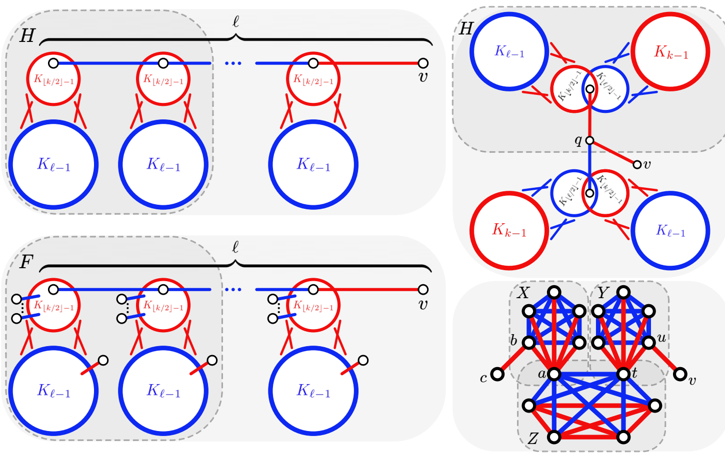

Fig. 8. $(k, \ell, k-1)$-red-transmitters for even k with $\ell > k$, odd k with $\ell > k+1$, and odd k with $\ell = k+1$ are shown on the top-left, bottom-left, and top-right, respectively. The latter construction does not work for the case where $k = 5$, so an alternative construction for a $(5, 6, 4)$-red-transmitter is shown on the bottom-right. The graphs (H and F) described in each case are circled so that the proofs are easier to follow. A good coloring is shown for each transmitter.

Proof of Lemma 3 when k is even. Let $k \geq 4$ be an even integer and $r = R(P_k, P_\ell) - 1$. In this case, by Theorem 7, only Type 1 colorings are allowed for K_r. The term $A1$-vertex (resp., $B1$-vertex) is used to refer to vertices belonging to set A (resp., B) in a K_r with a Type 1 coloring, as defined in Theorem 7. We first make an observation about the graph H, constructed by adding an edge (u, v) between two disjoint K_r's. Note that u must be an $A1$-vertex, otherwise the edge (u, v) would form a red P_{k-1} or blue $P_{\ell-1}$ when colored red or blue, respectively (Observation 1). Similarly, v must also be an $A1$-vertex. Note that (u, v) must be blue; otherwise, by Observation 1, a red $P_{k-2+k-2}$ is formed, which cannot exist in a good coloring when $k \geq 4$.

We define the $(k, \ell, k-1)$-red-transmitter, G, as the $(K_r, u, \ell-1)$-thread graph, where u is an arbitrary vertex in $V(K_r)$. The thread-end v of G is a strict endpoint of a red P_{k-1}. Let U_i and u_i be the sets and vertices of G as described in Definition 3. From our observation about H, we know that each edge (u_i, u_{i+1}) must be blue. Thus, $u_{\ell-1}$ must be the strict endpoint of a blue $P_{\ell-1}$, implying that $(u_{\ell-1}, v)$ must be red. Since $u_{\ell-1}$ is also a strict endpoint of a red P_{k-2} (Observation 1), v must be the strict endpoint of a red P_{k-1}.

For completeness, we must also show that G is (P_k, P_ℓ)-good. Let A_i and B_i be the sets A and B as defined in Theorem 7 for each U_i. Note that the only edges whose coloring we have not discussed are the edges in each $E(A_i)$. It is easy to see that if each edge in each $E(A_i)$ is colored red, the resulting coloring is (P_k, P_ℓ)-good. This is because introducing a red edge in $E(A_i)$ cannot form

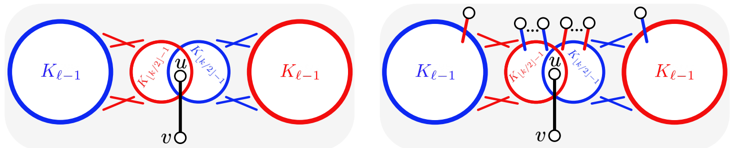

Fig. 9. $(k, k-1)$-transmitters for even k (left) and odd k (right).

a longer red path than is already present in the graph, i.e., any path going through an edge $(p, q) \in E(A_i)$ can be increased in length by selecting a vertex from $r \in E(B_i)$ using the edges (p, r) and (r, q) instead. This is always possible since $|E(B_i)|$ is sufficiently larger than $|E(A_i)|$. $\square$

Finally, we show how constructing (red-)transmitters where $x = k - 1$ is sufficient to show the existence of all defined transmitters.

Corollary 1. *For valid k, ℓ, and x, (k, ℓ, x)-blue-transmitters and (k, ℓ, x)-red-transmitters exist.*

Proof. Let H be a $(k, \ell, k-1)$-red-transmitter where $u \in V(H)$ is the strict endpoint of a red P_{k-1} in all of H's good colorings. For valid x, the $(H, u, x-1)$-thread graph G is a (k, ℓ, x)-blue-transmitter, where the thread-end v is the strict endpoint of a blue P_x in all good colorings of G; to avoid constructing red P_k's each edge along the path of u_i's is forced to be blue by the red P_{k-1} from H, where u_i is the vertex u in the i^{th} copy of H as defined in Definition 3.

To construct a (k, ℓ, x)-red-transmitter, we use a similar construction. Let H be a $(k, \ell, \ell-1)$-blue-transmitter where $u \in V(H)$ is the strict endpoint of a blue $P_{\ell-1}$ in all good colorings of H. For valid x, the (H, u, x)-thread graph G is a $(k, \ell, x-1)$-red-transmitter, where the thread-end v is the strict endpoint of a red P_x in all good colorings of G. $\square$

Corollary 2. *For valid k and x, (k, x)-transmitters exist.*

Proof. Let H be a $(k, k-1)$-transmitter where $u \in V(H)$ is the strict endpoint of a red/blue P_{k-1} in all of H's good colorings. For valid x, the $(k, u, x-1)$-thread graph G is a (k, x)-transmitter, where the thread-end v is the strict endpoint of a red or blue P_x in all good colorings of G. Let u_i be the vertex as defined in Definition 3. Each u_i is the strict endpoint of P_{k-1} of the same color; otherwise, the edge between two u's cannot be colored without forming a red or blue P_k. Thus, each such edge must be colored red (resp., blue) by the blue (resp., red) P_{k-1} coming from H. $\square$

5 Conclusion and Future Work

A major and very difficult goal is to classify the complexity for (F, H)-Arrowing for all fixed F and H. We conjecture that in this much more general case a dichotomy theorem still holds, with these problems being either in P or coNP-complete. This seems exceptionally difficult to prove. To our knowledge, all known dichotomy theorems for graphs classify the problem according to one fixed graph, and the polynomial-time characterizations are much simpler than in our case. We see this paper as an important first step in accomplishing this goal.

Acknowledgments

This work was supported in part by grant NSF-DUE-1819546. We would like to thank the anonymous reviewers for their valuable comments.

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Appendix

Clause Gadget for (P_3, P_ℓ)- and (P_4, P_4)-Nonarrowing

In Figure 10, we present the clause gadget for (P_3, P_ℓ)-Nonarrowing. In Figure 11, we present the clause gadget for (P_4, P_4)-Nonarrowing, inspired by the gadget used in Rutenburg's hardness proof [17] for the same problem.

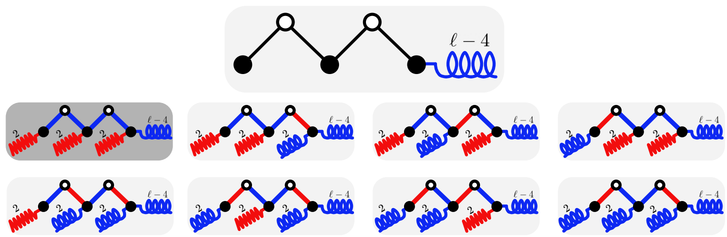

Fig. 10. The clause gadget for (P_3, P_ℓ)-Nonarrowing. The format is similar to Figure 2.

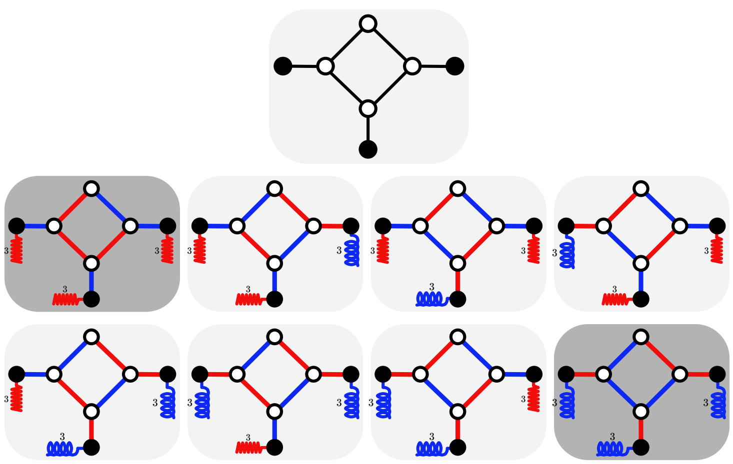

Fig. 11. The clause gadget for (P_4, P_4)-Nonarrowing. This is similar to the gadget used in Rutenburg's hardness proof [17], wherein the clause gadget was just the C_4.

Existence of Transmitters

We first state Hook's theorem for the symmetric case (where $k = \ell$) and state an observation about the properties of said colorings.

Theorem 8 (Hook [11]). *Let $k \geq 4$ and $r = R(P_k, P_k) - 1$. The possible (P_k, P_k)-good colorings of K_r can be categorized into two types. In each case, $V(G)$ is partitioned into sets A and B. The types are defined as follows:*

- *Type 1a. Let $|A| = \lfloor k/2 \rfloor - 1$ and $|B| = k - 1$. Each edge in $E(B)$ must be blue, and each edge in $E(A, B)$ must be red. Any coloring of $E(A)$ is allowed.*
- *Type 1b. Like Type 1a, but $E(B)$ is red, and $E(A, B)$ is blue.*
- *Type 2a. Let $|A| = \lfloor k/2 \rfloor - 1$ and $|B| = k - 1$, and let $b \in E(B)$. Each edge in $E(B) \setminus \{b\}$ must be blue, and each edge in $E(A, B) \cup \{b\}$ must be red. Any coloring of $E(A)$ is allowed.*
- *Type 2b. Like Type 2a, but $E(B) \setminus b$ is red, and $E(A, B) \cup b$ is blue.*

If k is even, then K_r can only have Type 1a/b colorings. If k is odd, then K_r can have all types of colorings.

Observation 2 *Let $k \geq 4$ and $r = R(P_k, P_k) - 1$.*

- *In Type 1a/1b (P_k, P_k)-good colorings of K_r: (1) each vertex in B is a strict endpoint of a blue/red P_{k-1}, (2) when k is even (resp., odd), each vertex in B is a strict endpoint of a red/blue P_{k-1} (resp., P_{k-2}), and (3) when k is even (resp., odd), each vertex in A is a strict endpoint of a red/blue P_{k-2} (resp., P_{k-3}).*
- *In Type 2a/2b (P_k, P_k)-good colorings of K_r: (1) each vertex in B is a strict endpoint of a blue/red P_{k-1}, (2) each vertex in B is a strict endpoint of a red/blue P_{k-1}, and (3) each vertex in A is a strict endpoint of a red/blue P_{k-2}.*

We now present the proofs of Lemmas 3 and 4. The term Ai-vertex (resp., Bi-vertex) for $i \in \{1, 2, 3\}$ is used to refer to vertices belonging to set A (resp., B) in a K_r with a Type i coloring, as defined in Theorem 7.

Proof of Lemma 3. We first consider the case where $k = 3$. Let G be the graph constructed by attaching a leaf vertex, v, to a $K_{\ell-1}$. It is easy to see that every vertex in a (P_3, P_ℓ)-good coloring of $K_{\ell-1}$ is the strict endpoint of blue $P_{\ell-1}$. Thus, the leaf edge must be red, and v is the strict endpoint of a red P_2 is all good coloring of G. Now, suppose $k \geq 4$. Let $r = R(P_k, P_\ell) - 1$. We consider three cases.

Case 1 (k is even). Covered in the main text.

Case 2 (k is odd and $\ell > k + 1$). Type 1 and 2 colorings of K_r are allowed in this case. Let H be the graph constructed by attaching a leaf node to $\lfloor k/2 \rfloor$ vertices of a K_r. We refer to these leaf vertices as L-vertices. We now analyze the properties of H. First, note that at least one vertex in B must be an L-vertex, since $|A| = \lfloor k/2 \rfloor - 1$. Recall that in both Type 1 and 2 colorings, a Bi-vertex is a strict endpoint of blue $P_{\ell-1}$. Thus, the leaf edge must be red to avoid making a blue P_ℓ. This implies that the K_r in H must have a Type 1 coloring; otherwise, the red leaf edge and red P_{k-1} from the Type 2 coloring would form a red P_k. Note that if at least two L-vertices were adjacent to vertices in B, then two red leaf edges and the red P_{k-2} would form a red P_k. Thus, there is exactly one L-vertex adjacent to a vertex in B. The red leaf edge adjacent to a B-vertex, along with the red P_{k-3} (formed using edges in $E(A, B)$), makes each vertex in A the strict endpoint of a red P_{k-2}. H can now emulate the role of K_r with Type 1 colorings as in the previous case.

Let H' be the graph constructed by attaching a leaf node to $\lfloor k/2 \rfloor - 1$ vertices of an K_r. Consider the graph F, where two disjoint H''s are connected by a single edge (u, v), where u and v are members of a K_r and not adjacent to L-vertices. As in the previous case, it is easy to see that u and v must be $A1$-vertices, and (u, v) must be blue.

We define the $(k, \ell, k-1)$-red-transmitter, G, as the $(H', u, \ell-1)$-thread graph, where u is a member of a K_r not adjacent to any L-vertex. The argument from the previous case shows that the thread-end of G is the strict endpoint of a red P_{k-1}. Moreover, the coloring where each $E(A)$ is colored red and each edge between an L-vertex and an $A1$-vertex is colored blue is a (P_k, P_ℓ)-good coloring.

Case 3 (k is odd and $\ell = k+1$). Note that in this case, all three types of colorings of K_r are allowed. We consider two subcases.

Case 3.i ($k \geq 7$). Consider the graph H where two K_r's share a single vertex, p, and q is a leaf vertex connected to p. Let X and Y refer to the vertex set of each K_r. We first show that p cannot be a Bi-vertex for any $i \in \{1, 2, 3\}$ in both X and Y. We prove via contradiction: assume without loss of generality that p is a Bi-vertex in X. Recall that a Bi-vertex is the endpoint of a blue $P_{\ell-1}$ and red P_{k-2}. Thus, p must be an Aj-vertex in Y, for some $j \in \{1, 2, 3\}$. However, $A1$- and $A2$-vertices are endpoints of red P_{k-3}'s, and $A3$-vertices are endpoints of blue $P_{\ell-2}$'s. If p is an $A1$- or $A2$-vertex, then a red $P_{k-3+k-2-1}$ is formed, which is forbidden when $k \geq 6$, or a blue $P_{\ell-2+\ell-1-1}$ is formed, which is forbidden when $\ell \geq 4$. Thus, p cannot be a Bi-vertex in X or Y. Also note that if p is an $A1$- or $A2$-vertex in both X and Y, then a red $P_{k-3+k-3-1}$ is formed, which is forbidden when $k \geq 7$. Similarly, p cannot be an $A3$-vertex in both X and Y otherwise a blue $P_{\ell-2+\ell-2-1}$ is formed. Thus, p must be an $A1$- or $A2$-vertex in X, and an $A3$-vertex in Y, or vice versa. This implies that p must be the strict endpoint of a red P_{k-3} (or P_{k-2}) and a blue $P_{\ell-2}$.

Let G be the graph constructed as follows. Take two copies of H and contract the vertices labeled q. Then, attach a leaf vertex, v, to the contracted vertex. Let p_1 and p_2 be the vertices in the intersection of two K_r's. Observe that (p_1, q) and (p_2, q) must be different colors, otherwise a red $P_{k-3+k-3+1}$ or a blue $P_{\ell-2+\ell-2+1}$ is formed when both are red or blue, respectively. Assume without loss of generality that (p_1, q) is red and (p_2, q) is blue. Since p_1 and p_2 are the strict endpoints of blue $P_{\ell-2}$'s, q must be the strict endpoint of a blue $P_{\ell-1}$, forcing (q, v) to be red. Observe that (p_1, q) is the strict endpoint of a red P_{k-2} or P_{k-1} depending on whether p_1 is an $A1$ or $A2$-vertex in one of the K_r's. Clearly, p_1 must be an $A1$-vertex, otherwise a red P_k is formed with (q, v).

Therefore, v must be the strict endpoint of a red P_{k-1}. Finally, we describe a good coloring of G: color $E(A)$ red if the K_r has a Type 1 coloring, and blue if it has a Type 3 coloring.

Case 3.ii ($k < 7$). Note that $k = 5$ and $\ell = 6$ is the only possibility in this case. The transmitter for this is shown in Figure 8. Let X, Y, and Z be the vertex sets of the K_r's as in Figure 8. First, note that Z cannot have a Type 1 or Type 2 coloring. We prove via contradiction. Assume that Z has a Type 1 or 2 coloring. Since $k = 5$, $|A| = \lfloor 5/2 \rfloor = 1$ for Type 1 and 2 colorings, at least one vertex in $\{a, t\}$ must be a $B1$- or $B2$-vertex in Z. Assume without loss of generality that a is a $B1$ or $B2$-vertex. Note that a is the strict endpoint of a blue $P_{\ell-1}$ from Z (Observation 1). Thus, each edge connected to a in X must be red. Since a is not adjacent to any blue edge, it can only be an $A1$ or $A2$-vertex in X. Since $|A| = 1$ in Type 1 and 2 colorings, b must be a $B1$ or $B2$-vertex and be the strict endpoint of a blue $P_{\ell-1}$, implying that (b, c) is red. Since a is also a strict endpoint of a red P_{k-2} from Z, the edges (a, b) and (b, c) would form a red P_k. Thus, Z must have a Type 3 coloring. Note that a and t must be $A3$-vertices in Z, otherwise the edges (a, b) and (t, u) would form red P_k's or blue P_ℓ's. a (resp., t) must be an $A1$- or $A2$-vertex in X (resp., Y) because every other type of vertex is the strict endpoint of a blue $P_{\ell-2}$ in X (resp., Y). This would form a blue $P_{\ell-2+\ell-2}$ with the $P_{\ell-2}$ from Z. Since $|A| = 1$ in Type 1 and 2 colorings, b and u must be in $B1$- or $B2$-vertices. X and Y must have Type 1 colorings otherwise (b, c) and (u, v) make red P_k's, on account of $B2$-vertices being strict endpoints of red P_{k-1}'s and the leaf edges being red. Thus, b and u are $A1$-vertices and are strict endpoints of red P_{k-2}'s, making c and v strict endpoints of red P_{k-1}'s. Finally, (a, t) must be colored blue to obtain a good coloring of G. □

We now prove Lemma 4. Since the constructions are similar to that of the previous lemma, some details are skipped since the same arguments can be applied *mutatis mutandis.*

Proof of Lemma 4. When $k = 3$, K_2 is trivially a transmitter. Suppose $k \geq 4$. Let $r = R(P_k, P_k) - 1$. We consider two cases.

Case 1 (k is even). Recall that only Type 1a and 1b colorings are allowed in this case. As in Case 3.i of Lemma 3's proof, consider the graph G where two K_r's share a single vertex, u, and v is a leaf vertex connected to p. Let X and Y refer to the vertex set of each K_r. Note that p cannot be a $B1$-vertex in either K_r; a $B1$-vertex is the strict endpoint of a red and blue P_{k-1} (Observation 2), and this would form a red/blue P_k with (p, q). Thus, q is a Type 1 vertex in X and Y. Also, note that q must be a Type 1a vertex in X and a Type 1b vertex in Y (or vice versa). Otherwise, if both X and Y are of the same type, then a red/blue $P_{k-2+k-2}$ is formed, which is forbidden when $k \geq 4$. Thereby, q is the strict endpoint of a red and blue P_{k-2} in all good colorings of G. Moreover, v is the strict endpoint of a red/blue P_{k-1} depending on the color of (p, v). Thus, G is a $(k, k-1)$-transmitter where v is the strict endpoint of a red/blue P_{k-1} in all good colorings of G. Finally, note that G is (P_k, P_k)-good since all edges in $E(A)$ can be colored red (resp., blue) in the K_r with the Type 1a (resp., Type 1b) coloring.

Case 2 (k is odd). Let H be the graph constructed by attaching a leaf node to $\lfloor k/2 \rfloor - 1$ vertices of a K_r. Consider the graph G, constructed by taking two copies of H, contracting two vertices not adjacent to L-vertices, and attaching a leaf vertex, v, to the contracted vertex, denoted as u. Let X and Y refer to the vertex set of each K_r. As argued in Case 2 of Lemma 3's proof, both K_r's in G must have Type 1 colorings. Moreover, v is the strict endpoint red/blue P_{k-2} from both K_r's. As argued in the even case, X and Y must have different types of colorings to avoid a red/blue P_k going across both H's. Thus, G is $(k, k-1)$-transmitter, and v is the strict endpoint of a red/blue P_{k-1} in all good colorings of G. Finally, note that G is (P_k, P_k)-good using the coloring from the previous case. □

Proofs for Observations 1 and 2

The observations are simple corollaries of the following lemmas.

Lemma 5. *Let G be a graph whose vertices are partitioned into M and N such that (1) $|M| = k$ and $|N| \geq k+1$, (2) $E(M, N)$ includes all possible edges, and (3) N is an independent set. Then, each vertex in N is an endpoint of a P_{2k+1}. Moreover, this is the largest path in G that each $n \in N$ is the endpoint of.*

Proof. Let $m_i \in M$ and $n_i \in N$. We first show that each $n_i \in N$ is the endpoint of P_{2k+1}. Consider the path $n_1, m_1, n_2, \ldots, m_k, n_{k+1}$, which alternates between vertices in M and N. Clearly this path is of size $2k+1$, and must exist because $|M| \geq k+1$ and all edges in $E(M, N)$ exist. It is easy to see that the vertices may be relabeled so that any vertex in N can be the endpoint of this P_{2k+1}. We now show that no path larger than $2k+1$ exists in G, via contradiction. Assume there exists a path Q on $2k+2$ vertices in G. Since $|M| = k$, at least $2k+2-|M| = k+2$ vertices of Q must be in N. Recall that in any P_ℓ there are two vertices of degree one and $\ell - 2$ vertices of degree two, and there are a total of $\ell - 1$ edges. If at least $k+2$ vertices in N are in Q, then $1+1+2k$ edges of Q must be in $E(M, N)$, since N is an independent set. However, this implies that $Q = P_{2k+2}$ has at least $2k+2$ edges, which is a contradiction.

Lemma 6. *Let G be a graph who vertices are partitioned into M and N such that (1) $|M| = k$ and $|N| \geq k$, (2) $E(M, N)$ includes all possible edges, and (3) N is an independent set. Then, each vertex in M is an endpoint of a P_{2k}. Moreover, this is the largest path in G that each $m \in M$ is the endpoint of.*

Proof. Let $m_i \in M$ and $n_i \in N$. We first show that each $m_i \in M$ is the endpoint of P_{2k}. Consider the path $n_1, m_1, n_2, m_2, \ldots, n_k, m_k$, which alternates between vertices in M and N. Clearly this path is of length $2k$, and must exist because $|M| = k$ and all edges in $E(M, N)$ exist. It is easy to see that the vertices may be relabeled so that any vertex in M can be the endpoint of this P_{2k}. We now show that no path larger than $2k$ with an endpoint in M exists in G, via contradiction.

Assume there exists a path Q on $2k+1$ vertices in G, with one endpoint in M. Since $|M| = k$, at least $2k+1-|M| = k+1$ vertices of Q must be in N. Recall that in any P_ℓ there are two vertices of degree one and $\ell - 2$ vertices of degree two, and there are a total of $\ell - 1$ edges. If at least $k+1$ vertices in N are in Q and at most one of them may be an endpoint of Q, then $1+2k$ edges of Q must be in $E(M, N)$, since N is an independent set. However, this implies that $Q = P_{2k+1}$ has at least $2k+1$ edges, which is a contradiction.

Lemma 7. *For all $k \geq 1$, each vertex in K_k is an endpoint of a P_k.*

Proof. The statement is trivially true for $k = 1$. Assume it is true for all $k < n$. For $k = n$, let v be any vertex in $V(K_n)$. By the inductive hypothesis, there must be a P_{n-1} in $V(K_n) \setminus \{v\}$. v must be connected to an endpoint of said P_{n-1}, implying that v is an endpoint of a P_n.

Lemma 8. *For all $k \geq 4$, every vertex in $K_k - e$ is an endpoint of a P_k.*

Proof. Let v and w be the only two vertices in $K_k - e$ that do not share an edge. Let u be any vertex in $V(K_k - e) \setminus \{v, w\}$. Let $V' = V(K_k - e) \setminus \{u, v, w\}$. Since $k \geq 4$, $|V'| \geq 1$ and $G[V']$ is the complete graph K_{k-3}. Clearly, K_{k-3} must have a P_{k-3}. Observe that w, the P_{k-3} in $G[V']$, u, and v form a P_k with endpoints w and v. Moreover, u, v, the P_{k-3}, and w form a P_k with endpoints u and w.

In Type 1/1a/1b and Type 3 colorings, the observations made are simple applications of the lemmas above. For Type 2/2a/2b colorings, it is easy to see that the extra red edge in $E(B)$ increases the length of the red paths in K_r.