How Easy it is to Know How: An Upper Bound for the Satisfiability Problem

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Abstract. We investigate the complexity of the satisfiability problem for a modal logic expressing 'knowing how' assertions, related to an agent's abilities to achieve a certain goal. We take one of the most standard semantics for this kind of logics based on linear plans. Our main result is a proof that checking satisfiability of a 'knowing how' formula can be done in Σ_2^P. The algorithm we present relies on eliminating nested modalities in a formula, and then performing multiple calls to a satisfiability checking oracle for propositional logic.

Keywords: Knowing How · Complexity · Satisfiability.

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1 Introduction

The term 'Epistemic Logic' [15] encompasses a family of logical formalisms aimed at reasoning about the knowledge of autonomous agents about a given scenario. Originally, epistemic logics restricted their attention to so-called *knowing that*, i.e., the capability of agents to know about certain facts. More recently, several logics have been proposed to reason about alternative forms of knowledge (see [32] for a discussion). For instance, *knowing whether* is looked into in [7]; *knowing why* in [34]; and *knowing the value* in [12,3], just to mention a few. Finally, a novel approach focuses on *knowing how* –related to an agent's ability to achieve a goal [8]. This concept is particularly interesting, as it has been argued to provide a fresh way to reason about scenarios involving strategies in AI, such as those found in automated planning (see, e.g., [6]).

The first attempts to capture knowing how were through a combination of 'knowing that' and actions (see, e.g., [25,26,18,14]). However, it has been discussed, e.g., in [16,13], that this idea does not lead to an accurate representation of knowing how. In response, a new logic is presented in [31,33] featuring an original modality specifically tailored to model the concept of 'knowing how'. In a nutshell, an agent knows how to a achieve a goal φ under some initial condition ψ, written $\mathsf{Kh}(\psi, \varphi)$, if and only if there exists a 'proper' plan π, i.e., a finite sequence of actions, that unerringly leads the agent from situations in which ψ holds only to situations in which φ holds. A 'proper' plan is taken as one whose execution never aborts, an idea that takes inspiration from the notion of *strong executability* from contingent planning [29]. As discussed in, e.g., [17,13], the quantification pattern we just described cannot be captured using logics with 'knowing that' modalities and actions. For this reason, the new Kh modality from [31,33] has reached a certain consensus in the community as an accurate way of modelling 'knowing how'. Moreover, it has paved the way to a deep study of knowing how, and to a rich family of logics capturing variants of the initial reading. Some examples of which are a ternary modality of knowing how with intermediate constraints [21]; a knowing how modality with weak plans [19]; a local modality for strategically knowing how [9] (and some relatives, see [28,27]); and, finally, a knowing how modality which considers an epistemic indistinguishability relation among plans [1].

As witnessed by all the ideas it triggered, the foundational work in [31,33] greatly improved the understanding of 'knowing how' from a logical standpoint. The literature on logics of 'knowing how' explores a wide variety of results, such as axiom systems (in most of the works cited above), proof methods [23,20], and expressivity [10], just to name a few. Yet, if we consider 'knowing how' logics as suitable candidates for modelling problems in strategic reasoning, it is important to consider how difficult (or how easy) it is to use these logics for reasoning tasks. There have been some recent developments on the complexity of logics with 'knowing how' modalities. For instance, model-checking for the Kh modality above, and some of its variants, is investigated in [5]. The complexity of model-checking and the decidability status of satisfiability for the local 'knowing how' modality from [9], and some of its generalizations, is explored in [24]. These two problems are also explored for 'knowing how' with epistemic indistinguishability in [1]. Notwithstanding, the complexity of the satisfiability problem for the original Kh modality from [31,33] is still unknown ([22] presents only a decidability statement). In this work, we shed some light into this matter.

Our contribution is to provide an upper for the satisfiability problem of the knowing how logic from [31,33], called here $\mathsf{L_{Kh}}$. More precisely, we introduce an algorithm for deciding satisfiability that is in Σ_2^{P}, the second level of the polynomial hierarchy (PH) [30]. In short, this complexity class can be though as those problems invoking an NP oracle a polynomial number of times, and whose underlying problem is also in NP (see e.g. [2]). Currently, it is unknown whether PH collapses, or it is strictly contained in PSpace. This being said, having an algorithm in a lower level of PH is generally understood as a good indication that the problem is close to, e.g., NP or $\mathsf{Co\text{-}NP}$. It is easy to see that NP is a lower bound for checking satisfiability in $\mathsf{L_{Kh}}$, as it extends propositional logic. For an upper bound, a natural candidate is PSpace, as for instance the model-checking problem for $\mathsf{L_{Kh}}$ is PSpace-complete [5], a potentially higher complexity of what is proved here for satisfiability. We argue that this is due to the fact that in model-checking the full expressivity of the semantics is exploited (specially related to properties of regular languages), whereas for satisfiability, all this expressivity is completely hidden. Although our procedure does not lead to a tight complexity characterization, it gives us an interesting upper bound towards filling this gap.

We put forth that our result is not obvious. To obtain it, we combine techniques such as defining a normal form to eliminate nested modalities, calling an NP oracle to guess propositional valuations and computing a closure over a matrix of formulas to combine them, adapting the Floyd-Warshall algorithm [4].

The article is organized as follows. In Sec. 2 we introduce some notation and the basic definitions of the logic $\mathsf{L_{Kh}}$. Sec. 3 is devoted to incrementally show our result. Finally, in Sec. 4 we provide some remarks and future lines of research.

2 Knowing How Logic

From here onwards, we assume Prop is a denumerable set of *proposition symbols*, and Act is a denumerable set of *action symbols*. We refer to $\pi \in \mathsf{Act}^*$ as a *plan*.

Definition 1. *The* language $\mathsf{L_{Kh}}$ *is determined by the grammar:*

$$\varphi, \psi ::= p \mid \neg\varphi \mid \varphi \vee \psi \mid \mathsf{Kh}(\varphi, \psi),$$

where $p \in \mathsf{Prop}$*. We use* $\bot$*,* $\top$*,* $\varphi \wedge \psi$*,* $\varphi \to \psi$*, and* $\varphi \leftrightarrow \psi$ *as the usual abbreviations;* $\mathsf{A}\varphi$ *is defined as* $\mathsf{Kh}(\neg\varphi, \bot)$ *(see e.g. [31,33]), while* $\mathsf{E}\varphi$ *abbreviates* $\neg\mathsf{A}\neg\varphi$*. The elements of* $\mathsf{L_{Kh}}$ *are* formulas*.*

We read $\mathsf{Kh}(\varphi, \psi)$ as: *"the agent knows how to achieve* ψ *given* φ*"*. We call φ and ψ, the precondition and the postcondition of $\mathsf{Kh}(\varphi, \psi)$, respectively. We read $\mathsf{A}\varphi$ as: *"*φ *holds anywhere"*; and its dual $\mathsf{E}\varphi$ as: *"*φ *holds somewhere"*. As it is usually done, we refer to A and E as the *universal* and *existential* modalities [11].

Formulas of $\mathsf{L_{Kh}}$ are interpreted with respect to *labelled transition systems* over so-called *strongly executable plans*. Sometimes, we refer to LTS as *models*. We introduce their definitions below.

Definition 2. *A* labelled transition system (LTS) *is a tuple* $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V} \rangle$ *s.t.:*

(1) S *is a non-empty set of* states*;*
(2) $\mathrm{R} = \{\mathrm{R}_a \mid a \in \mathsf{Act}\}$ *is a collection of binary relations on* S*; and*
(3) $\mathrm{V} : \mathsf{Prop} \to 2^{\mathrm{S}}$ *is a* valuation function*.*

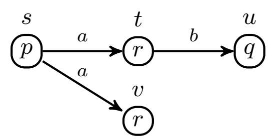
Definition 3. *Let* $\{\mathrm{R}_a \mid a \in \mathsf{Act}\}$ *be a collection of binary relations on* S*. Let* $\varepsilon \in \mathsf{Act}^*$ *be the empty plan. We define:* $\mathrm{R}_\varepsilon = \{(s, s) \mid s \in \mathrm{S}\}$*, and for every* $\pi \in \mathsf{Act}^*$*, and* $a \in \mathsf{Act}$*,* $\mathrm{R}_{\pi a} = \mathrm{R}_\pi \mathrm{R}_a$ *(i.e., their composition). For every relation* R_π*, and* $T \subseteq \mathrm{S}$*, define* $\mathrm{R}_\pi(T) = \{(s, t) \mid s \in T \text{ and } (s, t) \in \mathrm{R}_\pi\}$*, and* $\mathrm{R}_\pi(t) = \mathrm{R}_\pi(\{t\})$*.*

The notion of *strong executability* determines the "adequacy" of a plan. Strong executability takes inspiration from conformant planning [29], and its jusification is discussed at length in [31].

Definition 4. *Let $\pi = a_1 \dots a_n \in \mathsf{Act}^*$, and $1 \leq i \leq j \leq n$, we denote: $\pi_i = a_i$; $\pi[i,j] = a_i \dots a_j$; and $|\pi| = n$. Moreover, let $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V} \rangle$ be an LTS; we say that π is* strongly executable (SE) *at $s \in \mathrm{S}$, iff for all $i \in [1, |\pi| - 1]$ and all $t \in \mathrm{R}_{(\pi[1,i])}(s)$, it follows that $\mathrm{R}_{\pi_{(i+1)}}(t) \neq \emptyset$. The set of all states at which π is strongly executable is defined as $\mathrm{SE}(\pi) = \{s \mid \pi \text{ is SE at } s\}$. Note: $\mathrm{SE}(\varepsilon) = \mathrm{S}$.*

We illustrate the notions we just introduced with a simple example.

Example 1. Let $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V} \rangle$ be the LTS depicted below and $\pi = ab$. We have, $\mathrm{R}_\pi(s) = \{u\}$, and $\mathrm{R}_{\pi[1,1]}(s) = \mathrm{R}_a(s) = \{t, v\}$. It can be seen that $s \in \mathrm{SE}(a)$; while $s \notin \mathrm{SE}(\pi)$ –since $v \in \mathrm{R}_{\pi[1,1]}(s)$ and $\mathrm{R}_{\pi_{(2)}}(v) = \mathrm{R}_b(v) = \emptyset$. Finally, we have that $\mathrm{SE}(\varepsilon) = \mathrm{S}$, $\mathrm{SE}(a) = \{s\}$ and $\mathrm{SE}(ab) = \emptyset$.



We are now ready to introduce the semantics of $\mathsf{L_{Kh}}$, based on [31,33].

Definition 5. *Let $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V} \rangle$ be an* LTS*, we define $[\![\varphi]\!]^{\mathfrak{M}}$ inductively as:*

$$[\![p]\!]^{\mathfrak{M}} = \mathrm{V}(p) \qquad [\![\neg\varphi]\!]^{\mathfrak{M}} = \mathrm{S} \setminus [\![\varphi]\!]^{\mathfrak{M}} \qquad [\![\varphi \vee \psi]\!]^{\mathfrak{M}} = [\![\varphi]\!]^{\mathfrak{M}} \cup [\![\varphi]\!]^{\mathfrak{M}}$$

$$[\![\mathsf{Kh}(\varphi, \psi)]\!]^{\mathfrak{M}} = \begin{cases} \mathrm{S} & \textit{if exists } \pi \in \mathsf{Act}^* \textit{s.t. } [\![\varphi]\!]^{\mathfrak{M}} \subseteq \mathrm{SE}(\pi) \textit{ and } \mathrm{R}_\pi([\![\varphi]\!]^{\mathfrak{M}}) \subseteq [\![\psi]\!]^{\mathfrak{M}} \\ \emptyset & \textit{otherwise.} \end{cases}$$

We say that a plan $\pi \in \mathsf{Act}^*$ is a witness *for $\mathsf{Kh}(\varphi, \psi)$ iff $[\![\varphi]\!]^{\mathfrak{M}} \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi([\![\varphi]\!]^{\mathfrak{M}}) \subseteq [\![\psi]\!]^{\mathfrak{M}}$. We use $([\![\varphi]\!]^{\mathfrak{M}})^{\complement}$ instead of $\mathrm{S} \setminus [\![\varphi]\!]^{\mathfrak{M}}$. We write $\mathfrak{M} \Vdash \varphi$ as an alternative to $[\![\varphi]\!]^{\mathfrak{M}} = \mathrm{S}$; and $\mathfrak{M}, s \Vdash \varphi$ as an alternative to $s \in [\![\varphi]\!]^{\mathfrak{M}}$.*

Example 2. Let $\mathfrak{M}$ be the LTS from Ex. 1. From Def. 5, we have $[\![\mathsf{Kh}(p, r)]\!]^{\mathfrak{M}} = \mathrm{S}$ (using a as a witness), while $[\![\mathsf{Kh}(p, q)]\!]^{\mathfrak{M}} = \emptyset$ (there is no witness for the formula).

We included the universal modality A as abbreviation since formulas of the form $\mathsf{A}\varphi$ play a special role in our treatment of the complexity of the satisfiability problem for $\mathsf{L_{Kh}}$. It is proven in, e.g., [31,33], that $\mathsf{A}\varphi$ and $\mathsf{E}\varphi$ behave as the universal and existential modalities ([11]), respectively. Recall that $\mathsf{A}\varphi$ is defined as $\mathsf{Kh}(\neg\varphi, \bot)$, which semantically states that φ holds everywhere in a model iff $\neg\varphi$ leads always to impossible situations. Formulas of this kind are called here 'global'. Below, we formally restate the results just discussed.

Proposition 1. *Let $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V} \rangle$ and ψ and χ be formulas s.t. $[\![\chi]\!]^{\mathfrak{M}} = \emptyset$; then $[\![\mathsf{Kh}(\psi, \chi)]\!]^{\mathfrak{M}} = \mathrm{S}$ iff $[\![\neg\psi]\!]^{\mathfrak{M}} = \mathrm{S}$.*

Corollary 1. *Let $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V} \rangle$ and a formula φ, $\mathfrak{M}, s \Vdash \mathsf{A}\varphi$ iff $[\![\varphi]\!]^{\mathfrak{M}} = \mathrm{S}$.*

We introduce now Prop. 2, which is of use in the rest of the paper.

Proposition 2. *Let ψ, ψ', χ, χ' and φ be formulas, and $\mathfrak{M}$ an LTS; then:*

(1) $[\![\psi']\!]^{\mathfrak{M}} \subseteq [\![\psi]\!]^{\mathfrak{M}}$ *and* $[\![\chi]\!]^{\mathfrak{M}} \subseteq [\![\chi']\!]^{\mathfrak{M}}$ *implies* $[\![\mathsf{Kh}(\psi,\chi)]\!]^{\mathfrak{M}} \subseteq [\![\mathsf{Kh}(\psi',\chi')]\!]^{\mathfrak{M}}$*;*
(2) $[\![\psi]\!]^{\mathfrak{M}} \subseteq [\![\psi']\!]^{\mathfrak{M}}$ *implies* $([\![\mathsf{Kh}(\varphi,\psi)]\!]^{\mathfrak{M}} \cap [\![\mathsf{Kh}(\psi',\chi)]\!]^{\mathfrak{M}}) \subseteq [\![\mathsf{Kh}(\varphi,\chi)]\!]^{\mathfrak{M}}$*.*

We conclude this section with some useful definitions.

Definition 6. *A formula φ is* satisfiable*, written $\mathsf{Sat}(\varphi)$, iff there is $\mathfrak{M}$ s.t. $[\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset$. A finite set $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ of formulas is* satisfiable*, written $\mathsf{Sat}(\Phi)$, iff $\mathsf{Sat}(\varphi_1 \wedge \cdots \wedge \varphi_n)$. For convenience, we define $\mathsf{Sat}(\emptyset)$ as true. We use $\mathsf{Unsat}(\varphi)$ iff $\mathsf{Sat}(\varphi)$ is false; similarly, $\mathsf{Unsat}(\Phi)$ iff $\mathsf{Sat}(\Phi)$ is false. Finally, whenever $\mathsf{Sat}(\varphi)$ iff $\mathsf{Sat}(\varphi')$, we call φ and φ'* equisatisfiable*, and write $\varphi \equiv_{\mathsf{Sat}} \varphi'$.*

Definition 7. *The modal depth of a formula φ, written $\mathsf{md}(\varphi)$, is defined as:*

$$\mathsf{md}(\varphi) = \begin{cases} 0 & \text{if } \varphi \in \mathsf{Prop} \\ \mathsf{md}(\psi) & \text{if } \varphi = \neg\psi \\ \max(\mathsf{md}(\psi), \mathsf{md}(\chi)) & \text{if } \varphi = \psi \vee \chi \\ 1 + \max(\mathsf{md}(\psi), \mathsf{md}(\chi)) & \text{if } \varphi = \mathsf{Kh}(\psi, \chi). \end{cases}$$

We use $\mathsf{sf}(\varphi)$ to indicate the set of subformulas of φ. We say that $\mathsf{Kh}(\psi,\chi)$ is a leaf of φ iff $\mathsf{Kh}(\psi,\chi) \in \mathsf{sf}(\varphi)$ and $\mathsf{md}(\psi) = \mathsf{md}(\chi) = 0$ (i.e., $\mathsf{md}(\mathsf{Kh}(\psi,\chi) = 1)$).

In words, the modal depth of a formula is the length of the longest sequence of nested modalities in the formula; whereas a leaf is a subformula of depth one. Notice that, since $\mathsf{A}\varphi$ is a shortcut for $\mathsf{Kh}(\neg\varphi, \bot)$, we have $\mathsf{md}(\mathsf{A}\varphi) = 1 + \mathsf{md}(\varphi)$.

Example 3. Let $\varphi = \mathsf{Kh}(p, \mathsf{Kh}(\neg q, p \to q)) \vee \mathsf{Kh}(r, t)$; it can easily be checked that $\mathsf{md}(\varphi) = 2$ and that $\mathsf{Kh}(\neg q, p \to q)$ and $\mathsf{Kh}(r,t)$ are its modal leaves.

3 An Upper Bound for the Satisfiability Problem of L_{Kh}

In this section we establish an upper bound on the complexity of the satisfiability problem for L_{Kh}, which is the main result of our paper. We start with some preliminary definitions and results.

Proposition 3. *Let φ' be the result of replacing all occurrences of a leaf θ in φ by a proposition symbol $k \notin \mathsf{sf}(\varphi)$; it follows that $\varphi \equiv_{\mathsf{Sat}} (\varphi' \wedge (\mathsf{A}k \leftrightarrow \theta))$.*

We say that φ is in *leaf normal form* iff $\mathsf{md}(\varphi) \leq 1$. Prop. 4 tells us that we can put any formula into an equisatisfiable formula in leaf normal form. The function Flatten in Alg. 1 tells us how to do this in polynomial time.

Proposition 4. *Alg. 1 is in P; on input φ, it outputs φ_0 and φ_1 such that $\mathsf{md}(\varphi_0) = 0$, $\mathsf{md}(\varphi_1) = 1$, and $\varphi \equiv_{\mathsf{Sat}} (\varphi_0 \wedge \varphi_1)$.*

The result in Prop. 4 allows us to think of the complexity of the satisfiability problem for L_{Kh} by restricting our attention to formulas in leaf normal form. In turn, this enables us to work towards a solution in terms of subproblems. More precisely, given φ_0 and φ_1 in the leaf normal form that results from Flatten, the

Algorithm 1 Flatten

require: true

function FLATTEN(φ)
  $\varphi_0, \varphi_1 \leftarrow \varphi, \top$
  loop ▷ ***invariant:*** *$\varphi \equiv_{\mathsf{Sat}} (\varphi_0 \wedge \varphi_1)$ (see Prop. 3)*
    $\Theta \leftarrow$ the set of leaves of φ_0
    if $\Theta = \emptyset$ **then break end if** ▷ *loop guard*
    for all $\theta \in \Theta$ **do**
      $k \leftarrow$ a proposition symbol not in $\mathsf{sf}(\varphi_0 \wedge \varphi_1)$
      $\varphi_0 \leftarrow$ result of replacing all occurrences of θ in φ_0 for k
      $\varphi_1 \leftarrow \varphi_1 \wedge (\mathsf{A}k \leftrightarrow \theta)$
  return $\varphi_0 \wedge \varphi_1$

ensure: $\mathsf{md}(\varphi_0) = 0$ and $\mathsf{md}(\varphi_1) = 1$ and $\varphi \equiv_{\mathsf{Sat}} (\varphi_0 \wedge \varphi_1)$

subproblems are (i) determining the satisfiability of φ_0; and (ii) determining the satisfiability of φ_1 based on a solution to (i). The solution to (i) is well-known, φ_0 is a propositional formula. We split the solution of (ii) into (a) determining when formulas of the form $\mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$ are satisfiable, see Prop. 5; (b) determining when formulas of the form $\neg\mathsf{Kh}(\psi'_1, \chi'_1) \wedge \cdots \wedge \neg\mathsf{Kh}(\psi'_m, \chi'_m)$ are satisfiable, see Prop. 7; and (c) combining (a) and (b), see Prop. 11. We present (a), (b), and (c), in a way such that they incrementally lead to a solution to the satisfiability problem for L_{Kh}. Finally, in Prop. 12, we show how to combine (i) and (ii) to obtain an upper bound on the complexity of this problem.

Let us start by solving the first problem: checking whether a conjunction φ of positive formulas in leaf normal form are satisfiable altogether. In a nutshell, we show that solving this problem boils down to building a set I of the preconditions of those subformulas whose postconditions are falsified in the context of φ, and checking whether the formulas in I are satisfiable or not. Intuitively, the formulas in I correspond to 'global' formulas. We made precise these ideas in Prop. 5.

Proposition 5. *Let $\varphi = \mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$ be such that $\mathsf{md}(\varphi) = 1$; and let the sets $I_0, \ldots, I_n$ be defined as follows:*

$$I_i = \begin{cases} \{k \in [1,n] \mid \mathsf{Unsat}(\chi_k)\} & \text{if } i = 0, \\ I_{(i-1)} \cup \{k \in [1,n] \mid \mathsf{Unsat}(\{\neg\psi_{k'} \mid k' \in I_{(i-1)}\} \cup \{\chi_k\})\} & \text{if } i > 0, \end{cases}$$

where $i \in [0,n]$; further, let $I = I_n$. Then: (1) $\mathsf{Sat}(\varphi)$ iff (2) $\mathsf{Sat}(\bigwedge_{i \in I} \neg\psi_i)$.

Proof. ($\Rightarrow$) Suppose that $\mathsf{Sat}(\varphi)$ holds, i.e., exists $\mathfrak{M}$ s.t. $[\![\varphi]\!]^{\mathfrak{M}} = \mathrm{S}$. From this assumption, we know that, for every $j \in [1,n]$, $[\![\mathsf{Kh}(\psi_i, \chi_i)]\!]^{\mathfrak{M}} = \mathrm{S}$. The proof is concluded if $[\![\bigwedge_{i \in I} \neg\psi_i]\!]^{\mathfrak{M}} \neq \emptyset$. We obtain this last result with the help of the following auxiliary lemma:

$$(*) \text{ for all } k \in I_i,\ [\![\chi_k]\!]^{\mathfrak{M}} = \emptyset \text{ and } [\![\neg\psi_k]\!]^{\mathfrak{M}} = \mathrm{S}$$

The lemma is obtained by induction on the construction of I_i. The base case is direct. Let $k \in I_0$; from the definition of I_0, we get $\mathsf{Unsat}(\chi_k)$; this implies $[\![\chi_k]\!]^{\mathfrak{M}} = \emptyset$; which implies $\mathrm{S} = [\![\mathsf{Kh}(\psi_k, \chi_k)]\!]^{\mathfrak{M}} = [\![\mathsf{A}\neg\psi_k]\!]^{\mathfrak{M}} = [\![\neg\psi_k]\!]^{\mathfrak{M}}$. For the inductive step, let $k \in I_{(i+1)} \setminus I_i$. From the Inductive Hypothesis, for all $k' \in I_i$, $[\![\chi_{k'}]\!]^{\mathfrak{M}} = \emptyset$ and $[\![\neg\psi_{k'}]\!]^{\mathfrak{M}} = \mathrm{S}$. This implies ($\dagger$) $[\![\bigwedge_{k' \in I_i} \neg\psi_{k'}]\!]^{\mathfrak{M}} = \mathrm{S}$.

From the definition of $I_{(i+1)}$, $\mathsf{Unsat}(\{\neg\psi_{k'} \mid k' \in I_i\} \cup \{\chi_k\})$. This is equivalent to $[\![\bigwedge_{k' \in I_i} \neg\psi_{k'}]\!]^{\mathfrak{M}} \subseteq [\![\neg\chi_k]\!]^{\mathfrak{M}}$. From (†), $\mathrm{S} \subseteq [\![\neg\chi_k]\!]^{\mathfrak{M}} = \mathrm{S}$. Thus, $[\![\chi_k]\!]^{\mathfrak{M}} = \emptyset$ and $[\![\neg\psi_k]\!]^{\mathfrak{M}} = \mathrm{S}$. Since $I = I_n$; using (∗) we get $[\![\bigwedge_{i \in I} \neg\psi_i]\!]^{\mathfrak{M}} = \mathrm{S} \neq \emptyset$. This proves (2).

(⇐) The proof is by contradiction. Suppose (2) and $\mathsf{Unsat}(\varphi)$. Then, for all $\mathfrak{M}$, (†) $[\![\varphi]\!]^{\mathfrak{M}} = \emptyset$. Let $J = \{j \in [1,n] \mid \mathsf{Unsat}(\{(\bigwedge_{i \in I} \neg\psi_i), \psi_j\})\}$. Moreover, let $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}\rangle$ be s.t. S is the smallest set containing all valuations that make $(\bigwedge_{i \in I} \neg\psi_i)$ true. From (2), we know that $\mathrm{S} \neq \emptyset$ and $[\![\neg\psi_k]\!]^{\mathfrak{M}} = \mathrm{S}$ for all $k \in I$. By induction on the construction of $I = I_n$, we get that $[\![\chi_k]\!]^{\mathfrak{M}} = \emptyset$ for all $k \in I = \bigcup_{i=0}^{n} I_i$. The case for $k \in I_0$ is direct since $\mathsf{Unsat}(\chi_k)$, thus $[\![\chi_k]\!]^{\mathfrak{M}} = \emptyset$. For the inductive case, let $k \in I_i \setminus I_{i-1}$, then $\mathsf{Unsat}(\{\neg\psi_{k'} \mid k' \in I_{(i-1)}\} \cup \{\chi_k\})$. This is equivalent to say that the implication $((\bigwedge_{k' \in I_{(i-1)}} \neg\psi_{k'}) \to \neg\chi_k)$ is valid. Thus, $[\![\bigwedge_{k' \in I_{(i-1)}} \neg\psi_{k'}]\!]^{\mathfrak{M}} \subseteq [\![\neg\chi_k]\!]^{\mathfrak{M}}$. By hypothesis, $[\![\bigwedge_{k' \in I} \neg\psi_{k'}]\!]^{\mathfrak{M}} = \mathrm{S}$. Thus, $[\![\bigwedge_{k' \in I_{(i-1)}} \neg\psi_{k'}]\!]^{\mathfrak{M}} = \mathrm{S}$, and we get $[\![\neg\chi_k]\!]^{\mathfrak{M}} = \mathrm{S}$ and $[\![\chi_k]\!]^{\mathfrak{M}} = \emptyset$. In turn, for all $k \in J$, since $\mathsf{Unsat}(\{(\bigwedge_{i \in I} \neg\psi_i), \psi_k\})$ and $[\![\bigwedge_{i \in I} \neg\psi_i]\!]^{\mathfrak{M}} = \mathrm{S}$ we can conclude that $[\![\psi_k]\!]^{\mathfrak{M}} = \emptyset$. Thus, we have that $[\![\mathsf{Kh}(\psi_k, \chi_k)]\!]^{\mathfrak{M}} = [\![\mathsf{A}\neg\psi_k]\!]^{\mathfrak{M}} = \mathrm{S}$, for all $k \in I \cup J$. Then, from (†), exists $K = \{k \mid [\![\mathsf{Kh}(\psi_k, \chi_k)]\!]^{\mathfrak{M}} = \emptyset\}$ s.t. $\emptyset \subset K \subseteq [1,n] \setminus (I \cup J)$. For all $k \in K$, $[\![\psi_k]\!]^{\mathfrak{M}} \neq \emptyset$ since $\mathsf{Sat}(\{(\bigwedge_{i \in I} \neg\psi_i), \psi_k\})$; and $[\![\chi_k]\!]^{\mathfrak{M}} \neq \emptyset$ since $\mathsf{Sat}(\{\neg\psi_{k'} \mid k' \in I_{(i-1)}\} \cup \{\chi_k\})$ for all $i \geq 0$, even $I_{(i-1)} = I_n = I$. Without loss of generality, let $K = [1,m]$ and $\mathfrak{M}' = \langle \mathrm{S}, \mathrm{R}', \mathrm{V}\rangle$ be s.t. $\mathrm{R}' = \{\mathrm{R}'_{a_j} \mid a_j \in \mathsf{Act}\}$, where:

$$\mathrm{R}'_{a_j} = \begin{cases} [\![\psi_j]\!]^{\mathfrak{M}'} \times [\![\chi_j]\!]^{\mathfrak{M}'} & \text{if } j \in K \\ \mathrm{R}_{a_{(j-m)}} & \text{if } j \notin K. \end{cases}$$

In the definition of R', it is worth noticing that since $j \notin K$, $\mathrm{R}_{a_{(j-m)}}$ is defined, i.e., $\mathrm{R}_{a_{(j-m)}} \in \mathrm{R}$. Then clearly, for all $k \in K$, $[\![\mathsf{Kh}(\psi_k, \chi_k)]\!]^{\mathfrak{M}'} = \mathrm{S}$. The claim is that for all $k' \in I \cup J$, $[\![\mathsf{Kh}(\psi_{k'}, \chi_{k'})]\!]^{\mathfrak{M}'} = \mathrm{S}$. To prove this claim, consider a function $\sigma : \mathsf{Act}^* \to \mathsf{Act}^*$ s.t. $\sigma(\varepsilon) = \varepsilon$, and $\sigma(a_k\alpha) = a_{(k+m)}\sigma(\alpha)$. For all $\pi \in \mathsf{Act}^*$, if $[\![\psi_{k'}]\!]^{\mathfrak{M}} \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi([\![\psi_{k'}]\!]^{\mathfrak{M}}) \subseteq [\![\chi_{k'}]\!]^{\mathfrak{M}}$, then $[\![\psi_{k'}]\!]^{\mathfrak{M}'} \subseteq \mathrm{SE}(\sigma(\pi))$ and $\mathrm{R}_{\sigma(\pi)}([\![\psi_{k'}]\!]^{\mathfrak{M}'}) \subseteq [\![\chi_{k'}]\!]^{\mathfrak{M}'}$ –since the valuation functions for $\mathfrak{M}$ and $\mathfrak{M}'$ coincide, the truth sets in $\mathfrak{M}$ and $\mathfrak{M}'$ coincide for formulas with no modalities. Then, $[\![\mathsf{Kh}(\psi_{k'}, \chi_{k'})]\!]^{\mathfrak{M}'} = \mathrm{S}$. But we had assumed $\mathsf{Unsat}(\varphi)$. Thus, (1) follows.

The following example illustrates the result in Prop. 5.

Example 4. Let $\varphi = \mathsf{Kh}(p, \bot) \wedge \mathsf{Kh}(q, p)$, i.e., $\psi_1 = p$, $\psi_2 = q$, $\chi_1 = \bot$ and $\chi_2 = p$. It is clear that $\mathsf{Sat}(\varphi)$. Let us build the sets I_0, I_1 and I_2:

- $I_0 = \{1\}$, as $\mathsf{Unsat}(\chi_1)$ and $\mathsf{Sat}(\chi_2)$ hold;
- $I_1 = \{1, 2\}$, since it holds $\mathsf{Unsat}(\{\neg\psi_1, \chi_2\})$;
- $I_2 = \{1, 2\} = I$, as I_1 already contains all the indices in $[1, 2]$.

Thus (as it can be easily checked) we get $\mathsf{Sat}(\{\neg\psi_1, \neg\psi_2\})$ (i.e., $\mathsf{Sat}(\{\neg p, \neg q\})$).

Algorithm 2 $\textsc{Sat}^{+}_{\mathsf{Kh}}$

require: $\varphi = \mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$ and $\mathsf{md}(\varphi) = 1$
function $\textsc{Global}(\varphi)$
  $I, \Psi \leftarrow \emptyset, \emptyset$
  for $i \leftarrow 0$ **to** n **do**
    $K \leftarrow \emptyset$
    for $k \leftarrow 1$ **to** n **do**
      if not $\mathsf{Sat}(\Psi \cup \{\chi_k\})$ **then** $K \leftarrow K \cup \{k\}$ **end if**
    $I \leftarrow I \cup K$
    $\Psi \leftarrow \Psi \cup \{\neg\psi_k \mid k \in K\}$
  return I
ensure: $\textsc{Global}(\varphi) = I_0 \cup \cdots \cup I_n$ where I_i is as in Prop. 5

require: $\varphi = \mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$ and $\mathsf{md}(\varphi) = 1$
function $\textsc{Sat}^{+}_{\mathsf{Kh}}(\varphi)$
  return $\mathsf{Sat}(\{\neg\psi_i \mid i \in \textsc{Global}(\varphi)\})$
ensure: $\textsc{Sat}^{+}_{\mathsf{Kh}}(\varphi)$ iff $\mathsf{Sat}(\varphi)$

Interestingly, the result in Prop. 5 tells us that the satisfiability of a formula $\mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$ depends solely on the joint satisfiability of its 'global' subformulas (cf. Prop. 1); i.e., subformulas $\mathsf{Kh}(\psi_i, \chi_i)$ whose postconditions χ_i are falsified in the context of φ. The satisfiability of the 'global' subformulas provides us with the universe, i.e., set of states, on which to build the plans that witness those formulas that are not in I, and that are not 'trivially' true as a result of their preconditions being falsified in this universe.

Building on Prop. 5, the function $\textsc{Sat}^{+}_{\mathsf{Kh}}$ in Alg. 2 gives us a way of checking whether a formula $\varphi = \mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$ is satisfiable. The algorithm behind this function makes use of a (propositional) Sat oracle, and the function $\textsc{Global}$. The Sat oracle tests for pre and postconditions of Kh formulas, as these are propositional formulas. Intuitively, $\textsc{Global}$ iteratively computes the indices in the sets I_i in Prop. 5, each of them corresponding to the 'global' subformulas of the input. Once this is done, $\textsc{Sat}^{+}_{\mathsf{Kh}}$ checks the joint satisfiability of the negation of the preconditions of 'global' subformulas.

Proposition 6. *Let φ be as in Prop. 5; Alg. 2 solves $\mathsf{Sat}(\varphi)$.*

Let us now move to determining the satisfiability conditions of a formula $\neg\mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \neg\mathsf{Kh}(\psi_n, \chi_n)$ in leaf normal form. Prop. 7 establishes that, for any such a formula, it is enough to check whether each conjunct $\psi_i \wedge \neg\chi_i$ is individually satisfiable. Note that this satisfiability check is purely propositional.

Proposition 7. *Let $\varphi = \neg\mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \neg\mathsf{Kh}(\psi_n, \chi_n)$ be s.t. $\mathsf{md}(\varphi) = 1$; it follows that $\mathsf{Sat}(\varphi)$ iff for all $i \in [1, n]$, $\mathsf{Sat}(\psi_i \wedge \neg\chi_i)$.*

Proof. ($\Rightarrow$) The proof is by contradiction. Suppose that (†) $\mathsf{Sat}(\varphi)$ and for some $i \in [1, n]$ we have (‡) $\mathsf{Unsat}(\psi_i \wedge \neg\chi_i)$. Let $\mathfrak{M}$ be a model such that $[\![\varphi]\!]^{\mathfrak{M}} \neq \emptyset$, which exists by (†). Then, $[\![\mathsf{Kh}(\psi_i, \chi_i)]\!]^{\mathfrak{M}} = \emptyset$. From this, we get $[\![\psi_i]\!]^{\mathfrak{M}} \neq \emptyset$; otherwise $[\![\mathsf{Kh}(\psi_i, \chi_i)]\!]^{\mathfrak{M}} = \mathrm{S}$. From (‡), we know that $[\![\psi_i]\!]^{\mathfrak{M}} \subseteq [\![\chi_i]\!]^{\mathfrak{M}}$. Since $\varepsilon \in \mathsf{Act}^*$, we have $[\![\psi_i]\!]^{\mathfrak{M}} \subseteq \mathrm{SE}(\varepsilon) = \mathrm{S}$ and $[\![\psi_i]\!]^{\mathfrak{M}} = \mathrm{R}_\varepsilon([\![\psi_i]\!]^{\mathfrak{M}}) \subseteq [\![\chi_i]\!]^{\mathfrak{M}}$. But this means $[\![\mathsf{Kh}(\psi_i, \chi_i)]\!]^{\mathfrak{M}} = \mathrm{S}$; which is a contradiction. Thus, $\mathrm{R}_\varepsilon[\![\psi_i]\!]^{\mathfrak{M}} \not\subseteq [\![\chi_i]\!]^{\mathfrak{M}}$; i.e., $[\![\psi_i]\!]^{\mathfrak{M}} \not\subseteq [\![\chi_i]\!]^{\mathfrak{M}}$. This means $[\![\psi_i \wedge \neg\chi_i]\!]^{\mathfrak{M}} \neq \emptyset$. This establishes $\mathsf{Sat}(\psi_i \wedge \neg\chi_i)$.

Algorithm 3 $\textsc{Sat}^{-}_{\mathsf{Kh}}$

require: $\varphi = \neg\mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \neg\mathsf{Kh}(\psi_n, \chi_n)$ and $\mathsf{md}(\varphi) = 1$

```
function SAT⁻_Kh(φ)
    r ← ⊤
    for i ← 1 to n do
        r ← r and Sat(ψ_i ∧ ¬χ_i)
    return r
```

ensure: $\textsc{Sat}^{-}_{\mathsf{Kh}}(\varphi)$ iff $\mathsf{Sat}(\varphi)$

($\Leftarrow$) Suppose that (†) for all $i \in [1, n]$, $\mathsf{Sat}(\psi_i \wedge \neg\chi_i)$. Let $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V} \rangle$ where: (‡) S is s.t. for all i, $[\![\psi_i \wedge \neg\chi_i]\!]^{\mathfrak{M}} \neq \emptyset$; and (§) for all $\mathrm{R}_a \in \mathrm{R}$, $\mathrm{R}_a = \emptyset$. From (†), we know that at least one S exists, as every ψ_i and χ_i are propositional; thus, each satisfiable conjunction can be sent to a different $s \in \mathrm{S}$. From (§), we know for all $\pi \in \mathsf{Act}^*$, $\mathrm{SE}(\pi) \neq \emptyset$ iff $\pi = \varepsilon$. From (‡) and (§), we know that $[\![\psi_i]\!]^{\mathfrak{M}} = \mathrm{R}_\varepsilon[\![\psi_i]\!]^{\mathfrak{M}} \not\subseteq [\![\chi_i]\!]^{\mathfrak{M}}$. This means that $[\![\mathsf{Kh}(\psi_i, \chi_i)]\!]^{\mathfrak{M}} = \emptyset$, for all $i \in [1, n]$. Hence $[\![\varphi]\!]^{\mathfrak{M}} = \mathrm{S}$ which implies $\mathsf{Sat}(\varphi)$.

The key idea behind Prop. 7 is to build a discrete universe to force the only possible witness of a formula of the form $\mathsf{Kh}(\psi_i, \chi_i)$ to be the empty plan. If in this discrete universe we always have at hand a state which satisfies $\psi_i \wedge \neg\chi_i$, then, the empty plan cannot be a witness for $\mathsf{Kh}(\psi_i, \chi_i)$. If the latter is the case, then the satisfiability of $\neg\mathsf{Kh}(\psi_i, \chi_i)$ is ensured. Building on this result, we define, in Alg. 3, a function $\textsc{Sat}^{-}_{\mathsf{Kh}}$ to check the satisfiability of a formula $\neg\mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \neg\mathsf{Kh}(\psi_n, \chi_n)$ in leaf normal form. The function proceeds by traversing each subformula $\mathsf{Kh}(\psi_i, \chi_i)$ and checking the satisfiability of $\psi_i \wedge \neg\chi_i$.

Proposition 8. *Let φ be as in Prop. 7; Alg. 3 solves* $\mathsf{Sat}(\varphi)$.

We are now ready to extend the results in Props. 5 and 7 to work out the joint satisfiability of a formula of the form $\varphi^+ = \mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$, and a formula of the form $\varphi^- = \neg\mathsf{Kh}(\psi'_1, \chi'_1) \wedge \cdots \wedge \neg\mathsf{Kh}(\psi'_m, \chi'_m)$, both in leaf normal form. The main difficulty is how to "build" witnesses for the subformulas $\mathsf{Kh}(\psi_i, \chi_i)$ of φ^+ in a way such that they do not yield witnesses for the subformulas $\neg\mathsf{Kh}(\psi'_j, \chi'_j)$ of φ^-. We show that the key to the solution hinges on "composition". We start with a preliminary definition.

Definition 8. *Let $\varphi = \mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$ and ψ be a formula; we define $\Pi(\varphi, \psi) = \bigcup_{i \geq 0} \Pi_i$ where:*

$$\Pi_0 = \{(x, x) \mid x \in [1, n]\}$$
$$\Pi_{(i+1)} = \Pi_i \cup \{(x, z) \mid (x, y) \in \Pi_i \text{ , } z \in [1, n], \text{ and } \mathsf{Unsat}(\{\psi, \chi_y, \neg\psi_z\})\}.$$

In words, $\Pi(\varphi, \psi)$ captures the notion of composition of formulas $\mathsf{Kh}(\psi, \chi)$ and $\mathsf{Kh}(\psi', \chi')$ into a formula $\mathsf{Kh}(\psi, \chi')$. This composition is best explained by recalling the validity of $(\mathsf{Kh}(\psi, \chi) \wedge \mathsf{A}(\chi \to \psi') \wedge \mathsf{Kh}(\psi', \chi')) \to \mathsf{Kh}(\psi, \chi')$ (see, e.g. [31,33]). The definition of $\Pi(\varphi, \psi)$ records the conjuncts of φ which can be composed in this sense. Below, we list some properties of $\Pi(\varphi, \psi)$.

Algorithm 4 PLANS

require: φ^+ is as in Def. 9
function PLANS(φ^+, ψ)
  $\Pi \leftarrow$ FALSE(n, n) ▷ *Π is an $n \times n$ matrix whose entries are all set to $\bot$ (false)*
  for $x \leftarrow 1$ **to** n **do**
    $\Pi(x, x) \leftarrow \top$
  for $x \leftarrow 1$ **to** n **do** ▷ *compute all $(x, z) \in \Pi(\varphi^+, \psi)$*
    for $z \leftarrow 1$ **to** n **do**
      for $y_0 \leftarrow 1$ **to** n **do**
        for $y_1 \leftarrow 1$ **to** n **do**
          $\Pi(x, z) \leftarrow \Pi(x, z)$ or $(\Pi(x, y_0)$ and $\Pi(y_1, z)$ and not $\mathsf{Sat}(\{\psi, \chi_{y_0}, \neg\psi_{y_1}\}))$
  return $\{(x, y) \mid \Pi(x, y) = \top\}$
ensure: PLANS$(\varphi^+, \psi) = \Pi(\varphi^+, \psi)$

Proposition 9. *Let φ and ψ be as in Def. 8; if $(x, y) \in \Pi(\varphi, \psi)$, then, for any model $\mathfrak{M}$, it holds $[\![\varphi \wedge \mathsf{A}\psi]\!]^{\mathfrak{M}} \subseteq [\![\mathsf{Kh}(\psi_x, \chi_y)]\!]^{\mathfrak{M}}$.*

Proof. We start by stating and proving an auxiliary lemma: $(*)$ $(x, y) \in \Pi_i$ iff there is a non-empty sequence π of indices in $[1, n]$ s.t.:

(†) $x = \pi_1$ and $y = \pi_{|\pi|}$; and
(‡) for all $j \in [1, |\pi| - 1]$, $\mathsf{Unsat}(\{\psi, \chi_{\pi_j}, \neg\psi_{\pi_{(j+1)}}\})$.

The proof of this lemma is by induction on i. The base case for $(*)$ is $i = 0$. We know that $(x, x) \in \Pi_0$, the sequence containing just x satisfies (†) and (‡). Conversely, we know that any sequence π of indices in $[1, n]$ s.t. $|\pi| = 1$ satisfies (†) and (‡); it is immediate that $(\pi_1, \pi_1) \in \Pi_0$. This proves the base case. For the inductive step, let $(x, z) \in \Pi_{(i+1)}$, $(x, y) \in \Pi_i$, $z \in [1, n]$, and $\mathsf{Unsat}(\{\psi, \chi_y, \neg\psi_z\})$. From the Inductive Hypothesis, there is π that satisfies (†) and (‡). Immediately, $\pi' = \pi z$ also satisfies (†) and (‡).

It is easy to see that, if there is π satisfying (†) and (‡), then, (§) for every model $\mathfrak{M}$ and $j \in [1, |\pi| - 1]$, $[\![\mathsf{A}\psi]\!]^{\mathfrak{M}} = \mathrm{S}$ implies $[\![\chi_{\pi_j}]\!]^{\mathfrak{M}} \subseteq [\![\psi_{\pi_{(j+1)}}]\!]^{\mathfrak{M}}$.

Let us now resume with the main proof. Let $(x, y) \in \Pi(\varphi, \psi)$ and $\mathfrak{M}$ be any model. The result is direct if $[\![\varphi \wedge \mathsf{A}\psi]\!]^{\mathfrak{M}} = \emptyset$. Thus, consider $[\![\varphi \wedge \mathsf{A}\psi]\!]^{\mathfrak{M}} \neq \emptyset$; i.e., s.t. $[\![\varphi \wedge \mathsf{A}\psi]\!]^{\mathfrak{M}} = \mathrm{S}$. From $(*)$, we know that exists a sequence π of indices in $[1, n]$ that satisfies (†) and (‡). Then, for all $j \in [1, |\pi| - 1]$, $[\![\chi_{\pi_j}]\!]^{\mathfrak{M}} \subseteq [\![\psi_{\pi_{(j+1)}}]\!]^{\mathfrak{M}}$. Using Prop. 3, $[\![\varphi \wedge \mathsf{A}\psi]\!]^{\mathfrak{M}} \subseteq \bigcap_{j=1}^{|\pi|} [\![\mathsf{Kh}(\psi_{\pi_j}, \chi_{\pi_j})]\!]^{\mathfrak{M}} \subseteq [\![\mathsf{Kh}(\psi_x, \chi_y)]\!]^{\mathfrak{M}}$.

Proposition 10. *Let $\varphi = \mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$ and ψ be a formula; $\Pi(\varphi, \psi)$ is the smallest set s.t.: (1) for all $x \in [1, n]$, $(x, x) \in \Pi(\varphi, \psi)$; and (2) if $\{(x, y_0), (y_1, z)\} \subseteq \Pi(\varphi, \psi)$ and $\mathsf{Unsat}(\{\psi, \chi_{y_0}, \neg\psi_{y_1}\})$, then, $(x, z) \in \Pi(\varphi, \psi)$.*

The function PLANS in Alg. 4 can be used to compute the set $\Pi(\varphi, \psi)$ in Def. 8. This function looks into whether a pair of indices belongs to this set using the result in Prop. 10.

Example 5. Let $\varphi = \mathsf{Kh}(p, p \wedge q) \wedge \mathsf{Kh}(q, r) \wedge \mathsf{Kh}(r \vee s, t)$ and $\psi = \top$; in this case we have: $\psi_1 = p$, $\chi_1 = p \wedge q$, $\psi_2 = q$, $\chi_2 = r$, $\psi_3 = r \vee s$, and $\chi_3 = t$. We can easily verify that $\Pi(\varphi, \psi) = \{(1, 1)(1, 2)(1, 3)(2, 2)(2, 3)(3, 3)\}$. Indeed, in the initial

	χ_1	χ_2	χ_3
ψ_1	$\top$	$\bot$	$\bot$
ψ_2	$\bot$	$\top$	$\bot$
ψ_3	$\bot$	$\bot$	$\top$

initial step

	χ_1	χ_2	χ_3
ψ_1	$\top$	$\top$	$\bot$
ψ_2	$\bot$	$\top$	$\bot$
ψ_3	$\bot$	$\bot$	$\top$

$x = 1$, $y_0 = 1$
$z = 2$, $y_1 = 2$

	χ_1	χ_2	χ_3
ψ_1	$\top$	$\top$	$\top$
ψ_2	$\bot$	$\top$	$\bot$
ψ_3	$\bot$	$\bot$	$\top$

$x = 1$, $y_0 = 2$
$z = 3$, $y_1 = 3$

	χ_1	χ_2	χ_3
ψ_1	$\top$	$\top$	$\top$
ψ_2	$\bot$	$\top$	$\top$
ψ_3	$\bot$	$\bot$	$\top$

$x = 2$, $y_0 = 2$
$z = 3$, $y_1 = 3$

Fig. 1: A Run of Plans for $\varphi = \mathsf{Kh}(p, p \wedge q) \wedge \mathsf{Kh}(q, r) \wedge \mathsf{Kh}(r \vee s, t)$ and $\psi = \top$.

step we get $\Pi_0 = \{(1,1)(2,2)(3,3)\}$. The pairs of indices correspond to those of the pre/post conditions of the subformulas $\mathsf{Kh}(\psi_i, \chi_i) \in \mathsf{sf}(\varphi)$. Then, since we have $\{(1,1)(2,2)\} \subseteq \Pi_0$, $\mathsf{Unsat}(\{\chi_1, \neg\psi_2\})$, and $\mathsf{Unsat}(\{\chi_2, \neg\psi_3\})$, it follows that $\Pi_1 = \Pi_0 \cup \{(1,2)(2,3)\}$. The new pairs of indices can intuitively be taken as the formulas $\mathsf{Kh}(\psi_1, \chi_2)$ and $\mathsf{Kh}(\psi_2, \chi_3)$. In this case, note the connection between $\mathsf{Kh}(\psi_1, \chi_2)$ and $(\mathsf{Kh}(\psi_1, \chi_1) \wedge \mathsf{A}(\chi_1 \to \psi_2) \wedge \mathsf{Kh}(\psi_2, \chi_2)) \to \mathsf{Kh}(\psi_1, \chi_2)$, and $\mathsf{Kh}(\psi_2, \chi_3)$ and $(\mathsf{Kh}(\psi_2, \chi_2) \wedge \mathsf{A}(\chi_2 \to \psi_3) \wedge \mathsf{Kh}(\psi_3, \chi_3)) \to \mathsf{Kh}(\psi_2, \chi_3)$. Finally, since we have $(1,2) \in \Pi_2$ and $\mathsf{Unsat}(\{\chi_2, \neg\psi_3\})$, then $\Pi_2 = \Pi_1 \cup \{(1,3)\}$. The justification for the pair $(1,3)$ is similar to the one just offered. In Fig. 1 we illustrate a run of Plans which computes this set (only the steps in which the matrix is updated are shown).

The composition of formulas $\mathsf{Kh}(\psi, \chi)$ and $\mathsf{Kh}(\psi', \chi')$ has an impact if we wish to add a formula $\neg\mathsf{Kh}(\psi'', \chi'')$ into the mix. The reason for this is that witness plans π and π' for $\mathsf{Kh}(\psi, \chi)$ and $\mathsf{Kh}(\psi', \chi')$, respectively, yield a witness plan $\pi'' = \pi\pi'$ for $\mathsf{Kh}(\psi, \chi')$. In adding $\neg\mathsf{Kh}(\psi'', \chi'')$ we need to ensure π'' is not a witness for $\mathsf{Kh}(\psi'', \chi'')$, as such a plan renders $\neg\mathsf{Kh}(\psi'', \chi'')$ unsatisfiable. We make these ideas precise in the definition of *compatible* below.

Definition 9. *Let φ^+ and φ^- be formulas s.t.:* $\mathsf{md}(\varphi^+) = 1$ *and* $\mathsf{md}(\varphi^-) = 1$*;* $\varphi^+ = \mathsf{Kh}(\psi_1, \chi_1) \wedge \cdots \wedge \mathsf{Kh}(\psi_n, \chi_n)$*; and* $\varphi^- = \neg\mathsf{Kh}(\psi'_1, \chi'_1) \wedge \cdots \wedge \neg\mathsf{Kh}(\psi'_m, \chi'_m)$*. Moreover, let $I, J \subseteq [1, n]$ be as in Prop. 5 and $\psi = \bigwedge_{i \in I} \neg\psi_i$. We say that φ^+ and φ^- are* compatible *iff the following conditions are met:*

(1) $\mathsf{Sat}(\psi)$*;*
(2) for all $\mathsf{Kh}(\psi'_{k'}, \chi'_{k'}) \in \mathsf{sf}(\varphi^-)$*,*
  (a) $\mathsf{Sat}(\{\psi, \psi'_{k'}, \neg\chi'_{k'}\})$*; and*
  (b) for all $(x, y) \in \Pi(\varphi^+, \psi)$,
    if $x \notin J$ and $\mathsf{Unsat}(\{\psi, \psi'_{k'}, \neg\psi_x\})$*, then,* $\mathsf{Sat}(\{\psi, \chi_y, \neg\chi'_{k'}\})$*.*

Def. 9 aims to single out the conditions under which the formulas φ^+ and φ^- can be jointly satisfied. Intuitively, (1) tells us φ^+ must be individually satisfied (cf. Prop. 5). In turn, (2.a) tells us φ^- must be individually satisfied (cf. Prop. 7), while (2.b) tells us φ^+ and φ^- can be satisfied together if no composition of subformulas in φ^+ contradicts a subformula in φ^-. Such a contradiction would originate only as a result of strengthening the precondition and/or weakening the postcondition of a composition of subformulas in φ^+, in a way such that they would result in the opposite of a subformula in φ^-. Prop. 11 states that the conditions in Def. 9 guarantee the satisfiability of a combination of φ^+ and φ^-.

Proposition 11. *It follows that φ^+ and φ^- are compatible iff* $\mathsf{Sat}(\varphi^+ \wedge \varphi^-)$.

Proof. ($\Rightarrow$) Suppose that φ^+ and φ^- are compatible. Let $\mathfrak{M} = \langle \mathrm{S}, \mathrm{R}, \mathrm{V}\rangle$ be s.t. S contains all valuations that make ψ true; and $\mathrm{R} = \{\mathrm{R}_{a_k} \mid a_k \in \mathsf{Act}\}$ where

$$\mathrm{R}_{a_k} = \begin{cases} [\![\psi_k]\!]^{\mathfrak{M}} \times [\![\chi_k]\!]^{\mathfrak{M}} & \text{if } k \in K \\ \emptyset & \text{otherwise,} \end{cases}$$

for $K = [1, n] \setminus (I \cup J)$. From (1), we know $\mathrm{S} \neq \emptyset$. It is not difficult to see that $[\![\varphi^+]\!]^{\mathfrak{M}} = \mathrm{S}$ (cf. Prop. 5). The proof is concluded if $[\![\varphi^-]\!]^{\mathfrak{M}} = \mathrm{S}$. We proceed by contradiction. Let $k' \in [1, m]$ be s.t. $[\![\mathsf{Kh}(\psi'_{k'}, \chi'_{k'})]\!]^{\mathfrak{M}} = \mathrm{S}$; i.e., ($*$) exists $\pi \in \mathsf{Act}^*$ s.t. $[\![\psi'_j]\!]^{\mathfrak{M}} \subseteq \mathrm{SE}(\pi)$ and $\mathrm{R}_\pi([\![\psi'_j]\!]^{\mathfrak{M}}) \subseteq [\![\chi'_j]\!]^{\mathfrak{M}}$. We consider the following cases.

($\pi = \varepsilon$) From (2.a), we know $[\![\psi'_{k'} \wedge \neg\chi'_{k'}]\!]^{\mathfrak{M}} \neq \emptyset$; i.e., $[\![\psi'_{k'}]\!]^{\mathfrak{M}} \not\subseteq [\![\chi'_{k'}]\!]^{\mathfrak{M}}$. This implies $[\![\psi'_{k'}]\!]^{\mathfrak{M}} = \mathrm{R}_\varepsilon([\![\psi'_{k'}]\!]^{\mathfrak{M}}) \not\subseteq [\![\chi'_{k'}]\!]^{\mathfrak{M}}$.

($\pi \neq \varepsilon$ and $\pi = a_{k_1}, \ldots, a_{k_{|\pi|}}$ with $k_j \in K$ and $j \in [1, |\pi|]$) In this case we have:
- (a) $\emptyset \neq [\![\psi'_{k'}]\!]^{\mathfrak{M}} \subseteq \mathrm{SE}(\pi) \subseteq \mathrm{SE}(a_{k_1}) = [\![\psi_{k_1}]\!]^{\mathfrak{M}}$;
- (b) $[\![\chi_{k_j}]\!]^{\mathfrak{M}} = \mathrm{R}_{a_{k_j}}([\![\psi_{k_j}]\!]^{\mathfrak{M}}) \subseteq [\![\psi_{k_{(j+1)}}]\!]^{\mathfrak{M}}$; and
- (c) $[\![\chi_{k_{|\pi|}}]\!]^{\mathfrak{M}} = \mathrm{R}_\pi([\![\psi'_{k'}]\!]^{\mathfrak{M}}) \subseteq [\![\chi'_{k'}]\!]^{\mathfrak{M}}$.

Since S contains all valuations that make ψ true; from (a)–(d) we get:
- (d) $\mathsf{Unsat}(\{\psi, \psi'_{k'}, \neg\psi_{k_1}\})$ –from (a);
- (e) $\mathsf{Unsat}(\{\psi, \chi_{k_j}, \neg\psi_{k_{(j+1)}}\})$ –from (b);
- (f) $\mathsf{Unsat}(\{\psi, \chi_{k_{|\pi|}}, \neg\chi'_k\})$ –from (c).

From (e) and π, we obtain a sequence $k_1 \ldots k_{|\pi|}$ that satisfies the conditions (†) and (‡) in the proof of Prop. 9. Then, $(k_1, k_{|\pi|}) \in \Pi(\varphi^+, \psi)$. From (a) and (2.a), $k_1 \notin J$. We are in an impossible situation: $(k_1, k_{|\pi|}) \in \Pi(\varphi^+, \psi)$; $k_1 \notin J$; and $\mathsf{Unsat}(\{\psi, \chi_{k_{|\pi|}}, \neg\chi'_k\})$. This contradicts (2.b); meaning that φ^+ and φ^- are not compatible.

(π is none of the above) It is clear that $[\![\psi'_{k'}]\!]^{\mathfrak{M}} \not\subseteq \mathrm{SE}(\pi)$.

In all the cases above we have: $[\![\psi'_{k'}]\!]^{\mathfrak{M}} \not\subseteq \mathrm{SE}(\pi)$ or $\mathrm{R}_\pi([\![\psi'_{k'}]\!]^{\mathfrak{M}}) \not\subseteq [\![\chi'_{k'}]\!]^{\mathfrak{M}}$; i.e., $[\![\mathsf{Kh}(\psi'_{k'}, \chi'_{k'})]\!]^{\mathfrak{M}} = \emptyset$, a contradiction. Then, $[\![\varphi^-]\!]^{\mathfrak{M}} = \mathrm{S}$; and so $\mathsf{Sat}(\varphi^+ \wedge \varphi^-)$.

($\Leftarrow$) Suppose $\mathsf{Sat}(\varphi^+ \wedge \varphi^-)$; i.e., exists (†) $\mathfrak{M}$ s.t. $[\![\varphi^+ \wedge \varphi^-]\!]^{\mathfrak{M}} = \mathrm{S}$. From (†) we get $[\![\varphi^+]\!]^{\mathfrak{M}} = \mathrm{S}$. Using Cor. 1, we get $[\![\mathsf{A}\psi]\!]^{\mathfrak{M}} = \mathrm{S}$. This establishes (1). The proof of (2.a) is by contradiction. Let $\mathsf{Kh}(\psi'_{k'}, \chi'_{k'}) \in \mathsf{sf}(\varphi^-)$ be s.t. $\mathsf{Unsat}(\{\psi, \psi'_{k'}, \neg\chi'_{k'}\})$. Then, $[\![\psi'_{k'}]\!]^{\mathfrak{M}} \subseteq [\![\chi'_{k'}]\!]^{\mathfrak{M}}$. Choosing $\pi = \epsilon$, we obtain $[\![\mathsf{Kh}(\psi'_{k'}, \chi'_{k'})]\!]^{\mathfrak{M}} = \mathrm{S}$. This contradicts $[\![\varphi^-]\!]^{\mathfrak{M}} = \mathrm{S}$. The proof of (2.b) is also by contradiction. Let $\mathsf{Kh}(\psi'_{k'}, \chi'_{k'}) \in \mathsf{sf}(\varphi^-)$, ($*$) $(x, y) \in \Pi(\varphi^+, \psi)$, (†) $\mathsf{Unsat}(\{\psi, \psi'_{k'}, \neg\psi_x\})$, and (‡) $\mathsf{Unsat}(\{\psi, \chi_y, \neg\chi'_{k'}\})$. From (†) and (‡), $[\![\psi'_{k'}]\!]^{\mathfrak{M}} \subseteq [\![\psi_x]\!]^{\mathfrak{M}}$ and $[\![\chi_y]\!]^{\mathfrak{M}} \subseteq [\![\chi'_{k'}]\!]^{\mathfrak{M}}$. At the same time, from ($*$) and Prop. 9, $\mathrm{S} = [\![\varphi^+]\!]^{\mathfrak{M}} \subseteq [\![\mathsf{Kh}(\psi_x, \chi_y)]\!]^{\mathfrak{M}}$. Then, using Prop. 3, $[\![\mathsf{Kh}(\psi'_j, \chi'_j)]\!]^{\mathfrak{M}} = \mathrm{S}$. This also contradicts $[\![\varphi^-]\!]^{\mathfrak{M}} = \mathrm{S}$. Thus, φ^+ and φ^- are compatible.

Having at hand the result in Prop. 11, we proceed to define an algorithm for checking the satisfiability of compatible formulas φ^+ and φ^-. This is done in two stages. In the first stage, we build the set $\Pi(\varphi^+, \psi)$, where ψ is the conjunction

Algorithm 5 COMPATIBLE

require: φ^+ and φ^- are as in Def. 9
1: **function** COMPATIBLE(φ^+, φ^-)
2: $\Psi \leftarrow \{\neg\psi_i \mid i \in \text{GLOBAL}(\varphi^+)\}$
3: $r \leftarrow \mathsf{Sat}(\Psi)$ ▷ *check for condition (1)*
4: **for** $k' \leftarrow 1$ **to** m **do** ▷ *check for condition (2.a)*
5: $r \leftarrow r$ and $\mathsf{Sat}(\Psi \cup \{\psi'_{k'}, \neg\chi'_{k'}\})$
6: $\Pi \leftarrow \text{PLANS}(\varphi^+, \bigwedge\Psi)$
7: **for** $k' \leftarrow 1$ **to** m **do** ▷ *check for condition (2.b)*
8: **for** $x \leftarrow 1$ **to** n **do**
9: **for** $y \leftarrow 1$ **to** n **do**
10: **if** $(x, y) \in \Pi$ and $\mathsf{Sat}(\Psi \cup \{\psi_x\})$ and not $\mathsf{Sat}(\Psi \cup \{\psi'_{k'}, \neg\psi_x\})$ **then**
11: $r \leftarrow r$ and $\mathsf{Sat}(\Psi \cup \{\chi_y, \neg\chi'_{k'}\})$
12: **return** r

ensure: COMPATIBLE(φ^+, φ^-) iff φ^+ and φ^- are compatible

of the negation of the precondition of the 'global' subformulas in φ^+. This task is encapsulated in the function PLANS in Alg. 4. Notice that the set $\Pi(\varphi^+, \psi)$ corresponds to a matrix which is computed using the result in Prop. 10. The second stage is encapsulated in the function COMPATIBLE in Alg. 5. In this function, lines 2 and 3 check condition (1) in Def. 9, i.e., whether φ^+ is individually satisfiable, by verifying the joint satisfiability of the 'global' subformulas in φ^+ (cf. Alg. 2). In turn, lines 4 to 6 in COMPATIBLE check condition (2.a) of Def. 9, i.e., whether φ^- is individually satisfiable, by verifying the individual satisfiability of the subformulas in φ^+ (cf. Alg. 3). Lastly, in lines 7 to 18 in COMPATIBLE, we check whether the result of composing subformulas in φ^+ contradicts any of the subformulas in φ^-. We carry out this task by making use of the result of the function PLANS which computes such compositions.

Notice that the function COMPATIBLE in Alg. 5 makes a polynomial number of calls to a propositional Sat solver. From this fact, we get the following result.

Proposition 12. *Let φ^+, φ^- be as in Def. 9; it follows that Alg. 5 solves $\mathsf{Sat}(\varphi^+ \wedge \varphi^-)$ and is in P^{NP} (i.e., Δ^{P}_2 in PH).*

Proof. By Prop. 11 we get that the function COMPATIBLE in Alg. 5 solves $\mathsf{Sat}(\varphi^+ \wedge \varphi^-)$. Moreover, it makes a polynomial number of calls to a Sat solver for formulas of modal depth 0. Thus, it runs in polynomial time with access to a Sat oracle. Therefore, $\mathsf{Sat}(\varphi^+ \wedge \varphi^-)$ is in P^{NP}, i.e., in Δ^{P}_2.

Prop. 12 is the final step we need to reach the main result of our work.

Theorem 1. *The satisfiability problem for $\mathsf{L_{Kh}}$ is in $\mathsf{NP}^{\mathsf{NP}}$ (i.e., Σ^{P}_2 in PH).*

Proof. Let φ be a $\mathsf{L_{Kh}}$-formula. By Alg. 1, we can obtain, in polynomial time, a formula $\varphi' = \varphi_0 \wedge (\mathsf{A}p_1 \leftrightarrow \mathsf{Kh}(\psi_1, \chi_1)) \wedge \cdots \wedge (\mathsf{A}p_n \leftrightarrow \mathsf{Kh}(\psi_n, \chi_n))$ in leaf normal form such that $\varphi \equiv_{\mathsf{Sat}} \varphi'$. We know $\mathsf{md}(\varphi_0) = 0$ and $\mathsf{md}(\mathsf{Kh}(\psi_i, \chi_i)) = 1$. Let $Q = \{q_1 \dots q_m\} \subseteq \mathsf{Prop}$ be the set of proposition symbols in φ'. To check $\mathsf{Sat}(\varphi')$, we start by guessing a propositional assignment $v : Q \to \{0, 1\}$ that makes φ_0 true. Then, we define sets $P^+ = \{i \mid v(p_i) = 1\}$ and $P^- = \{i \mid v(p_i) = 0\}$, from which we build formulas

$$\varphi^{+} = \bigwedge_{i \in P^{+}} \mathsf{Kh}(\psi_i, \chi_i) \qquad \varphi^{-} = \left(\bigwedge_{i \in P^{-}} \neg\mathsf{Kh}(\psi_i, \chi_i)\right) \wedge \neg\mathsf{Kh}(\varphi_0, \bot)$$

(recall that $\neg\mathsf{Kh}(\varphi_0, \bot) = \neg\mathsf{A}\neg\varphi_0 = \mathsf{E}\varphi_0$.) Finally, we use Alg. 5 to check $\mathsf{Sat}(\varphi^{+} \wedge \varphi^{-})$. Since Alg. 5 is in P^{NP} (Prop. 12), the whole process is in $\mathsf{NP}^{\mathsf{NP}}$.

We conclude this section with an example of how to check the satisfiability of a formula using the procedure in the proof of Thm. 1.

Example 6. Let $\psi = \mathsf{Kh}(p \wedge q, r \wedge t) \vee \mathsf{Kh}(p, r)$. By applying Alg. 1, we get $(k_1 \vee k_2) \wedge (\mathsf{A}k_1 \leftrightarrow \mathsf{Kh}(p \wedge q, r \wedge t)) \wedge (\mathsf{A}k_2 \leftrightarrow \mathsf{Kh}(p, r))$. Suppose that we set k_1 to true and k_2 to false. Based on this assignment, we build formulas $\varphi^{+} = \mathsf{Kh}(p \wedge q, r \wedge t)$ and $\varphi^{-} = \neg\mathsf{Kh}(p, r) \wedge \neg\mathsf{Kh}(k_1 \wedge \neg k_2, \bot)$. Using Alg. 5, we can check that they are not compatible (and hence not satisfiable; we have $\mathsf{Sat}(p \wedge q)$ and $\mathsf{Unsat}(\{(p \wedge q), \neg p\})$ but not $\mathsf{Sat}(\{r \wedge t, \neg r\})$). However, if we set both k_1 and k_2 to true, then, $\varphi^{+} = \mathsf{Kh}(p \wedge q, r \wedge t) \wedge \mathsf{Kh}(p, r)$ and $\varphi^{-} = \neg\mathsf{Kh}(k_1 \wedge k_2, \bot)$. In this case, Alg. 5 returns they are compatible, and thus satisfiable.

4 Final Remarks

We provided a satisfiability-checking procedure for L_{Kh}, the 'knowing how' logic with linear plans from [31,33], obtaining a Σ_2^P upper bound. Although not a tight bound (as the best lower bound known is NP), we argue this is an interesting result, as our bound is (unless PH collapses) below the PSpace-complete complexity of model-checking [5]. We argue that, this unusual situation is a consequence of that in model-checking the full expressive power is exploited, while here we showed that plans are almost irrelevant for the satisfiability of a formula.

Interestingly also, our procedure uses a polynomial transformation into a normal form without nested modalities, and calls to an NP oracle (i.e., to a propositional Sat solver). It is well-known that modern Sat solvers are able to efficiently deal with large formulas (having millions of variables), and usually support the exploration of the solution state space. Thus, the ideas presented in this paper can be used to implement a Sat solver for knowing-how logics relying on modern propositional Sat solving tools. We consider this as part of the future work to undertake. Also, we would like to obtain a tight bound for the satisfiability problem. In this regard, we will explore the possibility of providing a reduction from the problem of checking the truth of Quantified Boolean Formula (TQBF) with a single $\exists\forall$ quantification pattern (called $\Sigma_2\mathsf{Sat}$ in [2]), which is known to be Σ_2^P-complete.

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