Minwise-Independent Permutations with Insertion and Deletion of Features

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Abstract In their seminal work, Broder *et. al.* [7] introduces the minHash algorithm that computes a low-dimensional sketch of high-dimensional binary data that closely approximates pairwise Jaccard similarity. Since its invention, minHash has been commonly used by practitioners in various big data applications. Further, the data is dynamic in many real-life scenarios, and their feature sets evolve over time. We consider the case when features are dynamically inserted and deleted in the dataset. We note that a naive solution to this problem is to repeatedly recompute minHash with respect to the updated dimension. However, this is an expensive task as it requires generating fresh random permutations. To the best of our knowledge, no systematic study of minHash is recorded in the context of dynamic insertion and deletion of features. In this work, we initiate this study and suggest algorithms that make the minHash sketches adaptable to dynamic insertion and deletion of features. We show a rigorous theoretical analysis of our algorithms and complement it with extensive experiments on several real-world datasets. Empirically we observe a significant speed-up in the running time while simultaneously offering comparable performance with respect to running minHash from scratch. Our proposal is efficient, accurate, and easy to implement in practice.

Keywords Sketching algorithms · Jaccard similarity estimation · Streaming algorithms · Locality sensitive hashing (LSH).

1 Introduction

The seminal work of Broder *et. al.* [7] suggests the minHash algorithm that computes a low-dimensional representation (or *sketch*) of the high-dimensional binary data that closely approximates the underlying pairwise Jaccard similarity. The Jaccard similarity between two binary data points $X, Y \in \{0, 1\}^d$ is defined as $|S(X) \cap S(Y)|/|S(X) \cup S(Y)|$ where $S(X) = \{i : x_i = 1\}$ and x_i is the *i*-th feature of X. For a *d*-dimensional permutation π chosen uniformly at random, and a data point $X \in \{0, 1\}^d$, the minHash [7] is defined as follows:

$$\min \operatorname{Hash}_{\pi}(X) = \min_{s \in S(X)} \pi(s).$$

For a pair of points $X, Y \in \{0, 1\}^d$, minHash [7] offers the following guarantee

$$\Pr[\min \text{Hash}_{\pi}(X) = \min \text{Hash}_{\pi}(Y)] = \frac{|S(X) \cap S(Y)|}{|S(X) \cup S(Y)|}.$$

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We note that binary vectors and sets give two equivalent representations of the same data object. We elaborate on it as follows: Consider our data elements are a subset of a fixed universe. In the corresponding binary representation, we can generate a vector whose dimension is the size of the universe, where for each possible element of the universe, a feature position is designated. To represent a set into a binary vector, we label each element's location with 1 if it is present in the set, and 0 otherwise.

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In fact, the same guarantee holds for a restricted class of permutations called min-wise independent permutations [7, 18, 26]. The above characteristic demonstrates the locality-sensitive nature (LSH) [19] of minHash, and as a consequence, it can be effectively used for the approximate nearest neighbour search problem. minHash is successfully applied in several real life applications such as computing document similarity [5], itemset mining [2, 13], faster de-duplication [6], all-pair similarity search [1], document clustering [8], building recommendation engine [15], near-duplicate image detection [12], web-crawling [17, 25], genomics [3, 4, 9, 29], large scale graph hashing [16, 24] to name a few.

This work considers the scenario where features are dynamically inserted and/or deleted from the input. We emphasize that this natural setting may arise in many applications. Consider the "Bag-of-Word" (BoW) representation of text, where first, a dictionary is created using the important words present in the corpus such that each word present in the dictionary corresponds to a feature in the representation. Consequently, the embedding of each document is generated using this dictionary based on the frequency of the words present. Consider the downstream application where the task is to compute pairwise Jaccard similarities between these documents, and the dimensionality of the BoW representation is high due to the large dictionary size. We can use minHash to compute the low-dimensional sketch of input documents. It is quite natural to assume that the dictionary is evolving; new words are inserted into the dictionary, and unused words are deleted. One evident approach to handle such a dynamic scenario is to run the minHash from scratch on the updated dictionary, which is expensive since it involves generating fresh min-wise independent (random) permutations. Note that during the insertion/deletion of features in the dataset, we consider inserting/deleting the same features in all the data points. To clarify this further, let $\mathcal{D} = \{X_i\}_{i=1}^n$ be our dataset, where $X_i \in \{0,1\}^d$. Considering the addition/removal of the *j*-th feature, the *j*-th feature gets inserted/deleted in the point X_i . Similarly, the corresponding *j*-th feature is inserted/deleted in all the remaining points in \mathcal{D} . Note that we don't consider the case when data points are dynamically inserted or deleted in the dataset.

Problem statement: minHash for dynamic insertion and deletion of features: In this work, we focus on making minHash adaptable to dynamic feature insertions and deletions of features. We note that the insertion/deletion of features dynamically leads to the expansion/shrink of the data dimension.

We note that in practice a d dimensional permutation required for minHash is generated via the universal hash function $h_d(i) = ((ai + b) \mod p) \mod d$, where p is a large prime number, and a, b are randomly sampled from $\{0, 1, \ldots p - 1\}$; typically $((ai + b) \mod p) > d^{-1}$. This hash function generates permutations via mapping each index $i \in [d]$ to another index [d] that can be used to compute the minHash sketch. We note that in the case of dynamic insertions/deletion of features, even using universal hash functions to compute the minHash sketch doesn't give an efficient solution. We illustrate it as follows. Suppose we have a minHash sketch of data points using the hash function $h_d(.)$. Consider the case of feature insertion, where the dimension d increases to d + 1, and therefore, we require a hash function $h_{d+1}(.)$ to generate a (d + 1)-dimensional permutation. Note that the permutation generated via $h_{d+1}(.)$ can potentially be different on several values of $i \in [d + 1]$. Therefore, just computing $h_{d+1}(d + 1)$, taking the corresponding input feature, and taking the minimum of this quantity with the previous minHash would not suffice to compute minHash after feature insertion. This re-computation seems to take O(d) in the worst case if implemented naively. A similar argument also holds in the case of feature deletion.

1.1 Our Contribution:

In this work, we consider the problem of making minHash adaptable to dynamic insertions and deletions of features. We focus on cases where features are inserted/deleted at randomly chosen positions from 1 to d. We argue that this is a natural assumption that commonly occurs in practice. For example, in the context of BoW, a word's position in the dictionary is determined via a random hash function that randomly maps it to a position from 1 to d. Therefore, when a new word is added to the dictionary, its final position in the representation appears as a random position (from 1 to d). A similar argument is also applicable for feature deletion. With this motivation and context, we summarize our contributions as follows:

- Contribution 1: We present algorithms (Section 3) that makes minHash sketch adaptable to single/multiple feature insertions. Our algorithm takes the current permutation and the corresponding minHash sketch; values and positions of the inserted features as input and outputs the minHash sketch corresponding to the updated dimension.

¹ These hash functions are called universal hash functions. Readers may refer to Chapter 11 of [14] for details.

 Contribution 2: We also suggest algorithms (Section 4) that makes minHash sketch adaptable for single/multiple feature deletions. It takes the data points, current sketch, and permutations used to generate the same; positions of the deleted features and outputs the minHash sketch corresponding to the updated dimension.

Our work leaves the possibility of some interesting open questions: to propose algorithms when features are inserted or deleted adversarially (rather than uniformly at random from 1 to d, as considered in this work). We hope that our techniques can be extended to handle this situation.

1.2 Our techniques and their advantages:

A major benefit of our results is that they do not require generating fresh random permutations corresponding to the updated dimension (after feature insertions/deletions) to compute the updated sketch. We implicitly generate a new permutation (required to compute the sketch after feature insertion/deletion) using the old *d*-dimensional permutation, and also show that it satisfies the min-wise independence property. We further give simple and efficient update rules that take the value and position of inserted/deleted features, and output the updated minHash sketch. To show the correctness of our result, we prove that the sketch obtained via our update rule is the same as obtained via computing minHash from scratch using the implicitly generated permutation as mentioned above.

For both insertions and deletion cases, our algorithms give significant speedups in dimensionality reduction time while offering almost comparable accuracy with respect to running minHash from scratch. We validate this by running extensive experiments on several real-world datasets (Section 5 and Table 3). We want to emphasize that our algorithms can also be easily implemented when permutations are generated via random hash functions.

1.3 Applicability of our result in other sketching algorithms for Jaccard similarity:

We note that there are several improved variants of minHash are known such as one-permutation hashing [21, 30], b-bit minwise hashing [20, 22], oddsketch [28] that offer space/time efficient sketches. We would like to highlight that our algorithms can be easily adapt to these improved variants of minHash, in case of dynamic insertion and deletion of features. We briefly discuss it as follows: One permutation hashing divides the permuted columns evenly into k bins. For each data point, the sketch is computed by picking the smallest nonzero feature location in each bin. In the case of dynamic settings, our algorithms can be applied in the bin where features are getting inserted/deleted. Both b-bit minwise hashing [20] and oddsketch [28] are two-step sketching algorithms. In their first step, the minHash sketch of the data points is computed using their proposed hashing algorithm. As both of these results compute the minHash sketch in case of feature insertion/deletion. This will make their algorithms adaptable to dynamic feature insertions and deletions.

Recently, some hashing algorithms have been proposed that closely estimate the pairwise Jaccard similarity [10, 11, 27] without computing their minHash sketch. However, to the best of our knowledge, their dynamic versions (that can handle dynamic insertions/deletions of features) are unknown. Several improvements of the LSH algorithm [31] have been proposed that are adaptable to the dynamic/streaming framework. However, a significant difference is in the underlying problem statement. These results aim to handle dynamic insertion and deletions of data points, whereas we focus on dynamic insertions and deletions of the features.

Organization of the paper: In Section 2 we present the required technical background. In Section 3 and Section 4 we present our algorithms for feature insertion and feature deletion respectively. We summarize the results of our experiments in Section 5. Finally we note the concluding remarks in Section 6.

2 Background

Definition 1 (Minwise Independent Permutations [7]) Let S_d be the set of all permutation on [d]. We say that $F \subseteq S_d$ (the symmetric group) is min-wise independent if for any set $U \subseteq [d]$ and any $u \in U$,

Table 1 Notations

Data dimension	d	Input data point $\{0,1\}^d$ or input set	X
Set $\{1, \ldots, d\}$	[d]	Data point after feature insertion $\{0,1\}^{d+1}$	X'
Position of the inserted feature	m	Original d-dim. permutation (a_1, \ldots, a_d) s.t. $a_i \in [d]$	π
Value of the inserted feature	b	Lifted $(d+1)$ -dim. permutation (a'_1, \ldots, a'_{d+1}) s.t. $a'_i \in [d+1]$	π'_m
No. of 1's in X	X	Set of non-zero indices of X, <i>i.e.</i> , $\{i x_i = 1\}$	J
Size of the set J	J	minHash of X with π , <i>i.e.</i> , minHash _{π} (X)	h_{old}

when π is chosen at random in F, we have

$$\Pr[\min\{\pi(U)\} = \pi(u)] = 1/|U|.$$
(1)

For a permutation $\pi \in F$ chosen at random and a set $U \subseteq [d]$, Broder *et.al.* [7] define minHash as follows minHash $\pi(U) = \arg \min_u \pi(u)$ for $u \in U$. For two data points, $U, V \subseteq [d]$, and π is chosen at random in F, due to minHash we have

$$\Pr[\min \operatorname{Hash}_{\pi}(U) = \min \operatorname{Hash}_{\pi}(V)] = |U \cap V| / |U \cup V|.$$
(2)

3 Algorithm for feature insertion

We first give our algorithm for a single feature insertion. We discuss it in the following subsection.

3.1 One feature insertion at a time – liftHash:

The liftHash (Algorithm 2) is our main algorithm for updating the sketch of data points consisting of binary features. It takes a d dimensional permutation π and the corresponding minHash sketch $h_{old} \pi$ as input. In addition, it takes an index m and a bit value b, corresponding to the position and the value of the binary feature to be inserted, respectively, and outputs updated hash value h_{new} . We show that h_{new} corresponds to a minHash sketch of the updated feature vector. In order to show this, we use liftPerm (Algorithm 1) which extends the original permutation π to a (d + 1) dimensional min-wise independent permutation. Note that the liftPerm algorithm is used solely for the proof and not required in the liftHash algorithm.

The main intuition of our algorithm is that we can (implicitly) generate a new (d + 1)-dimensional permutation by reusing the old d-dimensional permutation (Algorithm 1), and can update the corresponding minHash w.r.t. the new (d + 1)-dimensional permutation via a simple update rule (Algorithm 2). Consider a d dimensional input vector $X = (x_1, x_2, \ldots, x_d)$. A permutation π of $\{1, 2, \ldots, d\}$ can be thought of as imposing the following ordering on the indices of $X: \pi(1), \pi(2), \ldots, \pi(d)$. After feature insertion, we want the (implicit) liftPerm algorithm to generate a new permutation π' of $\{1, 2, \ldots, d + 1\}$ that still maintains the ordering that was imposed by π . We show that such an extension is achievable with high probability assuming (i) feature insertion is happening at a random position and (ii) our binary feature vector is sparse. This helps us guarantee (with high probability) that π' is min-wise independent if π is min-wise independent (see Theorem 2). Finally, we show that the sketch obtained by the liftHash algorithm is the same one produced by applying the minHash with respect to the output π' of the liftPerm algorithm (see Theorem 3).

We illustrate our algorithm with the following example and then state its proof of correctness.

Example 1. We illustrate our Algorithms using the following example. We assume that the index count starts with 1. Let X = [1, 0, 0, 1, 0, 1, 0] be the data point, and $\pi = [6, 3, 1, 7, 2, 5, 4]$ be the original permutation. Then minHash_{π}(X) is 5. Further, let us assume that we insert the value b = 1 at the index m = 2. Therefore $a_m = \pi(m) = 3$. The updated value X' = [1, 1, 0, 0, 1, 0, 1, 0] and due to Algorithm 1 by setting r = m = 2, we obtain $\pi'_m = [7, 3, 4, 1, 8, 2, 6, 5]$. We calculate the value of h_{new} outputted by Algorithm 2: as $h_{old} = 5 > a_m = 3$ and b = 1, then we have $h_{new} = \text{liftHash}(\pi, m, b, h_{old}) = a_m = 3$. Further, minHash_{π'_m}(X') = 3. Therefore, we have $h_{new} = \text{minHash}_{\pi'_m}(X')$.

The following theorem gives proof of correctness of Algorithm 1, and shows that the permutation π' outputted by the algorithms satisfies the minwise independent property (Definition 1), with high probability. At a high-level proof of Theorem 2 relies on showing the bijection between the ordering on the indices of X by the original d-dimensional permutation π , and (d + 1)-dimensional permutation π' . We show that this bijection holds with probability 1 when inserted feature value b = 0, and holds with a high probability when b = 1.

Algorithm 1: liftPerm (π, \mathbf{r}) .

1 Input: d-dim permutation π , a number r. 2 **Output:** (d+1)-dim. permutation π' . for $i \in \{1, ..., d+1\}$ do з if $i \leq r$ then 4 $\pi'(i) = \pi(i)$ 5 6 else $| \quad \pi'(i) = \pi(i-1)$ 7 8 end 9 end $i \in \{1, \ldots, d+1\}/\{r\}$ do 10 for if $\pi'(i) \geq \pi'(r)$ then 11 $\pi'(i) = \pi'(i) + 1$ 12 end 13 14 end 15 return π

Algorithm 2: liftHash (π, m, b, h_{old}) .

```
1 Input: h_{old} := \min \operatorname{Hash}_{\pi}(X), \pi, m \in [d], b \in \{0, 1\}.
 2 Output: h_{new} := \text{liftHash}(\pi, m, b, h_{old}).
 3 Denote a_m = \pi(m).
       m is the position of the inserted feature \ast/
 4
    if h_{old} < a_m then
        h_{new} = h_{old}
 5
 6
    else
 7
         if b = 1 then
             h_{new} = a_m
 8
          end
 9
         if b = 0 then
10
11
              h_{new} = h_{old} + 1
         e^{\mathbf{n}}\mathbf{d}
12
13 end
14 return h_{new}
```

Theorem 2 Let $\pi = (a_1, \ldots, a_d)$ be a minwise independent permutation, where $a_i \in [d]$, and r be a random number from [d]. Let π and r be the input to Algorithm 1. Then for any $X \in \{0, 1\}^d$ with $|X| \leq k$, the permutation $\pi' = (a'_1, \ldots, a'_{d+1})$, where $a'_i \in [d+1]$, obtained from Algorithm 1 satisfies the condition stated in Equation (1) of Definition 1, with probability at least 1 - O(k/d).

Theorem 3 gives a proof of correctness of Algorithm 2. We show that the sketch outputted by Algorithm 2 is the same as obtained by running minHash using the (d + 1)-dimensional permutation obtained by Algorithm 1 on the updated data point after one feature insertion.

Theorem 3 Let π'_m be the (d+1)-dimensional permutation outputted by Algorithm 1 by setting r = m. Then, the sketch obtained from Algorithm 2 is exactly the same to the sketch obtained with the permutation π'_m on X', that is, $h_{new} := \text{liftHash}(\pi, m, b, h_{old}) = \min \text{Hash}_{\pi'_m}(X')$.

Remark 1 We remark that in order to compute the minHash sketch of X', Algorithm 2 requires only h_{old} , b, m, the value of $\pi(m)$. Whereas vanilla minHash requires a fresh (d + 1) dimensional permutation to compute the same.

We first give a proof of Theorem 2 using Propositions 1, 2, 3, 4 and 5.

Proposition 1. In Algorithm 1, we have the following: if i < r, then $\pi'(i) = \pi(i) + \mathbb{1}_{\{\pi(i) \ge \pi(r)\}}$; if i = r, then $\pi'(r) = \pi(r)$; if i > r, then $\pi'(i) = \pi(i-1) + \mathbb{1}_{\{\pi(i-1) > \pi(r)\}}$.

Proof. Note that Algorithm 1 initially sets all $\pi'(i) = \pi(i)$ if $i \le r$, and $\pi'(i) = \pi(i-1)$ if i > r. If i < r, then only if the condition in line 11 of Algorithm 1 is satisfied we increment $\pi'(i)$ by 1, which happens when $\mathbb{1}_{\{\pi'(i) \ge \pi(r)\}} = 1$. If i = r, then $\pi'(r)$ is never updated by Algorithm 1. If i > r, then $\pi'(i)$ is initialized in line 7 of the algorithm to $\pi(i-1)$. Then it gets updated in line 12 only when the condition in line 11 is satisfied which happens when $\mathbb{1}_{\{\pi(i) \ge \pi(r)\}} = 1$.

Proposition 2. If $X \in \{0,1\}^d$ with $|X| \leq k$ and r is chosen uniformly at random from $\{1,2,\ldots,d\}$, then with probability at least 1 - O(k/d) we have: $x_r = 0$ and $x_{r+1} = 0$.

Proof. Let $J = \{j : x_j = 1\}$ be the set of non-zero indices of X. Since r is chosen uniformly at random from $\{1, \ldots, d+1\}$, probability that $r-1, r, r+1 \in J$ is at most O(k/d). Thus $x_r = 0$ and $x_{r+1} = 0$ with probability at least 1 - O(k/d).

Proposition 3. Let J be the set of non-zero indices of X, and J' be the set of non-zero indices of X'. Assume that $r - 1, r, r + 1 \notin J$. If b = 0, then there is a bijection from J to J' defined as following mapping from $j \in J$ to $j' \in J'$: if j < r, then j' = j and j > r, then j' = j + 1. If b = 1 then the same bijection holds from $J \cup \{r\}$ to J' by additionally mapping r to r.

Proof. If b = 0, then from Proposition 1, we get a bijection from J to J' defined as follows: if j < r, then j' = j; if j > r then j' = j + 1. If b = 1, then again from Proposition 1 and the fact that $r - 1, r, r + 1 \notin J$, we get a bijection from $J \cup \{r\}$ to J' defined as follows: if j < r then j' = j; if j = r then j' = j; if j > r then j' = j + 1.

Note that if $r \in J$ then $J \cup \{r\} = J$. Since |J| = k and |J'| = k + 1, a bijection between $J \cup \{r\}$ and J would not be possible.

Proposition 4. Let $j_{min} \in J$ be the index such that $\pi(j_{min})$ is minimum among $\{\pi(j) : j \in J\}$. If b = 0, then $\pi'(j'_{min})$ is minimum among all $\{\pi'(j') : j' \in J'\}$, where j'_{min} is obtained from j_{min} via the bijection from Proposition 3.

Proof. Let $\pi(j_{min}) = \min\{\pi(j) : j \in J\}$. Recall that we have a bijection from J to J' given by $j \mapsto j'$, where $j \in J$ and $j' \in J'$. We want to show that $\pi'(j'_{min}) = \min\{\pi'(j') : j' \in J'\}$.

Recall from Proposition 1 that for any $j' \in J'$, we have: $\pi'(j') = \pi(j) + \mathbb{1}_{\{\pi(j) \ge \pi(r)\}}$. In particular, we have $\pi'(j'_{min}) = \pi(j_{min}) + \mathbb{1}_{\{\pi(j_{min}) \ge \pi(r)\}}$. Also note that $\pi'(j')$ is either $\pi(j)$ or $\pi(j) + 1$.

If $\mathbb{1}_{\{\pi(j_{min}) \ge \pi(r)\}} = 0$, then $\pi'(j'_{min}) = \pi(j_{min})$. Since for any other $j', \pi'(j')$ is either $\pi(j)$ or $\pi(j) + 1$, $\pi'(j'_{min})$ still remains the minimum.

If $\mathbb{1}_{\{\pi(j_{min}) \ge \pi(r)\}} = 1$ then $\pi'(j'_{min}) = \pi(j_{min}) + 1$. Moreover, for any other $j' \in J'$, since $\pi(j) \ge \pi(j_{min}) \ge \pi(r)$, the indicator $\mathbb{1}_{\{\pi(j) \ge \pi(r)\}} = 1$ holds. Hence $\pi'(j') = \pi(j) + 1$ also holds. Thus we have $\pi'(j'_{min}) = \min\{\pi'(j') : j' \in J'\}$.

Claim With probability at least 1 - O(k/d) we have: $r - 1, r, r + 1 \notin J$.

Proposition 5. Let $j_{min} \in J \cup \{r\}$ be the index such that $\pi(j_{min})$ is minimum among $\{\pi(j) : j \in J \cup \{r\}\}$. If b = 1, and $r - 1, r, r + 1 \notin J$, then we have that $\pi'(j'_{min})$ is minimum among all $\{\pi'(j') : j' \in J'\}$, where j'_{min} is obtained from j_{min} via the bijection from Proposition 3.

Proof. Let $\pi(j_{min}) = \min\{\pi(j) : j \in J \cup \{r\}\}$. Therefore, we have a bijection from $J \cup \{r\}$ to J' given by $j \mapsto j'$, where $j \in J$ and $j' \in J'$. We want to show that $\pi'(j'_{min}) = \min\{\pi'(j') : j' \in J'\}$. Recall from Proposition 1 that for any $j' \in J'$, we have: $\pi'(j') = \pi(j) + \mathbb{1}_{\{\pi(j) \ge \pi(r)\}}$ when $j' \ne r$ and $\pi'(r) = \pi(r)$. Note that $\pi'(j')$ is either $\pi(j)$ or $\pi(j) + 1$. If $j_{min} = r$, then since $\pi'(r) = \pi(r)$ we have $j'_{min} = r$.

If $j_{min} \neq r$, then we have $\pi'(j'_{min}) = \pi(j_{min}) + \mathbb{1}_{\{\pi(j_{min}) \geq \pi(r)\}}$.

If $\mathbb{1}_{\{\pi(j_{min}) \ge \pi(r)\}} = 0$, then $\pi'(j'_{min}) = \pi(j_{min})$. Since for any other $j', \pi'(j')$ is either $\pi(j)$ or $\pi(j) + 1$, $\pi'(j'_{min})$ still remains the minimum.

If $\mathbb{1}_{\{\pi(j_{min}) \ge \pi(r)\}} = 1$, then $\pi'(j'_{min}) = \pi(j_{min}) + 1$. Moreover, for any other $j' \in J'$, since $\pi(j) \ge \pi(j_{min}) \ge \pi(r)$, the indicator $\mathbb{1}_{\{\pi(j) \ge \pi(r)\}} = 1$ holds. Hence $\pi'(j') = \pi(j) + 1$ also holds. Thus, we have $\pi'(j'_{min}) = \min\{\pi'(j') : j' \in J'\}$.

We complete a proof of Theorem 2 as follows.

Proof of Theorem 2:

Proof. We split in the following cases:

Case 1: when b = 0. Let $\pi(j_{min}) = \min\{\pi(j) : j \in J\}$. From Proposition 4, we know that $\pi'(j'_{min}) = \min\{\pi'(j') : j' \in J'\}$. Since π is minwise independent, we have j_{min} is uniformly distributed across J. Since $j \mapsto j'$ is a bijection, we have that j'_{min} is uniformly distributed across J'. Hence minwise independence for π' holds.

Case 2: when b = 1. Let $\pi(j_{min}) = \min\{\pi(j) : j \in J \cup \{r\}\}$. Since r is chosen uniformly random from $\{1, \ldots, d\}$, from Proposition 2, with probability at least 1 - O(k/d), we have: $x_r = 0$ and $x_{r+1} = 0$, i.e., $r \notin J$. Recall that we have a bijection from $J \cup \{r\} \mapsto J'$.

Let $\pi(J \cup \{r\}) = \{\pi(j) : j \in J \cup \{r\}\}$. Since $x_{r+1} = 0$, we have that $\pi(J \cup \{r\})$ consists of $\pi(j_1), \pi(j_2), ..., \pi(j_{k+1})$ for k+1 distinct indices. Also $\pi'(J')$ consists' of $\pi'(j'_1), \pi'(j'_2)..., \pi'(j'_{k+1})$ where $\pi(j')$ is either $\pi(j)$ or $\pi(j) + 1$. for k+1 distinct values of j.

Note that if $x_{r+1} = 1$, then $\pi'(r+1) = \pi(r) + 1$ and $\pi'(r) = \pi(r)$. So the distinctness of values of j would not hold. However with probability at least 1 - O(k/d) we have: $x_{r+1} = 0$.

Since π is minwise independent, we have that the minimum index j_{min} is uniformly distributed across $J \cup \{r\}$. Hence from Proposition 5, we can conclude that j'_{min} is uniformly distributed across J'. Hence π' is minwise independent.

We now give a proof of Theorem 3. To do so, we require the following Propositions 6, 7, 8. We divide the proof into two main cases based on whether the permutation value of the inserted index is greater than the hash value or not. The case when the permutation value is less than or equal to the hash value is further divided into two cases based on whether the inserted bit is 0 or 1.

Proposition 6. If $h_{old} < \pi(m)$, then $h_{new} = h_{old} = \min \operatorname{Hash}_{\pi'_m}(X')$.

Proof. Note that if $h_{old} < \pi(m)$, then Algorithm 2 outputs $h_{new} = h_{old}$. Let j_{min} be the index such that $\pi(j_{min}) = h_{old}$. From Proposition 1, we have: $\pi'_m(j'_{min}) = \pi(j_{min}) + \mathbb{1}_{\{\pi(j_{min}) \ge \pi(m)\}}$.

Since $\pi(j_{min}) = h_{old} < \pi(m)$, we have that $\mathbb{1}_{\{\pi(j_{min}) \ge \pi(m)\}} = 0$. Hence we have $\pi'_m(j'_{min}) = \pi(j_{min}) = h_{old}$. Note that since $\pi'(m) = \pi(m) > \pi'_m(j_{min})$ we can conclude $\pi'(m)$ can not be new minimum. From Proposition 4 and 5, we have $\pi'_m(j'_{min}) = \min \operatorname{Hash}_{\pi'_m}(X')$.

Proposition 7. If $h_{old} \ge \pi(m)$ and b = 1, then $h_{new} = \pi(m) = \min \operatorname{Hash}_{\pi'_m}(X')$.

Proof. If $h_{old} = \pi(m)$ then Algorithm 2 outputs $h_{new} = \pi(m)$ in line 8. Also note that from Proposition 1, we have for any $j' \in J', \pi'_m(j') \ge \pi(j) \ge \pi(m)$. Hence, minHash $\pi'_m(X') = \pi(m) = h_{new}$.

If $h_{old} > \pi(m)$ then $\pi(m) = \min\{\pi(j) : j \in J \cup \{m\}\}$. Therefore, from Proposition 5, we have $\min \operatorname{Hash}_{\pi'_m}(X') = \pi'_m(m) = \pi(m)$. From line 8 of Algorithm 2, we have $h_{new} = \pi(m) = \min \operatorname{Hash}_{\pi'_m}(X')$.

Proposition 8. If $h_{old} \ge \pi(m)$ and b = 0, then $h_{new} = h_{old} + 1 = \min \operatorname{Hash}_{\pi'_m}(X')$.

Proof. Since b = 0, note that $\pi'_m(m)$ is not a candidate for the minHash $_{\pi'_m}(X')$. From Proposition 1, we know that $\pi'_m(j') = \pi(j) + \mathbb{1}_{\{\pi(j) \ge \pi(m)\}}$ for any $j \ne m$. Since $h_{old} \ge \pi(m)$ we know that for any $j \in J$, we have $\mathbb{1}_{\{\pi(j) \ge \pi(m)\}} = 1$. Hence $\min\{\pi'_m(j') : j' \in J'\} = \min\{\pi(j) + 1 : j \in J\} = h_{old} + 1$. Also note that Algorithm 2, in line 11, outputs, $h_{new} = h_{old} + 1$, if $h_{old} \ge \pi(m)$ and b = 0. Hence, we have $h_{new} = h_{old} + 1 = \min Hash_{\pi'_m}(X')$.

Proof of Theorem 3: Propositions 6, 7, and 8 completes a proof of Theorem 3.

Remark 2 We can extend our results for multiple feature insertion by repeatedly applying Theorem 2, and Theorem 3 along with the probability union bound. However, the time complexity of the algorithm obtained by sequentially inserting n features will grow linearly in n as observed in the empirical results (Figure 1, Section 5). In the next subsection, we present an algorithm that performs multiple insertions in parallel, which helps us achieve much better speedups.

Table 2 Notations

No. of inserted features	n	Position of inserted features $\{m_i\}_{i=1}^n, m_i \in [d+1]$	M
X after n features insertion $\{0,1\}^{d+n}$	X'	Set of inserted bits $\{b_1, \ldots, b_n\}$ with $b_i \in \{0, 1\}$	B
multipleLiftHash (M, π, B, h_{old})	h_{new}	Lifted $(d+n)$ -dim. permutation	π'_M

3.2 Algorithm for multiple feature insertions – multipleLiftHash:

Results presented in this subsection are extensions to that of Subsection 3.1. The intuition of our proposal is that we can (implicitly) generate a new (d + n)-dimensional permutation (*n* is the number of inserted features), using the old *d*-dimensional permutation. By exploiting the sparsity of input, and the fact that inserted bits are random positions, we show that the updated permutation satisfies the min-wise independent property with high probability. Further, we suggest a simple update rule aggregating the existing minHash sketch and the minHash restricted to inserted position and outputs the updated sketch.

Algorithm 3: partialMinHash (π, M, B)

- **1 Input:** Permutation π , a sorted set of indices $M = \{m_1, ..., m_n\}$, and set of inserted bits $B = \{b_1, ..., b_n\}$
- **2** Output: The min value of π (with appropriate shift) restricted to only those indices m_i of M that correspond to non-zero b_i .
- **3** $\pi_{M,B} = \{\pi(m_i) \mid i \in \{1, \ldots, n\} \text{ and } b_i = 1\}$
- 4 return $\min\{\pi_{M,B}(k) + 1\}$

Algorithm 4: multipleLiftPerm (π, R) .

1 Input: Permutation π ; R with |R| = n. 2 Output: (d + n)-dim. permutation π' . 3 $R \leftarrow \text{sorted}(R)$ /* sorting array R in ascending order */ 4 for $i \in \{1, 2, ..., n\}$ do 5 |R[i] = R[i] + i - 16 end 7 $\pi' = \pi$ /* Initialization */ 8 for $i \in \{1, ..., n\}$ do 9 $|\pi' = \text{liftPerm}(\pi', R[i])$ /* Calling Algorithm 1 with $\pi = \pi'$ and r = R[i] */ 10 end 11 return π'

Algorithm 5: multipleLiftHash (M, π, B, h_{old}) .

- **1 Input:** $h_{old} := \min \operatorname{Hash}_{\pi}(X)$, permutation π , M and B.
- 2 **Output:** $h_{new} :=$ multipleLiftHash (M, π, B, h_{old}) .
- **3** Let $\pi_M := \{\pi(m) : m \in M\}.$
- 4 $a_M = \text{partialMinHash}(\pi, M, B)$
- 5 $h_{new} = \min(h_{old} + |\{x \mid x \in \pi_M \text{ and } x \leq h_{old}\}|, a_M) /*$ Picking the minimum between partialMinHash and shifted value of h_{old} . */

Algorithm 5 takes h_{old} , M, B, and π as input, and outputs the updated sketch h_{new} . Algorithm 5 uses Algorithm 3 to obtain the value of partialMinHash – minimum π value restricted to the inserted indices only with inserted bit value 1, from which it obtains multipleLiftHash for the updated input. Algorithm 4 is implicit and is used to prove the correctness of Algorithm 5. Algorithm 4 takes the permutation π and M as input, and outputs a (d + n)-dimensional permutation π'_M which satisfies the condition stated in Equation (1) for X, with $|X| \leq k$. We show this in Theorem 5. Then in Theorem 6, we show that $h_{new} = \min \operatorname{Hash}_{\pi'_M}(X')$. As π'_M satisfies the condition stated in Equation (1) for sparse X, then due to Equation (2) and [7] the sketch of data points obtained from Algorithm 5 approximates the Jaccard similarity.

Example 4. Suppose X = [1, 0, 0, 1, 0, 1, 0] and $\pi = [6, 3, 1, 7, 2, 5, 4]$ are input point and original permutation, respectively. Then the value of h_{old} is 5. Let M = [2, 4] and B = [0, 1]. Thus, in this case $\pi'_M = [7, 3, 4, 1, 8, 9, 2, 6, 5]$ and X' = [1, 0, 0, 0, 1, 1, 0, 1, 0]. Consequently we have, partial MinHash $(\pi, M, B) = 2 < h_{old} + |\{x \mid x \in \pi_M \text{ and } x \le h_{old}\}| = 5 + 1 = 6$. Therefore, minHash $\pi'_M(X') = 2$.

⁶ return h_{new}

We have the following theorems for the correctness of the algorithms presented in this subsection. A proof of the Theorem 5 follows similarly to the proof of Theorem 2 along with the probability union bound, and the proof of Theorem 6 is a generalization of proof of Theorem 3.

Theorem 5 Let π be a minwise independent permutation. Let $M = \{m_1, \ldots, m_n\}$ such that m_i is chosen uniformly at random from $\{1, \ldots, d\}$. Then for any $X \in \{0, 1\}^d$ with $|X| \leq k$, the permutation π'_M obtained from Algorithm 4 satisfies the condition stated in Equation (1) of Definition 1, with probability 1 - O(kn/d).

Theorem 6 Let π'_M be the (d+n)-dimensional permutation outputted by Algorithm 4, if we set R = M. Then, the sketch obtained from Algorithm 5 is exactly the same as the sketch obtained with the permutation π'_M on X', that is, multipleLiftHash $(\pi, M, B, h_{old}) = \min \operatorname{Hash}_{\pi'_M}(X')$.

We require the following propositions in order to prove the Theorem 6.

Proposition 9. The bit whose index is h_{old} in X_{π} has index $h_{old} + |\{x \mid x \in \pi_M \text{ and } x \leq h_{old}\}|$ in $X'^{\pi'_M}$.

Proof. While extending from X^{π} to $X'^{\pi'_M}$ we know that we have added the bits B at positions M. So the index in X^{π} where h_{old} existed has been shifted by a number of units to the right to form $X'^{\pi'_M}$. The number of units it has been shifted will be equal to the number of bits in π'_M that is less than h_{old} , and hence the result follows.

Proposition 10. The partial MinHash (π, M, B) represents the minHash value of $X'^{\pi'_M}$ restricted to only non-zero indices of B, if only the elements at newly added indices M were taken into account, i.e. after shifting the corresponding π values by the number of insertions.

Proof. We take into account the number of insertions that have happened before the insertion at index i for $i \in \{1, \ldots, n\}$ and determine the π'_M value for each newly added bit. Then, we calculate the minimum among all such π'_M values where a 1 has been inserted in X' which in other words is minHash in terms of just the inserted indices.

Proposition 11. The minHash value of X' with respect to the permutation π'_M is given by the minimum of partialMinhash (π, M, B) and $h_{old} + |\{x \mid x \in \pi_M \text{ and } x \leq h_{old}\}|$.

Proof. We have proved that partialMinhash (π, M, B) returns the minHash value with respect to only the newly added indices (assuming bits at old indices are all 0). And if the newly added elements were assumed to be 0 then the minHash will be $h_{old} + |\{x \mid x \in \pi_M \text{ and } x \leq h_{old}\}|$ (due to the shift we showed earlier).

Now minHash is the first time we see 1 while traversing through the indices of $X'^{\pi'_M}$ from left to right. So the first time 1 will occur will either happen in the new indices or the old indices. If it occurs in the old indices, then it's bound to be at $h_{old} + |\{x \mid x \in \pi_M \text{ and } x \leq h_{old}\}|$ and if it happens at the new indices it will happen at partialMinhash (π, M, B) . So the first time it occurs will be at the minimum of the two values.

Proof of Theorem 6: Propositions 9, 10, 11 completes a proof of the theorem.

4 Algorithm for feature deletion

We first give our result for one feature deletion.

4.1 One feature deletion at a time – dropHash :

We denote $X' \in \{0,1\}^{d-1}$ as the data point after one feature deletion. The intuition of our algorithm is that we can (implicitly) generate a new (d-1)-dimensional permutation by creating a bijection between the input indices before and after feature deletion. This preserves the distribution of the minimum index with respect to permutation, and ensures the minwise independent property stated in Equation (1). We discuss this in Algorithm 6 that takes π , and m as input, and outputs a (d-1) dimensional permutation π'_m . We give its proof of correctness in Theorem 8, where we show that π'_m satisfies the minwise independent property stated in Equation (1). Further, the corresponding sketch updation w.r.t. the new permutation is done via a simple update rule mentioned in Algorithm 7. The algorithm takes h_{old} , the position of the deleted feature m, and the corresponding value b as input, and outputs the updated sketch h_{new} . We give a proof of correctness of Algorithm 7 in Theorem 9, where we show that $h_{new} = \min \text{Hash}_{\pi'_m}(X')$. Therefore, due to Theorems 8, 9, and Equation (2) (and [7]) the sketch of data points obtained after Algorithm 7 approximates the pairwise Jaccard similarity. We illustrate our algorithm with the following example, and then we state its proof of correctness in Theorems 8, 9.

Algorithm 6: dropPerm (π, r) .

1 Input: d-dimensional permutation π , index position r. **2 Output:** (d-1)-dimensional permutation π' . **3** for $i \in \{1, \ldots, d-1\}$ do if i < r then 4 $\pi'(i) = \pi(i)$ 5 6 else $\mathbf{7}$ $\pi'(i) = \pi(i+1)$ 8 end 9 end 10 for $i \in \{1, \dots, d-1\}$ do 11 if $\pi'(i) > \pi(r)$ then $\mathbf{12}$ $\pi'(i) = \pi'(i) - 1$ $\mathbf{13}$ end 14 end 15 return π'

Algorithm 7: dropHash (m, X, π, h_{old}) .

```
1 Input: h_{old}, \pi, X, m \in [d], \text{ and } b \in \{0, 1\}.
 2 Output: h_{new} := dropHash(m, X, \pi, h_{old}).
 3 Let X^{\pi}[i] := X[j], s.t. \pi(j) = i and a_m := \pi(m)
    /* X^{\pi} is values of X permuted according to \pi. */
 4 if h_{old} < a_m then
 \mathbf{5}
        h_{new} = h_{old}
 6
    else
         if h_{old} > a_m then
 7
             h_{new} = h_{old} - 1
 8
         \mathbf{end}
 9
        if h_{old} = a_m then
10
               * Recalculate minHash from scratch. */
             for i \in \{a_m + 1, ..., d\} do
11
                  if X^{\pi}[i] = 1 then
12
                      h_{new} = i - 1 and exit;
13
                  \mathbf{end}
\mathbf{14}
             \mathbf{end}
15
16
        end
17 end
18 return h_{new}
```

Example 7. Let X = [1, 0, 0, 1, 0, 1, 0] be the input, and $\pi = [6, 2, 1, 7, 3, 5, 4]$ be the original permutation. The value of $h_{old} = 5$. Suppose that we delete the feature at the index 5. Then X' = [1, 0, 0, 1, 1, 0] and due to Algorithm 6 the value of $\pi'_m = [5, 2, 1, 6, 4, 3]$. We have $\pi(m) = a_m = 3$. We calculate the value of

 h_{new} outputted by Algorithm 7: as $h_{old} = 5 > a_m = 3$, we have $h_{new} = h_{old} - 1 = 5 - 1 = 4$. Further, minHash $\pi'_m(X') = 4$. Therefore, we have $h_{new} = \min \operatorname{Hash}_{\pi'_m}(X')$.

Theorem 8 Let π be a d-dimensional minwise independent permutation. Then, for any index position $r \in \{1, \ldots d\}$, the (d-1)-dimensional permutation π' obtained from Algorithm 6 satisfies the condition of being minwise independent permutation mentioned in Definition 1).

Proof. Let $J \subseteq \{1, 2, ..., d\}$ denote the set of the non-zero indices of d dimensional binary vector X, and let $J' \subseteq \{1, 2, ..., d-1\}$ denote the set of non-zero indices of d-1 dimensional binary vector X' referred in Algorithm 7.

Consider the bijection from $J - \{r\}$ to J' defined as follows:

 $j \mapsto j'$, where $j' = j + \mathbb{1}_{\{j > r\}}$.

Note that from line 11 and 12 of Algorithm 6, we have:

$$\pi'(j') = \pi(j) - \mathbb{1}_{\{\pi(j) > \pi(r)\}}.$$
(3)

We want to show that $\min(\pi'(J')) := \min\{\pi'(j') : j' \in J'\}$ is uniformly distributed across J'. Let $J'_{\pi(j)>\pi(r)} := \{j' : \mathbb{1}_{\{\pi(j)>\pi(r)\}} = 1\}$ and $J'_{\pi(j)<\pi(r)} := \{j' : \mathbb{1}_{\{\pi(j)>\pi(r)\}} = 1\}$. Note that since we have removed x_r , we do not have $\pi(j) = \pi(r)$ corresponding to any j'. Let $J_{\pi(j)>\pi(r)} := \{j : \mathbb{1}_{\{\pi(j)>\pi(r)\}} = 1\}$ and $J_{\pi(j)<\pi(r)} := \{j : \mathbb{1}_{\{\pi(j)<\pi(r)\}} = 1\}$.

Now

 $\min(\pi'(J')) = \min\{\min(\pi'(J'_{\pi(j)>\pi(r)}), \min(\pi'(J'_{\pi(j)<\pi(r)}))\}.$

From Equation 3, we have

$$\min(\pi'(J'_{\pi(j)<\pi(r)})) = \min\{\pi(j): j' \in J'_{\pi(j)<\pi(r)}\}.$$
(4)

$$\min(\pi'(J'_{\pi(j)>\pi(r)})) = \min\{\pi(j) - 1 : j' \in J'_{\pi(j)>\pi(r)}\}.$$
(5)

Since π is minwise independent, RHS in Equation 4 is uniformly distributed across the set. Also, RHS in Equation 5 is uniformly distributed across the set.

Moreover, since $J'_{\pi(j)<\pi(r)}$ and $J'_{\pi(j)>\pi(r)}$ are disjoint and $\{j : j' \in J'_{\pi(j)<\pi(r)}\}$ and $\{j : j' \in J'_{\pi(j)<\pi(r)}\}$ are also disjoint, by minwise independent of π we can conclude that $\min(\pi'(J'))$ is uniformly distributed across J'.

Theorem 9 Let π'_m be the (d-1)-dimensional permutation outputted by Algorithm 6 by setting r = m. Then, the sketch obtained from Algorithm 7 is exactly the same as the sketch obtained via the permutation π'_m on X', that is, dropHash $(m, X, \pi, h_{old}) = \min \operatorname{Hash}_{\pi'_m}(X')$.

We require Propositions 12, 13, 14 to prove the theorem.

Proposition 12. If $h_{old} < a_m$ then $h_{new} = h_{old} = \min \operatorname{Hash}_{\pi'_m}(X')$.

Proof. We know that h_{old} is the minimum index at which we see a 1 while iterating through the features of X in order permutation of π , which is the same as going through the elements of X^{π} in the order $\{1, ..., d\}$. We delete $a_m = \pi(m)$ at the index m of X which occurs after index h_{old} of X^{π} . Here X^{π} and $X'^{\pi'_m}$ will look as follows:

$$X^{\pi} = (0, \dots, X^{\pi}[h_{old} - 1] = 0, X^{\pi}[h_{old}] = 1, \dots, X^{\pi}[d]), \text{ and}$$
$$X'^{\pi'_{m}} = (0, \dots, X^{\pi}[h_{old}] = 1, \dots, X^{\pi}[a_{m} - 1], X^{\pi}[a_{m} + 1] \dots, X^{\pi}[d]).$$

Therefore, the minimum index at which we see a 1 in $X'^{\pi'_m}$ remains at index h_{old} , and the desired result follows.

Proposition 13. If $h_{old} > a_m$, then $h_{new} = h_{old} - 1 = \min \operatorname{Hash}_{\pi'_m}(X')$.

Proof. We know that h_{old} is the minimum index at which we see a 1 while iterating through the features of X in order of permutation π , which is the same as going through the elements of X^{π} in the order $\{1, ..., d\}$. We have deleted an element $a_m = \pi(m)$ at the index m of X which occurs before index h_{old} of X^{π} . Here X^{π} and $X'^{\pi'}$ will look as follows:

$$X^{\pi} = (0, \dots, X^{\pi}[h_{old} - 1] = 0, X^{\pi}[h_{old}] = 1, \dots, X^{\pi}[d]), \quad \text{and},$$
$$X'^{\pi'} = (0, \dots, X^{\pi}[a_m - 1], X^{\pi}[a_m + 1], \dots, X^{\pi}[h_{old} - 1] = 1, \dots, X^{\pi}[d]).$$

Therefore, the minimum index at which we see a 1 in $X'^{\pi'}$ is at index $h_{old} - 1$. Since index a_m has been deleted, the desired result follows.

Proposition 14. If $h_{old} = a_m$, then $h_{new} = \min \operatorname{Hash}_{\pi'_m}(X')$.

Proof. Note that line 11- of Algorithm 7 finds the next index where 1 occurs, in a brute-force way. \Box

Proof of Theorem 9: Propositions 12, 13, 14 completes a proof of the theorem.

Remark 3 The only expensive case for Algorithm 7 happens when $h_{old} = a_m$ considered in line (10-18) In this case, the algorithm has to compute the sketch in a brute-force way. However, as we choose m uniformly at random, this case happens with a probability of 1/d.

Remark 4 We can extend our results for multiple feature deletion by repeatedly applying Theorem 8, and Theorem 9 using the probability union bound. However, the time complexity of this approach grows linearly in n as also observed in the empirical results (Figure 2, Section 5). In the following subsection, we present an algorithm that performs multiple parallel deletions that helps achieve much better speedups.

4.2 Algorithm for multiple feature deletion – multipleDropHash:

Algorithm 8: multipleDropPerm (π, R) .

```
1 Input: Permutation \pi, array R with |R| = n.

2 Output: (d - n)-dimensional permutation \pi'.

3 R \leftarrow \text{sorted}(R)

/* sorting array R in the ascending order */

4 for i = 1 to n do

5 | R'[i] = R[i] - (i - 1)

6 end

7 \pi = \pi' /* Initialization Step */

8 for i \in \{1, 2, ..., n\} do

9 | \pi' = \text{dropPerm}(\pi', R'[i])

| /* Calling Algorithm 6 with \pi = \pi' and r = R'[i] */

10 end

11 return \pi'
```

Algorithm 9: multipleDropHash (M, X, π, h_{old})

```
1 Input: h_{old} := \min \operatorname{Hash}_{\pi}(X), \pi, M, X. Output: h_{new} := \operatorname{multipleDropHash}(M, X, \overline{\pi, h_{old}}).
 2 Let \pi(M) := {\pi(m) : m \in M}
 3 if h_{old} < \min(\pi_M) then
        h_{new} = h_{old}
 4
 5 else
         if h_{old} \notin \pi_M then
 6
             h_{new} = h_{old} - |\{x : x \in \pi_M \text{ and } x \le h_{old}\}|
 7
 8
         else
              /* Compute minHash from scratch. */
               h_{new} = \min\{\pi(i) : x_i = 1 \text{ and } \pi(i) \notin \pi_M\} - |\{x : x \in \pi_M \text{ and } x \le h_{old}\}|
 9
10
         end
11
         return h_{new}
12 end
```

The results presented in this subsection give algorithms for multiple feature deletion and are generalizations of the result presented in Subsection 4.1. We consider two algorithms for enabling minHash for multiple feature deletion. Suppose we have a data point $X \in \{0, 1\}^d$ and its minHash with permutation π is h_{old} . Let $X' \in \{0, 1\}^{d-n}$ be the data point after deleting *n* features. Algorithm 9 takes h_{old} , positions of the deleted feature $M = \{m_1, m_2, \ldots, m_n\}$ and outputs the updated sketch h_{new} . Algorithm 8 is implicit and is used to prove the correctness of Algorithm 9. Algorithm 8 takes the permutation π , and M as input, and outputs a (d - n) dimensional permutation π'_M . We show in Theorem 11 that π'_M satisfies the condition stated in Equation (1). Then in Theorem 12, we show that $h_{new} = \min \operatorname{Hash}_{\pi'_M}(X')$. Therefore, due to Theorems 11, 12, and Equation (2) (and [7]) the sketch of data points obtained after Algorithm 9 approximates the pairwise Jaccard similarity.

Example 10. Suppose our input, original permutation, and the list of deleted features are X = [1, 0, 0, 1, 0, 1, 0], $\pi = [6, 3, 1, 7, 2, 5, 4]$, and M = [2, 4], respectively. Thus after deletion X' = [1, 0, 0, 1, 0] and $\pi' = [5, 1, 2, 4, 3]$. The value of $h_{old} = 5$. We can calculate $\pi(M) = [3, 7]$ and therefore $\min(\pi(M)) = 3$. Thus, in this case $h_{old} = 5 > 3 = \min(\pi(M))$ and also $h_{old} \notin \pi(M)$. Therefore, the $\min \operatorname{Hash}_{\pi'_M}(X')$ is $h_{old} - |\{x \mid x \in \pi_M \text{ and } x \le h_{old}\}| = 5 - 1 = 4$.

Theorem 11 Let π be a d-dimensional minwise independent permutation. Then for every R, the (d-n) dimensional permutation π' obtained from Algorithm 8 satisfies the condition of being minwise independent permutation.

Proof. A proof follows by repeated application of Theorem 8.

Theorem 12 Let π'_M be the (d-n)-dimensional permutation outputted by Algorithm 8 by setting R = M. Then, the sketch obtained from Algorithm 9 is exactly the same as the sketch obtained with the permutation π'_M on X', that is, multipleDropHash $(\pi, M, B, h_{old}) = \min \operatorname{Hash}_{\pi'_M}(X')$.

We require the following propositions to prove the theorem.

Proposition 15. If $h_{old} < \min(\pi(M))$, then $h_{new} = h_{old} = \min \operatorname{Hash}_{\pi'_M}(X')$.

Proof. Recall that $X^{\pi}[i] := X[j]$, where $\pi(j) = i$. We know that h_{old} is the minimum index at which we see a 1 while going through the points of X in order permutation π , which is the same as going through the elements of X^{π} in the order $\{1, ..., d\}$. Now we have deleted a set of indices $M = \{m_1, ..., m_n\}$ which occurs after h_{old} . Here X_{π} and $X'_{\pi'_M}$ will look as follows:

$$X^{\pi} = (0, \dots, X^{\pi}[h_{old} - 1] = 0, X^{\pi}[h_{old}] = 1, \dots, X^{\pi}[d]), \quad \text{and} \\ X'^{\pi'_{M}} = (0, \dots, X^{\pi}[h_{old}] = 1, \dots, X^{\pi}[d - n]).$$

So we see that the first time we see a 1 remains at index h_{old} , and the desired result follows.

Proposition 16. If $h_{old} \notin \pi(M)$ and $h_{old} > \min(\pi(M))$, then $h_{new} = h_{old} - |\{x \mid x \in \pi_M \text{ and } x \leq h_{old}\}| = \min \operatorname{Hash}_{\pi'_M}(X')$.

Proof. We know that h_{old} is the minimum index at which we see a 1 while going through the points of X in order permutation π , which is the same as going through the elements of X^{π} in the order $\{1, ..., d\}$. Now we have deleted an element at indices $M = \{m_1, \ldots, m_n\}$ which does not contain the index containing h_{old} . So in $X'^{\pi'}$ h_{old} will be shifted to the left by the number of indices deleted before h_{old} which is given by $|\{x \mid x \in \pi_M \text{ and } x \leq h_{old}\}|$. Therefore, X_{π} and $X'_{\pi'}$ will look as follows:

$$X^{\pi} = (0, \dots, X^{\pi}[h_{old} - 1] = 0, X^{\pi}[h_{old}] = 1, \dots, X^{\pi}[d]), \text{ and}$$
$$X'^{\pi'_{M}} = (0, \dots, X^{\pi}[h_{old} - |\{x \mid x \in \pi_{M} \text{ and } x \le h_{old}\}| = 1, \dots, X^{\pi}[d - n]).$$

So we see that the first time we see a 1 is at index $h_{old} - |\{x \mid x \in \pi_M \text{ and } x \leq h_{old}\}|$ since index a_m has been deleted and the desired result follows.

Proposition 17. If h_{old} is in $\pi(M)$, then $h_{new} = \min \operatorname{Hash}_{\pi'_M}(X')$.

Proof. Since h_{old} has been deleted note that line number 9 of Algorithm 9 finds the next index where 1 occurs in a brute-force way.

Proof of Theorem 12:

Proof. Propositions 15, 16, 17 completes a proof of the theorem.

Remark 5 Since π is minwise-independent, the probability that h_{old} is in $\pi(M)$ is O(|M|/d). Thus the expensive part of Algorithm 9 (line 9) happens rarely.

5 Experiments

Hardware description: CPU model name: Intel(R) Xeon(R) CPU @ 2.20GHz; RAM:12.72GB; Model name: Google Colab.

Datasets and baselines: We perform our experiments on "*Bag-of-Words*" representations of text documents [23]. We use the following datasets: NYTimes news articles (number of points = 300000, dimension = 102660), Enron emails (number of points = 39861, dimension= 28102), and KOS blog entries (number of points = 3430, dimension = 6960).

We consider the binary version of the data, where we focus on the presence/absence of a word in the document. For our experiments, we considered a random sample of 500 points from the NYTimes and 2000 points for Enron and KOS. We compare the performance of our algorithms multipleLiftHash and multipleDropHash with respect to running minHash from scratch on the updated dimension, and we refer to it as vanilla minHash. We also note the performance of sequential versions of single feature insertion/deletion algorithms – liftHash and dropHash, respectively. We give implementation details of the baseline algorithms as the following link https://tinyurl.com/y98yh6k3.

Table 3 Speedup of our algorithms w.r.t their vanilla minHash version.

Experiment	Method	Max.	Гimes Avg.	En Max.	ron Avg.	Max.	OS Avg.
Feature Insertions	multipleLiftHash liftHash	$\begin{vmatrix} 54.91 \times \\ 91.23 \times \end{vmatrix}$	$51.96 \times 87.38 \times$	$\begin{array}{c} 9.61\times\\ 13.96\times\end{array}$	$9.17 \times 12.66 \times$	$\begin{vmatrix} 24.4 \times \\ 35.00 \times \end{vmatrix}$	$\begin{array}{c} 23.11\times\\ 35.50\times\end{array}$
Feature Deletions	multipleDropHash dropHash	$\begin{vmatrix} 109.5 \times \\ 78.34 \times \end{vmatrix}$	$\begin{array}{c} 105.31\times\\ 72.79\times\end{array}$	$18.6 \times 15.95 \times$	$\begin{array}{c} 17.01\times\\ 14.89\times\end{array}$	$\begin{vmatrix} 46.02 \times \\ 38.24 \times \end{vmatrix}$	$\begin{array}{c} 43.94\times\\ 35.71\times\end{array}$



Figure 1 Comparison among liftHash, multipleLiftHash, and vanilla minHash on the task of feature insertions. Vanilla minHash corresponds to computing minHash on the updated dimension. We iteratively run liftHash n times, where n is the number of inserted features.



Figure 2 Comparison among dropHash, multipleDropHash, and vanilla minHash on feature deletions. We iteratively run dropHash n times, where n is the number of deleted features.

5.1 Experiments for feature insertions:

We use two metrics for evaluation: a) RMSE: to examine the quality of the sketch, and b) running time: to measure the efficiency. For each dataset, we first create a 500 dimensional minHash sketch using 500 independently generated permutations. Consider that we have a set of n random indices representing the locations where features need to be inserted.For each position, we insert the bit 1 with probability 0.1 and 0 with probability 0.9. We then run the liftHash algorithm (Algorithm 2) after each feature insertion, we repeat this step until n feature insertions are done. This gives a minHash sketch corresponding to the liftHash algorithm. We again run our multipleLiftHash algorithm (Algorithm 5) on the initial 500 dimensional sketch with the parameter n. We compare our methods with vanilla minHash by generating a 500 dimensional sketch corresponding to the updated datasets after feature insertions.

For computing the RMSE, our ground truth is the pairwise Jaccard similarity on the original fulldimensional data. We measure it by computing the square root of the mean (over all pairs of sketches) of the square of the difference between the pairwise ground truth similarity and the corresponding similarity estimated from the sketch. A lower RMSE is an indication of better performance. We compare the RMSE of our methods with that of vanilla minHash by generating a fresh 500 dimensional sketch. We summarise our results in Figure 1.

Insights: Both of our algorithms offer comparable performance (under RMSE) with respect to running minHash from scratch on the updated dimension. That is, our estimate of the Jaccard similarity is as accurate as the one obtained by computing minHash from scratch on the updated dimension. Simultaneously, we obtain significant speedups in running time compared to running minHash from scratch. In particular, the speedup for multipleLiftHash is noteworthy (Table 3).

5.2 Experiments for feature deletion:

We use the same metric as feature insertion experiments – RMSE and running time. For each dataset, we first create a 500 dimensional minHash sketch using minHash. Suppose we have a list of n indices that denote the position where features need to be deleted. We then run dropHash algorithm (Algorithm 7) after each feature deletion. We repeat this step n times. This gives a minHash sketch corresponding to the dropHash algorithm. We again run our multipleDropHash algorithm (Algorithm 9) on the initial 500 dimensional sketch with the parameter n. We compare our results with vanilla minHash by generating a fresh 500 dimensional sketch on the updated dataset. We note the RMSE and running time as above. We summarise our results in Figure 2.

Insights: Again, both our algorithms offer comparable performance (under RMSE) with respect to running minHash from scratch. Similar to the previous case, we obtained a significant speedup in running time w.r.t. computing minHash from scratch. In particular, the speedup obtained in multipleDropHash is quite prominent. We summarise a numerical speedup in Table 3.

Remark 6 Our current implementation of multipleLiftHash makes multiple passes over indices to be inserted, whereas multipleDropHash makes only one pass over the deleted indices. This is reflected in higher speedup values for multipleDropHash in Table 3. We believe an optimized implementation for multipleLiftHash would further improve the speedup.

6 Conclusion and open questions

We present algorithms that make minHash adaptable to dynamic feature insertions and deletions of features. Our proposals' advantage is that they do not require generating fresh permutations to compute the updated sketch. Our algorithms take the current permutation (or its representation using universal hash function [14]), minHash sketch, position, and the corresponding values of inserted/deleted features and output updated sketch. The running time of our algorithms remains linear in the number of inserted/deleted features. We comprehensively analyse our proposals and complement them with supporting experiments on several real-world datasets. Our algorithms are simple, efficient, and accurately estimate the underlying pairwise Jaccard similarity. Our work leaves the possibility of several interesting open questions:

- extending our results for dense datasets in the case of feature insertions;
- extending our algorithms for the case when features are inserted/deleted adversely;
- improving our algorithms when we have prior information about the distribution of features; for example features distribution follows Zipf's law etc;
- improving theoretical guarantees and obtaining further speedups by optimizing our algorithms.

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