Robustness in Metric Spaces over Continuous Quantales and the Hausdorff-Smyth Monad

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Abstract. Generalized metric spaces are obtained by weakening the requirements (e.g., symmetry) on the distance function and by allowing it to take values in structures (e.g., quantales) that are more general than the set of non-negative real numbers. Quantale-valued metric spaces have gained prominence due to their use in quantitative reasoning on programs/systems, and for defining various notions of behavioral metrics.

We investigate imprecision and robustness in the framework of quantale-valued metric spaces, when the quantale is continuous. In particular, we study the relation between the robust topology, which captures robustness of analyses, and the Hausdorff-Smyth hemi-metric. To this end, we define a preorder-enriched monad P_S , called the Hausdorff-Smyth monad, and when Q is a continuous quantale and X is a Q-metric space, we relate the topology induced by the metric on $P_S(X)$ with the robust topology on the powerset P(X) defined in terms of the metric on X.

Keywords: Quantale · Robustness · Monad · Topology · Enriched category

Introduction

In the 1970s, Lawvere [20] proposed viewing metric spaces as small categories enriched over the monoidal category \mathbb{R}_+ , whose objects are the extended non-negative real numbers, where there is an arrow $x \to y$ if and only if $x \ge y$, and + and 0 provide the monoidal structure. In this way, one recovers most notions and results about metric spaces as instances of those about enriched categories [18].

Enrichment over arbitrary monoidal categories, however, is unnecessarily general for studying metric phenomena. Indeed, the base of enrichment for Lawvere's metric spaces belongs to the class of small (co)complete posetal categories, where the tensor commutes with colimits. These categories are called *quantales* and small categories enriched over a quantale Q are dubbed Q-metric spaces. Quantales are a useful compromise between arbitrary monoidal categories and the specific case of \mathbb{R}_+ [9,15,5]. Beside a substantial simplification of the theory, restricting to quantales allows to use well-known order-theoretic notions which do not have obvious counterparts in arbitrary monoidal categories, but are crucial to relating Q-metric spaces to other structures such as topological spaces.

Quantale-valued metric spaces are also increasingly used for *quantitative reasoning* on programs/systems, and for defining various notions of *behavioral metrics* [10,3,7,25,27,11]. The use of quantitative methods is important in coping with the uncertainty/imprecision that arises in the analysis of, e.g., probabilistic programs or systems interacting with physical processes. In these contexts, quantales provide a flexible framework which allows choosing the most suitable notion of distance for the specific analysis one is interested in.

Quantales arise naturally also in analysis of algorithms, namely, *costs* are values in certain quantales (see Example 2), but researchers in this area usually consider only subsets of these quantales and their partial order.

Motivations. the notions of imprecision and robustness are relevant in the context of software tools for the analysis of hybrid/continuous systems. These tools manipulate (formal descriptions of) mathematical models. A mathematical model is usually a simplified description of the system (and its environment), with the requirement that the simplification should be *safe*, i.e., if the analysis says that the model satisfies a property, then the system also satisfies that property. Usually, safe simplification is achieved by injecting *non-determinism* in the model (non-determinism is useful also to model known unknowns in the environment and don't care in the model). For hybrid/continuous systems there is another issue: *imprecision* in observations. In fact, predictions based on a mathematical

model and observations on a real system can be compared only up to the precision of measurements on the real system. We say that an analysis is *robust* when it can cope with *small amounts* of imprecision in the model, i.e., if a robust analysis says that a model M has a property, then it says so also for models that have a bit more non-determinism than M. Working with metric spaces makes it possible to define imprecision formally and to quantify the amount of non-determinism added to a model.

Following [22], given a metric space X, we can identify analyses with monotonic maps on the complete lattice P(X) of subsets of X ordered by reverse inclusion.³ However, even when imprecision is made arbitrarily small, two subsets with the same closure are *indistinguishable*. Therefore, analyses should be considered over the complete lattice C(X) of closed subsets, rather than that of arbitrary subsets, and should cope with *small amounts* of imprecision in the input. Formally, this property was defined as continuity with respect to the *robust topology* [21, Def. A.1] on C(X). This yields a functor from metric spaces to T_0 -topological spaces, which maps a metric on X to the robust topology on C(X). An anonymous referee suggested that the robust topology might be related to the *Hausdorff-Smyth hemi-metric* in [13, Proposition 1], and thus the functor from metric spaces (aka, Lawvere's metric spaces).

Contributions. This paper studies the link between the robust topology and the Hausdorff-Smyth hemi-metric—as suggested by an anonymous referee of [8]—and in doing so, addresses also more general issues, namely:

- 1. The notion of *imprecision* and the definition of *robust topology* are generalized to Q-metric spaces when Q is a *continuous quantale*, and the results in [22] are extended to this wider setting (see Section 4.1).
- 2. Indistinguishability is investigated in the context of \mathcal{Po} -enriched categories⁴ and the notion of separated object is introduced. In Section 5, we prove that, under certain conditions, every \mathcal{Po} -enriched monad can be *transformed* into one that factors through the full sub-category of separated objects. The conditions that allow this transformation hold in many \mathcal{Po} -enriched categories, such as that of Q-metric spaces and that of topological spaces.
- 3. The Hausdorff-Smyth $\mathcal{P}o$ -enriched monad P_S is defined on the category of Q-metric spaces, with Q an arbitrary quantale (see Section 6). When Q is a continuous quantale, the topology induced by the metric on $\mathsf{P}_S(X)$ is shown to coincide with a topology on $\mathsf{P}(X)$, called *-robust, defined in terms of the metric on X. In general, the *-robust topology is included in the robust topology, but they coincide when Q is linear and non-trivial (e.g., \mathbb{R}_+).

Although we apply the construction in Section 5 only to the monad defined in Section 6, it is applicable to other monads definable on Q-metric spaces (see Section 7) or on other $\mathcal{P}o$ -enriched categories.

Summary. The rest of the paper is organized as follows:

- Section 1 contains the basic notation and mathematical preliminaries.
- Section 2 introduces the category Qnt of quantales and lax-monoidal maps, and states some properties of continuous quantales.
- Section 3 defines the $\mathcal{P}o$ -enriched category $\mathcal{M}et_Q$ of Q-metric spaces and short maps for a quantale Q, and gives some of its properties.
- Section 4 introduces two topologies associated with a Q-metric space when Q is continuous, and characterizes the open and closed subsets.
- Section 5 defines separated objects in a $\mathcal{P}o$ -enriched category \mathcal{A} , and shows that, under certain assumptions on \mathcal{A} satisfied by $\mathcal{M}et_Q$, every $\mathcal{P}o$ -enriched monad on \mathcal{A} can be transformed (in an optimal way) into one that factors through the full sub-category of separated objects.
- Section 6 defines the Hausdorff-Smyth distance d_S and a related \mathcal{Po} -enriched monad on \mathcal{Met}_Q , characterizes the preorder induced by d_S and, when Q is continuous, also the topology induced by d_S .
- Section 7 contains an overview of related work and some concluding remarks.
- Omitted proofs appear in Appendix A.
- ³ The category of complete lattices and monotonic maps is the framework proposed in [6] for abstract interpretations.

⁴ $\mathcal{P}o$ denotes the category of preorders and monotonic maps.

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1 Mathematical Preliminaries

In this section, we present the basic mathematical notation used throughout the paper. We assume basic familiarity with order theory [14]. We write $\sqcup S$ to denote the join (aka lub) of a set S, and write $\sqcap S$ to denote the meet (aka glb) of S. Binary join and meet of two elements x and y are written as $x \sqcup y$ and $x \sqcap y$, respectively. We write \bot and \top to denote the bottom and top element of a partial order Q, respectively, when they exist.

We also assume basic familiarity with category theory [4]. In this article:

- Set denotes the category of sets and functions (alias maps).
- $\mathcal{P}\!o$ denotes the category of preorders and monotonic maps.
- $-\mathcal{Po}_0$ denotes the full (reflective) sub-category of \mathcal{Po} consisting of posets.
- ${\mathcal T}\!{op}$ denotes the category of topological spaces and continuous maps.
- $-\mathcal{T}op_0$ denotes the full (reflective) sub-category of $\mathcal{T}op$ consisting of T_0 -spaces.

All categories above have small limits and colimits. Set, $\mathcal{P}o$ and $\mathcal{P}o_0$ have also exponentials, thus they are examples of symmetric monoidal closed categories [18]. $\mathcal{P}o$ and $\mathcal{T}op$ (and their sub-categories) can be viewed as $\mathcal{P}o$ -enriched categories [18], e.g., the hom-set $\mathcal{P}o(X, Y)$ of monotonic maps from X to Y can be equipped with the pointwise preorder induced by the preorder Y.

Other categories introduced in subsequent sections are \mathcal{Po} -enriched, and this additional structure is relevant when defining adjunctions and equivalences between two objects of a \mathcal{Po} -enriched category.

Definition 1 (Adjunction). Given a pair of maps $X \xleftarrow{f}{\leftarrow} g \xrightarrow{f} Y$ in a Po-enriched category \mathcal{A} , we say that they form:

- 1. an adjunction (notation $f \dashv g$) \iff $f \circ g \leq id_Y$ and $id_X \leq g \circ f$, in which f and g are called left- and right-adjoint, respectively.
- 2. an equivalence $\stackrel{\Delta}{\iff}$ id_Y $\leq f \circ g \leq id_Y$ and id_X $\leq g \circ f \leq id_X$.

We use ' \in ' for set membership (e.g., $x \in X$), but we use ':' for membership of function types (e.g., $f:X \to Y$) and to denote objects and arrows in categories (e.g., X:Top and f:Top(X,Y)). The powerset of a set X is denoted by P(X). Subset inclusion is denoted by \subseteq , whereas strict (proper) subset inclusion is denoted by \subset . The finite powerset (i.e., the set of finite subsets) of X is denoted by $P_f(X)$, and $A \subseteq_f B$ denotes that A is a finite subset of B.

We denote with ω the set of natural numbers, and identify a natural number with the set of its predecessors, i.e., $0 = \emptyset$ and $n = \{0, ..., n - 1\}$, for any $n \ge 1$.

2 Quantales

Conceptually, a quantale [23,24,2] is a degenerate case of monoidal category [18], in the same way that a partial order is a degenerate case of category.

Definition 2 (Quantale). A quantale $(Q, \sqsubseteq, \otimes)$ is a complete lattice (Q, \sqsubseteq) with a monoid structure (Q, \otimes, \mathbf{u}) satisfying the following distributive laws:

$$x \otimes (\sqcup S) = \sqcup \{ x \otimes y \mid y \in S \} \quad and \quad (\sqcup S) \otimes x = \sqcup \{ y \otimes x \mid y \in S \},$$

for any $x \in Q$ and $S \subseteq Q$. A quantale is **trivial** when $\bot = u$ (which implies that $\forall x \in Q$. $\bot = x$), **affine** when $u = \top$, **linear** when \sqsubseteq is a linear order, and **commutative** when \otimes is commutative (in this case the two distributive laws are inter-derivable). A **frame**⁵ is a quantale where $\otimes = \sqcap$ (thus, necessarily commutative and affine).

The complete lattice (Q, \sqsubseteq) amounts to a complete and cocomplete category, while the distributivity laws imply that:

- \otimes is monotonic. Thus, (Q, \otimes, u) makes (Q, \sqsubseteq) a (strict) monoidal category.
- \otimes (viewed as a functor) preserves colimits, in particular $\perp \otimes x = \perp = x \otimes \perp$.

⁵ Alternative names for frame are locale and Heyting algebra, see [17].

These properties imply that the functors $x \otimes -$ and $- \otimes y$, have right-adjoints $x \setminus -$ and -/y, i.e., $x \otimes y \sqsubseteq z \iff y \sqsubseteq x \setminus z$ and $x \otimes y \sqsubseteq z \iff x \sqsubseteq z/y$, called left- and right-residual, respectively. In commutative quantales (i.e., degenerate examples of symmetric monoidal closed categories) $x \setminus z = z/x$ is denoted as [x, z] and is given by $[x, z] = \sqcup \{y \mid x \otimes y \sqsubseteq z\}$.

Example 1. We present some examples of quantales. The first four examples describe linear, commutative and affine quantales (some are frames). The last two items (excepts in degenerate cases) give non-linear, non-commutative and non-affine quantale. The construction Q/u always returns an affine quantale and preserves the linearity and commutative properties, while $\prod_{j \in J} Q_j$ and Q^P preserve the affine and commutative properties.

- 1. The quantale \mathbb{R}_+ of [20] is the set of non-negative real numbers extended with ∞ , with $x \sqsubseteq y \iff x \ge y$ and $x \otimes y \triangleq x + y$. Therefore, $\sqcup S = \inf S$, $\sqcap S = \sup S$, $\bot = \infty$, $\mathsf{u} = \top = 0$, [x, z] = z x if $x \le z$ else 0.
- 2. \mathbb{R}_{\sqcap} is similar to \mathbb{R}_+ , but $x \otimes y \stackrel{\triangle}{=} x \sqcap y = \max(x, y)$. Thus, \mathbb{R}_{\sqcap} is a frame, $\mathsf{u} = 0$, [x, z] = z if $x \leq z$ else $0 \ (\top, \bot, \sqcup S)$, and $\sqcap S$ are the same as in \mathbb{R}_+).
- 3. \mathbb{N}_+ is the *sub-quantale* of \mathbb{R}_+ whose carrier is the set of natural numbers extended with ∞ . \mathbb{N}_{\sqcap} is the *sub-frame* of \mathbb{R}_{\sqcap} with the same carrier as \mathbb{N}_+ .
- 4. Σ is the *sub-quantale* of \mathbb{R}_+ whose carrier is $\{0, \infty\}$. Σ is a frame.
- 5. Q/u is the sub-quantale of Q whose carrier is $\{x \in Q \mid x \sqsubseteq u\}$. Thus, u is the top element of Q/u.
- 6. $\prod_{i \in J} Q_i$ is the product of the quantales Q_i , with \sqsubseteq and \otimes defined pointwise.
- 7. Q^P is the quantale of monotonic maps from the poset P to the quantale Q, with \sqsubseteq and \otimes defined pointwise.
- 8. $(\mathsf{P}(M), \subseteq, \otimes)$ is the quantale (actually a boolean algebra) of subsets of the monoid (M, \cdot, e) , with $\mathsf{u} = \{e\}$ and $A \otimes B \stackrel{\Delta}{=} \{a \cdot b \mid a \in A, b \in B\}$.
- 9. $(\mathsf{P}(X^2), \subseteq, \otimes)$ is the quantale (boolean algebra) of relations on the set X, with $\mathsf{u} = \{(x, x) \mid x \in X\}$ and:

$$R \otimes S \stackrel{\vartriangle}{=} \{ (x, z) \mid \exists y \in X. (x, y) \in R, (y, z) \in S \}.$$

Example 2. We consider some quantales arising in the analysis of algorithms. We identify algorithms with multi-tape deterministic Turing Machines (TM), which accept/reject strings written in a finite input alphabet A. In this context, one is interested in quantale-valued cost functions $X \to Q$, rather than distances.

- The size s(w) of an input w for a TM is a value in the quantale \mathbb{N}_+ , namely the length of the string w. In particular, the size of an infinite string is ∞ , and the size of the concatenation of two strings is the sum of their sizes.
- The time (i.e., the number of steps) taken by a TM on a specific input w is again a value in \mathbb{N}_+ . In particular, a TM failing to terminate on w takes time ∞ , and the time taken for executing sequentially two TMs on w is the sum of the times taken by each TM (plus a linear overhead for copying w on two separate tapes, so that the two TMs work on disjoint sets of tapes).

The time complexity associated to a TM typically depends on the input (or its size), thus it cannot be a cost in \mathbb{N}_+ . Such cost should be drawn from a quantale reflecting this dependency, namely a *higher-order* quantale.⁶ We now describe some of such quantales from the most precise to the most abstract.

- 1. The most precise quantale is $\mathbb{N}_{+}^{A^*}$ (i.e., the product of A^* copies of \mathbb{N}_{+}). A $t \in \mathbb{N}_{+}^{A^*}$ maps each finite input $w \in A^*$ to the time taken by a TM on w.
- 2. A first abstraction is to replace $t \in \mathbb{N}^{A^*}_+$ with $T \in \mathbb{N}^{\omega}_+$, where T(n) is the best upper-bound for the time taken by a TM on inputs of size n, i.e., $T(n) = \max\{t(w) \mid s(w) = n\}$.
- 3. In practice (by the linear speed-up theorem), time complexity is given in O-notation, i.e., $T \in \mathbb{N}^{\omega}_{+}$ is replaced with the subset O(T) of \mathbb{N}^{ω}_{+} such that $T' \in O(T) \iff \forall n \geq n_0.T'(n) \leq C * T(n)$ for some n_0 and C in ω . If we replace \mathbb{N}^{ω}_{+} with the partial order L_O of O-classes O(T) ordered by reverse inclusion, we get a distributive lattice (i.e., binary meets distribute over finite joins, and conversely): the top is

⁶ This resembles higher-order distances used to compare functional programs [7,25].

O(0), the bottom is $O(\infty)$, the join $O(T_1) \sqcup O(T_2)$ is $O(T_1) \cap O(T_2) = O(T_1 \sqcup T_2) = O(\min(T_1, T_2))$, the meet $O(T_1) \cap O(T_2)$ is $O(T_1 \cap T_2) = O(\max(T_1, T_2)) = O(T_1 + T_2)$.

The lattice L_O is distributive, because the complete lattice underlying \mathbb{N}^{ω}_+ is distributive, but it is not a frame (as it fails to have arbitrary joins). However, there is a general construction, see [17, page 69], which turns a distributive lattice L into the *free frame* I(L) over L. More precisely, I(L) is the poset of *ideals* in L ordered by inclusion, and the embedding $x \mapsto \downarrow x$ from L to I(L)preserves finite meets and joins.

4. A simpler way to obtain a frame is to take the subset of L_O consisting of the $O(n^k)$ with $k \in [0, \infty]$. This linear frame is isomorphic to \mathbb{N}_{\Box} , namely $k \in \mathbb{N}_{\Box}$ corresponds to $O(n^k)$.

There are several notions of morphism between quantales, we consider those corresponding to lax and strict monoidal functors.

Definition 3. A monotonic map $h: Q \to Q'$ between quantales is called:

 $- \textit{lax-monoidal} \iff \mathsf{u'} \sqsubseteq' h(\mathsf{u}) \textit{ and } \forall x, y \in Q.h(x) \otimes' h(y) \sqsubseteq h(x \otimes y);$

 $- \ \textit{strict-monoidal} \ \Longleftrightarrow \ \mathsf{u}' = h(\mathsf{u}) \ \textit{and} \ \forall x, y \in Q.h(x) \otimes' h(y) = h(x \otimes y).$

Qut denotes the \mathcal{Po}_0 -enriched category of quantales and lax-monoidal maps, where $\operatorname{Qut}(Q,Q')$ has the pointwise order induced by the order on Q'.

We give some examples of monotonic maps between quantales.

Example 3. In the following diagram we write \longrightarrow for lax- and \longrightarrow for strict-monoidal maps, 1 for the trivial quantale (with only one element *), $!_Q$ for the unique map from Q to 1, and $f \dashv g$ for "f is left-adjoint to g":

$$1 \xrightarrow[]{q}{\stackrel{}{\longrightarrow}} Q \xrightarrow[]{q}{\stackrel{}{\longrightarrow}} Q/\mathsf{u} \xrightarrow[]{q}{\stackrel{}{\longrightarrow}} \Sigma \qquad \mathbb{N}_+ \xrightarrow[]{q}{\stackrel{}{\longleftarrow}} \mathbb{R}_+ \xleftarrow[]{id}{\stackrel{}{\longrightarrow}} \mathbb{R}_{\sqcap}$$

 $- \top_Q \text{ maps } * \text{ to } \top;$

- f is the inclusion of Q/\mathfrak{u} into Q, and g maps x to $x \sqcap \mathfrak{u}$;

-f' maps \perp to \perp and \top to \top , and g' maps \top to \top and $x \sqsubset \top$ to \perp ;

-i is the inclusion, $c(x) = \lceil x \rceil$ is integer round up, and *id* is the identity.

The frames for measuring the time complexity of TMs (see Example 2) are related by obvious monoidal maps going from the more precise to the more abstract frame:

$$\mathbb{N}_{+}^{A^{*}} \xrightarrow{f} \mathbb{N}_{+}^{\omega} \xrightarrow{g} I(L_{O}) \xrightarrow{h} \mathbb{N}_{\square}$$

- f maps $t \in \mathbb{N}_{+}^{A^*}$ to $T \in \mathbb{N}_{+}^{\omega}$ such that $T(n) = \max\{t(w) \mid s(w) = n\};$
- g maps $T \in \mathbb{N}^{\omega}_+$ to the principal ideal $\downarrow O(T) \in I(L_O)$;
- h maps $X \in I(L_O)$ to $n \in \mathbb{N}_{\square}$ such that $n = \min\{k \mid \forall A \in X.A \subseteq O(n^k)\}$.

2.1 Continuous Quantales

To reinterpret in quantale-valued metric spaces the common ϵ - δ definition of continuous maps, and relate such spaces to topological spaces, we restrict to *continuous quantales*, i.e., quantales whose underlying lattices are continuous. Note that linear quantales are always continuous. We recall the definition of a continuous lattice and related notions. More details may be found in [12,1,14].

Definition 4. Given a complete lattice (Q, \sqsubseteq) and $x, y \in Q$, we say that:

- 1. $D \subseteq Q$ is directed $\iff \forall x, y \in D. \exists z \in D. x \sqsubseteq z \text{ and } y \sqsubseteq z$.
- 2. x is **way-below** y (notation $x \ll_Q y$, or $x \ll y$ when Q is clear from the context) \iff for any directed subset D of Q, $y \sqsubseteq \sqcup D \implies \exists d \in D.x \sqsubseteq d$.
- 3. x is compact $\iff x \ll x$.

We write $\downarrow y$ for $\{x \in Q \mid x \ll y\}$, and Q_0 for the set of compact elements in Q.

The following are some basic properties of the way-below relation.

Proposition 1. In any complete lattice (Q, \sqsubseteq) , and for all $x, x_0, x_1 \in Q$:

1. $x_0 \ll x_1 \implies x_0 \sqsubseteq x_1$. 2. $x'_0 \sqsubseteq x_0 \ll x_1 \sqsubseteq x'_1 \implies x'_0 \ll x'_1$. 3. $\perp \ll x$. 4. $\downarrow x \text{ is directed. In particular, } x_0, x_1 \ll x \implies x_0 \sqcup x_1 \ll x$.

Definition 5 (Continuous Lattice). Given a complete lattice Q, we say that:

1. Q is continuous $\Leftrightarrow^{\Delta} \forall x \in Q.x = \sqcup \downarrow x.$

2. $B \subseteq Q$ is a **base** for $Q \iff \forall x \in X.B \cap \downarrow x$ is directed and $x = \sqcup (B \cap \downarrow x)$.

3. Q is ω -continuous \Leftrightarrow^{Δ} Q has a countable base.

4. Q is algebraic $\Leftrightarrow^{\Delta} Q_0$ is a base for Q.

A complete lattice Q is continuous exactly when it has a base. Any base for Q must includes Q_0 . The set Q_0 is a base only when Q is algebraic and the bottom element \perp is always compact. Continuous lattices enjoy the following interpolation property (see [1, Lemma 2.2.15]):

Lemma 1. For any continuous lattice Q and $q_1, q_2 \in Q$, $q_1 \ll q_2 \implies \exists q \in Q.q_1 \ll q \ll q_2$.

Continuous quantales enjoy a further interpolation property:

Lemma 2. In every continuous quantale, $q_1 \ll q_2 \implies \exists q \ll u.q_1 \ll q_2 \otimes q$ and $q_1 \ll q_2 \implies \exists q \ll u.q_1 \ll q \otimes q_2$.

Proof. Appendix A.1.

Example 4. The quantales in Example 1 have the following properties:

- $-\mathbb{N}_+, \mathbb{N}_{\sqcap}$, and Σ are ω -algebraic. More precisely, all elements in these quantales are compact, and $x \ll y \iff x \ge y$ (or equivalently $x \sqsubseteq y$).
- $-\mathbb{R}_+$ and \mathbb{R}_{\sqcap} are ω -continuous, e.g., the set of rational numbers with ∞ is a base, $x \ll y \iff (x = \infty \lor x > y)$, and ∞ is the only compact element.
- $\mathsf{P}(M)$ and $\mathsf{P}(X^2)$ are algebraic, the sets of compact elements are $\mathsf{P}_f(M)$ for $\mathsf{P}(M)$ and $\mathsf{P}_f(X^2)$ for $\mathsf{P}(X^2)$, and $A \ll B \iff A \subseteq_f B$.

Continuous lattices (and quantales) have the following closure properties:

Proposition 2. Continuous (algebraic) lattices are closed under small products. ω -continuous lattices are closed under countable products.

Proof. The claims follow from the fact that if $\forall j \in J.B_j$ is a base for Q_j , then $\{x \in \prod_{j \in J} B_j \mid \exists J_0 \subseteq_f J.\forall j \in J - J_0.x_j = \bot_j\}$ is a base for $\prod_{j \in J} Q_j$.

We conclude by observing that linear quantales are always continuous.

Proposition 3. Every linear quantale is continuous.

Proof. Use [12, Exercise 1.7], where linearly ordered complete lattices are called complete chains. \Box

3 Quantale-valued Metric Spaces

In [20], Lawvere views metric spaces as \mathbb{R}_+ -enriched categories, and shows that several definitions and results on metric spaces are derivable from general results on \mathcal{V} -enriched categories, where \mathcal{V} is a symmetric monoidal closed category (see [18]). We replace \mathbb{R}_+ with a quantale Q, and consider the $\mathcal{P}o$ -enriched category of Q-metric spaces and short maps, whose objects are Q-enriched small categories and whose arrows are Q-enriched functors.

Definition 6 (Met_Q). Given a quantale Q, the Po-enriched category Met_Q of Q-metric spaces and short maps is given by:

- **objects** are pairs (X,d) with $d:X^2 \to Q$ satisfying $d(x,y) \otimes d(y,z) \sqsubseteq d(x,z)$ and $u \sqsubseteq d(x,x)$; d induces on X the d-preorder $x \leq_d y \iff u \sqsubseteq d(x,y)$.
- arrows in $\mathcal{M}et_Q((X,d), (X',d'))$ are $f: X \to X'$ satisfying $\forall x, y \in X.d(x,y) \sqsubseteq d'(f(x), f(y))$ with hom-preorder $f \leq f' \Leftrightarrow \forall x \in X.f(x) \leq_{d'} f'(x)$.

An arrow $f: \mathcal{M}et_Q((X,d),(X',d'))$ is said to be an **isometry** when $\forall x, y \in X.d(x,y) = d'(f(x),f(y))$.

In comparison with the properties of a standard metric d, we have that:

- the triangular inequality $d(x, z) \leq d(x, y) + d(y, z)$ becomes $d(x, y) \otimes d(y, z) \sqsubseteq d(x, z)$. Note that, in \mathbb{R}_+ , the order \sqsubseteq is \geq , and $\otimes = +$;
- $-d(x,y) = 0 \iff x = y$ is replaced by the weaker property $\mathbf{u} \sqsubseteq d(x,x)$, which corresponds to d(x,x) = 0. Note that in \mathbb{R}_+ , we have $\mathbf{u} = 0 = \top$;
- symmetry d(x, y) = d(y, x) is unusual in (enriched) category theory.

In the absence of symmetry, separation, i.e., $d(x, y) = 0 \implies x = y$, should be recast as $(d(x, y) = 0 \land d(y, x) = 0) \implies x = y$, which in a quantale setting becomes $(\mathsf{u} \sqsubseteq d(x, y) \land \mathsf{u} \sqsubseteq d(y, x)) \implies x = y$. The objects with this property are exactly the (X, d) such that the preorder \leq_d is a poset. Section 5 gives a more abstract definition of separated object in a $\mathcal{P}o$ -enriched category.

Example 5. We relate $\mathcal{M}et_Q$ for some quantales Q to more familiar categories:

- 1. \mathbb{R}_{\square} -metric spaces generalize ultrametric spaces, i.e., spaces where the metric satisfies $d(x, z) \leq \max(d(x, y), d(y, z))$.
- 2. $\mathcal{M}et_{\Sigma}$ is (isomorphic to) the $\mathcal{P}o$ -enriched category $\mathcal{P}o$ of preorders and monotonic maps, and the *separated objects* of $\mathcal{M}et_{\Sigma}$ are the posets.
- 3. $\mathcal{M}et_1$ is the category $\mathcal{S}et$ of sets and functions, with the chaotic preorder on $\mathcal{S}et(X,Y)$, i.e., $f \leq g$ for every $f, g; \mathcal{S}et(X,Y)$, and the *separated objects* of $\mathcal{M}et_1$ are the sets with at most one element.

We summarize some properties of $\mathcal{M}et_Q$, which ignore the $\mathcal{P}o$ -enrichment, proved in [18] for a generic complete and cocomplete symmetric monoidal closed category in place of a quantale Q.

Proposition 4. For any quantale Q, the category Met_Q has small products, small sums, equalizers and coequalizers.

Proof. Appendix A.2.

Lax-monoidal maps induce $\mathcal{P}o$ -enriched functors.

Definition 7. Given a lax monoidal map h: Qnt(P,Q), the **Po**-enriched functor $h: Met_P \to Met_Q$ is such that $h(X,d) \triangleq (X, h \circ d)$ and is the identity on arrows.

4 Topologies on *Q*-metric spaces

When Q is a continuous quantale, one can establish a relation between $\mathcal{M}et_Q$ and $\mathcal{T}op$, thereby generalizing the open ball topology induced by a standard metric. In general, to a Q-metric d on X one can associate at least two topologies on X. When Q is ω -continuous—a restriction desirable from a computational viewpoint (see [26])—convergence can be defined in terms of sequences.

Definition 8. Given a continuous quantale Q and (X, d): \mathcal{Met}_Q , the open ball with center $x \in X$ and radius $\delta \ll u$ is $B(x, \delta) \triangleq \{y \in X \mid \delta \ll d(x, y)\}$. The open ball topology τ_d is the topology generated by the family of open balls.

One can define also the **dual open ball** $B^o(x, \delta) \triangleq \{y \in X \mid \delta \ll d(y, x)\}$, and the corresponding **dual open ball topology** τ_d^o .

When d is symmetric, i.e., d(x, y) = d(y, x), the two notions of open ball agree. In the rest of this section, we focus on open balls only, but the results hold *mutatis mutandis* also for the dual notion. The following proposition implies that open balls form a *base* for τ_d , i.e., every open in τ_d is a union of open balls.

Proposition 5. Open balls satisfy the following properties:

 $\begin{array}{ll} 1. \ x \in B(x,\delta). \\ 2. \ \delta \sqsubseteq \delta' \implies B(x,\delta') \subseteq B(x,\delta). \\ 3. \ y \in B(x,\delta) \implies \exists \delta' \ll \mathsf{u}.B(y,\delta') \subseteq B(x,\delta). \\ 4. \ y \in B(x_1,\delta_1) \cap B(x_2,\delta_2) \implies \exists \delta' \ll \mathsf{u}. \ B(y,\delta') \subseteq B(x_1,\delta_1) \cap B(x_2,\delta_2). \end{array}$

Proof. Appendix A.3.

We show that, for continuous quantales, continuity with respect to the open ball topology can be recast in terms of the usual *epsilon-delta* formulation:

Lemma 3. If (X, d): $\mathcal{M}et_Q$, with Q continuous, and $O \subseteq X$, then $O \in \tau_d \iff$

$$\forall x \in O. \exists \delta \ll \mathsf{u}. B(x, \delta) \subseteq O. \tag{1}$$

Proof. Appendix A.4.

The following result characterizes the closed subsets for the topology τ_d . Informally, the closure of a subset A can be described as the set of points *from* which one can reach a point in A within any arbitrarily small distance.

Lemma 4. If (X,d): $\mathcal{M}et_Q$, with Q continuous, and $A \in \mathsf{P}(X)$, then the closure of A in the topological space (X, τ_d) is given by:

$$\overline{A} = \{ y \in X \mid \forall \delta \ll \mathsf{u}. \exists x \in A. \delta \ll d(y, x) \}.$$
⁽²⁾

Proof. To prove that \overline{A} is the closure of A, we show that $z \notin \overline{A} \iff$ exists $\delta \ll \mathfrak{u}$ such that $B(z, \delta)$ and A are disjoint. The claim follows from the equivalences: $z \notin \overline{A} \iff \exists \delta \ll \mathfrak{u}. \forall x \in A.\delta \ll d(z, x) \iff \exists \delta \ll \mathfrak{u}. B(z, \delta) \cap A = \emptyset$.

Theorem 1. Given a continuous quantale Q_i and an object (X_i, d_i) : Met_{Q_i} for each $i \in \{1, 2\}$, if $f: X_1 \to X_2$, then $f: Top((X_1, \tau_{d_1}), (X_2, \tau_{d_2})) \iff$

$$\forall x \in X_1. \forall \epsilon \ll \mathsf{u}_2. \exists \delta \ll \mathsf{u}_1. f(B(x, \delta)) \subseteq B(f(x), \epsilon).$$
(3)

Proof. Appendix A.5.

The above characterization of continuous maps suggests a variant of $\mathcal{T}op$ in which the objects are Q-metric spaces (for some continuous quantale Q) instead of topological spaces, while the rest is unchanged (see [5]):

Definition 9. The Po-enriched category Met_c of metric spaces and continuous maps is given by:

objects are the triples (X, d, Q) with Q continuous quantale and (X, d): \mathcal{Met}_Q ; arrows in $\mathcal{Met}_c((X, d, Q), (X', d', Q'))$ are $f: \mathcal{Top}((X, \tau_d), (X', \tau_{d'}))$, or equivalently $f: X \to X'$ satisfying $\forall x \in X. \forall \epsilon \ll u'. \exists \delta \ll u. f(B(x, \delta)) \subseteq B(f(x), \epsilon)$.

Similarly, one can define the sub-category $\mathcal{M}et_u$ of $\mathcal{M}et_c$ with the same objects, but whose arrows are the *uniformly continuous* maps, i.e., $f: X \to X'$ satisfying $\forall \epsilon \ll \mathsf{u}'. \exists \delta \ll \mathsf{u}. \forall x \in X. f(B(x, \delta)) \subseteq B(f(x), \epsilon).$

4.1 Imprecision and Robustness

We extend the notions of imprecision and robustness, that in [21,22] are defined for standard metric spaces, to Q-metric spaces for a continuous quantale Q^7 . Since a Q-metric may fail to be symmetric, we must consider the "direction" along which the distance is measured. In particular, in the presence of imprecision, two subsets are indistinguishable when they have the same closure in the dual topology τ_d^o , rather than in the topology τ_d (Proposition 7). This difference cannot be appreciated when d is symmetric, because the two topologies coincide.

⁷ It is possible to relax the assumption of continuity of Q along the lines of [5].

- 1. $B_R(A, \delta) \triangleq \{y \in X \mid \exists x \in A. \delta \ll d(x, y)\} = \bigcup_{x \in A} B(x, \delta) \subseteq X$ is the set of points belonging to $A \subseteq X$ with precision greater than $\delta \ll u.^8$
- 2. $A_{\delta} \stackrel{\Delta}{=} \overline{B_R(A, \delta)}^o \subseteq X$ is the δ -flattening of $A \subseteq X$ with $\delta \ll \mathfrak{u}$, where \overline{A}^o is the closure of A in τ_d^o (see Lemma 4).

Proposition 6. The subsets $B_R(A, \delta)$ have the following properties:

- 1. $A \subseteq B_R(A, \delta) \subseteq B_R(A', \delta')$ when $A \subseteq A' \subseteq X$ and $\delta' \sqsubseteq \delta \ll u$.
- 2. $B_R(B_R(A, \delta_1), \delta_2) \subseteq B_R(A, \delta)$ when $\delta_1, \delta_2 \ll \mathsf{u}$ and $\delta \ll \delta_1 \otimes \delta_2[\sqsubseteq \delta_i]$.
- 3. $\overline{A}^o = \bigcap_{\delta \ll u} B_R(A, \delta)$ for every $A \subseteq X$.
- 4. $B_R(\overline{A}^o, \delta) = B_R(A, \delta)$ for every $A \subseteq X$ and $\delta \ll u$, i.e., A and \overline{A}^o are indistinguishable in the presence of imprecision.
- 5. $B_R(A, \delta) \subseteq A_\delta \subseteq B_R(A, \delta')$ when $A \subseteq X$ and $\delta' \ll \delta \ll u$.

Proof. Appendix A.6.

Example 6. Consider the Q-metric space (X, d), where $Q = X = \mathbb{R}_+$ and $d(x, y) \triangleq y - x$ if $x \leq y$ else 0. If A = [a, b] and $\delta \in (0, +\infty)$, then $\overline{A} = [a, +\infty]$, $\overline{A}^\circ = [0, b]$, and $B_R(\overline{A}^\circ, \delta) = B_R(A, \delta) = [0, b + \delta)$, as depicted in Fig. 1.

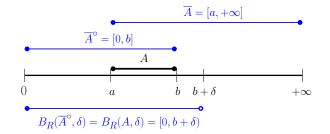


Fig. 1. Graphic recast of Example 6.

We can generalize to this wider setting also the definition of robust topology in [21, Definition A.1]. We define such topology on P(X), rather than on the set of closed subsets in the topology τ_d^o , since the restriction to the set of closed subsets amounts to replacing a topological space with an equivalent *separated* topological space (see Section 5).

Definition 11. Given a Q-metric space (X, d), with Q continuous, the **robust topology** $\tau_{d,R}$ on $\mathsf{P}(X)$ is given by:

$$U \in \tau_{d,R} \iff \forall A \in U. \exists \delta \ll \mathsf{u}.\mathsf{P}(B_R(A,\delta)) \subseteq U.$$

Finally, we characterize the specialization preorder $\leq_{\tau_{d,R}}$ induced by the robust topology $\tau_{d,R}$ on $\mathsf{P}(X)$. As a consequence, we have that two subsets are indistinguishable in $\tau_{d,R}$ exactly when they have the same closure in τ_d^o .

Proposition 7. Let (X, d) be a Q-metric space with Q continuous, and $A, B \subseteq X$. Then, we have $A \leq_{\tau_{d,R}} B \iff B \subseteq \overline{A}^o$.

Proof. Appendix A.7.

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⁸ The terminology used in [22] is "with imprecision less than δ ".

5 Separation in Preorder-enriched Categories

Structures like preorders and topologies have a notion of *indistinguishability* between elements. Informally, in such structures, *separation* can be understood as the property requiring that indistinguishable elements are equal.

In this section, we define and study this notion in the setting of $\mathcal{P}o$ -enriched categories. We also show that the definition separation in this abstract setting subsumes many *set-theoretic* definitions within specific categories, in particular the category of Q-metric spaces.

Definition 12 (Separation). Given a $\mathcal{P}o$ -enriched category \mathcal{A} , we say that:

- 1. $f, g \in \mathcal{A}(X, Y)$ are equivalent (notation $f \sim g) \iff f \leq g \wedge g \leq f$.
- 2. the hom-preorder $\mathcal{A}(X,Y)$ is separated \iff it is a poset.
- 3. the object $Y \in \mathcal{A}$ is separated $\iff \mathcal{A}(X,Y)$ is separated for every $X \in \mathcal{A}$.
- 4. \mathcal{A} is separated \Leftrightarrow Y is separated for every $Y \in \mathcal{A}$, i.e., \mathcal{A} is \mathcal{Po}_0 -enriched.

Remark 1. The definition of " $\mathcal{A}(X,Y)$ is separated" can be recast in terms of equivalence, i.e., $f \sim g \implies f = g$, for every $f, g: \mathcal{A}(X,Y)$. There is a similar recast also for the definition of " \mathcal{A} is separated", i.e., $f \sim g \implies f = g$, for every pair (f,g) of parallel arrows in \mathcal{A} . In some $\mathcal{P}o$ -enriched categories, separated objects have a set-theoretic characterization that does not refer to arrows:

- 1. in $\mathcal{P}o$, separated objects are posets.
- 2. In Top, separated objects are T_0 -spaces.
- 3. In $\mathcal{M}et_Q$, separated objects are separated Q-metric spaces (see Section 3).

Recall from [18] that a $\mathcal{P}o$ -enriched functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ is full&faithful (notation $F: \mathcal{A} \longrightarrow \mathcal{B}$) when the maps $F_{X,Y}: \mathcal{A}(X,Y) \to \mathcal{B}(FX,FY)$ are iso in $\mathcal{P}o$, and a $\mathcal{P}o$ -enriched sub-category \mathcal{A} of \mathcal{B} is full when the $\mathcal{P}o$ -enriched inclusion functor is full&faithful.

Definition 13. If \mathcal{A} is a $\mathcal{P}o$ -enriched category, then \mathcal{A}_0 denotes the full sub-category of separated objects in \mathcal{A} .

If every object in \mathcal{A} is separated, then \mathcal{A}_0 is equal to \mathcal{A} . A weaker property is that every object in \mathcal{A} is *equivalent* (in the sense of Definition 1) to one in \mathcal{A}_0 . This weaker property holds in $\mathcal{P}o$, $\mathcal{T}op$, and $\mathcal{M}et_Q$.

Proposition 8. In $\mathcal{M}et_{\mathcal{Q}}$, every object is equivalent to a separated one.

Proof. Appendix A.8.

If every object in \mathcal{A} is equivalent to a separated one, then every $\mathcal{P}o$ -enriched endofunctor on \mathcal{A} can be *transformed* into one that factors through \mathcal{A}_0 . This *transformer* lifts to the category of $\mathcal{P}o$ -enriched monads on \mathcal{A} .

Definition 14. Given a Po-enriched category \mathcal{A} , we denote by $Mon(\mathcal{A})$ the category of Po-enriched monads on \mathcal{A} and monad maps, *i.e.*

objects: Po-enriched monads on \mathcal{A} , i.e., triples $\hat{M} = (M, \eta, -^*)$, where:

- -M is a function on the objects of \mathcal{A} ,
- $-\eta$ is a family of arrows $\eta_X: \mathcal{A}(X, MX)$ for $X: \mathcal{A}$,
- -* is a family of monotonic maps $\mathcal{A}(X, MY) \rightarrow \mathcal{A}(MX, MY)$ between hom-preorders for $X, Y: \mathcal{A},$

and satisfy the equations:

$$\eta_X^* = \mathsf{id}_{MX} \quad , \quad f^* \circ \eta_X = f \quad , \quad g^* \circ f^* = (g^* \circ f)^*. \tag{4}$$

arrows: θ from \hat{M} to \hat{M}' are families of maps $\theta_X: \mathcal{A}(MX, M'X)$ for $X: \mathcal{A}$ satisfying the equations:

$$\theta_X \circ \eta_X = \eta'_X \quad , \quad \theta_Y \circ f^* = (\theta_Y \circ f)^{*'} \circ \theta_X.$$
(5)

A basic **monad transformer** on $Mon(\mathcal{A})$ is a pair (T, in), where T is function on the objects of $Mon(\mathcal{A})$ and in is a family of monad maps $in_{\hat{M}}$ from \hat{M} to $T\hat{M}$.

Remark 2. The category $\mathsf{Mon}(\mathcal{A})$ can be made \mathcal{Po} -enriched. The enrichment is relevant for defining equivalence of monads. For our purposes, however, it suffices to relate (by a monad map) a generic \mathcal{Po} -enriched monad on \mathcal{A} to one that factors through \mathcal{A}_0 .

We use the simplest form of monad transformer among those in the taxonomy of [16], i.e., *basic transformer*. However, the monad transformer described in the following theorem can be shown to be a *monoidal transformer*.

Theorem 2. If \mathcal{A} is a $\mathcal{P}o$ -enriched category and $(r_X: X \to RX \mid X: \mathcal{A})$ is a family of arrows in \mathcal{A} such that:

$$RX:\mathcal{A}_0 \text{ and } (r_X, s_X) \text{ is an equivalence for some } s_X:RX \to X,$$
 (6)

then (T,in) defined below is a monad transformer on $\mathsf{Mon}(\mathcal{A})$:

- T is the function mapping $\hat{M} = (M, \eta, -^*)$ to $T\hat{M} = (M', \eta', -^{*'})$, where
 - $M'X \stackrel{\scriptscriptstyle \Delta}{=} R(MX)$
 - $\eta'_X \stackrel{\vartriangle}{=} r_{MX} \circ \eta_X : \mathcal{A}(X, M'X)$
 - if $f: \mathcal{A}(X, M'Y)$, then $f^{*'} \triangleq r_{MY} \circ (s_{MY} \circ f)^* \circ s_{MX}: \mathcal{A}(M'X, M'Y)$.
- in is the family of monad maps such that $\inf_{\hat{M},X} \stackrel{\Delta}{=} r_{MX}: \mathcal{A}(MX, M'X).$

Moreover, the definition of T is independent of the choice of s_X .

Proof. Appendix A.9.

6 The Hausdorff-Smyth Monad

In this section, we introduce a \mathcal{Po} -enriched monad P_S on \mathcal{Met}_Q , related to the Hausdorff-Smyth hemi-metric in [13], which *extends* the powerset monad P on \mathcal{Set} to Q-metric spaces. By applying the monad transformer T defined in Section 5, one obtains a separated version of P_S , which amounts to partitioning $\mathsf{P}(X)$ into equivalence classes, for which we define canonical representatives. Finally, we investigate the relation between P_S and the robust topology in Definition 11.

Recall that the monad $(\mathsf{P}, \eta, -^*)$ on \mathcal{Set} is given by $\eta_X: \mathcal{Set}(X, \mathsf{P}(X))$ and $-^*: \mathcal{Set}(X, \mathsf{P}(X')) \to \mathcal{Set}(\mathsf{P}(X), \mathsf{P}(X'))$, where:

$$\eta(x) = \{x\},\$$

$$f^*(A) = \bigcup_{x \in A} f(x).$$

Definition 15 (The P_S **monad).** Let P_S be the function on *Q*-metric spaces such that $\mathsf{P}_S(X,d) = (\mathsf{P}(X), d_S)$, where $d_S: \mathsf{P}(X)^2 \to Q$ is given by:

$$d_S(A,B) = \sqcap_{y \in B} \sqcup_{x \in A} d(x,y).$$

The rest of the monad structure for P_S , i.e., the unit η and the Kleisli extension $-^*$, is inherited from that for P . In particular, $\eta_{(X,d)} = \eta_X$.

We now prove that what we have defined is a $\mathcal{P}o$ -enriched monad on $\mathcal{M}et_{O}$.

Proposition 9. The triple $(\mathsf{P}_S, \eta, -^*)$ is a *Po*-enriched monad on $\mathcal{M}et_Q$, i.e.

- 1. $(\mathsf{P}(X), d_S)$: $\mathcal{M}et_Q$, *i.e.*, $\mathsf{u} \sqsubseteq d_S(A, A)$ and $d_S(A, B) \otimes d_S(B, C) \sqsubseteq d_S(A, C)$.
- 2. η : $\mathcal{M}et_Q(X, \mathsf{P}_S(X))$.
- 3. $f: \mathcal{M}et_Q(X, \mathsf{P}_S(X')) \text{ implies } f^*: \mathcal{M}et_Q(\mathsf{P}_S(X), \mathsf{P}_S(X')).$
- 4. $f \leq g$ in $\mathcal{M}et_Q(X, \mathsf{P}_S(X'))$ implies $f^* \leq g^*$ in $\mathcal{M}et_Q(\mathsf{P}_S(X), \mathsf{P}_S(X'))$.

Moreover, $(\mathsf{P}_S, \eta, -^*)$ satisfies the equations (4) for a monad.

Proof. Appendix A.10.

The Hausdorff-Smyth metric d_S induces a preorder \leq_{d_S} and an equivalence \sim_{d_S} on $\mathsf{P}(X)$. In the following, we define the canonical representative for the equivalence class of $A \subseteq X$ with respect to \sim_{d_S} , called the *-closure of A, which turns out to be the biggest subset of X in the equivalence class.

Definition 16. Given a Q-metric space (X, d), we define:

1. $d(A, y) \triangleq \sqcup_{x \in A} d(x, y) \in Q$ the *-distance from $A \subseteq X$ to $y \in X$. 2. $\widetilde{A} \triangleq \{y \in X \mid u \sqsubseteq d(A, y)\}$ the *-closure of $A \subseteq X$.

Proposition 10. For every Q-metric space (X, d) the following properties hold:

1. $d_S(A, B) = \sqcap_{y \in B} d(A, y)$ and $d(A, y) = d_S(A, \{y\})$. 2. $A \leq_{d_S} B \iff B \subseteq \widetilde{A}$. 3. $A \sim_{d_S} B \iff \widetilde{A} = \widetilde{B}$.

Proof. For each property we give a proof hint.

- 1. The two equalities follow easily from the definition of d_s .
- 2. We have the following chain of equivalences: $-A \leq_{d_S} B \iff \mathsf{u} \sqsubseteq d_S(A, B) \iff$ $-\forall y \in B. \mathsf{u} \sqsubseteq d(A, y) \iff$ $-\forall y \in B. y \in \widetilde{A} \iff B \subseteq \widetilde{A}.$
- 3. Immediate by the characterization of \leq_{d_S} .

6.1 Hausdorff-Smyth and *-Robust Topology

We give a characterization of the topology τ_{d_S} on $\mathsf{P}(X)$ using a topology $\tau_{d,S}$ defined by analogy with the robust topology $\tau_{d,R}$ of Section 4.1. In summary, we have that $\tau_{d_S} = \tau_{d,S} \subseteq \tau_{d,R}$ when Qis continuous, and $\tau_{d,S} = \tau_{d,R}$ when Q is linear and non-trivial.

Definition 17. Given a Q-metric space (X, d), with Q continuous, we define the topology $\tau_{d,S}$ on $\mathsf{P}(X)$:

- 1. $B_S(A, \delta) \triangleq \{y \in X \mid \delta \ll d(A, y)\} \subseteq X$ is the set of points belonging to $A \subseteq X$ with *-precision greater than $\delta \ll u$.
- 2. the *-robust topology $\tau_{d,S}$ on $\mathsf{P}(X)$ is given by: $U \in \tau_{d,S} \iff \forall A \in U. \exists \delta \ll \mathsf{u}.\mathsf{P}(B_S(A, \delta)) \subseteq U.$

Lemma 5. The subsets $B_S(A, \delta)$ have the following properties:

1. $B_R(A, \delta) \subseteq B_S(A, \delta) \subseteq B_S(A', \delta')$ when $A \subseteq A' \subseteq X$ and $\delta' \sqsubseteq \delta \ll \mathfrak{u}$. 2. $\delta \sqsubseteq d_S(A, B_S(A, \delta))$ for every $A \subseteq X$ and $\delta \ll \mathfrak{u}$.

Proof. Appendix A.11.

Proposition 11. For every Q-metric space (X, d) with Q continuous:

$$\tau_{d_S} = \tau_{d,S} \subseteq \tau_{d,R}$$

Proof. Appendix A.12.

Lemma 6. For every (X, d): $\mathcal{M}et_Q$ with Q continuous, $A \subseteq X$, $y \in X$, and $\delta \in Q$:

$$\delta \ll d(A, y) \iff \exists A_0 \subseteq_f A.\delta \ll d(A_0, y).$$

Moreover, if Q is linear and $\perp \neq \delta$, then:

$$\delta \ll d(A, y) \iff \exists x \in A. \delta \ll d(x, y).$$

Proof. Appendix A.13.

Proposition 12. If Q is a linear non-trivial quantale, then $\tau_{d,R} = \tau_{d,S}$.

Proof. Appendix A.14.

Remark 3. Propositions 12 and 7 ensure that, when Q is linear and non-trivial, by applying the monad transformer T of Section 5, we get a monad mapping a Q-metric space (X, d) to the separated Q-metric space of closed subsets of X with respect to the dual topology τ_d^o with the Hausdorff-Smyth metric. In this way, we recover the setting of [22] as a special case.

Example 7. When the quantale Q is not linear, the robust topology $\tau_{d,R}$ can be strictly finer than the *-robust topology $\tau_{d,S}$. For instance, consider the Q-metric space (X,d), in which $Q = \mathbb{R}_+ \times \mathbb{R}_+$, $X = \mathbb{R}^2$, and the distance is given by d((x,y),(x',y')) = (|x-x'|,|y-y'|). Let $\delta_0 \triangleq (1,1)$ and note that $\mathbf{u} = (0,0)$. Take $A \triangleq \{(0,2),(2,0)\} \subseteq \mathbb{R}^2$, $p \triangleq (2,2) \in \mathbb{R}^2$, and consider the set $O \triangleq \bigcup_{\delta_0 \ll \delta' \ll \mathbf{u}} \mathsf{P}(B_R(A,\delta'))$. The set O is in $\tau_{d,R}$, but it is not open in the *-robust topology $\tau_{d,S}$. The reason is that $d(A,p) = (0,0) = \mathbf{u}$. Hence, for any $\delta \ll \mathbf{u}$, the set $B_S(A,\delta)$ must contain p. But, the point p is not included in any set in O, because $\forall p' \in A$. $(1,1) \not\ll d(p',p)$.

7 Concluding Remarks

Related work. Flagg and Kopperman define \mathcal{V} -continuity spaces [9, Def 3.1] and \mathcal{V} -domains, with \mathcal{V} a value quantale [9, Def 2.9], i.e., the dual \mathcal{V}^o of \mathcal{V} is (in our terminology) a commutative affine quantale, whose underlying complete lattice is completely distributive—hence, by [1, Thm. 7.1.1], continuous—and satisfies additional properties formulated using a stronger variant \ll of the waybelow relation \ll , called the *totally-below* relation, namely $p \ll q$ iff for any $A \subseteq Q$, if $q \sqsubseteq \sqcup A$, then $\exists a \in A.p \sqsubseteq a$ (in contrast with the definition of \ll , the set A is not required to be directed). Thus, a \mathcal{V} -continuity space (X, d) is what we call a \mathcal{V}^o -metric space, while a \mathcal{V} -domain is a separated \mathcal{V}^o -metric space satisfying further properties. The metric d_U in [9, Sec 6] corresponds to our d_S , and [9, Thm 6.1] characterizes those B such that $d_U(A, B) = 0$ as the subsets of the closure of A in the topology τ_d^o , under the stronger assumption that \mathcal{V} is a value quantale. The upper powerdomain U(X) defined in [9, Sec 6] is almost the separated object equivalent to $\mathsf{P}_S(X)$, as its carrier consists of the closed subsets in the topology τ_d^o , except the empty one.

Although not every topology is induced by a classical metric, Kopperman [19] showed that all topologies come from generalized metrics. Cook and Weiss [5] present a more nuanced discussion of this fact, with constructions that avoid the shortcomings of Kopperman's original construction. Their focus, however, is on comparing various topologies that arise from a given generalized metric, i.e., those generated by open sets, closed sets, interior, and exterior systems. Although the four topologies coincide in classical metric spaces, they may be different in quantale valued metric spaces. In particular, they consider three conditions on a quantale, which are named Kuratowski, Sierpiński, and triangularity conditions [5, Def. 3]. When a commutative affine quantale Q satisfies these three conditions, it can be shown that all the four topologies coincide for the metric spaces valued in Q. Cook and Weiss [5] use the totally-below relation \ll , which is included in the way-below relation \ll . Under the three conditions they impose on quantales, one can show that for every $\delta \ll \mathbf{u}$ there exists $\delta' \ll \mathbf{u}$ such that $\delta \sqsubseteq \delta'$. Therefore, the topology generated by open balls with radius $\delta' \ll \mathbf{u}$ coincide with that generated by the open balls with radius $\delta \ll \mathbf{u}$.

The main drawback of value quantales and the quantales considered in [5] is that they are not closed under product, which is crucial for multi-dimensional quantitative analyses. On the other hand, a continuous quantale Q may not satisfy the Kuratowski condition, and therefore the four topologies considered in [5] for a Q-metric space may not coincide. Specifically, $d_S(A, \{x\}) = u$ may not entail that x is in the closure of A under the open ball topology.

Future work. The results of the current article may be regarded as the first steps towards a framework for robustness analysis with respect to perturbations that are measured using generalized metrics. As such, more remains to be done for development of the framework. Our future work will include study of effective structures on *Q*-metric spaces.

In [13], Goubault-Larrecq defines the Hausdorff-Hoare and the Hutchinson hemi-metrics. We plan to investigate if they scale-up to $\mathcal{P}o$ -enriched monads (or endofunctors) on the category of Q-metric spaces, and in this case apply to them the monad transformer defined in Section 5.

We also plan to study the impact of imprecision on probability distributions on (Q-)metric spaces, i.e., to which extent they are indistinguishable in the presence of imprecision, by applying our monad transformer to probability monads.

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A Proofs

A.1 Proof of Lemma 2

We prove only the first implication. If $q_1 \ll q_2$, then

 $q_1 \ll q_2 = q_2 \otimes \mathsf{u}$ (by continuity of Q) = $q_2 \otimes \sqcup \downarrow \mathsf{u}$ (by distributivity for \otimes) = $\sqcup \{q_2 \otimes q \mid q \ll \mathsf{u}\}.$

Hence, for some $q \ll u$, we must have $q_1 \ll q_2 \otimes q$, because $\{q_2 \otimes q \mid q \ll u\}$ is directed and $\{q'_2 \in Q \mid q_1 \ll q'_2\}$ is Scott open [1, Proposition 2.3.6].

A.2 Proof of Proposition 4

Given a family $((X_i, d_i) | i \in I)$ of objects in $\mathcal{M}et_Q$, the metric on the product $\Pi_{i \in I} X_i$ (computed in $\mathcal{S}et$) is $d_{\Pi}(x, y) = \prod_{i \in I} d_i(x_i, y_i)$, and the metric on the sum $\Sigma_{i \in I} X_i$ is $d_{\Sigma}((j, x), (j', x')) = d_j(x, x')$ if j = j' else \bot .

Given a pair of short maps $f, g: \mathcal{Met}_Q((X, d), (X', d'))$, the equalizer is obtained by taking the equalizer $\iota: X_e \to X$ in \mathcal{Set} , i.e., $X_e = \{x \in X \mid f(x) = g(x)\}$ and ι is the inclusion of X_e into X, and endowing X_e with the restriction of d to it. Then, ι is obviously short. Dually, the coequalizer is obtained by taking the coequalizer $\pi: X' \to X' / \approx$ in \mathcal{Set} , i.e., \approx is the smallest equivalence relation on X' including the relation $\{(f(x), g(x)) \mid x \in X\}$ and π is the quotient map $x \mapsto [x]$, and endowing $X' \approx$ with the metric d'_{\approx} given by $d'_{\approx}([x], [y]) = \sqcup_{x' \in [x], y' \in [y]} d'(x', y')$.

A.3 Proof of Proposition 5

For each property we give a proof hint.

- 1. follows from $\delta \ll \mathsf{u} \sqsubseteq d(x, x)$
- 2. follows from $\delta \sqsubseteq \delta' \ll d(x, y) \implies \delta \ll d(x, y)$
- 3. $y \in B(x, \delta)$ is equivalent to $\delta \ll d(x, y)$, thus (by Lemma 2) $\delta \ll d(x, y) \otimes \delta'$ for some $\delta' \ll u$. Moreover, $B(y, \delta') \subseteq B(x, \delta)$ is equivalent to $\delta' \ll d(y, z) \implies \delta \ll d(x, z)$. If $\delta' \ll d(y, z)$, then $\delta \ll d(x, y) \otimes \delta' \sqsubseteq d(x, y) \otimes d(y, z) \sqsubseteq d(x, z)$, which implies (by Proposition 1) $\delta \ll d(x, z)$.
- 4. By item 3, for $i \in \{1, 2\}$, $y \in B(x_i, \delta_i)$ implies $B(y, \delta'_i) \subseteq B(x, \delta_i)$ for some $\delta'_i \ll \mathfrak{u}$. Let $\delta' = \delta'_1 \sqcup \delta'_2$, then $\delta'_i \sqsubseteq \delta' \ll \mathfrak{u}$ (by Proposition 1). Thus, $B(y, \delta') \subseteq B(y, \delta'_i) \subseteq B(x, \delta_i)$ (by item 2). \Box

A.4 Proof of Lemma 3

The (\Leftarrow) direction is straightforward as property (1) states that O is a union of open balls. Thus, $O \in \tau_d$. To prove the (\Rightarrow) direction, note that every $O \in \tau_d$ is a union of finite intersections of open balls. Therefore, to prove that property (1) holds for all $O \in \tau_d$, it suffices to prove that, for any $n \in \mathbb{N}$ and any (finite) sequence $(B_i \mid i \in n)$ of open balls, property (1) holds for $\cap_{i \in n} B_i$, which we prove by induction on n:

- base case 0: We note that $\cap \emptyset = X$ and $X = B(x, \bot)$ for any $x \in X$. Thus, by choosing $\delta = \bot$, $\cap \emptyset$ satisfies property (1) by item 2 of Proposition 5;
- inductive step n + 1: by induction hypothesis $O = \bigcap_{i \in n} B_i$ satisfies property (1). Thus, for any $x \in O \cap B_n$, we have $B(x, \delta) \subseteq O$ for some $\delta \ll u$. In particular, $x \in B(x, \delta) \cap B_n$. Therefore, by item 4 of Proposition 5, there exists $\delta' \ll u$ such that $B(x, \delta') \subseteq B(x, \delta) \cap B_n \subseteq O \cap B_n$. \Box

A.5 Proof of Theorem 1

For the (\Rightarrow) direction, assume that $f: \mathcal{T}op((X_1, \tau_{d_1}), (X_2, \tau_{d_2}))$, which means $\forall O \in \tau_{d_2}. f^{-1}(O) \in \tau_{d_1}$. Let O be the open ball $B(f(x), \epsilon) \in \tau_{d_2}$. Then, by Lemma 3, there exists a $\delta \ll \mathfrak{u}_1$ such that $B(x, \delta) \subseteq f^{-1}(O)$, which is equivalent to $f(B(x, \delta)) \subseteq O = B(f(x), \epsilon)$.

For the (\Leftarrow) direction, assume that $O \in \tau_{d_2}$ and f satisfies property (3). We must prove that $O' = f^{-1}(O) \in \tau_{d_1}$, or equivalently (by Proposition 5) for any $x \in O'$, there exists $\delta \ll u$ such that $B(x, \delta) \subseteq O'$. If $x \in O'$, then $f(x) \in O$. Hence, by Lemma 3, $B(f(x), \epsilon) \subseteq O$ for some $\epsilon \ll u_2$. By property (3), there exists a $\delta \ll u_1$ such that $f(B(x, \delta)) \subseteq B(f(x), \epsilon) \subseteq O$, which implies $B(x, \delta) \subseteq O'$.

A.6 Proof of Proposition 6

For each property we give a proof hint.

- 1. Follows easily from the definition of $B_R(A, \delta)$.
- 2. First, under the assumption $\delta_1, \delta_2 \ll u$, one has $\delta_1 \otimes \delta_2 \sqsubseteq \delta_1 \otimes u = \delta_1 \ll u$ and $\delta_1 \otimes \delta_2 \sqsubseteq u \otimes \delta_2 = \delta_2 \ll u$. If $z \in B_R(B_R(A, \delta_1), \delta_2)$, then $\delta_2 \ll d(y, z)$ for some $y \in B_R(A, \delta_1)$, thus $\delta_2 \ll d(y, z)$ and $\delta_1 \ll d(x, y)$ for some $x \in A$, thus $\delta \ll \delta_1 \otimes \delta_2 \sqsubseteq d(x, y) \otimes d(y, z) \sqsubseteq d(x, z)$.
- 3. Follows easily from Lemma 4 and the definition of $B_R(A, \delta)$.
- 4. It suffices to prove the inclusion $B_R(\overline{A}^o, \delta) \subseteq B_R(A, \delta)$. If $z \in B_R(\overline{A}^o, \delta)$, then $\delta \ll d(y, z)$ for some $y \in \overline{A}^o$. Choose $\epsilon \ll u$ such that $\delta \ll \epsilon \otimes d(y, z)$ and $x \in A$ such that $\epsilon \ll d(x, y)$, then $\delta \ll \epsilon \otimes d(y, z) \sqsubseteq d(x, y) \otimes d(y, z) \sqsubseteq d(x, z)$.
- 5. The first inclusion follows easily from the definition of A_{δ} , For the second inclusion, since $A_{\delta} \subseteq \bigcap_{\epsilon \ll \mathsf{u}} B_R(B_R(A, \delta), \epsilon)$, it suffices to choose $\epsilon \ll \mathsf{u}$ such that $\delta' \ll \delta \otimes \epsilon$, then $B_R(B_R(A, \delta), \epsilon) \subseteq B_R(A, \delta')$.

A.7 Proof of Proposition 7

For the (\Leftarrow) direction, consider $U \in \tau_{d,R}$ such that $A \in U$. By definition of $\tau_{d,R}$, we have $\mathsf{P}(B_R(A,\delta)) \subseteq U$, for some $\delta \ll \mathsf{u}$. By hypothesis and Proposition 6 we get $B \subseteq \overline{A}^o \subseteq B_R(\overline{A}^o, \delta) = B_R(A, \delta)$. Thus, $B \in U$, as required.

For the (\Rightarrow) direction, we proceed by contraposition, namely, we prove the logically equivalent $B \not\subseteq \overline{A}^o \Rightarrow (\exists U \in \tau_{d,R}.A \in U \land B \notin U)$. If $B \not\subseteq \overline{A}^o$, then there is $x \in B$ such that $x \notin \overline{A}^o$. By Proposition 6, we have $\overline{A}^o = \bigcap_{\delta \ll u} B_R(A, \delta)$. Thus, there is $\delta \ll u$ such that $x \notin B_R(A, \delta)$, and consequently, $x \notin B_R(A, \delta')$ for every δ' such that $\delta \ll \delta' \ll u$. We define an open subset $U \in \tau_{d,R}$ such that $A \in U$ and $B \notin U$. Let $U = \bigcup_{\delta \ll \delta' \ll u} \mathsf{P}(B_R(A, \delta'))$. Clearly, $A \in U$, because by Lemma 1 there is at least one δ' such that $\delta \ll \delta' \ll u$, and $B \notin U$, since $x \notin B_R(A, \delta')$ for every δ' such that $\delta \ll \delta' \ll u$.

It remains to prove that $U \in \tau_{d,R}$, namely, that for every $A' \in U$, i.e., $A' \subseteq B_R(A, \delta_1)$ for some $\delta \ll \delta_1 \ll u$, there exists $\delta_2 \ll u$ such that $\mathsf{P}(B_R(A', \delta_2)) \subseteq U$. By Lemma 1 and 2, there are $\delta', \delta_2 \ll u$ such that $\delta \ll \delta' \ll \delta_1 \otimes \delta_2 \ll u$. Hence, by Proposition 6 we get $B_R(A', \delta_2) \subseteq$ $B_R(B_R(A, \delta_1), \delta_2) \subseteq B_R(A, \delta')$. Therefore, we have $\mathsf{P}(B_R(A', \delta_2)) \subseteq \mathsf{P}(B_R(A, \delta')) \subseteq U$. \Box

A.8 Proof of Proposition 8

Given a Q-metric space (X, d), denote by \sim_d the equivalence induced by the preorder \leq_d on X, i.e., $x \sim_d y \iff u \sqsubseteq d(x, y) \land u \sqsubseteq d(y, x)$. Let X_0 be the quotient X/\sim_d and define $d_0: X_0 \times X_0 \to Q$ as $d_0([x], [y]) = d(x, y)$. Since $x \sim_d x' \land y \sim_d y' \implies d(x, y) = d(x', y')$, d_0 is a well-defined Q-metric on X_0 .

Let $r:X \to X_0$ be the map such that r(x) = [x], which is clearly an isometry from (X, d) to (X_0, d_0) . Since r is surjective, there is a section $s:X_0 \to X$, which chooses a representative from each equivalence class $[x] \in X_0$. Thus, $s([x]) \sim_d x$ for every $x \in X$. Therefore, s in an isometry from (X_0, d_0) to (X, d).

To prove that (r,s) is an equivalence in $\mathcal{M}et_Q$, i.e., $r \circ s \sim id_{X_0}$ and $s \circ r \sim id_X$, where \sim on $\mathcal{M}et_Q((X,d),(X',d'))$ is the pointwise extension of $\sim_{d'}$, it suffices to observe that r(s([x])) = [x] and $s(r(x)) \sim x$ for every $x \in X$.

A.9 Proof of Theorem 2

All the maps on hom-preorders used in the definition of T are monotonic, thus they preserve \sim . Therefore, to prove that two arrows $f, g \in \mathcal{A}(X, Y)$ defined by different monotonic constructions are equal, it suffices to prove that they are equivalent (i.e., $f \sim g$), if Y is separated. For the same reason, if in a monotonic construction, one can replace s_X with another s'_X such that (r_X, s'_X) is an equivalence, the results will be equivalent, because $s_X \sim s'_X$.

$$\begin{aligned} - \mathsf{T}\hat{M} &= (M', \eta', -^{*'}) \text{ satisfies equations (4) for a monad, namely:} \\ \bullet (\eta'_X)^{*'} &= \mathsf{id}_{M'X}: R(MX) \to R(MX), \text{ because:} \\ \eta'_X^{*'} &= r_{MX} \circ (s_{MX} \circ r_{MX} \circ \eta_X)^* \circ s_{MX} \\ &\sim r_{MX} \circ \eta^*_X \circ s_{MX} = r_{MX} \circ s_{MX} = \mathsf{id}_{M'X}. \end{aligned}$$
$$\bullet f^{*'} \circ \eta'_X &= f: X \to R(MY) \text{ when } f: X \to R(MY), \text{ because:} \\ f^{*'} \circ \eta'_X &= r_{MY} \circ (s_{MY} \circ f)^* \circ s_{MX} \circ r_{MX} \circ \eta_X \\ &\sim r_{MY} \circ (s_{MY} \circ f)^* \circ \eta_X = r_{MY} \circ s_{MY} \circ f = f. \end{aligned}$$
$$\bullet g^{*'} \circ f^{*'} &= (g^{*'} \circ f)^{*'}: R(MX) \to R(MZ) \text{ when } f: X \to R(MY) \text{ and } g: Y \to R(MZ), \text{ because:} \\ g^{*'} \circ f^{*'} &= r_{MZ} \circ (s_{MZ} \circ g)^* \circ s_{MY} \circ r_{MY} \circ (s_{MY} \circ f)^* \circ s_{MX} \\ &\sim r_{MZ} \circ (s_{MZ} \circ g)^* \circ (s_{MY} \circ f)^* \circ s_{MX} \\ &= r_{MZ} \circ ((s_{MZ} \circ g)^* \circ s_{MY} \circ f)^* \circ s_{MX} \\ &= r_{MZ} \circ ((s_{MZ} \circ g)^* \circ s_{MY} \circ f)^* \circ s_{MX} \\ &= r_{MZ} \circ (s_{MZ} \circ g)^* \circ s_{MY} = (g^{*'} \circ f)^* \circ s_{MX} \\ &= r_{MZ} \circ (s_{MZ} \circ g)^* \circ s_{MX} = (g^{*'} \circ f)^* \circ s_{MX} \end{aligned}$$

 $- in_{\hat{M}}$ satisfies equations (5) for a monad map from \hat{M} to $\mathsf{T}\hat{M}$, namely:

• $\operatorname{in}_{\hat{M},X} \circ \eta_X = \eta'_X : X \to R(MX)$, because:

$$\inf_{\hat{M}_{\cdot}X} \circ \eta_X = r_{MX} \circ \eta_X = \eta'_X \cdot \eta_X$$

• $\operatorname{in}_{\hat{M},Y} \circ f^* = (\operatorname{in}_{\hat{M},Y} \circ f)^{*'} \circ \operatorname{in}_{\hat{M},X}: MX \to R(MY)$ when $f: X \to MY$, because:

$$\begin{split} & \operatorname{in}_{\hat{M},Y} \circ f^* = r_{MY} \circ f^* \\ & \sim r_{MY} \circ (s_{MY} \circ r_{MY} \circ f)^* \circ s_{MX} \circ r_{MX} \\ & = (\operatorname{in}_{\hat{M},Y} \circ f)^{*'} \circ \operatorname{in}_{\hat{M},X}. \end{split}$$

A.10 Proof of Proposition 9

We prove each of the four properties in sequence.

1. $\mathbf{u} \sqsubseteq d_S(A, A)$ means $\forall y \in A.\mathbf{u} \sqsubseteq \sqcup_{x \in A} d(x, y)$. It holds because $\mathbf{u} \sqsubseteq d(y, y) \sqsubseteq \sqcup_{x \in A} d(x, y)$ for any $y \in A$. The inequality $d_S(A, B) \otimes d_S(B, C) \sqsubseteq d_S(A, C)$ is equivalent to $\forall z \in C.d_S(A, B) \otimes d_S(B, C) \sqsubseteq \sqcup_{x \in A} d(x, z)$, which holds by the following chain of \sqsubseteq for any $z \in C$

$$d_{S}(A, B) \otimes d_{S}(B, C) \sqsubseteq \text{ by monotonicity of } \otimes d_{S}(A, B) \otimes \sqcup_{y \in B} d(y, z) = \text{ by distributivity} \\ \sqcup_{y \in B} (d_{S}(A, B) \otimes d(y, z)) \sqsubseteq \text{ by monotonicity of } \otimes \\ \sqcup_{y \in B} (\sqcup_{x \in A} d(x, y)) \otimes d(y, z) = \text{ by distributivity} \\ \sqcup_{y \in B} \sqcup_{x \in A} (d(x, y) \otimes d(y, z)) \sqsubseteq \text{ by triangular inequality} \\ \sqcup_{y \in B} (\sqcup_{x \in A} d(x, z)) \sqsubseteq \text{ because } \sqcup_{j \in J} q \sqsubseteq q \\ \\ \sqcup_{x \in A} d(x, z).$$

- 2. The property follows from $d_S({x}, {x}) = d(x, x)$. Actually η is an isometry.
- 3. The implication amounts to proving $d_S(A, B) \sqsubseteq d'_S(f^*(A), f^*(B))$ from the assumption $\forall x, y \in X.d(x, y) \sqsubseteq d'_S(f(x), f(y))$. But:

$$d_S(A,B) \sqsubseteq d'_S(f^*(A), f^*(B))$$

means $\forall y \in B. \forall y' \in f(y).d_S(A, B) \sqsubseteq \sqcup_{x' \in f^*(A)} d'(x', y')$. Thus, it holds by the following chain of \sqsubseteq for $y \in B$ and $y' \in f(y)$:

$$d_{S}(A, B) \sqsubseteq \text{ because } \forall k \in J.(\sqcap_{j \in J} q_{j}) \sqsubseteq q_{k}$$
$$\sqcup_{x \in A} d(x, y) \sqsubseteq \text{ by the assumption}$$
$$\sqcup_{x \in A} d'_{S}(f(x), f(y)) \sqsubseteq \text{ because } \forall k \in J.(\sqcap_{j \in J} q_{j}) \sqsubseteq q_{k}$$
$$\sqcup_{x \in A} (\sqcup_{x' \in f(x)} d'(x', y')) = \text{ by definition of } f^{*}$$
$$\sqcup_{x' \in f^{*}(A)} d'(x', y').$$

4. The implication amounts to proving $\forall A \in \mathsf{P}(X).\mathsf{u} \sqsubseteq d'_S(f^*(A), g^*(A))$ from the assumption $\forall y \in X.\mathsf{u} \sqsubseteq d'_S(f(y), g(y))$, i.e., $\forall y \in A. \forall y' \in g(y).\mathsf{u} \sqsubseteq \sqcup_{x' \in f(y)} d'(x', y')$. But $\mathsf{u} \sqsubseteq d'_S(f^*(A), g^*(A))$ means $\forall y \in A. \forall y' \in g(y).\mathsf{u} \sqsubseteq \sqcup_{x' \in f^*(A)} d'(x', y')$. Thus, it holds by the following chain of \sqsubseteq for any $y \in A$ and $y' \in g(y)$:

 $\mathsf{u} \sqsubseteq \text{ by the assumption} \\ \sqcup_{x' \in f(y)} d'(x', y') = \text{ by definition of } f^* \\ \sqcup_{x' \in f^*(A)} d'(x', y').$

Since the unit η and Kleisli extension $-^*$ for P_S are *equal* to those for the monad P on \mathcal{Set} , they necessarily satisfy the required equational properties.

A.11 Proof of Lemma 5

For each property we give a proof hint.

- 1. The first inclusion follows from $d(x, y) \sqsubseteq d(A, y)$ for every $x \in A$, while the second follows from $d(A, y) \sqsubseteq d(A', y)$ when $A \subseteq A'$.
- 2. Let $B = B_S(A, \delta)$. Then, $\forall y \in B.\delta \ll d(A, y)$. Thus:

$$\delta \sqsubseteq \sqcap_{y \in B} \delta \sqsubseteq \sqcap_{y \in B} d(A, y) = d_S(A, B).$$

A.12 Proof of Proposition 11

 $\tau_{d,S} \subseteq \tau_{d,R}$ follows from $B_R(A,\delta) \subseteq B_S(A,\delta)$ when $A \subseteq X$ and $\delta \ll \mathfrak{u}$.

Let $B(A, \delta)$ be the open ball with center $A \subseteq X$ and radius $\delta \ll \mathfrak{u}$ for the metric d_S . To prove $\tau_{d_S} \subseteq \tau_{d,S}$, we show that every open ball $B(A, \delta)$ belongs to $\tau_{d,S}$. Since $B(A, \delta)$ is downwards closed, it suffices to prove that $B \in B(A, \delta) \implies \exists \epsilon \ll \mathfrak{u}.B_S(B, \epsilon) \in B(A, \delta)$. Choose $\epsilon \ll \mathfrak{u}$ such that $\delta \ll d_S(A, B) \otimes \epsilon$, then $\delta \ll d_S(A, B) \otimes \epsilon \sqsubseteq d_S(A, B) \otimes d_S(B, B_S(B, \epsilon)) \sqsubseteq d_S(A, B_S(B, \epsilon))$. To prove $\tau_{d,S} \subseteq \tau_{d_S}$, we show that $B(A, \delta) \subseteq \mathsf{P}(B_S(A, \delta))$ for every $A \subseteq X$ and $\delta \ll \mathfrak{u}$. In fact, $d_S(A, B) = \sqcap_{y \in B} d(A, y) \sqsubseteq d(A, y)$ when $B \subseteq X$ and $y \in B$.

A.13 Proof of Lemma 6

Choose (using Lemma 1) δ' such that $\delta \ll \delta' \ll d(A, y)$, then we have the chain of equivalences:

- $-\delta \ll d(A, y) \iff$ by definition of d(A, y)
- $-\delta \ll \delta' \ll \sqcup_{x \in A} d(x, y) \iff$ by definition of \ll
- $\exists A_0 \subseteq_f A.\delta \ll \delta' \sqsubseteq \sqcup_{x \in A_0} d(x, y) = d(A_0, y).$

If Q is linear and $\perp \Box \delta$, then $\perp \Box d(A_0, y)$, thus $\emptyset \subset A_0 \subseteq_f A$. This implies that $\{d(x, y) \mid x \in A_0\}$ has a maximum, thus $d(A_0, y) = d(x, y)$ for some $x \in A_0$.

A.14 Proof of Proposition 12

 $\tau_{d,S} \subseteq \tau_{d,R}$ follows from Proposition 11. For the other inclusion we prove that $\forall \delta \ll \mathfrak{u}.\exists \epsilon \ll \mathfrak{u}.B_S(A,\epsilon) \subseteq B_R(A,\delta)$. By Lemma 6, $B_S(A,\delta) = B_R(A,\delta)$ when $\bot \sqsubset \delta$. $B_S(A,\bot) = X = B_R(A,\bot)$ when $\emptyset \subset A$. $B_R(\emptyset,\bot) = \emptyset = B_S(\emptyset,\delta)$ for any $\delta \ll \mathfrak{u}$ such that $\bot \sqsubset \delta$, which exists because Q is not trivial. \Box