Delaying Decisions and Reservation Costs

Elisabet Burjons¹ Fabian Frei²

Daniel $Mock^3$

Matthias Gehnen³ Peter Rossmanith³ Henri Lotze³

¹ York University, Canada burjons@yorku.ca

² ETH Zürich, Switzerland fabian.frei@inf.ethz.ch

³ RWTH Aachen University, Germany {gehnen,lotze,mock,rossmani}@cs.rwth-aachen.de

Abstract

We study the FEEDBACK VERTEX SET and the VERTEX COVER problem in a natural variant of the classical online model that allows for *delayed decisions* and *reservations*. Both problems can be characterized by an obstruction set of subgraphs that the online graph needs to avoid. In the case of the VERTEX COVER problem, the obstruction set consists of an edge (i.e., the graph of two adjacent vertices), while for the FEEDBACK VERTEX SET problem, the obstruction set contains all cycles.

In the *delayed-decision* model, an algorithm needs to maintain a valid partial solution after every request, thus allowing it to postpone decisions until the current partial solution is no longer valid for the current request.

The *reservation* model grants an online algorithm the new and additional option to pay a so-called reservation cost for any given element in order to delay the decision of adding or rejecting it until the end of the instance.

For the FEEDBACK VERTEX SET problem, we first analyze the variant with only delayed decisions, proving a lower bound of 4 and an upper bound of 5 on the competitive ratio. Then we look at the variant with both delayed decisions and reservation. We show that given bounds on the competitive ratio of a problem with delayed decisions impliy lower and upper bounds for the same problem when adding the option of reservations. This observation allows us to give a lower bound of min $\{1 + 3\alpha, 4\}$ and an upper bound of min $\{1 + 5\alpha, 5\}$ for the FEEDBACK VERTEX SET problem. Finally, we show that the online Vertex Cover problem, when both delayed decisions and reservations are allowed, is min $\{1 + 2\alpha, 2\}$ -competitive, where $\alpha \in \mathbf{R}_{\geq 0}$ is the reservation cost per reserved vertex.

1 Introduction

In contrast to classical offline problems, where an algorithm is given the entire instance it must then solve, an online algorithms has no advance knowledge about the instance it needs to solve. Whenever a new element of the instance is given, some irrevocable decision must be taken before the next piece is revealed.

An online algorithm tries to optimize an objective function that is dependent on the solution set formed by its decisions. The *strict competitive ratio* of an algorithm, as defined by Sleator and Tarjan [11], is the worst-case ratio of the performance of an algorithm compared to that of an optimal solution computed by an offline algorithm for the given instance, over all instances. The competitive ratio of an online *problem* is then the best competitive ratio over all online algorithms. For a general introduction to online problems, we refer to the books of Borodin and Ran El-Yaniv [4] and of Komm [9].

Not all online problems admit a competitive algorithm (i.e., one whose competitive ratio is bounded by a constant) under the classical model. In particular, this is the case for the problems VERTEX COVER and FEEDBACK VERTEX SET discussed in this paper.

The goal in the general VERTEX COVER problem is, given a graph G = (V, E), to find a minimum set of vertices $S \subseteq V$ such that $G[V \setminus S]$ contains no edges, i.e., the the obstruction set is a path of length 1.

In the classical online version of VERTEX COVER, the graph is revealed vertex by vertex, including all induced edges, and an online algorithm must immediately and irrevocably decide for each vertex whether to add it to the proposed vertex cover or not.

The goal of the FEEDBACK VERTEX SET problem is, given a graph G = (V, E), to find a minimum set of vertices $S \subseteq V$ such that $G[V \setminus S]$ contains no cycles. In this case, the obstruction set contains all cycles.

In both problems, the non-competitiveness is easy to see: If the first vertex is added to the solution set, the instance stops and thus leaving a single-vertex instance with an optimal solution size of zero. On the other hand, not selecting the first vertex will lead to an instance where this vertex becomes a central vertex, either of a star at VERTEX COVER, or of a friendship graph at FEEDBACK VERTEX SET.

These adversarial strategies are arguably pathological and unnatural, as decisions are enforced that are not based in the properties of the very problem to be solved: We need to start constructing a vertex cover before any edge is presented or a feedback vertex set without it being clear if there are even any cycles in the instance. To address this issue in general online Node- and Edge-Deletion problems, Komm et al. [10] introduced the *preemptive* online model, which was re-introduced by Chen et al. [7] as the *delayed-decision* model. This model allows an online algorithm to remain idle until a "need to act" occurs, which in our case means waiting until a graph from the obstruction set appears in the online graph. The online algorithm may then choose to delete any vertices in the current online graph. The main remaining restriction is that an online algorithm may not undo any of these deletions.

Definition 1. Let G be an online graph induced by its nodes $V(G) = \{v_1, \ldots, v_n\}$, ordered by their occurrence in an online instance. The DELAYED VERTEX COVER problem is to select, for every i, a subset of vertices $S_i \subseteq \{v_1, \ldots, v_i\}$ with $S_1 \subseteq \ldots \subseteq S_n$ such that the induced subgraph $G[\{v_1, \ldots, v_i\} \setminus S_i]$ contains no edge. The goal is to minimize $|S_n|$.

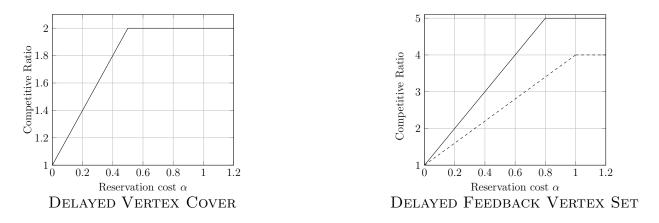


Figure 1: Upper and lower bounds on the competitive ratio of DELAYED VERTEX COVER (left) and DELAYED FEEDBACK VERTEX SET (right), each with reservations.

The definition of the DELAYED FEEDBACK VERTEX SET problem is identical, except that "contains no edge" is replaced by "is cycle-free."

A constant competitive ratio of 2 for the DELAYED VERTEX COVER problem is simple to prove and given in the introduction of the paper by Chen et al. [7]. The DELAYED FEEDBACK VERTEX SET problem, in contrast, is more involved. We show that no algorithm can admit a competitive ratio better than 4 and adapt results by Bar-Yehuda et al. [1] to give an algorithm that is strictly 5-competitive as an upper bound.

We also consider the model where decisions can be delayed even further by allowing an algorithm to *reserve* vertices (or edges) of an instance. If removing the reserved vertices from the instance would mean that a valid solution is maintained, the instance continues. Once an instance has ended, the algorithm can freely select the vertices to be included in the final solution (in addition to the already irrevocably chosen ones) among all presented vertices, regardless of their reservation status. This reservation is not free: When computing the final competitive ratio, the algorithm has to pay a constant $\alpha \in \mathbf{R}_{\geq 0}$ for each reserved vertex; these costs are then added to the size of the chosen solution set.

Definition 2. Let $\alpha \in \mathbf{R}_{\geq 0}$ be a constant and G an online graph induced by its nodes $V(G) = \{v_1, \ldots, v_n\}$, ordered by their occurrence in an online instance. The DELAYED VERTEX COVER problem with reservations is to select, for every i, vertex subsets $S_i, R_i \subseteq \{v_1, \ldots, v_i\}$ with $S_1 \subseteq \ldots \subseteq S_n$ and $R_1 \subseteq \ldots \subseteq R_n$ such that $G[\{v_1, \ldots, v_i\} \setminus (S_i \cup R_i)]$ contains no edge. The goal is to minimize the sum $|S_n| + |T| + \alpha |R_n|$, where $T \subseteq V(G)$ is a minimal vertex subset such that $G - (S_n \cup T)$ contains no edge.

Again, the definition for the DELAYED FEEDBACK VERTEX SET problem with reservations is identical, except for replacing "contains no edge" with "is cycle-free."

For reservation costs of $\alpha = 0$, the problem becomes equivalent to the offline version, whereas for $\alpha \ge 1$ taking an element directly into the solution set becomes strictly better than reserving it, rendering this reservation option useless. The results for DELAYED VERTEX COVER and DELAYED FEEDBACK VERTEX SET, each with reservations, are depicted in Figure 1. The reservation model is still relatively new and has been applied to the simple knapsack problem [3] and the secretary problem [6]. We note that the two cited papers consider relative reservation costs, while for the two problems in the present paper the cost per item are fixed.

The online VERTEX COVER problem has not received a lot of attention in the past years. Demange and Paschos [8] analyzed the online VERTEX COVER problem with two variations of how the online graph is revealed: either vertex by vertex or in clusters, per induced subgraphs of the final graph. The proven competitive ratios are functions on the maximum degree of the graph. Zhang et al. [14] looked at a variant called the Online 3-Path VERTEX COVER problem, where every induced path on three vertices needs to be covered. In this setting, the competitive ratio is again dominated by the maximum degree of the graph. Buchbinder and Naor [5] considered online integral and fractional covering problems formulated as linear programs where the covering constraints arrive online. As these are a strong generalization of the online VERTEX COVER problem, they achieve only logarithmic and not constant competitive ratios.

There has been some work on improving upon the bound of 2 for some special cases of VERTEX COVER in the model with delayed decisions (under different names). For the VERTEX COVER problem on *bipartite* graphs where one side is offline, Wang and Wong [13] give an algorithm achieving a competitive ratio of $\frac{1}{1-1/e} \approx 1.582$. Using the same techniques they achieve a competitive ratio of 1.901 for the full online VERTEX COVER problem for bipartite graphs and for the online fractional VERTEX COVER problem on general graphs.

To the best of our knowledge, the FEEDBACK VERTEX SET problem has received no attention in the online setting so far, most likely due to the fact that there is no competitive online algorithm for this problem in the classical setting. The offline FEEDBACK VERTEX SET problem, however, has been extensively studied, especially in the approximation setting. One notable algorithm is the one in the paper of Bar-Yehuda et al. [1], yielding an approximation ratio of 4 - 2/n on an undirected, unweighted graph. We adapt their notation in Section 2, and our delayed-decision algorithm with a competitive ratio of 5 is based on their aforementioned approximation algorithm. The currently best known approximation ratio of 2 by an (offline) polynomial-time algorithm

was given by Becker and Geiger [2].

The paper is organized as follows. We first look at the DELAYED FEEDBACK VERTEX SET problem, giving a lower bound of 4 and an upper bound of 5 on the competitive ratio. Then, we discuss how bounds on obstruction set problems without reservation imply bounds on the equivalent problems with reservations and vice versa, and how this applies to the DELAYED FEEDBACK VERTEX SET problem. Finally, we consider the DELAYED VERTEX COVER *problem with reservations*, giving tight bounds dependent on the reservation costs.

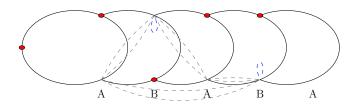


Figure 2: Sketch of the lower bound graph, revealed from left to right. The marked vertices are deleted by an algorithm. Then dashed edges (gray) and finally the self-loops (blue) force more deletions. The competitive ratio tends to 4 with increasing instance size.

2 Feedback Vertex Set with Delayed Decisions

In this section, we consider the DELAYED FEEDBACK VERTEX SET problem, which is concerned with finding the smallest subset of the vertices of a graph such that their removal yields a cycle-free graph. We give almost matching bounds on the competitive ratio in the delayed decision model.

Theorem 1 (Lower Bound). Given an $\varepsilon > 0$, there is no algorithm for DELAYED FEEDBACK VERTEX SET achieving a competitive ratio of $4 - \varepsilon$.

Proof. The adversarial strategy depicted in Figure 2 provides the lower bound.

First, a cycle is presented, which forces any algorithm to delete a single vertex. The adversary now repeats the following scheme n times: Identify a pair of vertices of the latest added (half) cycle that are still connected and close a cycle between those two vertices by adding a half cycle between them.

Every time such a half cycle is added, any algorithm has to delete a vertex from it. This is sketched in Figure 2 by the black cycles with red dots for exemplary deletions of some algorithm. We assume w.l.o.g. that an algorithm chooses one of the vertices of degree 3 between two cycles for deletion, we call these vertices of degree 3 branchpoints. If an algorithm deletes vertices of degree 2 instead, the adversary can force the algorithm to delete even more vertices.

After following this scheme, we split the remaining branchpoints (one for each new cycle) into two sets A and B alternatingly. We now take each (a, b)-pair with $a \in A$ and $b \in B$ and connect the two vertices with two paths, forming a new cycle between each pair of branching vertices. In order to remove all cycles with the lowest amount of deletions, any algorithm has to delete either all vertices from the set A or all vertices from the set B. Once such a set is chosen, say A, the adversary adds "self-loops", i.e., new cycles each only connected to a vertex of B, forcing the deletion of all (branchpoint) vertices of B.

The optimal solution consists only of the vertices of B and possibly, and depending on whether n is even and on the choice of A or B, another vertex from the first and the last cycle.

Thus, any online algorithm has to delete at least 4n - 2 vertices, while an optimal algorithm only deletes at most n + 2 vertices. The resulting competitive ratio is hence worse than $4 - \varepsilon$ for large enough values of n.

Next, we present Algorithm 1, which guarantees a competitive ratio of 5. It is a modified version of the 4-approximative algorithm for FEEDBACK VERTEX SET by Bar-Yehuda et al. [1]. Algorithm 1 maintains a maximal so-called 2-3-subgraph H in the presented graph G and selects a feedback vertex set F for G within H. It is important to note that H is not necessarily an *induced* subgraph of G.

Definition 3 (2-3-subgraph). A 2-3-subgraph of a graph G is a subgraph H of G where every vertex has degree exactly 2 or 3 in H.

Given a 2-3-subgraph H, a vertex is called a *branchpoint of* H if it has exactly degree 3 in H. We say a vertex v is a *linkpoint in* H if it has exactly degree 2 in H and there is a cycle in G whose only intersection with H is v. A cycle is called *isolated in* H if every vertex of this cycle (in G) is contained in H and has exactly degree 2 in H.

Algorithm 1 adds every branchpoint, linkpoint and one vertex per isolated cycle without linkpoints in H to the solution set F.

Note that it is possible that vertices, that were once added to F as linkpoints or due to isolated cycles, can still become branchpoints of degree 3 in H due to new vertices in G. Also later on in the analysis we consider them as branchpoints.

Algorithm 1 Online SubG-2-3
$H \leftarrow \emptyset, F \leftarrow \emptyset$
for every new edge in G do
if H is not a maximal 2-3-subgraph then
Extend H to a maximal 2-3-subgraph, converting linkpoints to branchpoints
whenever possible.
$F \leftarrow F \cup$ All new branchpoints
$F \leftarrow F \cup$ All new linkpoints
$F \leftarrow F \cup A$ set of one vertex per new isolated cycle without linkpoints in H
return F

Lemma 1 (Correctness of Online SubG-2-3). Algorithm 1 returns a feedback vertex set for the given input graph G.

Proof. By contradiction, assume there is a cycle C in G without a vertex in F.

If C contains no point of H, then the complete cycle C can be added to the subgraph H as an isolated cycle, thus H is not maximal, which contradicts the procedure of Algorithm 1.

Therefore, we can assume that there is at least one vertex of C in H. If the cycle contains no branchpoints, either the cycle is an isolated cycle, where all vertices are in H and one vertex is added to F, or it intersects with H at just a single vertex. This vertex is, thus, a linkpoint and part of F as well. Or, a third option is that C intersects H on two or more points and none of them are branchpoints, in which case, we can extend the 2-3-subgraph, which also contradicts the procedure of Algorithm 1.

Thus, every cycle in G has to contain at least one vertex of F, which is a feedback vertex set at the end. \Box

Proving that Algorithm 1 is 5-competitive is more tricky. First, note that an optimal feedback vertex set does not change if we consider a *reduction graph* G' instead of a graph G.

Definition 4 (Reduction Graph). A reduction graph G' of a graph G is obtained by deleting vertices of degree 1 and their incident edges, and by deleting vertices of degree 2, connecting the two neighbors directly (unless the vertex is self-looped).

The following lemma bounds the size of a reduced graph by its maximum degree and the size of any feedback vertex set. This will be used in the analysis of Algorithm 1. The lemma is due to Voss [12] and is used in the proof of 4-competitiveness by Bar-Yehuda et al. [1].

Lemma 2. Let G be a graph where no vertex has degree less than 2. Then for every feedback vertex set F, that contains all vertices of degree 2,

$$|V(G)| \le (\Delta(G) + 1)|F| - 2$$

holds, where $\Delta(G)$ is the maximum degree of G. In particular, if every vertex has a degree of at most 3, $|V(G)| \leq 4|F| - 2$ holds.

Theorem 2. Algorithm 1 achieves a strict competitive ratio of 5 - 2/|V(G)| for the online FEEDBACK VERTEX SET with delayed decisions.

Proof. At the end of the instance, after running Algorithm 1, call the set of branchpoints $B \subseteq F$, the set of linkpoints $L \subseteq F$, and the set of vertices added due to isolated cycles I. Again note that vertices can become branchpoints in later steps, even if they were added to F as a linkpoint or for an isolated cycle. Let μ be the size of an optimal feedback vertex set for the graph G.

The vertices in I are part of pairwise independent cycles. This follows from the fact that every cycle, that was handled as an isolated cycle, was added completely to H, thus new isolated cycles cannot contain a vertex of one already added to H. Therefore, $|I| \leq \mu$, since an optimal solution must contain at least one vertex for each of the pairwise independent cycles.

Moreover, the set of cycles due to to which the vertices of L were added to F are also pairwise independent. This is only true due to the possibility of relabeling linkpoints to branchpoints: For a contradiction, first assume that two cycles overlap at a vertex v and and intersect with H at linkpoints ℓ_1 and ℓ_2 . Then, H could be extended with a path from ℓ_1 through v to ℓ_2 , which would make ℓ_1 and ℓ_2 branchpoints, contradicting the assumption. Since ℓ_1 and ℓ_2 are not branchpoints, both have degree 2 in H and a new path through the two cycles can be added to H, if there are no other points of H on this path. In this case the path would be added to H, therefore ℓ_1 and ℓ_2 would have become branchpoints. This contradicts the assumption.

If there are other vertices in H on the mentioned path, call the first of those vertices along the path x and assume, w.l.o.g. that it is part of the same cycle as ℓ_1 . If x is connected to either ℓ_1 or ℓ_2 via H and the degree of x in H is at most 2 at any point while it was part of H, then we have a contradiction since H can be extended and one of the linkpoints ℓ_1 or ℓ_2 become a branchpoint. The only case that remains is where x immediately had degree 3 with respect to H when it was first added to H. In this case x is a branchpoint. Note that ℓ_1 was already deleted as a linkpoint at this time, since by definition the cycle would not have caused any deleted linkpoints otherwise. Since the adversary presents the instance vertex-wise, there must be a vertex s such that there was no possibility to add x to H before s was presented, but such that x immediately became a branchpoint when s was added to G. In particular, x cannot have been added immediately to H when it was presented. Therefore, there must be at least two independent paths in $G \setminus H$ from x to s. But since the algorithm could also add two independent paths from x to s to H, and one path from x to ℓ_1 , and the algorithm is forced to convert linkpoints to branchpoints whenever possible, it adds the path from x to ℓ_1 first. Note that priority is unambiguous, since there are no paths from x to some other linkpoints in $G \setminus H$: Otherwise ℓ_1 and the other linkpoint would already be connected within H, thus making them branchpoints. Thus we have $|L| \leq \mu$. This also shows that there cannot be a branchpoint inside a cycle that was used to delete a linkpoint, without also converting the linkpoint to a branchpoint.

It follows that $|L| + |I| \le 2\mu$. If $|B| \le 2|L|$, then we have $|F| = |I| + |L| + |B| \le 3|L| + |I| \le 4\mu$, which proves the statement.

In every other case, assume |B| > 2|L|. We now consider a reduction graph H' of the graph $H \setminus L$,

and delete every component consisting of only a single vertex. Every vertex in the resulting graph has degree 3. In the graph H we can have up to 2|L| more branchpoints than in the resulting graph here.

By Lemma 2, |B| - 2|L| is less than $4\mu(H') - 2$, where $\mu(H')$ is the optimal size of a feedback vertex set for the graph H'.

The size of an optimal feedback vertex set of the original graph G must be at least $\mu(H') + |L|$ since every linkpoint in G is part of a cycle that is not intersecting H'.

The inequality chain $|F| = |I| + |L| + |B| \le |I| + 3|L| + |B| - 2|L| \le |I| + 4|L| + 4\mu(H') - 2 \le |I| + 4\mu(G) - 2 \le 5\mu(G) - 2$ concludes the proof.

One could think that the reason Algorithm 1 does not match the competitive ratio of 4 is because the algorithm deletes vertices even in cases where it is not necessary. However, this is not the case. In the appendix we prove that Algorithm 1 cannot be better than 5-competitive even if vertices in F are only deleted whenever they are part of a completed cycle.

Lemma 3. The competitive ratio of Algorithm 1 is at least 5 - 2/|V(G)|, even if vertices are only deleted whenever necessary.

3 Adding Reservations to The Delayed-Decision Model

We now extend the previous results for the delayed-decision model without reservations by presenting two general theorems that translate both upper and lower bounds to the model with reservation.

The delayed-decision model allows us to delay decisions free of cost as long as a valid solution is maintained. Combining delayed decisions with reservations, we have to distinguish minimization and maximization: For a minimization problem the default is that the union of selected vertices and reserved ones constitutes a valid solution at any point. For a maximization problem, in contrast, it is that the selected vertices without the reserved ones always constitute a valid solution.

Whether the goal is minimizing or maximizing, any algorithm for a problem without reservations is also an algorithm for the problem with reservations, it just never uses this third option. We present a slightly smarter approach for any *minimization* problem such as the DELAYED FEEDBACK VERTEX SET problem.

Theorem 3. In the delayed-decision model, a c-competitive algorithm for a minimization problem without reservation yields a $\min\{c, 1 + c\alpha\}$ -competitive algorithm for the variant with reservation.

Proof. Modify the *c*-competitive algorithm such that it reserves whatever piece of the input it would usually have immediately selected until the instance ends – incurring an additional $\cot c \cdot \alpha \cdot Opt$ – and then pick an optimal solution for a cost of Opt. This already provides an upper bound of $1 + c\alpha$ on the competitive ratio. But running the algorithm without modification still yields an upper bound of *c* of course. The algorithm can now choose the better of these two options based on the given α , yielding an algorithm that is $\max\{c, 1+c\alpha\}$ -competitive.

Corollary 1. There is a $\min\{5, 1+5\alpha\}$ -competitive algorithm for the DELAYED FEEDBACK VERTEX SET problem with reservations.

Theorem 4. In the delayed-decision model, a lower bound of c on the competitive ratio for a minimization problem without reservations yields a lower bound of $\min\{1 + (c-1)\alpha, c\}$ on the competitive ratio for the problem with reservation.

Proof. The statement is trivial for $\alpha \geq 1$; we thus consider now the case of $\alpha < 1$. Assume that we have an algorithm with reservations with a competitive ratio better than $1+(c-1)\alpha$. Even though this algorithm has the option of reserving, it must select a definitive solution, at the latest when the instance ends. This definitive solution is of course at least as expensive as the optimal solution. Achieving a competitive ratio better than $1 + (c-1)\alpha$ is thus possible only if less the algorithm is guaranteed to reserve fewer than c - 1 input pieces in total. But in this case, we can modify the algorithm with reservation such that it immediately accepts whatever it would have only reserved otherwise. This increases the incurred costs by $1 - \alpha$ for each formerly reserved input piece, yielding an algorithm without reservations that achieves a competitive ratio better than $1 + (c-1)(1 - \alpha) = c$.

Corollary 2. There is no algorithm solving the DELAYED FEEDBACK VERTEX SET problem with reservations that can achieve a lower bound better than $\min\{1 + 3\alpha, 4\}$.

4 Vertex Cover

As already mentioned, the DELAYED VERTEX COVER has a competitive ratio of 2 without reservation. We now present tight bounds for all reservation-cost values α , beginning with the upper bound.

Theorem 5. There is an algorithm for the DELAYED VERTEX COVER problem with reservations that achieves a competitive ratio of $\min\{1 + 2\alpha, 2\}$ for any reservation value.

These upper bounds, depicted in Figure 1, have matching lower bounds. We start by giving a lower bound for $\alpha \leq \frac{1}{2}$.

Theorem 6. Given an $\varepsilon > 0$, there is no algorithm for the DELAYED VERTEX COVER problem with reservations achieving a competitive ratio of $1 + 2\alpha - \varepsilon$ for any $\alpha \leq \frac{1}{2}$.

Proof. We present the following exhaustive set of adversarial instances as depicted in Figure 4, deferred to the appendix. First an adversary presents two vertices u_1 and v_1 connected to each other. Any algorithm for online VERTEX COVER with reservations is forced to cover this edge either by irrevocably choosing one vertex for the cover or by placing one of the vertices in the temporary cover (i.e., reserving it). In the first case, assume w.l.o.g. that v_1 is the chosen vertex for the cover. The adversary then presents a vertex v_2 connected only to u_1 and ends the instance. Such an algorithm would have a competitive ratio of 2 as it must then cover the edge (u_1, v_2) by placing one of its endpoints in the cover. Choosing vertex u_1 alone would have been optimal, however. This is the same lower bound as given for the model without reservations.

If, again w.l.o.g., the vertex v_1 is temporarily covered instead, the adversary still presents a vertex v_2 connected to u_1 . Now an algorithm has four options to cover the edge (u_1, v_2) : Each of the two vertex u_1 or v_2 can be either irrevocably chosen or temporarily reserved. If u_1 or v_2 are temporarily covered, the instance will end here and the reservation costs of the algorithm will be 2α . Both the algorithm and the optimal solution will end up choosing only vertex u_1 , which implies a final competitive ratio of $1 + 2\alpha$ in this case. If vertex v_2 is chosen, the instance will also end, yielding a competitive ratio worse than 2.

Thus, the only option remaining is to irrevocably cover the vertex u_1 . In this case, the adversary presents a vertex u_2 connected to v_2 . An algorithm can then irrevocably or temporarily take v_2 or u_2 respectively. If an algorithm temporarily takes v_2 or u_2 , the adversary will present one more vertex u_0 connected to v_1 and end the instance. This results in a graph that can be minimally covered by the vertices v_1 and v_2 . The algorithm, however, will have 3 vertices in the cover and additional reservation costs of 2α for the temporarily chosen vertices. Thus it will have a competitive ratio of $\frac{3}{2} + \alpha$, which is larger than $1 + 2\alpha$ for the considered values of α . If an algorithm irrevocably takes u_2 , the same vertex u_0 will be presented and then the instance will end with another auxiliary vertex a_2 connected to v_2 . An optimal vertex cover would take vertices v_1 and v_2 . Any algorithm that has already irrevocably chosen u_1 and u_2 , however, will have to choose two more vertices in order to cover the edges $\{v_1, u_0\}$ and $\{v_2, a_2\}$; thus, its competitive ratio will be worse than 2.

Again, the only remaining option is to irrevocably choose the vertex v_2 , after which an adversary presents a vertex v_3 connected to u_2 . An algorithm may choose to irrevocably or

temporarily take the vertex u_2 or v_3 . If an algorithm decides to temporarily take any vertex or irrevocably choose v_3 , then the adversary presents an auxiliary vertex b_2 connected to u_2 and ends the request sequence. An optimal vertex cover in this case has size two, containing only the vertices u_1 and u_2 . In the best case, however, such an algorithm has a vertex cover of size 3 and two temporary covers, thus its competitive ratio will be at best $\frac{3}{2} + \alpha$, which again is worse than $1 + 2\alpha$, as already observed.

In general, after irrevocably choosing u_1, \ldots, u_{k-1} and v_2, \ldots, v_{k-1} , and temporarily choosing v_1 , the adversary presents the vertex u_k connected to v_k . If an algorithm chooses to reserve any of the endpoints or irrevocably selects u_i , then the adversary presents the vertex u_0 and ends the request sequence. In this case an optimal vertex cover only contains the vertices v_i for every $i = 1, \ldots, k$, thus it has size k. The algorithm, however, will have to take v_1 and v_k in addition to the previously irrevocably taken vertices, thus obtaining a vertex cover of size 2k - 1 at best together with two temporarily taken vertices. Thus the competitive ratio is $\frac{2k-1+2\alpha}{k} = 2 - \frac{1-2\alpha}{k} \ge 1+2\alpha$, where the inequality holds for every $k \ge 1$.

In the other case, after irrevocably choosing vertices u_1, \ldots, u_{k-1} and v_2, \ldots, v_k , the adversary presents the vertex v_{k+1} connected to u_k . If an algorithm chooses to reserve one of the two endpoints or irrevocably chooses v_{k+1} , then the adversary stops the request sequence. An optimal vertex cover of such a graph consists of the vertices u_i for every $i = 1, \ldots, k$ and it has size k. The algorithm will have to choose u_i in order to obtain a vertex cover at all, obtaining a vertex cover of size 2k - 1 at best together with at least one reservation, thus achieving a competitive ratio of $\frac{2k-1+\alpha}{k} \ge 1+2\alpha-\varepsilon$ for any $k \ge \frac{1}{\varepsilon}$.

For larger values of α the same adversarial strategy holds, but it gives us the following lower bound.

Theorem 7. For $\alpha > 1/2$, no algorithm for the DELAYED VERTEX COVER problem with reservations is better than 2-competitive.

Proof. The lower bound of Theorem 6 for $\alpha = 1/2$ is $2 - \varepsilon$. For larger values of α the same adversarial strategy will give us a lower bound of 2. This is because, at all points during the analysis, either the value of the competitive ratio for each strategy was at least 2, or it had a positive correlation with the value of α , meaning that for larger values of α any algorithm following that strategy obtains strictly worse competitive ratios.

5 Conclusion

We have shown that some problems that are non-competitive in the classical model become competitive in modified, but natural variations of the classical online model. Some questions remain open, such as the best competitive ratio for the DELAYED FEEDBACK VERTEX SET problem, which we believe to be 4.

It may be worthwhile to investigate which results can be found for restricted graph classes. For example, it is easy to see that the online version of DELAYED FEEDBACK VERTEX SET is 2-competitive on graphs with maximum degree three.

In addition we also introduced the reservation model on graphs, providing an upper and a lower bound for general graph problems. It would be interesting try to find matching bounds, also on specific graph problems.

References

- Reuven Bar-Yehuda, Dan Geiger, Joseph Naor, and Ron M. Roth. Approximation algorithms for the feedback vertex set problem with applications to constraint satisfaction and bayesian inference. SIAM J. Comput., 27(4):942–959, 1998.
- [2] Ann Becker and Dan Geiger. Optimization of pearl's method of conditioning and greedy-like approximation algorithms for the vertex feedback set problem. Artif. Intell., 83(1):167–188, 1996.
- [3] Hans-Joachim Böckenhauer, Elisabet Burjons, Juraj Hromkovič, Henri Lotze, and Peter Rossmanith. Online simple knapsack with reservation costs. In STACS 2021, volume 187 of LIPIcs, pages 16:1–16:18, 2021.
- [4] Allan Borodin and Ran El-Yaniv. Online computation and competitive analysis. Cambridge University Press, 1998.
- [5] Niv Buchbinder and Joseph Naor. Online primal-dual algorithms for covering and packing. *Math. Oper. Res.*, 34(2):270–286, 2009.
- [6] Elisabet Burjons, Matthias Gehnen, Henri Lotze, Daniel Mock, and Peter Rossmanith. The secretary problem with reservation costs. In COCOON 2021, volume 13025 of LNCS, pages 553–564, 2021.
- [7] Li-Hsuan Chen, Ling-Ju Hung, Henri Lotze, and Peter Rossmanith. Online node- and edge-deletion problems with advice. *Algorithmica*, 83(9):2719–2753, 2021.
- [8] Marc Demange and Vangelis Th. Paschos. On-line vertex-covering. Theoretical Computer Science, 332(1):83–108, 2005.
- [9] Dennis Komm. An Introduction to Online Computation Determinism, Randomization, Advice. Texts in Theoretical Computer Science. Springer, 2016.
- [10] Dennis Komm, Rastislav Královič, Richard Královič, and Christian Kudahl. Advice Complexity of the Online Induced Subgraph Problem. In *MFCS 2016*, volume 58 of *LIPIcs*, pages 59:1–59:13, 2016.
- [11] Daniel Dominic Sleator and Robert Endre Tarjan. Amortized efficiency of list update and paging rules. Commun. ACM, 28(2):202–208, 1985.
- [12] Heinz-Jürgen Voss. Some properties of graphs containing k independent circuits. Theory of Graphs. Proceedings of Colloquium Tihany, pages 321–334, 1968.
- [13] Yajun Wang and Sam Chiu-wai Wong. Two-sided online bipartite matching and vertex cover: Beating the greedy algorithm. In *ICALP 2015*, volume 9134 of *LNCS*, pages 1070–1081, 2015.
- [14] Yubai Zhang, Zhao Zhang, Yishuo Shi, and Xianyue Li. Algorithm for online 3-path vertex cover. *Theory Comput. Syst.*, 64(2):327–338, 2020.

Appendix

Deferred Proofs

For convenience, we restate all statements before proving them.

Lemma 3. The competitive ratio of Algorithm 1 is at least 5 - 2/|V(G)|, even if vertices are only deleted whenever necessary.

Proof. The following adversarial strategy, illustrated in Figure 3, provides the desired lower bound.

First, the adversary presents n independent cycles $C_1, ..., C_n$. Algorithm 1 deletes at least one vertex per cycle.

Second, the adversary connects each pair of neighboring cycles C_i and C_{i+1} for $i \leq n-1$ with two independent paths. This adds 4n - 4 branchpoints. Everything that is presented is part of the 2-3-subgraph, marked in blue in Figure 3.

Now a vertex c (the top vertex in Figure 3) is added and connected to every branchpoint with two edges. This new vertex will not be part of the 2-3-subgraph. Therefore, even a modified Algorithm 1 that only deletes branchpoints whenever necessary has to delete every branchpoint. Here, Algorithm 1 deletes n + 4n - 4 vertices, whereas an optimal solution consists of only one vertex per independent cycle and the vertex c.

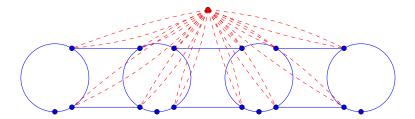


Figure 3: The 2-3-subgraph is given in blue, marked vertices will be deleted by Algorithm 1. The top vertex cannot be added to the 2-3-subgraph, so it will not be deleted.

Theorem 0. There is an algorithm for the DELAYED VERTEX COVER problem with reservations that achieves a competitive ratio of min $\{1 + 2\alpha, 2\}$ for any reservation value.

Proof. We present an algorithm that achieves a competitive ratio of $1+2\alpha$ and – together with the algorithm by Chen et al. [7], which does not reserve vertices – we achieve a competitive ratio of min $\{1 + 2\alpha, 2\}$.

Our algorithm is itself an adaptation of the classical 2-approximation for VERTEX COVER. Given a new vertex, the algorithm considers every edge and whenever an edge is uncovered the algorithm temporarily covers both endpoints by reserving the two vertices. Given a graph G with a minimal vertex cover of size k, this algorithm incurs reservation costs of $\alpha \cdot 2k$, as the algorithm selects at most as many vertices as $2 \cdot Opt$, where Opt is the size of the optimal vertex cover. However, because the final decision on the vertex cover is completely left to the very last step and no vertex is permanently chosen during the running of the algorithm, once the whole instance is presented the algorithm can choose a minimal vertex cover for G as the final solution. Thus, its competitive ratio is $\frac{k+\alpha \cdot 2k}{k} = 1 + 2\alpha$. For $\alpha > 1/2$, the online algorithm without reservations by Chen et al. [7] has a competitive ratio of 2, beating the ratio of $1 + 2\alpha$.

Deferred Illustration

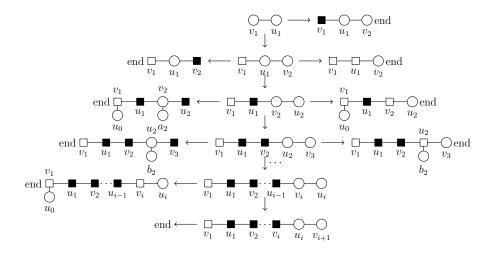


Figure 4: Illustration for the proof of Theorem 0. Adversarial strategy for $\alpha \leq \frac{1}{2}$. Black squares are vertices irrevocably included into the vertex cover, white square vertices are reserved to be temporarily in the vertex cover, and round vertices are not yet chosen.