Lower Bounds of Functions on Finite Abelian Groups

Jiangting Yang Key Lab of Mathematics Mechanization, AMSS University of Chinese Academy of Sciences Beijing, 100190, China yangjianting@amss.ac.cn Ke Ye Key Lab of Mathematics Mechanization, AMSS University of Chinese Academy of Sciences Beijing,100190, China keyk@amss.ac.cn Lihong Zhi Key Lab of Mathematics Mechanization, AMSS University of Chinese Academy of Sciences Beijing, 100190, China Izhi@mmrc.iss.ac.cn

ABSTRACT

The problem of computing the optimum of a function on a finite set is an important problem in mathematics and computer science. Many combinatorial problems such as MAX-SAT and MAXCUT can be recognized as optimization problems on the hypercube $C_2^n =$ $\{-1, 1\}^n$ consisting of 2^n points. It has been noticed that if a finite set is equipped with an abelian group structure, then one can efficiently certify nonnegative functions on it by Fourier sum of squares (FSOS). Motivated by these works, this paper is devoted to developing a framework to find a lower bound of a function on a finite abelian group efficiently. We implement our algorithm by the SDP solver SDPNAL+ for computing the verified lower bound of f on G and test it on the MAX-2SAT and MAX-3SAT benchmark problems from the Max-SAT competitions in 2009 and 2016. Beyond that, we also test our algorithm on random functions on C_3^n . These experiments demonstrate the advantage of our algorithm over previously known methods.

KEYWORDS

Abelian group, Fourier sum of squares, Semidefinite programming, Sparse Gram matrices, SAT, MAX-SAT

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1 INTRODUCTION

The problem of computing the optimum of a function on a finite set is an important but challenging problem in mathematics and computer science. Examples include the Knapsack problem[30, 31], the set cover problem[11, 23], the travelling salesman problem[19, 39], the vehicle routing problem[9, 45], the *k*-SAT problem [6, 21] and their numerous variants. Although each of these problems can be formulated as an integer programming, there do not exist polynomial time algorithms for most of them unless **P=NP** [21, 23, 24, 34, 35, 45]. Therefore various approximation algorithms

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are employed to resolve the issue [5, 15, 28, 42, 46, 47]. Semidefinite programming (SDP) is one of the most powerful and extensively studied technique to design and analyze an approximation algorithm[3, 4, 13, 20, 22, 25, 27, 37, 43, 46, 54]. Among those successful applications of the SDP technique, the most well-known ones are MAX-2SAT[16], MAX-3SAT[20] and MAX-CUT[22]. On the other side, it is noticed in [12, 41, 52, 53] that if a finite set is equipped with an abelian group structure, then one can efficiently certify nonnegative functions on it by Fourier sum of squares (FSOS). Motivated by these works, this paper is concerned with establishing a framework to solve the following problem by FSOS and SDP.

Problem 1.1 (lower bound by FSOS). *Given a function f on a finite abelian group, find a lower bound of f efficiently.*

Let $S \subseteq \mathbb{C}^n$ be an algebraic variety. Algebraically, identifying $\mathbb{C}[S]$ with $\mathbb{C}[z_1, \ldots, z_n]/I(S)$ is a favourable perspective as the latter ring is endowed with rich geometric and algebraic structures. For computational purposes, however, regarding a function as an equivalence class is not convenient, on account of the fact that an equivalence can be represented by two different polynomials. If S = G is a finite group, then there is an alternative algebraic structure on $\mathbb{C}[G]$ which is extremely useful for computations [12, 41, 52, 53]. Namely, one can identify $\mathbb{C}[G]$ with the group ring of G via the Fourier transform [14]. The advantage of such a point of view is that a function f on G can be expanded as $f = \sum_{\chi \in \widehat{G}} \widehat{f}(\chi)\chi$, where \widehat{G} is the dual group of G and $\widehat{f}(\chi)$ is the Fourier coefficient of f at $\chi \in \widehat{G}$. This well-known viewpoint enables us to introduce analytic tools to solve Problem 1.1.

related works and our contributions

Recently, a method for general-purpose polynomial optimization called the TSSOS hierarchy is proposed in [49]. The new method follows the well-known methodology established in [26], but it exploits the sparsity of polynomials to reduce the size of SDP. Combing the TSSOS hierarchy and the method in [48] for correlative sparsity, [50] introduces the CS-TSSOS hierarchy for large scale sparse polynomial optimization. In particular, both TSSOS and CS-TSSOS hierarchies are applicable to optimization problems on finite abelian groups.

Many combinatorial problems, such as MAX-SAT and MAX-CUT, can be recognized as optimization problems on the hypercube $C_2^n = \{-1, 1\}^n$. Due to its great importance in computer science, there are various solvers for MAX-SAT. For instance, in [46], the quotient structure of $\mathbb{C}[C_2^n]$ is explored for support selection strategies, from which one can solve MAX-SAT by SDP; by combing

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several optimization techniques specifically designed for MAX-SAT, [51] provides an efficient SDP-based MIXSAT algorithm; based on the resolution refutation, a solver called MS-builder is proposed in [36].

On the one hand, TSSOS and CS-TSSOS hierarchies can handle general polynomial optimization problems, while specially designed solvers such as MIXSAT and MS-builder can only deal with MAX-SAT problems. On the other hand, however, it is natural to expect that these specially designed solvers would outperform generalpurpose methods on MAX-SAT problems.

It is our framework of optimization by FSOS that balances the universality and efficiency. Indeed, our method is applicable to any optimization problems on finite abelian groups, including the hypercube C_2^n , cyclic group \mathbb{Z}_N and their product.

The main contribution of this paper is the approximation algorithm for the minimum of a function f on an abelian group G. To be more specific, for each positive integer k, we propose an SDP (Lemma 3.2) associated with some $S_f \subseteq \widehat{G}$ (Algorithm 1) where $|S_f| = k$. A solution to the SDP provides a good lower bound for f(Theorem 3.5). Since the size of the SDP is $k \times k$, our approximation algorithm is applicable to functions on a large abelian group. Another feature of our method is the support selection strategy for the SDP. Unlike the strategy suggested in [46], which is almost the same for all functions on the hypercube, our strategy provides different bases for different functions by exploiting the magnitudes of their Fourier coefficients. Moreover, we provide two rounding methods based on the nullspace of the Gram matrix and the lower rank approximation of the moment matrix, respectively. We test our algorithm (Algorithm 2) and rounding methods on numerical examples in Appendix A, from which one may clearly see the advantage of our algorithm over aforementioned methods.

applications

Below we mention some potential applications of our algorithm. Although we can test our algorithm on all these interesting problems and present numerical experiments in Appendix A, we only concentrate on MAX-2SAT and MAX-3SAT, due to the page limit.

SAT problems. The characteristic function (cf. Subsection 2.3) of a Boolean formula in a *k* Conjunctive Normal Form (*k*-CNF¹) with *n* variables can be recognized as a non-negative function on the group C_2^n . As a consequence, our framework provides an approximation algorithm for MAX-*k*SAT² and its variants such as UN-SAT and MIN-SAT.

MAX-CUT problems. Let G = (V, E) be an undirected graph where |V| = n with edge weights $\{w_{ij} : \{i, j\} \in E\}$. The MAX-CUT problem for *G* is a partition $V = S \bigsqcup T$ such that the sum of weights of edges between *S* and *T* is as large as possible. Thus it can be

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formulated as an optimization problem on \mathbb{Z}_2^n .

$$\max \sum_{\{i,j\} \in E} w_{ij} x_i (1 - x_j)$$

s.t. $x_i \in \mathbb{Z}_2 = \{0, 1\}, 1 \le i \le n$

ground states of a lattice model. A lattice model is a fixed set of locations in \mathbb{R}^3 on which interacting particles are distributed. A ground state of a lattice model is a configuration of these particles which minimizes the Hamiltonian. Mathematically, computing the ground state of a lattice model may be rephrased as an optimization problem on $\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_n}$ where *n* is the number of particles and d_j is determined by the type of the *j*-th particle. In [18], the case where $d_1 = \cdots = d_n = 2$ is considered.

2 PRELIMINARIES

In this section, we first review the Fourier analysis on abelian groups. After that we provide a brief introduction to Fourier sum of squares (FSOS) on abelian groups. Lastly, we define characteristic functions of (weighted) k-CNF formulae, which are important examples of integer-valued functions on abelian groups.

2.1 Fourier analysis on groups

We briefly summarize fundamentals of group theory and representation theory in this subsection. For more details, we refer interested readers to [14, 33, 40].

Let *G* be a finite abelian group. A *character* of *G* is a group homomorphism $\chi : G \to \mathbb{C}^{\times}$. Here \mathbb{C}^{\times} is $\mathbb{C} \setminus \{0\}$ endowed with the multiplication of complex numbers as the group operation. The set of all characters of *G* is denoted by \widehat{G} , called the *dual group*³ of *G*.

According to [14, Chapter 1], any function f on G admits the *Fourier expansion*:

$$f = \sum_{\chi \in \widehat{G}} \widehat{f}(\chi) \chi,$$

where $\widehat{f}(\chi) := \frac{1}{|G|} \sum_{g \in G} f(g) \overline{\chi(g)}$ is called the *Fourier coefficient* of *f* at $\chi \in \widehat{G}$. The *support* of *f* is

$$\operatorname{supp}(f) \coloneqq \left\{ \chi \in \widehat{G} : \widehat{f}(\chi) \neq 0 \right\}.$$

As an example, the dual group of the hypercube $C_2^n = \{-1, 1\}^n$ is

$$\widehat{C_2^n} = \{ z^\alpha : \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_2^n \} \simeq \mathbb{Z}_2^n.$$

Here $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ is the additive group and $z^{\beta} \coloneqq z_1^{\beta_1} \dots z_n^{\beta_n}$ for each $\beta \in \mathbb{N}^n$. Thus a function $f : C_2^n \to \mathbb{C}$ can be expressed as a linear combination of multilinear monomials:

$$f = \sum_{\alpha \in \mathbb{Z}_2^n} f_\alpha z^\alpha$$

2.2 Fourier sum of squares on finite abelian groups

This subsection concerns with the theory of FSOS developed in [12, 41, 52]. Let *f* be a nonnegative function on a finite abelian group *G*. An *FSOS certificate* of *f* is a finite family $\{h_i\}_{i \in I}$ of complex valued functions on *G* such that $f = \sum_{i \in I} |h_i|^2$. The *sparsity* of

¹A k-CNF formula is a CNF formula in which each clause contains at most k literals. ²Here we adopt the most commonly used definition [24] of MAX-kSAT, which requires one to compute the maximum number of simultaneously satisfiable clauses in a k-CNF formula. There are other definitions of MAX-kSAT, though. For instance, it is defined in [2] to be the NP-hard problem of finding a maximal satisfiable assignment. In [35], MAX-kSAT means the NP-complete problem which determines the existence of an assignment satisfying at least a given number of clauses.

³Since G is an abelian group, \widehat{G} is indeed a group and $\widehat{G} \simeq G$.

 $\{h_i\}_{i \in I}$ is defined to be $|\bigcup_{i \in I} \operatorname{supp}(h_i)|$. We say that *f* is *sparse* if f admits an FSOS certificate of low sparsity. The proposition that follows ensures the existence of FSOS certificates.

Proposition 2.1. [12, Proposition 2] A nonnegative function f on a finite abelian group admits an FSOS certificate.

A function f on G is nonnegative if and only if there exists a Hermitian positive semidefinite matrix $Q = (Q_{\chi,\chi'})_{\chi,\chi'\in\widehat{G}} \in$ $\mathbb{C}^{|G| \times |G|}$ such that:

$$\sum_{\chi'\in\widehat{G}}Q_{(\chi')^{-1}\chi,\chi}=f_{\chi},\quad \forall\chi\in\widehat{G}.$$
(1)

Any $|G| \times |G|$ complex matrix $Q \ge 0$ satisfying (1) is called a *Gram matrix* of f. Gram matrices are of great importance in both the theoretical and computational study of FSOS. In fact, if Q is a Gram matrix of *f* and $Q = M^*M$ for some matrix

$$M = (M_{j,\chi})_{1 \le j \le r,\chi \in \widehat{G}} \in \mathbb{C}^{r \times |G|}$$

then we have $f = \sum_{j=1}^{r} \left| \sum_{\chi \in \widehat{G}} M_{j,\chi} \chi \right|^2$. Clearly this construction provides us a one-to-one correspondence between (sparse) Gram matrices and (sparse) FSOS certificates of f. Thus the problem of computing a sparse FSOS certificate of f is equivalent to computing a sparse Gram matrix of f.

THEOREM 2.2. [52, Lemma 3.6] Let f be a nonnegative, nonzero function on a finite abelian group G. Suppose

$$\sqrt{f} = \sum_{\chi \in \widehat{G}} a_{\chi} \chi$$

is the Fourier expansion of \sqrt{f} . Then the optimal solution of the convex relaxation of

$$\min_{\substack{Q: gram matrix of f}} \|Q\|_0$$

is the rank-one positive semidefinite matrix $H \in \mathbb{C}^{|\widehat{G}| \times |\widehat{G}|}$ where

$$H_{\chi,\chi'} = \overline{a_{\chi}} a_{\chi'}.$$

Here again we respectively index rows and columns of H by elements in Ĝ.

By fast Fourier transform (FFT) and the inverse fast Fourier transform (iFFT), \sqrt{f} can be computed in quasi-linear time in |G|. However, if group $G = C_2^n$, then the complexity of computing \sqrt{f} is $O(n \cdot 2^n)$ which is exponential in *n*. In Section 3, we present a method to estimate \sqrt{f} efficiently.

integer-valued functions and characteristic 2.3 functions of weighted k-CNF formulae

Let G be a finite abelian group. In this paper, an *integer-valued function*⁴ on *G* is a function $f : G \to \mathbb{Z}$ such that for all $g \in G$, $|f(g)| = O (\operatorname{polylog}(|G|)).$

Typical examples of integer-valued functions are characteristic functions of weighted k-CNF formulae. A weighted k-CNF formula ϕ consists of logic clauses

$$\left\{ \left(\lor_{i \in S_{j}^{+}} x_{i} \right) \lor \left(\lor_{i \in S_{j}^{-}} \neg x_{i} \right) \right\}_{j \in J}$$

and integer weights $\{w_i\}_{i \in I}$. Here x_i 's are logic variables with values in {True, False} and \lor , \neg are logical operations OR and NOT respectively. We remark that if $w_j = 1$ for all $j \in J$, then ϕ is simply a k-CNF [10]:

$$\sum_{j\in J} \left(\lor_{i\in S_j^+} x_i \right) \lor \left(\lor_{i\in S_j^-} \neg x_i \right).$$

We define the *characteristic function* of ϕ by

$$f_{\phi}(z_1,\ldots,z_n) = \sum_{j\in J} \frac{w_j}{2^l} \left(\prod_{i\in S_j^+} (1+z_i) \cdot \prod_{i\in S_j^-} (1-z_i) \right),$$

where $(z_1, \ldots, z_n) \in C_2^n$. It is clear that the weighted MAX-SAT problem for ϕ is equivalent to the problem of computing the minimum of f_{ϕ} .

If we define

$$\tau$$
: {False, True} $\rightarrow C_2$, τ (False) = 1, τ (True) = -1,

then characteristic functions of k-CNF formulae have the following properties.

Proposition 2.3. [52] Let ϕ be a k-CNF formula in n variables with *m* clauses and let f_{ϕ} be the characteristic function of ϕ . Then

- (i) For $x_1, x_2, \dots, x_n \in \{\text{False, True}\}, f_{\phi}(\tau(x_1), \tau(x_2), \dots, \tau(x_n))$ is the number of clauses of ϕ falsified by x_1, x_2, \cdots, x_n .
- (ii) The degree of f_{ϕ} is at most k.
- (iii) The cardinality of supp (f_{ϕ}) is at most min $\left\{2^{k}m, \sum_{i=0}^{k} \binom{n}{i}\right\}$.
- (iv) f_{ϕ} can be computed by $O(2^k km)$ operations. In particular, if we fix k then f_{ϕ} can be computed by O(m) operations.
- (v) For each $x \in \{False, True\}^n$, $f_{\phi}(\tau(x))$ is the number of clauses of ϕ falsified by x.
- (vi) The image set of f_{ϕ} is contained in $\{0, \ldots, m\}$.

Moreover, the following statements are equivalent:

- (a) The number of satisfiable clauses of ϕ is at most (m L).
- (b) The number of falsified clauses of ϕ is at least L.
- $\begin{array}{l} (c) \ f_{\phi}-L+\delta \geq 0 \ for \ some \ function \ \delta: C_2^n \rightarrow [0,1). \\ (d) \ f_{\phi}-L+\delta \geq 0 \ for \ any \ function \ \delta: C_2^n \rightarrow [0,1). \end{array}$

According to Proposition 2.3, solving MAX-SAT for a CNF formula ϕ is equivalent to solving Problem 1.1 for f_{ϕ} .

LOWER BOUNDS BY FSOS 3

In this section, we present our solution to Problem 1.1 for an integervalued function f on G. Before we proceed, it worths a remark that if a lower bound L of f is given, then a better lower bound of f can be obtained by computing a positive lower bound of f - L unless Lis already the minimum of f.

Our solution to Problem 1.1 is based on the lemma that follows.

Lemma 3.1. Let $f : G \mapsto \mathbb{R}$ be a function on a finite abelian group *G* and let α_0 be a real number. Then α_0 is a lower bound of *f* if and only if there exists a finite family $\{h_i\}_{i \in I}$ of complex-valued functions

 $^{{}^{4}\}mathrm{The}$ assumption on the bound of values of f is used to control the complexity of Algorithm 1. Except for this, all statements in this paper actually hold true for any $f: G \to \mathbb{Z}.$

on G such that $f - \alpha_0 = \sum_{i \in I} |h_i|^2$. As a consequence, there is a one to one correspondence between the set of lower bounds of f and the feasible set of

$$\max_{\substack{h_i:G \mapsto \mathbb{C}, i \in I}} \alpha,$$

s.t. $f - \alpha = \sum_{i \in I} |h_i|^2.$ (2)

3.1 support selection

We notice that the inclusion of subsets of \widehat{G} induces a filtration on the feasible set of (2). Namely, for each $S \subseteq \widehat{G}$, we define

$$F_S := \left\{ \{h_i\}_{i \in I} : f - \alpha = \sum_{i \in I} |h_i|^2 \text{ for some } \alpha, \bigcup_{i \in I} \operatorname{supp}(h_i) \subseteq S \right\}.$$

Then we have

feasible set of (2) =
$$F_{\widehat{G}} := \bigcup_{S \subseteq \widehat{G}} F_S$$
 (3)

and $F_{S_1} \subseteq F_{S_2}$ if $S_1 \subseteq S_2 \subseteq \widehat{G}$. Therefore for a given $S \subseteq \widehat{G}$, we may consider the following problem:

$$\max_{\{h_i\}_{i\in I}\in F_S}\alpha.$$
 (4)

By the next lemma, (4) can be reformulated as an SDP.

Lemma 3.2. For each $S \subseteq \widehat{G}$, (4) is equivalent to

$$\max_{Q \in \mathbb{C}^{S \times S}} \quad \hat{f}(\chi_0) - \operatorname{trace}(Q), \tag{5}$$

s.t.
$$\sum_{\chi'\in\widehat{G}}Q_{(\chi')^{-1}\chi,\chi}=\widehat{f}(\chi), \ \chi\neq\chi_0\in\widehat{G}$$
(6)

$$Q \ge 0. \tag{7}$$

Here $\chi_0 \in \widehat{G}$ denotes the identity element in \widehat{G} and $Q \in \mathbb{C}^{S \times S}$ means Q is a matrix of size |S| whose rows and columns are indexed by elements in S.

PROOF. Conditions (6) and (7) ensure that Q is a Gram matrix of $f - \widehat{f}(\chi_0) + \operatorname{trace}(Q)$. Let Q be a feasible solution of (5) and let $Q = H^*H$ where $H \in \mathbb{C}^{r \times S}$. We set $h_i = \sum_{\chi \in S} H(i, \chi)\chi$, $i = 1, \ldots, r$. Then $\{h_i\} \in F_S$ is a feasible solution of (4), with $\alpha = \widehat{f}(\chi_0) - \operatorname{trace}(Q)$. On the contrary, if $\{h_i\}_{i \in I}$ is a feasible solution of (4), then we may define $H \in \mathbb{C}^{I \times S}$ where $H(i, \chi) = \widehat{h}_i(\chi)$. Thus $Q = H^*H$ is a feasible solution of (4), with $\widehat{f}(\chi_0) - \operatorname{trace}(Q) = \alpha$.

According to Lemma 3.1 and (3), each optimal solution of (5) supplies a lower bound of f. The quality of such a lower bound depends on the choice of S. For instance, if $S \supseteq \bigcup_{i \in I} \operatorname{supp}(h_i)$ for some optimal solution $\{h_i\}_{i \in I}$ of (2), then clearly the lower bound given by an optimal solution of (5) is the minimum of f. Moreover, the size of (5) is completely determined by |S|. Thus to efficiently compute a good lower bound of f, we need to choose small S which contains as more elements in $\bigcup_{i \in I} \operatorname{supp}(h_i)$ as possible, for some optimal solution $\{h_i\}_{i \in I}$ of (2). The rest of this section is devoted to the choice of $S \subseteq \widehat{G}$, under the guidance of this principal.

Let *f* be a nonnegative function on an abelian group *G*. We may expand \sqrt{f} as:

$$\sqrt{f} = \sum_{i=1}^{|G|} a_i \chi_i,$$

where $|a_1| \ge |a_2| \ge \cdots \ge |a_{|G|}|$ and $\widehat{G} = \{\chi_i\}_{i=1}^{|G|}$. By Theorem 2.2, $S = \{\chi_i\}_{i=1}^k$ is already a desired choice of *S* for (5). Here $k \le n$ is some given positive integer. However, as we point out at the end of Subsection 2.2, the complexity of computing the Fourier expansion of \sqrt{f} is quasi-linear in |G|, which leads to an exponential complexity in *n* if $G = C_2^n$. Thus to reduce the complexity, we may approximate the Fourier expansion of \sqrt{f} instead of computing it exactly. To this end, an estimate of the approximation error is necessary, which is the content of the next proposition.

Proposition 3.3. [53, Proposition 4.12] Let f be a nonnegative function on G and let $\varepsilon > 0$ be a fixed number. If p is a univariate polynomial such that $\max_{g \in G} \left| p(f(g)) - \sqrt{f(g)} \right| \le \varepsilon$, then we have

$$\max_{\chi \in \widehat{G}} \left| \widehat{p \circ f}(\chi) - \widehat{f}(\chi) \right| \le \varepsilon.$$

According to the above discussions, we may construct *S* of cardinality |S| = k by Algorithm 1. If *f* is integer-valued and $l \le f \le m$

Algorithm 1 Selection of support

Input nonnegative function f on G and parameters $l, m, d, k \in \mathbb{N}$. **Output** $S \subseteq \widehat{G}$ for (5) with |S| = k.

- approximate √t by a polynomial p(t) of degree at most d at integer points in [l, m].
- 2: compute the composition $p \circ f = \sum_{i=1}^{|G|} a_i \chi_i$, where $|a_1| \ge \cdots \ge |a_{|G|}|$ and $\deg(\chi_i) \ge \deg(\chi_{i+1})$ if $|a_i| = |a_{i+1}|, 1 \le i \le |G| 1$. **return** $S = \{\chi_1, \dots, \chi_k\}$.

for some $0 < l \le m$, then one can easily compute the polynomial p in step 1 of Algorithm 1 by solving a linear programming [53]. We remind the reader that $f : G \to \mathbb{Z}$ satisfies f(g) = O(polylog(|G|)) (cf. Subsection 2.3). Thus step 1 of Algorithm 1 can be accomplished in O(polylog(|G|)) time.

By Proposition 3.3, $p \circ f$ is an estimate of \sqrt{f} whose coefficient error is bounded above by $\max_{l \le i \le m} |p(i) - \sqrt{i}|$. This ensures that the set *S* obtained by Algorithm 1 indeed consists of first *k* terms of \sqrt{f} , if p(t) approximates \sqrt{t} sufficiently good at integer points between *l* and *m*. Furthermore, it is sufficient to take *d* to be small in practice. In fact, if *f* is the characteristic function of some CNF formula, then d = 1 or 2 already provides us good lower bounds, as we will see in Appendix A.

Let f_{\min} be the minimum of f on G. We notice that on the one hand, for each $a \leq f_{\min}$, we may apply Algorithm 1 to f - a to obtain an $S_a \subseteq \widehat{G}$. On the other hand, according to (2), (4) and (5), we can recover f_{\min} from the optimal solution of (5) if and only if S contains $S_{f_{\min}}$. An implication of the next proposition is that for each $a \leq f_{\min}$, S_a contains elements in $S_{f_{\min}}$ whose coefficients in f are sufficiently large. In particular, if f is nonnegative, then we can even take S_0 to serve as an estimate of $S_{f_{\min}}$.

Proposition 3.4. Let f be a function on G and let $a > \tilde{a} \ge 0$ be lower bounds of f. Assume that c_{χ} is the coefficient of χ in $\sqrt{f-a}$. If $|c_{\chi}| > \sqrt{a-\tilde{a}}$, then $\chi \in \text{supp}(\sqrt{f-\tilde{a}})$. Moreover, if $G = C_2^n$ and $f \le m$ for some $m \in \mathbb{R}$, then $|c_{\chi}| > \frac{a-\tilde{a}}{2} \left(\frac{1}{\sqrt{a-\tilde{a}}} - \frac{1}{\sqrt{m-\tilde{a}}+\sqrt{m-\tilde{a}}}\right)$ implies $\chi \in \text{supp}(\sqrt{f-\tilde{a}})$. PROOF. We recall that $c_{\chi} := \frac{1}{|G|} \sum_{g \in G} \overline{\chi(g)} \sqrt{f(g) - a}$. Similarly, we may write $\sqrt{f - \tilde{a}} = \sum_{\chi \in \widehat{G}} \widetilde{c_{\chi}} \chi$. For each $\chi \in \widehat{G}$, we notice that

$$\begin{aligned} |c_{\chi} - \widetilde{c_{\chi}}| &= \frac{1}{|G|} \left| \sum_{g \in G} \overline{\chi(g)} \left(\sqrt{f(g) - a} - \sqrt{f(g) - \widetilde{a}} \right) \right. \\ &= \frac{a - \widetilde{a}}{|G|} \left| \sum_{g \in G} \frac{\overline{\chi(g)}}{\sqrt{f(g) - a} + \sqrt{f(g) - \widetilde{a}}} \right| \\ &\leq \sqrt{a - \widetilde{a}}. \end{aligned}$$

Therefore, if $|c_{\chi}| > \sqrt{a - \tilde{a}}$ then $\tilde{c_{\chi}} \neq 0$.

If $G = C_2^n$, then $\chi \in \widehat{G} \simeq \mathbb{Z}_2^n$ is a multilinear monomial. Thus $\chi(g) = \pm 1$ for any $g \in C_2^n = \{-1, 1\}^n$. Moreover, it is straightforward to verify that we can construct a bijective map $\psi : C_2^n \to C_2^n$ such that $\chi(g) = -\chi(\psi(g))$ and $\psi^2 = \text{Id}$. Thus we have

$$\begin{aligned} |c_{\chi} - \widetilde{c_{\chi}}| &= \frac{a - \widetilde{a}}{|G|} \left| \sum_{g \in G} \frac{\chi(g)}{\sqrt{f(g) - a} + \sqrt{f(g) - \widetilde{a}}} \right| \\ &\leq \frac{a - \widetilde{a}}{|G|} \sum_{g \in G, \chi(g) = 1} \left(\frac{1}{\sqrt{a - \widetilde{a}}} - \frac{1}{\sqrt{m - a} + \sqrt{m - \widetilde{a}}} \right) \\ &= \frac{a - \widetilde{a}}{2} \left(\frac{1}{\sqrt{a - \widetilde{a}}} - \frac{1}{\sqrt{m - a} + \sqrt{m - \widetilde{a}}} \right). \end{aligned}$$

3.2 FSOS with error

We discuss in this subsection a remarkable feature of our solution (cf. Section 3) to Problem 1.1. That is, a solution to the SDP problem (5) which violates conditions (6) and (7) can still provide us a tight lower bound of f. This is the content of the next Theorem.

THEOREM 3.5 (FSOS WITH ERROR). Let G be a finite abelian group and let S a subset of \widehat{G} . Given a function $f : G \mapsto \mathbb{R}$ and a Hermitian matrix $Q \in \mathbb{C}^{S \times S}$, we have

$$\min_{g \in G} f(g) \ge - \|\widehat{e}\|_1 + \lambda |S|,$$

where λ is the minimal eigenvalue of Q, $e = f - v_S^* Q v_S$ and $v_S = (\chi)_{\chi \in S}$ is the vector consisting of all characters in S.

PROOF. For any $g \in G$, we have

$$f(g) - e(g) = v_S(g)^* Q v(g) \ge \lambda v_S(g)^* v(g).$$

We observe that $|\chi(g)| = 1$ for each $g \in G$ and $\chi \in \widehat{G}$. Thus $v_S(g)^*v_S(g) \le |S|$ and $|e(g)| \le \|\widehat{e}\|_1$ for any $g \in G$. This implies

$$f(g) \ge -\|\widehat{e}\|_1 + \lambda |S|, \quad g \in G.$$

According to Theorem 3.5, $-\|\hat{e}\|_1 + \lambda|S|$ is a lower bound of f even if $Q \neq 0$ or $e \neq 0$. The example that follows indicates that our method can give a tight lower bound of f even if it does not admit an FSOS supported on *S*.

Example 3.6. Let
$$f : C_2^3 \to \mathbb{R}$$
 be the function defined by
 $f(z_1, z_2, z_3) = 4 + z_1 + z_2 + z_3 + z_1 z_2 z_3.$

We can check that the SDP (5) has no feasible solution for $S = \{1, z_1, z_2, z_3\}$, i.e., f has no FSOS supported on S. However, if we let $e(z_1, z_2, z_3) := z_1 z_2 z_3$, then

$$f - e = 1 + \frac{1}{2} \sum_{i=1}^{3} (1 + z_i)^2,$$

from which we obtain $f \ge 1 - \|\widehat{e}\|_1 = 0$.

Below we record an algebraic version of Theorem 3.5, which is the form we need in Section 4.

Corollary 3.7. Let f, S, Q, e, λ be defined as above. Then $Q - \lambda \operatorname{Id} \geq 0$ is a Gram matrix of the function $f - e - \lambda |S|$.

We observe that f may be represented by the polynomial function $F_{Q-\lambda \text{Id}} + F_{\lambda \text{Id}}$, where $F_M(z) = z^T M z$ is the non-negative polynomial determined by a symmetric matrix M. Since $Q - \lambda \text{Id} \ge 0$, we obtain that the minimum of $F_{\lambda \text{Id}}$ supplies a lower bound of f. Unfortunately, when $z \in \mathbb{R}^n$, the minimum of $F_{\lambda \text{Id}}$ is $-\infty$ if $\lambda < 0$. This phenomenon distinguishes our method from the traditional method of polynomial optimization.

For instance, we consider $f = 2z_1z_2$ on C_2^2 so that Q is

$$egin{array}{ccc} z_1 & z_2 \ z_1 & \left[egin{array}{ccc} 0 & 1 \ z_2 \end{array}
ight] \ z_2 & \left[egin{array}{ccc} 1 & 0 \end{array}
ight] \end{array}$$

and the minimal eigenvalue of Q is -1. According to Corollary 3.7, we conclude that $Q + \text{Id} \ge 0$ is the Gram matrix of f + 2 and $f \ge -2$ on C_2^2 . As a comparison, f is represented by the polynomial

$$F_{Q+\mathrm{Id}} + F_{-\mathrm{Id}} = (z_1 + z_2)^2 - (z_1^2 + z_2^2)$$

and the minimum of $F_{-Id} = -(z_1^2 + z_2^2)$ is $-\infty$, which fails to provide a non-trivial lower bound of f.

We conclude this section by briefly summarizing the main advantages of FSOS with error. Interested readers are referred to Appendix A for numerical examples.

- (1) Early termination: existing methods [46, 51] need to wait the algorithm converges to a solution satisfying conditions (6) and (7). According to Theorem 3.5, our method can find a lower bound even these conditions are not satisfied. This feature enables us to set different time limits on solving SDP problems. Clearly the longer time limit we set, the better lower bound we obtain.
- (2) Adaptivity to more SDP solvers: since the condition (7) is not required to be satisfied exactly, we can use the more efficient SDP solver SDPNALplus [44] to solve the problem in Lemma 3.2. As a consequence, MAX-kSAT problems of much larger sizes from the benchmark set^{5 6} can be solved successfully, with which SOS based algorithms proposed in [46] fail to deal.
- (3) Size reduction: given a monomial basis S, algorithms in [46] can not get a lower bound if supp(f) ⊈ {χχ' : χ, χ' ∈ S}, since f has no FSOS supported on S. However, according to Theorem 3.5, a feasible solution of (5) that fails to satisfy (6) can still provide us a lower bound. This leads to a reduction on sizes of our SDP problems.

⁶http://www.maxsat.udl.cat/16/index.html

⁵http://www.maxsat.udl.cat/09/index.php

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4 COMPUTATION OF LOWER BOUNDS

This section is concerned with turning discussions in Section 3 into an algorithm for computing a lower bound of a function f on a finite abelian group G. To begin with, we recall that every function h on G admits the Fourier expansion:

$$h = \sum_{\chi \in \widehat{G}} \widehat{h}(\chi) \chi$$

For simplicity, we denote $h_0 := \hat{h}(1)$, where 1 is the identity element in \hat{G} .

4.1 our algorithm

Let *S* be a subset of \widehat{G} and let $Q \in \mathbb{C}^{S \times S}$ be a Hermitian matrix. We define $e := f - f_0 - v_S^* Q v_S$ where v_S is the |S|-dimensional column vector consisting of characters in *S*. Applying Theorem 3.5 to $f - f_0$, we obtain

$$f(g) \ge f_0 + e_0 - \|\widehat{e - e_0}\|_1 + \lambda_{\min}(Q)|S|, \quad g \in G$$

where $\lambda_{\min}(Q)$ is the smallest eigenvalue of Q. We also observe that

$$e_0 = -\sum_{\chi = \chi'} Q(\chi, \chi') = -\operatorname{trace}(Q)$$

Therefore, we can find a lower bound of f by solving the following unconstrained convex optimization problem:

$$\min_{Q \in \mathbb{C}^{S \times S}, \ Q = Q^*} F(Q), \tag{8}$$

where

$$F(Q) = \operatorname{trace}(Q) + \|E(Q)\|_1 - \lambda_{\min}(Q)|S|$$

and $E : \mathbb{C}^{S \times S} \mapsto \mathbb{C}^{|G|}$ is the affine map which maps Q to the (sparse) vector consisting of Fourier coefficients of $e - e_0$. We notice that the subgradient of F is

$$\partial F = \mathrm{Id} + \mathrm{sign}(E(Q))\partial E - |S|uu^*,$$

where ∂E is the gradient of E, sign(x) is the sign function and u is the unit eigenvector of Q corresponding to $\lambda_{\min}(Q)$. Thus we obtain Algorithm 2 for a lower bound of f.

Algorit	hm 2 Lower	Bounds o	f Functions	on Finite	Abelian	Groups
Input a	function f	on G. pos	itive integ	ers l. m. d	and k.	

Output a lower bound of *f*.

1: Select S with l, m, d and k .	⊳ by Algorithm 1
2: Solve (5) for Q_0 .	⊳ by SDPNAL+
3: Solve (8) for Q . \triangleright by gradient de	escent with initial point Q_0
4: return $f_0 - F(Q)$.	

4.2 rounding

Rounding is an important step in SDP-based algorithms for combinatorial optimization problems, especially when both the optimum value and optimum point are concerned. There exist several rounding techniques in the literature. Examples include rounding by random hyperplanes [15] together with its improved version [13], the skewed rounding procedure [29] and the randomized rounding technique [46]. Among all these rounding strategies, the one proposed in [46] can be easily adapted to our situation. We present two rounding methods, one is based on the null space of the Gram matrix, and the other is based on the lower rank approximation of the moment matrix.

 Let Q ∈ ℝ^{S×S} be a solution to (5). For a given f : G → ℝ and S ⊆ G containing all characters of degree at most one, we assume

$$f - \alpha = v_S^* Q v_S$$

where α is the minimum value of f, v_S is the |S|-dimensional column vector consisting of characters in S and $Q \ge 0$. If the null space of Q is one-dimensional, then by normalizing the first element of any null vector of Q to be one, we obtain the desired solution. Once we obtain a desired null vector $v \in \mathbb{R}^{|S|}$, we can recover $\tilde{g} \in G$ by rounding elements of v. If the null space of Q has dimension d > 1, then the normalization is no longer sufficient since elements of a null vector of Q might be algebraically inconsistent in this case. We can extract the solution based on the Stickelberger theorem, which is often used to solve polynomial systems in the literature [7, 8, 17, 32, 38].

(2) The moment matrix H ≥ 0 is the solution of the dual problem of (5). Let H = ∑_{i=1}^{|S|} µ_iu_iu_i^{*} be the eigenvalue decomposition of H, with µ₁ ≥ µ₂ ≥ µ₃, ... ≥ µ_{|S|}. Then H̃ = µ₁u₁u₁^{*} ∈ ℝ^{S×S} is the rank-one approximation of H. We can recover the solution x̃ from H̃ by setting x̃_i = H̃_{1,x_i}.

In practice, the matrix Q obtained in Algorithm 2 may not be positive semidefinite. We need to update it by $Q + \lambda_{\min}(Q)$ Id. It is a little surprise for us to notice that the numerical corank of $Q + \lambda_{\min}(Q)$ Id is 1 very often, which makes it possible to recover the optimal solution efficiently from its null vector.

To conclude this section, we illustrate the above rounding procedure by examples.

Example 4.1 (one-dimensional null space). In Example 3.6 we see that

$$f(z_1, z_2, z_3) = 4 + z_1 + z_2 + z_3 + z_1 z_2 z_3$$

is a non-negative function on C_2^3 . For

$$S = \{1, z_1, z_2, z_3, z_1 z_2 z_3\},\$$

we may obtain a Hermitian (but not positive semidefinite) matrix by SDPNAL+:

	1	z_1	z_2	z_3	$z_1 z_2 z_3$
1	1.9546	0.5000	0.5000	0.5000	0.5000
z_1	0.5000	0.4536	0.0000	0.0000	0.0000
$Q = z_2$	0.5000	0.0000	0.4536	0.0000	0.0000
z_3	0.5000	0.0000	0.0000	0.4536	0.0000
$z_1 z_2 z_3$	0.5000	0.0000	0.0000	0.0000	0.4536

whose eigenvalues are

$$-0.0462, 0.4536, 0.4536, 0.4536, 2.4544.$$

The normalized eigenvector corresponding to -0.0462 is

$$v = \begin{bmatrix} 1 & z_1 & z_2 & z_3 & z_1 z_2 z_3 \\ 1 & -1.000447 & -1.000447 & -1.000447 & -1.000447 \end{bmatrix}.$$

We recover $z_1 = z_2 = z_3 = -1$ by rounding the elements of v, which is the optimal solution of f = 0.

Lower Bounds of Functions on Finite Abelian Groups

Example 4.2 (higher dimensional null space). We consider the function $f(z_1, z_2) = (1+z_1+z_2+z_1z_2)^2$ on C_2^2 and $S = \{1, z_1, z_2, z_1z_2\}$. We notice that f has a Gram matrix

The null space of Q is spanned by

$$\begin{bmatrix} 1 & 0 & 0 & -1 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}^{\mathsf{T}}.$$

Using these null vectors, we form the multiplication matrix⁷ of $z_1 - z_2/2$:

$$\begin{bmatrix} 0 & 1 & -1/2 \\ 3/2 & 1/2 & 1/2 \\ -3/2 & -1 & -1 \end{bmatrix}$$

whose eigenvectors are

$$\begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^{\mathsf{T}}, \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^{\mathsf{T}}.$$

By normalizing the first element of these eigenvectors to be 1, we recover three solutions (1, -1), (-1, -1) and (-1, 1) of $f(z_1, z_2) = 0$.

A NUMERICAL EXPERIMENTS

In this section, we perform three numerical experiments to test Algorithm 2 and the rounding techniques discussed in Section 4.2. We implement our algorithm in Matlab (2016b) with SDPNAL+ [44]. The code is available on github.com/jty-AMSS/Fast-Lower-Bound-On-FAG. All experiments are performed on a desktop computer with Intel Core i9-10900X@3.70GHz CPU and 128GB RAM memory.

A.1 lower bounds of random functions

The goal of the first experiment is to exhibit the correctness and efficiency of Algorithm 2. We randomly generate non-negative integer-valued functions on two finite abelian groups and compute their lower bounds by Algorithm 2, TSSOS[49] and CS-TSSOS [50] respectively. Without loss of generality, we only consider functions whose minimum values are zero.

For comparison purposes, we record the running time and the result for each algorithm in Tables 1 and 2. This experiment indicates that Algorithm 2 is more efficient than the general-purpose methods TSSOS and CS-TSSOS on random examples.

A.1.1 experiment on C_2^{25} . We generate ten polynomials of degree three and sparsity at most 500 on C_2^{25} by randomly picking its non-constant coefficients from the set $\{m \in \mathbb{Z} : -5 \le m \le 5\}$. The constant terms of these polynomials are chosen so that their minimum values are zero.

For each of these ten polynomial f, we perform Algorithm 2 with the following parameters:

d = 2, k = 3 |supp(f)|, l = 0,

$$m =$$
sum of absolute value of coefficients of f .

We apply TSSOS and CS-TSSOS with Mosek [1] of relaxation order 2 to compute lower bounds of these functions. Results are shown in Table 1, where "sp" means the sparsity and "bound" means the lower bound obtained by the corresponding algorithm.

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No	cn	Algorithm 2		TSSOS		CS-TSSOS	
110	sp	bound	time	bound	time	bound	time
1	451	-6.6e-01	903	-2.75	407	9.0e-09	821
2	440	-1.1e-01	587	-3.95	1208	-3.44	1302
3	453	-4.9e-02	519	6.3e-08	1104	4.5e-09	1634
4	453	-1.6e-06	1464	-2.5	1091	-1.92	1510
5	451	-4.4e-01	804	-4.06	636	2.2e-06	1535
6	457	-6.7e-01	581	9.0e-11	1081	2.3e-10	2031
7	452	-2.2e-02	636	4.0e-10	847	7.0e-09	1895
8	455	-6.5e-03	775	4.8e-09	797	7.0e-11	1126
9	443	-1.3e-01	554	4.9e-10	716	2.4e-11	1284
10	454	-1.9e-02	561	2.0e-08	807	3.0e-09	1352

Table 1: Random examples on C_2^{25}

We notice that in Table 1, Algorithm 2 successfully recovers the minimum value zero for all these ten functions since our functions are integer-valued. As a comparison, TSSOS and CS-TSSOS fail to recover the minimum value on four and two instances, respectively. Moreover, Algorithm 2 is faster than TSSOS (resp. CS-TSSOS) on seven (resp. nine) out of ten functions.

A.1.2 experiment on C_3^{15} . We generate ten functions on C_3^{15} by the following procedure:

- (1) Set f = 0;
- (2) Randomly generate an integer-valued function *h* on C₃², such that 0 ≤ h ≤ 10;
- (3) Randomly pick a projection map $\tau : C_3^{15} \to C_3^2$;
- (4) Update $f \leftarrow f + h \circ \tau$;
- (5) Repeat steps (2)-(4) until the sparsity of f is greater than 190. (6) Update $f \leftarrow f - \min_{g \in C_\circ^{15}} f(g)$.

Functions obtained by the above procedure are integer-valued and their minimum values are zero. Although Algorithm 2 can handle functions on C_3^{15} of sparsity up to 350, we set the sparsity limit to be 190 because CS-TSSOS can only deal with functions of sparsity around 200.

We compute their lower bounds by Algorithm 2 with

$$d = 2, \quad k = 3 |\operatorname{supp}(f)|, \quad l = 0$$

m = sum of maximum value of h in step (2).

We also apply CS-TSSOS with Mosek of relaxation order 8 4 to these functions.

We do not test TSSOS in this experiment since it is only applicable to functions with real variables and real coefficients, and $C_3 = \{1, \omega, \omega^2\}$ where $\omega = \exp(2\pi i/3) \in \mathbb{C}$. Results are set out in Table 2, where "sp" denotes the sparsity of each function and "bound" means the lower bound computed by the corresponding algorithm.

As illustrated in Table 2, both Algorithm 2 and CS-TSSOS successfully recover the minimum value for all instances, but Algorithm 2 is much faster than CS-TSSOS.

We notice that all the ten functions in Table 2 have the same sparsity 191. After a little thought, one can realize that this is a result of our procedure to generate these functions. Indeed, the function $h \circ \tau$ in the procedure contains 8 monomials 1, z_i , z_j , z_i^2 , z_iz_j , $z_i^2z_j$, $z_iz_j^2$

⁷Any random linear combination of z_1 and z_2 works as well.

⁸CS-TSSOS fails to complete the computation if we use smaller relaxation order.

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No	en	Algorit	nm 2	CS-TSSOS			
INU	sp	bound time		bound	time		
1	191	-1.20e-01	237.0	-1.98e-05	3148.1		
2	191	-3.42e-12	224.8	2.38e-08	1923.3		
3	191	-3.53e-03	223.6	2.76e-09	444.5		
4	191	-1.74e-01	887.1	-8.55e-06	6623.8		
5	191	-3.31e-12	876.6	1.17e-07	1745.5		
6	191	-2.40e-12	221.7	5.34e-07	752.4		
7	191	-3.82e-12	225.3	3.92e-08	362.9		
8	191	-6.92e-02	402.5	8.33e-09	2144.8		
9	191	-1.53e-01	227.0	4.21e-07	1695.6		
10	191	-5.71e-12	404.3	4.61e-10	1105.3		

Table 2: Random examples on C_3^{15}

for some $1 \le i < j \le 15$. Once f involves all 15 variables in some iteration, it must contain all 31 monomials z_j^k , $1 \le j \le 15$, $0 \le k \le 2$. Thus each update $f + h \circ \tau$ only increases the sparsity by 0 or 4 and the sparsity of the resulting function is of the form 31 + 4k, $k \in \mathbb{N}$. Since we terminate the procedure if the sparsity exceeds 190, the sparsity is expected to be $191 = 31 + 4 \times 40$.

A.2 upper bounds of MAX-SAT problems

As an application, we test Algorithm 2 on MAX-SAT problems. We first recall from Proposition 2.3-(v) that for a given CNF formula ϕ in *n* variables with *m* clauses, the maximum number of simultaneously satisfiable clauses in ϕ is equal to $m - \min_{g \in C_2^n} f_{\phi}(g)$. Thus solving the MAX-SAT problem for ϕ is equivalent to computing the minimum value of its characteristic function f_{ϕ} on C_2^n .

Our testing MAX-SAT problems are drawn from the benchmark problem set in 2016 and 2009 MAX-SAT competitions⁹ ¹⁰. When applicable, we also apply TSSOS and CS-TSSOS to solve these problems. Results are summarized in Tables 3 and 4. From the former, we may conclude that Algorithm 2 is more effective than TSSOS and CS-TSSOS on MAX-SAT problems, while from the latter, we may see the flexibility of Algorithm 2.

A.2.1 experiment on unweighted MAX-2SAT problems. In this experiment, we consider randomly generated unweighted MAX-2SAT problems in 2016 MAX-SAT competition. Such a problem has 120 variables, in which the number of clauses ranges from 1200 to 2600. We apply Algorithm 2 to compute lower bounds of the corresponding characteristic functions on C_2^{120} , with the following parameters:

$$d = 1$$
, $k = |\text{supp}(f)|$, $l = 0$,
 $m =$ number of clauses.

We apply TSSOS and CS-TSSOS with Mosek of relaxation order 1 to these functions. It is worthwhile to point out that any higher order relaxations would result in a memory leak. This is because the size of a Gram matrix in a higher order relaxations increases exponentially.¹¹ For example, the size of a Gram matrix in the first relaxation is 121×121 while it is 7261×7261 in the second relaxation.

⁹http://www.maxsat.udl.cat/09/index.php

10 http://www.maxsat.udl.cat/16/index.html

Numerical results are reported in Table 3, in which "clause" denotes the number of clauses in each CNF formula, "min" is the minimum of the characteristic function and "bound" means the lower bound of the characteristic function obtained by each method.

From Table 3, we see that lower bounds obtained by Algorithm 2 are very close to minimum values of characteristic functions, which are better than those obtained by TSSOS and CS-TSSOS. In theory, one can expect an improvement in the quality of lower bounds by increasing the order of relaxations in TSSOS and CS-TSSOS. Unfortunately, as we point out, higher order relaxations are not possible due to the memory leak caused by huge sizes of MAX-2SAT problems we considered.

A.2.2 experiment on weighted MAX-3SAT problems. We test Algorithm 2 on weighted MAX-3SAT benchmark problems in 2009 MAX-SAT competition. These 3-CNF formulae are in 70 variables with 400 clauses. We remark that neither TSSOS nor CS-TSSOS is able to deal with these problems since their characteristic functions are of degree 3, forcing the relaxation order to be at least 2. This again causes a memory leak.

We apply Algorithm 2 to characteristic functions of these MAX-3SAT problems with the following two sets of parameters:

d = 1,	$k = \operatorname{supp}(f) ,$	l = 0,	m = sum of weights.
d = 2,	$k = 1.5 \left \operatorname{supp}(f) \right ,$	l = 0,	m = sum of weights.

The goal of this experiment is to exhibit the flexibility of Algorithm 2. Namely, the trade-off between the quality of the lower bound and the time cost can be controlled freely by parameters dand k. As a comparison, such a trade-off in other SOS-based algorithms such as TSSOS and CS-TSSOS is controlled by the order of relaxations, which may easily result in a memory leak (cf. A.2.1). Experimental results can be found in Table 4, where the meaning of labels are the same as those in Table 3.

A.3 rounding for MAX-2SAT problems

Let S_k be the support selected by Algorithm 1 with input l = 0, d = 2, m being the number of clauses, and k being selected such that the cardinality of $S = S_k \cup M_1$ is $|M_p|$, where M_1 is the set of monomials with degree at most 1, and M_p is the monomial basis containing M_1 and monomials $z_i z_j$ whenever logic variables x_i and x_j appear in the same clause, see [46, Definition 1]. We compare the rounding techniques presented in Section 4.2 with the rounding techniques with scaling factors ρ_i^N and $2^{-(i-1)}$ in [46] on randomly generated MAX-2SAT problems. For the rounding method based on the Gram matrix, the maximum number of iterations of SDPNAL+ is set to be |S|. For the rounding method based on the moment matrix, the maximum number of iterations of SDPNAL+ is set to be max(300, |S|).

A.3.1 experiment on random problems. In this experiment, we conduct the experiment in [46, Table 7]. For n = 25, 30, 35, 40 and m = 3n, 5n, 7n, we randomly generate 100 MAX-2SAT instances with *n* variables and *m* clauses, compare our rounding techniques with those given in [46], and record the frequencies of finding the optimum by each method. Results are presented in Table 5, in which

 $^{^{11}\}text{There}$ are $\sum_{j=0}^k \binom{120}{k}$ monomials involved in the k-th relaxation.

	No	clause	min	Algori	thm 2	TSS	OS	CS-TS	SOS
	140	clause		bound	time	bound	time	bound	time
	1	1200	161	159.5	370	146.7	45	146.7	52
	2	1200	159	156.7	327	143.1	49	143.1	55
	3	1200	160	159.0	362	146.8	46	146.8	64
	4	1300	180	177.5	450	162.4	52	162.4	73
	5	1300	172	170.6	417	156.2	47	156.2	65
	6	1300	173	171.6	432	158.8	44	158.8	58
	7	1400	197	194.8	506	179.8	46	179.8	75
	8	1400	191	189.3	499	174.3	51	174.3	87
	9	1400	189	187.2	504	172.1	58	172.1	78
	10	1500	211	209.9	589	194.5	53	194.5	93
	11	1500	213	210.1	573	194.4	50	194.4	88
	12	1500	207	205.7	531	191.3	50	191.3	87
	13	1600	233	231.2	668	215.6	50	215.6	90
	14	1600	239	235.0	668	218.7	48	218.7	85
	15	1600	233	230.5	617	215.2	52	215.2	68
	16	1700	257	255.1	745	238.3	48	238.3	91
	17	1700	248	245.7	749	229.2	53	229.2	105
	18	1700	239	238.9	738	225.3	50	225.3	86
	19	1800	291	285.8	788	268.4	56	268.4	87
	20	1800	262	261.3	896	244.2	53	244.2	99
	21	1800	279	277.5	836	259.8	50	259.8	99
	22	1900	293	292.1	1002	275.3	67	275.3	119
	23	1900	296	294.3	987	275.8	53	275.8	132
	24	1900	294	291.8	994	273.0	53	273.0	113
	25	2000	307	306.4	1103	288.8	57	288.8	149
	26	2000	321	318.1	1131	299.8	55	299.8	140
	27	2000	307	306.0	1240	288.3	58	288.3	139
ļ	28	2100	336	335.5	1202	317.7	68	317.7	141
	29	2100	336	332.2	1215	313.4	57	313.4	141
	30	2100	332	330.3	1224	311.3	57	311.3	104
ļ	31	2200	358	355.6	1467	334.6	54	334.6	163
	32	2200	371	365.8	1351	345.2	52	345.2	114
	33	2200	359	358.1	1381	338.3	59	338.3	130
	34	2300	380	377.6	1412	356.8	58	356.8	127
	35	2300	383	381.5	1535	359.2	61	359.2	136
	36	2300	365	364.7	1510	345.4	53	345.4	146
	37	2400	389	387.9	1686	368.7	56	368.7	151
	38	2400	402	400.0	1757	378.1	56	378.1	125
	39	2400	380	380.0	1951	363.0	69	363.0	153
	40	2500	418	416.4	1963	395.5	60	395.5	133
	41	2500	435	432.4	1876	411.5	55	411.5	143
	42	2500	425	424.2	1934	402.4	68	402.4	186
	43	2600	439	436.7	2004	414.0	57	414.0	148
	44	2600	458	455.8	2010	434.2	57	434.2	221
ļ	45	2600	440	439.1	1983	418.7	59	418.7	154

Lower Bounds of Functions on Finite Abelian Groups

Table 3: Unweighted MAX-2SAT problems

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No	min	$d = 1, k = \operatorname{supp}(f) $		d = 2, k = 3 supp(f) / 2		
INU		bound	time	bound	time	
1	12	6.2853	615	8.8433	2265	
2	19	10.7006	582	13.6655	2126	
3	5	4.2047	601	4.8462	2176	
4	20	12.1123	584	15.1769	2098	
5	14	7.4614	605	9.7210	2183	
6	12	8.0066	595	10.0030	2131	
7	17	10.1713	605	12.7487	2157	
8	7	5.4025	635	6.4714	2325	
9	17	10.6177	597	13.6209	2121	
10	12	8.2999	573	10.6198	2108	

Table 4: Weighted MAX-3SAT problems

"Vars" denotes the number of variables, "clause" denotes the number of clauses, "Gram" and "Moment" are the the frequencies of finding the optimum by methods based on the Gram matrix and the moment matrix repectively, ρ_i^N and $2^{-(i-1)}$ are the frequencies by recommended methods presented in [46]. Table 5 shows that the rounding method based on lower rank moment approximations has the highest frequency to find the optimum.

Vars	Clauses	Gram	Moment	ρ_i^N	$2^{-(i-1)}$
25	75	63	96	70	76
25	125	75	93	67	81
25	175	80	91	64	87
30	90	71	95	67	68
30	150	70	91	58	70
30	210	68	87	63	73
35	105	74	93	56	65
35	175	77	91	55	78
35	245	68	87	52	72
40	120	79	92	51	56
40	200	72	87	44	59
40	280	67	83	54	76

Table 5: rounding for random MAX-2SAT problems

A.3.2 experiment on benchmark problems. We test the rounding techniques respectively presented in Subsection 4.2 and [46] on randomly generated unweighted MAX-2SAT problems in 2016 MAX-SAT competition. Table 6 indicates that both of our rounding methods outperform the method presented in [46].

Furthermore, we remark that since the interior point method is used in [46] to solve SDP problems, their rounding techniques usually take at least 6000 seconds. In comparison, our methods take less than 1000 seconds, as our methods allow one to use SDPNAL+ to solve SDP problems.

REFERENCES

- MOSEK ApS. 2019. The MOSEK optimization toolbox for MATLAB manual. Version 9.3.11. https://docs.mosek.com/9.3/toolbox/index.html
- [2] Sanjeev Arora and Boaz Barak. 2009. Computational complexity: a modern approach. Cambridge University Press.

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No	clause	min	Gram	Moment	ρ_i^N	$2^{-(i-1)}$
1	1200	161	162	162	225	227
2	1200	159	159	164	215	194
3	1200	160	160	160	162	160
4	1300	180	180	185	226	243
5	1300	172	173	178	225	230
6	1300	173	173	174	245	253
7	1400	197	198	202	234	270
8	1400	191	192	199	255	246
9	1400	189	189	189	227	231

Table 6: rounding on MAX-2SAT benchmarks

- [3] Per Austrin. 2007. Balanced max 2-sat might not be the hardest. In Proceedings of the thirty-ninth annual ACM symposium on Theory of computing. 189–197.
- [4] Per Austrin. 2010. Towards sharp inapproximability for any 2-CSP. SIAM J. Comput. 39, 6 (2010), 2430-2463.
- [5] Nicos Christofides. 1976. Worst-case analysis of a new heuristic for the travelling salesman problem. Technical Report. Carnegie-Mellon Univ Pittsburgh Pa Management Sciences Research Group.
- [6] Stephen A Cook. 1971. The complexity of theorem-proving procedures. In Proceedings of the third annual ACM symposium on Theory of computing. 151– 158.
- [7] R. Corless, P. Gianni, and B. Trager. A reordered Schur factorization method for zero-dimensional polynomial systems with multiple roots. In *Proc. 1997 ISSAC'97*. 133–140.
- [8] R. Corless, P. Gianni, B. Trager, and S. Watt. The singular value decomposition for polynomial systems. In Proc. 1995 ISSAC'95. 96–103.
- [9] George B Dantzig and John H Ramser. 1959. The truck dispatching problem. Management science 6, 1 (1959), 80–91.
- [10] Martin Davis, Ron Sigal, and Elaine J Weyuker. 1994. Computability, complexity, and languages: fundamentals of theoretical computer science. Elsevier.
- [11] Irit Dinur and David Steurer. 2014. Analytical approach to parallel repetition. In STOC'14—Proceedings of the 2014 ACM Symposium on Theory of Computing. ACM, New York, 624–633.
- [12] Hamza Fawzi, James Saunderson, and Pablo A Parrilo. 2016. Sparse sums of squares on finite abelian groups and improved semidefinite lifts. *Mathematical Programming* 160, 1-2 (2016), 149–191.
- [13] U. Feige and M. Goemans. 1995. Approximating the value of two power proof systems, with applications to MAX 2SAT and MAX DICUT. In *Proceedings Third Israel Symposium on the Theory of Computing and Systems*. 182–189.
- [14] William Fulton and Joe Harris. 2013. Representation theory: a first course. Vol. 129. Springer Science & Business Media.
- [15] Michel X Goemans and David P Williamson. 1995. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. *Journal of the ACM (JACM)* 42, 6 (1995), 1115–1145.
- [16] Michel X. Goemans and David P. Williamson. 1995. Improved approximation algorithms for maximum cut and satisfiability problems using semidefinite programming. J. Assoc. Comput. Mach. 42, 6 (1995), 1115–1145.
- [17] Didier Henrion and Jean-Bernard Lasserre. 2005. Detecting Global Optimality and Extracting Solutions in GloptiPoly. Springer Berlin Heidelberg, Berlin, Heidelberg, 293–310.
- [18] Wenxuan Huang, Daniil A. Kitchaev, Stephen T. Dacek, Ziqin Rong, Alexander Urban, Shan Cao, Chuan Luo, and Gerbrand Ceder. 2016. Finding and proving the exact ground state of a generalized Ising model by convex optimization and MAX-SAT. Phys. Rev. B 94 (Oct 2016), 134424. Issue 13.
- [19] Anna R. Karlin, Nathan Klein, and Shayan Oveis Gharan. [2021] ©2021. A (slightly) improved approximation algorithm for metric TSP. In STOC'21-Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing. ACM, New York, 32–45.
- [20] H. Karloff and U. Zwick. 1997. A 7/8-approximation algorithm for MAX 3SAT?. In Proceedings 38th Annual Symposium on Foundations of Computer Science. 406–415.
- [21] Richard M Karp. 1972. Reducibility among combinatorial problems. In Complexity of computer computations. Springer, 85–103.
- [22] Subhash Khot, Guy Kindler, Elchanan Mossel, and Ryan O'Donnell. 2007. Optimal inapproximability results for MAX-CUT and other 2-variable CSPs? SIAM J. Comput. 37, 1 (2007), 319–357.
- [23] Bernhard H Korte, Jens Vygen, B Korte, and J Vygen. 2011. Combinatorial optimization. Vol. 1. Springer.
- [24] Mark W Krentel. 1988. The complexity of optimization problems. Journal of computer and system sciences 36, 3 (1988), 490-509.

- [25] Adam Kurpisz, Samuli Leppänen, and Monaldo Mastrolilli. 2016. Tight Sum-Of-Squares Lower Bounds for Binary Polynomial Optimization Problems. In 43rd International Colloquium on Automata, Languages, and Programming (ICALP 2016) (Leibniz International Proceedings in Informatics (LIPIcs)), Ioannis Chatzigiannakis, Michael Mitzenmacher, Yuval Rabani, and Davide Sangiorgi (Eds.), Vol. 55. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, 78:1–78:14.
- [26] Jean B. Lasserre. 2001. Global Optimization with Polynomials and the Problem of Moments. SIAM Journal on Optimization 11, 3 (2001), 796–817.
- [27] Jean B Lasserre. 2016. A MAX-CUT formulation of 0/1 programs. Operations Research Letters 44, 2 (2016), 158–164.
- [28] Monique Laurent. 2003. A comparison of the Sherali-Adams, Lovász-Schrijver, and Lasserre relaxations for 0–1 programming. *Mathematics of Operations Re*search 28, 3 (2003), 470–496.
- [29] Michael Lewin, Dror Livnat, and Uri Zwick. 2002. Improved Rounding Techniques for the MAX 2-SAT and MAX DI-CUT Problems. In *Integer Programming* and Combinatorial Optimization, William J. Cook and Andreas S. Schulz (Eds.). Springer Berlin Heidelberg, Berlin, Heidelberg, 67–82.
- [30] Silvano Martello and Paolo Toth. 1990. Knapsack problems: algorithms and computer implementations. John Wiley & Sons, Inc.
- [31] George B Mathews. 1896. On the partition of numbers. Proceedings of the London Mathematical Society 1, 1 (1896), 486–490.
- [32] H. Möller and H. Stetter. 1995. Multivariate Polynomial Equations with Multiple Zeros Solved by Matrix Eigenproblems. Numer. Math. 70 (1995), 311–329.
- [33] Ryan O'Donnell. 2014. Analysis of Boolean functions. Cambridge University Press, New York. xx+423 pages.
- [34] Christos H. Papadinitriou. 1977. The Euclidean traveling salesman problem is NP-complete. Theoret. Comput. Sci. 4, 3 (1977), 237-244.
- [35] Christos H. Papadimitriou. 1994. Computational complexity. Addison-Wesley Publishing Company, Reading, MA. xvi+523 pages.
- [36] Matthieu Py, Mohamed Sami Cherif, and Djamal Habet. 2021. A proof builder for Max-SAT. In International Conference on Theory and Applications of Satisfiability Testing. Springer, 488–498.
- [37] Prasad Raghavendra. 2008. Optimal algorithms and inapproximability results for every CSP? [extended abstract]. In STOC'08. ACM, New York, 245–254.
- [38] Greg Reid and Lihong Zhi. 2009. Solving polynomial systems via symbolicnumeric reduction to geometric involutive form. *Journal of Symbolic Computation* 44, 3 (2009), 280-291. Polynomial System Solving in honor of Daniel Lazard.
- [39] Julia Robinson. 1949. On the Hamiltonian game (a traveling salesman problem). Technical Report. Rand project air force arlington va.
- [40] Walter Rudin. 1962. Fourier analysis on groups. Vol. 121967. Wiley Online Library.
 [41] Shinsaku Sakaue, Akiko Takeda, Sunyoung Kim, and Naoki Ito. 2017. Exact semidefinite programming relaxations with truncated moment matrix for binary polynomial optimization problems. SIAM Journal on Optimization 27, 1 (2017),
- 565-582.
 [42] Petr Slavik. 1996. A tight analysis of the greedy algorithm for set cover. In Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996). ACM, New York, 435-441.
- [43] Lucas Slot and Monique Laurent. 2022. Sum-of-squares hierarchies for binary polynomial optimization. *Mathematical Programming* (2022), 1–40.
- [44] Defeng Sun, Kim-Chuan Toh, Yancheng Yuan, and Xin-Yuan Zhao. 2020. SDP-NAL+: A Matlab software for semidefinite programming with bound constraints (version 1.0). Optimization Methods and Software 35, 1 (2020), 87–115.
- [45] Paolo Toth and Daniele Vigo. 2002. The vehicle routing problem. SIAM.
- [46] H. van Maaren, L. van Norden, and M.J.H. Heule. 2008. Sums of squares based approximation algorithms for MAX-SAT. *Discrete Applied Mathematics* 156, 10 (2008), 1754–1779.
- [47] Vijay V Vazirani. 2001. Approximation algorithms. Vol. 1. Springer.
- [48] Hayato Waki, Sunyoung Kim, Masakazu Kojima, and Masakazu Muramatsu. 2006. Sums of squares and semidefinite program relaxations for polynomial optimization problems with structured sparsity. SIAM Journal on Optimization 17, 1 (2006), 218–242.
- [49] Jie Wang, Victor Magron, and Jean-Bernard Lasserre. 2021. TSSOS: A Moment-SOS hierarchy that exploits term sparsity. *SIAM Journal on Optimization* 31, 1 (2021), 30–58.
- [50] Jie Wang, Victor Magron, J. B. Lasserre, and Ngoc Hoang Anh Mai. 2022. CS-TSSOS: Correlative and Term Sparsity for Large-Scale Polynomial Optimization. ACM Trans. Math. Softw. 48, 4 (2022), 1–26.
- [51] Po-Wei Wang and J Zico Kolter. 2019. Low-rank semidefinite programming for the MAX2SAT problem. In Proceedings of the AAAI Conference on Artificial Intelligence, Vol. 33. 1641–1649.
- [52] Jianting Yang, Ke Ye, and Lihong Zhi. 2022. Computing sparse Fourier sum of squares on finite abelian groups in quasi-linear time. arXiv preprint arXiv:2201.03912 (2022).
- [53] Jianting Yang, Ke Ye, and Lihong Zhi. 2022. Short certificates for MAX-SAT via Fourier sum of squares. arXiv preprint arXiv:2207.08076 (2022).
- [54] Richard Y Zhang and Javad Lavaei. 2021. Sparse semidefinite programs with guaranteed near-linear time complexity via dualized clique tree conversion. *Mathematical programming* 188, 1 (2021), 351–393.