On Families of Planar DAGs with Constant Stack Number

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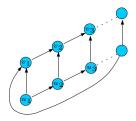
Abstract. A k-stack layout (or k-page book embedding) of a graph consists of a total order of the vertices, and a partition of the edges into k sets of non-crossing edges with respect to the vertex order. The stack number of a graph is the minimum k such that it admits a k-stack layout. In this paper we study a long-standing problem regarding the stack number of planar directed acyclic graphs (DAGs), for which the vertex order has to respect the orientation of the edges. We investigate upper and lower bounds on the stack number of several families of planar graphs: We improve the constant upper bounds on the stack number of single-source and monotone outerplanar DAGs and of outerpath DAGs, and improve the constant upper bound for upward planar 3-trees. Further, we provide computer-aided lower bounds for upward (outer-) planar DAGs.

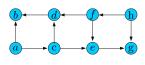
1 Introduction

Let G = (V, E) be a simple graph with n vertices and σ be a total order of the vertex set V. Two edges (u, v) and (w, z) in E with $u <_{\sigma} w$ cross if $u <_{\sigma} w <_{\sigma} v <_{\sigma} z$. A k-stack layout (k-page book embedding) of G is a total order of V and a partition of E into k subsets, called stacks or pages, such that no two edges in the same subset cross. The stack number (page number, book thickness) of G is the minimum k such that G admits a k-stack layout.

Heath et al. [18,19] extended the notion of stack number to directed acyclic graphs (DAGs for short) in a natural way: Given a DAG, G = (V, E), a book embedding of G is defined as for undirected graphs, except that the total order σ of V is now required to be a linear extension of the partial order of V induced by E. That is, if G contains a directed edge (u, v) from a vertex u to a vertex v, then $u <_{\sigma} v$ in any feasible total order σ of V. Heath et al. showed that DAGs with stack number 1 can be characterized and recognized efficiently; however, they proved that, in general, determining the stack number of a DAG is NP-complete.

The main problem raised by Heath et al. [18, 19] and studied in several papers [5,11,14,16,17] is whether every upward planar DAG has constant stack number. Recall that an *upward planar* DAG is a DAG that admits a drawing which is simultaneously *upward*, that is, each edge is represented by a curve monotonically increasing in the y-direction, and *planar*, that is, no two edges cross each other.





- (a) A DAG that requires n/2 stacks: edges (u_i, v_i) form an n/2-twist
- (b) An outerplanar DAG which is not upward planar [28]

Fig. 1: Planar DAGs that (a) need many stacks or (b) are not upward.

Open Problem 1 Is the stack number of every upward planar DAG bounded by a constant?

Notice that upward planarity is a necessary condition for the question: there exist DAGs which admit a planar non-upward embedding and that require $\Omega(n)$ stacks in any book embedding [19]; see Fig. 1a.

In its general form, Open Problem 1 is still unresolved. Heath et al. [18, 19] showed that directed trees and unicyclic DAGs have stack numbers 1 and 2, respectively. Mchedlidze and Symvonis [25] proved that N-free upward planar DAGs, which contain series-parallel digraphs, have stack number 2. Frati et al. [16] gave several conditions under which upward planar triangulations have bounded stack number. In particular, they showed that (i) maximal upward planar 3-trees have a constant stack number, and (ii) planar triangulations with a bounded (directed) diameter have a constant stack number. Notice that the graph in Fig. 1a, that requires $\Omega(n)$ stacks, is a partial planar 3-tree. Thus, it is reasonable to ask whether the stack number is bounded for (non-upward but directed acyclic) 2-trees or their subfamilies, outerplanar graphs, also known as simple 2-trees. This question has been first asked by Heath et al. [19] and recently highlighted by Bekos et al. [7].[‡] Bhore et al. [10] gave upper bounds for some upward outerplanar graphs, namely internally-triangulated outerpaths (16 stacks), cacti (6 stacks), and upward outerplanar graphs whose biconnected components are st-outerplanar (8 stacks).

We emphasize that directed acyclic 2-trees are planar but not necessarily upward, and thus, the results of Frati et al. [16] do not apply for this class of graphs. For example, the graph in Fig. 1b is a directed acyclic partial 2-tree (in fact, it is an outerpath DAG) but it cannot be drawn in an upward fashion.

[‡]Very recently, Jungeblut et al. [22] resolved the problem by proving that every outerplanar DAG has constant stack number, upper bounded by 24776, while there are directed acyclic (non upward planar) 2-trees with unbounded stack number. Their proof of the upper bound relies on Theorem 1b (see below) as a central tool and their second result solves an open question raised in a preprint version of this paper.

Our Contributions. We investigate upper and lower bounds for the stack number of upward planar DAGs and outerplanar DAGs (oDAGs for short). Throughout the paper, we express the bounds in terms of the maximum size of a twist in the vertex order, that is, the maximum number of mutually crossing edges. This parameter, also called the twist number of a graph, is tied to the stack number; analyzing the maximum twist size significantly simplifies the arguments at the cost of (slightly) worsened bounds for the stack number. We refer to Section 2 for details and formal definitions.

In Section 3, we present constant upper bounds for several prominent subclasses of outerplanar DAGs.

Theorem 1.

- **a.** Every single-source outerplanar DAG has a constant stack number with a vertex order whose twist size is at most 3.
- **b.** Every monotone outerplanar DAG has a constant stack number with a vertex order whose twist size is at most 4.
- c. Every outerpath DAG has a constant stack number with a vertex order whose twist size is at most 4.

The recent result of Davies [13] implies that every graph with a vertex order whose twist size is at most k, has stack number at most $2k \log_2 k + 2k \log_2 \log_2 k + 10k$ (and for k=3 Davies proves an upper bound of 19). It follows that single-source oDAGs have stack number at most 19, while monotone oDAGs and outerpath DAGs have stack number at most 64. We note that the stack assignment for the provided vertex orders can likely be improved. For example, we show an upper bound of 4 stacks for single-source oDAGs (refer to Lemma 2 in Section 3).

Our proof technique utilized for Theorem 1 can be applied to other classes of DAGs. In Section 4 we tighten the upper bound on the stack number of upward (maximal) planar 3-trees. Frati et al. [16] bound the stack number of upward planar 3-trees by a function of the size of the maximum twist size without providing an explicit bound. We strengthen their results by presenting an arguably simpler proof that yields an exact (small) bound of 5 on the maximum twist size (which by Davies' result [13] translates into a stack number of at most 85).

Theorem 2. Every upward planar 3-tree has a constant stack number with a vertex order whose twist size is at most 5.

The proofs of Theorem 1 and Theorem 2 are constructive and lead to lineartime algorithms for constructing the vertex orders.

Finally, we explore lower bounds on the stack number of planar DAGs in Section 5. They rely on computational experiments using a SAT formulation of the book embedding problem.

Theorem 3.

- a. There exists a single-source single-sink upward outerplanar DAG with stack number 3.
- **b.** There exists an upward outerplanar DAG with stack number 4.
- c. There exists an upward planar 3-tree DAG with stack number 5.

Other Related Work. Book embeddings of undirected graphs received a lot of attention due to their numerous applications. It is known that the graphs with stack number 1 are exactly outerplanar graphs, while graphs with stack number 2 are exactly the subhamiltonian graphs, which implies that it is NP-complete to decide whether a graph admits a 2-stack layout. More generally, every planar graph has stack number at most 4, and the bound is worst-case optimal [9, 34].

Stack numbers of directed acyclic graphs have also been extensively studied. Similarly to the undirected case, it is NP-complete to test whether the stack number of a DAG is at most k, even when k=2 [8]. Several works analyzed the stack number of partially ordered sets (posets), which can be viewed as upward planar DAGs without transitive edges. Nowakowski and Parker [27] asked whether the stack number of a planar poset is bounded by a constant. Notice that the question is a special case of Open Problem 1. Several works provide bounds for the stack number of special classes of posets and bounds in terms of various parameters (e.g., height or bump number) [4,20,33]. To our knowledge, there is no indication that the absence of transitive edges simplifies Open Problem 1.

As for the lower bounds on the stack number of DAGs and posets, not many results are known. It is easy to construct a planar poset with stack number 4 [4, 20], which for a long time has been the best known lower bound for the stack number of upward planar DAGs. Our Theorem 3 strengthens the result by showing that there exist (maximal) upward planar 3-trees with stack number 5. Merker [26] independently constructed a planar poset with stack number 5. Jungeblut et al. [21] further showed that upward planar graphs of constant width and height have a bounded stack number, and combined the two results to get an $\mathcal{O}(n^{2/3}\log(n)^{2/3})$ upper bound on the stack number of general upward planar graphs. Yet these results do not imply any upper bound for the graph classes considered in Section 3 (since oDAGs can be non-upward) or in Section 4 (since upward planar 3-trees can have linear width and height).

Due to space constraints, details of omitted/sketched proofs are in the appendix.

2 Preliminaries

Throughout the paper, G = (V, E) is a simple directed graph (digraph) with vertex set V and edge (arc) set E. A vertex order, σ , of a digraph G is a linear extension of V. That is, if G contains an edge from a vertex u to a vertex v, denoted $(u,v) \in E$, then $u <_{\sigma} v$ in any feasible vertex order σ of V. Let F be a set of $k \geq 2$ independent (that is, having no common endpoints) edges $(s_i,t_i), 1 \leq i \leq k$. If $s_1 <_{\sigma} \cdots <_{\sigma} s_k <_{\sigma} t_1 <_{\sigma} \cdots <_{\sigma} t_k$, then F is a k-twist. Two independent edges forming a 2-twist are called crossing. A k-stack layout of G is a pair $(\sigma, \{S_1, \ldots, S_k\})$, where σ is a vertex order of G and $\{S_1, \ldots, S_k\}$ is a partition of E into stacks, that is, sets of pairwise non-crossing edges. The minimum number of stacks in a stack layout of G is its stack number.

The size of the largest twist in a vertex order is tied to the number of stacks needed for the edges of the graph under the vertex order. In one direction, a vertex order with a k-twist needs at least k stacks, since each edge of a twist

must be in a distinct stack. In the other direction, a vertex order with no (k+1)-twist needs at most $\mathcal{O}(k \log k)$ stacks [13], which matches the lower bound of $\Omega(k \log k)$ [24]. An order without a 2-twist (that is, when k=1) corresponds to an outerplanar drawing of a graph, which is a 1-stack layout. For k=2 (an order without a 3-twist), 5 stacks are sufficient and sometimes necessary [1,23].

In the following we use notation $E(V_1 \to V_2)$ to indicate a subset of E between disjoint subsets $V_1, V_2 \subseteq V$, that is, $(x, y) \in E$ for $x \in V_1, y \in V_2$. Notation $E(V_1 \to V_2, V_3 \to V_4, \dots)$ indicates the union of the edge sets, that is, $E(V_1 \to V_2) \cup E(V_3 \to V_4) \cup \dots$ Similarly, we write $twist(V_1 \to V_2) \leq k$ to indicate that the maximum twist of the edges $E(V_1 \to V_2)$ is of size at most k. Slightly abusing the notation, we sometimes write $E(v \to V_1)$ or $E(V_1 \to v)$, where $v \in V \setminus V_1$ and $V_1 \subset V$. To specify a relative order between disjoint subsets of vertices, we use $\sigma = [V_1, V_2, \dots, V_r]$, where $V_i \subseteq V$ for $1 \leq i \leq r$. For the vertex order σ , it holds that $x <_{\sigma} y$ for all $x \in V_i, y \in V_j$ such that i < j.

3 Outerplanar DAGs

We study the stack number of oDAGs, that is, directed acyclic outerplanar graphs. We stress that such graphs are planar but not necessarily upward. For example, the graph in Fig. 1b cannot be drawn in an upward fashion. We assume oDAGs are maximal as it is straightforward to augment an oDAG to a maximal one, and the stack number is a monotone parameter under taking subgraphs.

It is well-known that every maximal outerplanar directed acyclic graph can be constructed from an edge, which we call the *base* edge, by repeatedly *stellating* edges [22]; that is, picking an edge, (s,t), on its outerface and adding a vertex x together with two edges connecting x with s and t; see Fig. 2a. In order to keep the graph acyclic, the directions of the new edges must be either *transitive*:

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O.1 (s,x) \in E and (x,t) \in E, or monotone:
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O.2
$$(s, x) \in E$$
 and $(t, x) \in E$, **O.3** $(x, s) \in E$ and $(x, t) \in E$.

We emphasize that every edge, including the base edge, in the construction sequence of outerplanar graphs can be stellated at most once; relaxing the condition yields a construction scheme for 2-trees.

We study subclasses of outerplanar DAGs that can be constructed using a subset of the three operations. First observe that so-called *transitive* oDAGs that are constructed from an edge by applying O.1 have a single source vertex, a single sink vertex, and an edge connecting the source with the sink. Such graphs are trivially embeddable in one stack. In Section 3.1 we observe that single-source oDAGs can be constructed using O.1 and O.2; similarly, single-sink oDAGs can be constructed by O.1 and O.3. We show that single-source (single-sink) oDAGs admit a layout in a constant number of stacks. Furthermore, using monotone operations (O.2 and O.3), one can construct outerplanar graphs with arbitrarily many sources and sinks. Such *monotone* oDAGs admit layouts in a constant number of stacks, as we prove in Section 3.2. Finally, we investigate *outerpath* DAGs, that is, oDAGs whose weak dual is a path. In Section 3.3 we describe

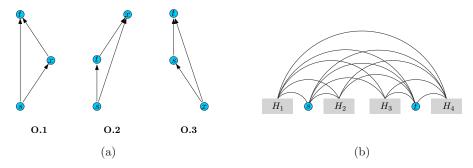


Fig. 2: (a) Possible ways of stellating base edge (s,t) with a vertex x for constructing oDAGs. (b) Vertex order utilized for the inductive schemes in Section 3.

a construction scheme for such graphs and prove that their stack number is constant.

All our proofs are based on an inductive scheme by decomposing a given oDAG into two subgraphs that can be embedded so that a list of carefully chosen invariants is maintained. Then we show how to combine the layouts of the subgraphs and verify the invariants. To this end, we consider a base edge $(s,t) \in E$ and define a vertex order consisting of six vertex-disjoint parts $\sigma = [H_1, s, H_2, H_3, t, H_4]$, where $H_i \subset V, 1 \le i \le 4$. For all the considered graph classes, we require that $E(H_2 \to H_3) = \emptyset$; see Fig. 2b. In all figures in the paper all edges are oriented from left to right unless the arrows explicitly indicate edge directions.

3.1 Single-Source oDAGs

Here we consider *single-source* (*single-sink*) outerplanar DAGs that contain only one source (sink) vertex. Single-source oDAGs can be constructed from an edge by applying two of the operations, O.1 and O.2. To this end, choose an edge incident to the source on the outerface of the graph as the base edge, and observe that applying O.3 would create a predecessor of the source or an additional source. Similarly, single-sink graphs can be constructed by two operations, O.1 and O.3.

Lemma 1. Every single-source (single-sink) outerplanar DAG admits an order whose twist size is at most 3.

Proof (sketch). Let G = (V, E) be a given oDAG with a unique source $s \in V$, and assume that $(s,t) \in E$ is the base edge in the construction sequence of G. We prove the claim by induction on the size of G by using the following invariant (see Fig. 3a): There exists an order of V consisting of four parts, $\sigma = [s, H_3, t, H_4]$ (that is, $H_1 = H_2 = \emptyset$), such that the following holds:

I.1
$$twist(s \rightarrow H_4, H_3 \rightarrow H_4) \leq 1$$
 I.2 $twist(E) \leq 3$

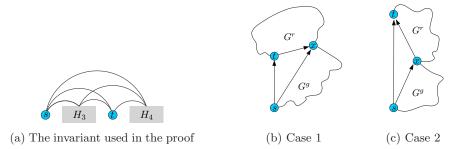


Fig. 3: An illustration for Lemma 1

Now we prove that these invariants can be maintained. If G is a single edge, then the base of the induction clearly holds. For the inductive case, we consider the base edge (s,t) of G. Let x be the unique neighbor of s and t. Since G is a single-source oDAG, there are two ways the edges between x and s,t are directed, corresponding to operations O.1 and O.2. Consider both cases.

Case 1. First assume $(s,x) \in E$ and $(t,x) \in E$. It is easy to see that G is decomposed into two edge-disjoint subgraphs sharing a single vertex x; denote the graph containing (s,x) by G^g and the graph containing (t,x) by G^r ; see Fig. 3b. Since G is a single-source oDAG, G^g is also a single-source oDAG with source s and base edge (s,x). Similarly, G^r is a single-source oDAG with source t and base edge (t,x). By the induction hypothesis, the graphs admit orders σ^g and σ^r satisfying the invariant. Next we combine the orders into a single one for G. Let $\sigma^g = [s, G_3, x, G_4]$ and $\sigma^r = [t, R_3, x, R_4]$. Then we set

$$\sigma = [s, t, R_3, G_3, x, G_4, R_4]$$

and observe that in the order $H_3 = \emptyset$, $H_4 = [R_3, G_3, x, G_4, R_4]$; see Fig. 4a. It is easy to see that σ is a linear extension of V, that is, $u <_{\sigma} v$ for all edges $(u, v) \in E$. Next we verify the conditions of the invariant.

- I.1. Since $H_3 = \emptyset$, we have $twist(s \to H_4, H_3 \to H_4) = twist(s \to H_4) \le 1$, where the inequality holds because all edges $E(s \to H_4)$ share a common vertex, s.
- I.2. Consider the maximum twist κ in G under vertex order $\sigma = [s, H_3, t, H_4]$, and suppose for contradiction that $|\kappa| \geq 4$. Observe that $H_3 = \emptyset$ in the considered case, and κ may contain at most one edge incident to s and at most one edge incident to t. Thus, at least two of the edges of κ are from $E(H_4 \to H_4)$; see Fig. 4a. Denote one of the two edges by $e \in \kappa$. Since $e \in E(H_4 \to H_4)$, both endpoints of e are in $R_3 \cup G_3 \cup \{x\} \cup G_4 \cup R_4$. Notice that if the two endpoints are in the same part (e.g., R_i or G_i for some i), then all edges of κ have at least one endpoint in that part (since they all cross e), and specifically all edges of κ are either in G^g or G^r , which implies that $|\kappa| \leq 3$ by the induction hypothesis. Hence, we assume that e belongs

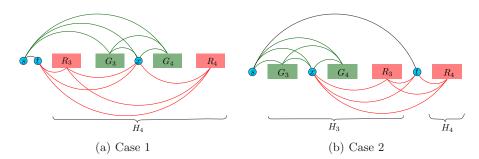


Fig. 4: An inductive step in the proof of Lemma 1

to $E(R_3 \to R_4, R_3 \to x, x \to R_4, G_3 \to G_4, G_3 \to x, x \to G_4)$ and that none of the edges of κ contains both endpoints in the same part R_i or G_i of V.

- If $e \in E(G_3 \to x, x \to G_4)$, then the only edges potentially crossing e are in $E(G_3 \to G_4, s \to G_4, s \to G_3)$, that is, they all belong to G^g . In that case $|\kappa| \leq 3$ by the hypothesis I.2 applied to G^g , a contradiction. Therefore, $\kappa \cap E(G_3 \to x, x \to G_4) = \emptyset$.
- If $e \in E(G_3 \to G_4)$, then the edges crossing e are either incident to s, or incident to x, or in $E(G_3 \to G_4)$. Since $twist(G_3 \to G_4) \le 1$, we have that $|\kappa| \le 3$. Therefore, $\kappa \cap E(G_3 \to G_4) = \emptyset$.
- If $e \in E(R_3 \to x)$, then the edges crossing e are in $E(s \to G_3, t \to R_3, R_3 \to R_4)$. Observe that each of the three subsets contributes at most one edge to κ ; thus, $|\kappa| \leq 3$. Therefore, $\kappa \cap E(R_3 \to x) = \emptyset$.
- If $e \in E(x \to R_4)$, then the edges crossing e are in $E(s \to G_4, t \to R_4, R_3 \to R_4)$. Each of the three subsets contributes at most one edge to κ ; thus, $|\kappa| \leq 3$. Therefore, $\kappa \cap E(x \to R_4) = \emptyset$.
- If $e \in E(R_3 \to R_4)$, then the edges crossing e are either adjacent to s or t, or in $E(R_3 \to R_4)$. Since $twist(R_3 \to R_4) \le 1$, we have that $|\kappa| \le 3$, contradicting our assumption.

Case 2. Now assume $(s, x) \in E$ and $(x, t) \in E$; see Fig. 3c. Again, G is decomposed into two edge-disjoint single-source subgraphs sharing a single vertex x; denote the graph containing (s, x) by G^g and the graph containing (x, t) by G^r , where s is the single source of G^g and x is the single source of G^r . By the induction hypothesis, the two graphs admit orders σ^g and σ^r satisfying the invariant. Let $\sigma^g = [s, G_3, x, G_4]$ and $\sigma^r = [x, R_3, t, R_4]$. Then $\sigma = [s, G_3, x, G_4, R_3, t, R_4]$, where $H_3 = [G_3, x, G_4, R_3]$, $H_4 = R_4$; see Fig. 4b. In Appendix A.1 we show that the invariants are maintained.

The recent result of Davies [13] implies that the stack number of single-source outerplanar DAGs is at most 48. We reduce this upper bound on the stack number to 4 via a similar argument that employs the same recursive decomposition as in Lemma 1. The proof of Lemma 2 is in the appendix.

Lemma 2. Every single-source outerplanar DAG admits a 4-stack layout.

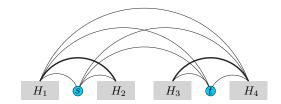


Fig. 5: The invariant used in the proof of Lemma 3

It is straightforward to extend Lemma 2 to oDAGs with a constant number of sources (sinks), that is, to construct a layout of an oDAG with 4s stacks, where s is the number of sources (sinks) in the graph. Partition the oDAG into s single-source subgraphs and embed each of them in a separate set of 4 stacks.

3.2 Monotone oDAGs

Here we consider *monotone* outerplanar DAGs that are constructed from an edge by applying operations O.2 and O.3. As in the previous section, we assume that the construction sequence along with the base edge is known.

Lemma 3. Every monotone outerplanar DAG admits an order whose twist size is at most 4.

Proof (sketch). We prove the claim by induction on the size of the given oDAG, G = (V, E), by using the following invariants (see Fig. 5): For a base edge $(s,t) \in E$, there exists a vertex order consisting of six parts, $\sigma = [H_1, s, H_2, H_3, t, H_4]$, such that the following holds:

- **I.1** $E(H_1 \rightarrow H_3) = E(H_2 \rightarrow H_3) = E(H_2 \rightarrow H_4) = \emptyset$
- **I.2** $twist(H_1 \cup \{s\} \rightarrow \{t\} \cup H_4) \le 1$
- **I.3** $twist(H_1 \to H_2 \cup \{t\} \cup H_4) \le 2$ **I.4** $twist(H_1 \cup \{s\} \cup H_3 \to H_4) \le 2$
- **I.5** $twist(E) \leq 4$

If G consists of a single edge, then the base of the induction clearly holds. For the inductive case, consider the base edge (s,t) of G and choose the unique common neighbor of s and t, denoted $x \in V$. Since G is monotone and acyclic, there are two ways the edges between x and s,t are directed: either $(s,x) \in E, (t,x) \in E$ (O.2) or $(x,s) \in E, (x,t) \in E$ (O.3). Observe that, since a (monotone) outerplanar DAG remains (monotone) outerplanar after reversing all edge directions and the described invariants are symmetric with respect to parts H_1, H_2 and parts H_3, H_4 , it is sufficient to study only one of the two cases. Therefore we investigate the former case, while the latter case follows from the symmetry.

Assume $(s, x) \in E$ and $(t, x) \in E$. It is easy to see that G is decomposed into two edge-disjoint monotone oDAGs sharing a vertex $x \in V$; denote the graph containing (s, x) by G^g and the graph containing (t, x) by G^r . By the induction

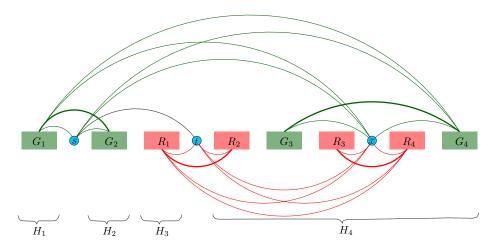


Fig. 6: An inductive step used in the proof of Lemma 3

hypothesis, the two graphs admit orders σ^g and σ^r satisfying the described invariant. Let $\sigma^g = [G_1, s, G_2, G_3, x, G_4]$ and $\sigma^r = [R_1, t, R_2, R_3, x, R_4]$. Then

$$\sigma = [G_1, s, G_2, R_1, t, R_2, G_3, R_3, x, R_4, G_4]$$

where $H_1 = G_1$, $H_2 = G_2$, $H_3 = R_1$, $H_4 = [R_2, G_3, R_3, x, R_4, G_4]$ in σ ; see Fig. 6. In Appendix A.2 we show that the invariants are maintained under σ . \square

3.3 Outerpath DAGs

Let G be an embedded (plane) graph. Recall that a weak dual is a graph whose vertices are bounded faces of G and edges connect adjacent faces of G. A graph is an outerpath if its weak dual is a path. Consider a face of an outerpath G = (V, E) that corresponds to a terminal of the path, and make an edge on the face adjacent to the outerface of G to be a base edge. It is easy to see that every outerpath can be constructed from such a base edge by repeatedly stellating edges such that the following holds (which keeps the weak dual to be a path): After stellating edge (u, v) with a vertex w, only one of the two newly added edges, $\{u, w\}$ and $\{v, w\}$, can be further stellated. In order to construct an outerpath DAG, the directions of the edges have to follow one of the operations, O.1, O.2, or O.3.

 ${\bf Lemma~4.~} \ Every \ outerpath \ DAG \ admits \ an \ order \ whose \ twist \ size \ is \ at \ most \ 4.$

Proof (sketch). We prove the claim by induction on the size of the given outerpath DAG, G = (V, E), by using the following invariants: For a base edge $(s, t) \in E$, there exists a vertex order consisting of six parts, $\sigma = [H_1, s, H_2, H_3, t, H_4]$, such that the following holds:

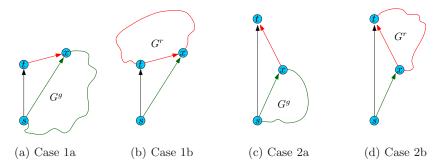


Fig. 7: Cases in Lemma 4: stellating base edge (s,t) with a vertex x

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I.1 H_2 = \emptyset or H_3 = \emptyset, that is, E(H_2 \rightarrow H_3) = \emptyset
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I.2
$$twist(H_2 \to t, H_2 \to H_4) \le 1$$
 I.3 $twist(H_1 \to H_3, s \to H_3) \le 1$

- **I.4** $twist(H_1 \cup \{s\} \cup H_2 \rightarrow H_3 \cup \{t\} \cup H_4) \le 2$
- **I.5** $twist(H_1 \rightarrow H_2, H_2 \rightarrow \{t\} \cup H_4) \leq 3$ **I.6** $twist(H_1 \cup \{s\} \rightarrow H_3, H_3 \rightarrow H_4) \leq 3$
- **I.7** $twist(H_1 \cup \{s\} \cup H_2 \cup H_3 \rightarrow H_4) \le 3$ **I.8** $twist(H_1 \rightarrow H_2 \cup H_3 \cup \{t\} \cup H_4) \le 3$
- **I.9** $twist(E) \leq 4$

If G consists of a single edge, then the base of the induction clearly holds. For the inductive case, consider a base edge $(s,t) \in E$ and choose the unique common neighbor of s and t, denoted $x \in V$. Although all three operations can be applied on (s,t), by symmetry, it is sufficient to study only O.1 and O.2. Depending on which edges of face $\langle s,t,x\rangle$ are utilized for the construction, we distinguish four cases; see Fig. 7. As in earlier proofs we denote the graphs constructed on (s,x) by G^g and the graph on (t,x) by G^r , and assume the graphs admit orders σ^g and σ^r satisfying the invariants. Notice however that, since G is an outerpath, only one of G^g , G^r contains more than two vertices.

Case 1a. Assume that $(s,x) \in E$, $(t,x) \in E$, $\sigma^g = [G_1, s, G_2, G_3, x, G_4]$, and $\sigma^r = [t,x]$. We set $\sigma = [G_1, s, G_2, t, G_3, x, G_4]$, where $H_1 = G_1$, $H_2 = G_2$, $H_3 = \emptyset$, $H_4 = [G_3, x, G_4]$ in σ ; see Fig. 8a.

Case 1b. Assume $(s, x) \in E$, $(t, x) \in E$, $\sigma^g = [s, x]$, and $\sigma^r = [R_1, t, R_2, R_3, x, R_4]$. We set $\sigma = [s, R_1, t, R_2, R_3, x, R_4]$, where $H_1 = H_2 = \emptyset$, $H_3 = R_1$, $H_4 = [R_2, R_3, x, R_4]$ in σ ; see Fig. 8b.

Case 2a. Assume $(s, x) \in E$, $(x, t) \in E$, $\sigma^g = [G_1, s, G_2, G_3, x, G_4]$, and $\sigma^r = [x, t]$. We set $\sigma = [G_1, s, G_2, G_3, x, G_4, t]$, where $H_1 = G_1$, $H_2 = [G_2, G_3, x, G_4]$, $H_3 = H_4 = \emptyset$ in σ ; see Fig. 8c.

Case 2b. Assume $(s, x) \in E$, $(x, t) \in E$, $\sigma^g = [s, x]$, and $\sigma^r = [R_1, x, R_2, R_3, t, R_4]$. The case is reduced to Case 2a by reversing all edge directions; see Fig. 8d. Appendix A.3 shows that the invariants are maintained in each of the cases. \Box

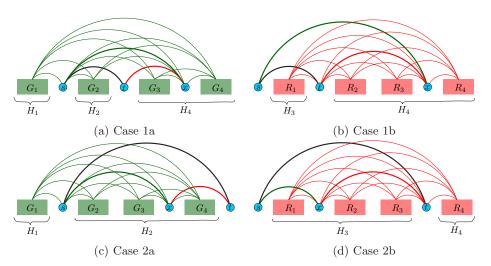


Fig. 8: An illustration for Lemma 4

4 Upward Planar 3-Trees

Theorem 2. Every upward planar 3-tree admits an order whose twist size is at most 5.

Proof (sketch). We prove the claim by induction on the size of a given upward planar 3-tree, G = (V, E), by using the following invariants (see Fig. 9a): For the outerface $\langle s, m, t \rangle$ of G, there exists a vertex order consisting of five parts, $\sigma = [s, H_1, m, H_2, t]$, where $H_1, H_2 \subset V$, and the following holds:

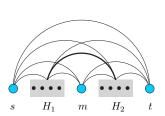
I.1
$$twist(\{s\} \cup H_1 \to H_2 \cup \{t\}) \le 2$$
 I.2 $twist(E) \le 5$

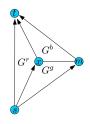
The base of the induction clearly holds when G is a triangle. For the inductive case, consider the outerface, $\langle s,m,t\rangle$, of G and identify the unique vertex, $x\in V$, adjacent to s,m,t. Since G is upward planar, we have $(s,x)\in E,\ (x,t)\in E;$ for the direction of the edge between x and m, there are two possible cases. We can reduce one case to another one by reversing edge directions, which preserves upward planarity of the graph. Therefore, we study only one of the cases.

Assume $(x,m) \in E$. Then G is decomposed into three upward planar subgraphs bounded by faces $\langle s, x, t \rangle$, $\langle s, x, m \rangle$, and $\langle x, m, t \rangle$; denote the graphs by G^r , G^g , and G^b , respectively; see Fig. 9b. By the induction hypothesis, the three graphs admit orders σ^r , σ^g , σ^b satisfying the described invariants. Let $\sigma^r = [s, R_1, x, R_2, t]$, $\sigma^g = [s, G_1, x, G_2, m]$, and $\sigma^b = [x, B_1, m, B_2, t]$. Then

$$\sigma = [s, R_1, G_1, x, G_2, R_2, B_1, m, B_2, t]$$

where $H_1 = [R_1, G_1, x, G_2, R_2, B_1]$ and $H_2 = B_2$; see Fig. 14 in Appendix B, where we show that the invariants are maintained under the vertex order.





- (a) The invariant used in the proof of Theorem 2
- (b) Decomposing an upward planar 3-tree into G^r , G^g , and G^b

Fig. 9: Bounding the twist size of upward planar 3-trees

5 Lower Bounds

We construct and computationally verify specific graphs that require a minimum number of stacks in every layout utilizing a SAT formulation of the linear layout problem [9,31]. Using a modern SAT solver, one can evaluate small and medium size instances (up to a few hundred of vertices) within a few seconds. An online tool and the source code of the implementation is available at [29].

Using the formulation, we identified a single-source single-sink upward oDAG that requires three stacks; see Fig. 10a. There are only two linear extensions of the graph, [a, b, c, d, e, f] and [a, b, d, c, e, f], and both require three stacks.

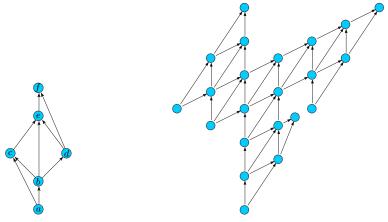
Next we found an upward outerplanar DAG with four sources and three sinks whose stack number is 4; see Fig. 10b. This oDAG is upward but not monotone, that is, it requires an addition of transitive edges via operation O.1.

Finally, we construct an upward planar 3-tree that requires five stacks; see Fig. 11a. The results are summarized in Theorem 3.

6 Conclusions

In this paper we studied the stack number of upward planar and outerplanar DAGs and provided improved upper and lower bounds for some interesting subclasses via their maximum twist sizes. With the recent results of Jungeblut et al. [22] one of the intriguing open questions is to decrease the gap between our lower bound of 4 and their upper bound of 24776 for oDAGs. Moreover, since our upper bounds are mostly based on bounding the twist number, they are likely too large and it would be interesting to decrease them further.

A queue layout of DAGs is a related concept, in which a pair of edges cannot nest. While two queues are sufficient for trees and unicyclic DAGs [19], there exist single-source single-sink upward oDAGs that require a linear number of queues; see Fig. 11b. This is in contrast with undirected planar graphs, which have a constant queue number [2, 15]. We suggest to investigate mixed stack-queue layouts in which every page is either a stack or a queue [6,12,30]. Another direction is to parameterize the queue number by a graph parameter that is tied to the queue number for undirected graphs, such as the width of a poset [3,32].



- (a) A singles-source single-sink outerplanar DAG that requires three stacks
- (b) An upward outerplanar DAG that requires four stacks

Fig. 10: Lower bound examples

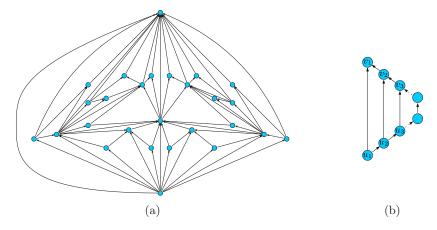


Fig. 11: (a) An upward planar DAG that require 5 stacks. (b) An upward outer-planar DAG that requires n/2 queues.

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Appendix

A Complete Proofs for Section 3

A.1 Single-Source oDAGs

Here we provide a complete proof of Lemma 1.

Lemma 1. Every single-source (single-sink) outerplanar DAG admits an order whose twist size is at most 3.

Proof. Let G = (V, E) be a given oDAG with a unique source $s \in V$, and assume that $(s,t) \in E$ is the base edge in the construction sequence of G. We prove the claim by induction on the size of G by using the following invariant (see Fig. 3a): There exists an order of V consisting of four parts, $\sigma = [s, H_3, t, H_4]$ (that is, $H_1 = H_2 = \emptyset$), such that the following holds:

I.1
$$twist(s \rightarrow H_4, H_3 \rightarrow H_4) \leq 1;$$

I.2 $twist(E) \leq 3.$

Now we prove that the described invariants can be maintained. If G consists of a single edge, then the base of the induction clearly holds. In order to prove the inductive case, we consider the base edge (s,t) of G and choose the unique neighbor of s and t, denoted $x \in V$. Since G is a single-source oDAG, there are two ways the edges between x and s,t are directed, corresponding to operations O.1 and O.2. Consider both cases.

Case 1. First assume $(s,x) \in E$ and $(t,x) \in E$. It is easy to see that G is decomposed into two edge-disjoint subgraphs sharing a single vertex x; denote the graph containing (s,x) by G^g and the graph containing (t,x) by G^r ; see Fig. 3b. Since G is a single-source oDAG, G^g is also a single-source oDAG with source s and base edge (s,x). Similarly, G^r is a single-source oDAG with source t and base edge (t,x). By the induction hypothesis, the two graphs admit orders σ^g and σ^r satisfying the described invariant. Next we show how to combine the orders into a single one for G.

Let
$$\sigma^g = [s, G_3, x, G_4]$$
 and $\sigma^r = [t, R_3, x, R_4]$. Then we set

$$\sigma = [s, t, R_3, G_3, x, G_4, R_4]$$

and observe that in the order $H_3 = \emptyset$, $H_4 = [R_3, G_3, x, G_4, R_4]$; see Fig. 4a.

It is easy to see that σ is a linear extension of V, that is, $u <_{\sigma} v$ for all edges $(u, v) \in E$. Next we verify the conditions of the invariant.

I.1. Since $H_3 = \emptyset$, we have $twist(s \to H_4, H_3 \to H_4) = twist(s \to H_4) \le 1$, where the inequality holds because all edges $E(s \to H_4)$ share a common vertex, s.

- 1.2. Consider the maximum twist κ in G under vertex order $\sigma = [s, H_3, t, H_4]$, and suppose for contradiction that $|\kappa| \geq 4$. Observe that $H_3 = \emptyset$ in the considered case, and κ may contain at most one edge incident to s and at most one edge incident to t. Thus, at least two of the edges of κ are from $E(H_4 \to H_4)$; see Fig. 4a. Denote one of the two edges by $e \in \kappa$. Since $e \in E(H_4 \to H_4)$, both endpoints of e are in $R_3 \cup G_3 \cup \{x\} \cup G_4 \cup R_4$. Notice that if the two endpoints are in the same part (e.g., R_i or G_i for some i), then all edges of κ have endpoints in that part (since they all cross e), and specifically all edges of κ are either in G^g or G^r , which implies that $|\kappa| \leq 3$ by the induction hypothesis. Hence, we assume that e belongs to $E(R_3 \to R_4, R_3 \to x, x \to R_4, G_3 \to G_4, G_3 \to x, x \to G_4)$ and that none of the edges of κ contains both endpoints in the same part R_i or G_i of V.
 - If $e \in E(G_3 \to x, x \to G_4)$, then the only edges potentially crossing e are in $E(G_3 \to G_4, s \to G_4, s \to G_3)$, that is, they all belong to G^g . In that case $|\kappa| \leq 3$ by the hypothesis I.2 applied to G^g , a contradiction. Therefore, $\kappa \cap E(G_3 \to x, x \to G_4) = \emptyset$.
 - If $e \in E(G_3 \to G_4)$, then the edges crossing e are either incident to s, or incident to x, or in $E(G_3 \to G_4)$. Since $twist(G_3 \to G_4) \le 1$, we have that $|\kappa| \le 3$. Therefore, $\kappa \cap E(G_3 \to G_4) = \emptyset$.
 - If $e \in E(R_3 \to x)$, then the edges crossing e are in $E(s \to G_3, t \to R_3, R_3 \to R_4)$. Observe that each of the three subsets contributes at most one edge to κ ; thus, $|\kappa| \leq 3$. Therefore, $\kappa \cap E(R_3 \to x) = \emptyset$.
 - If $e \in E(x \to R_4)$, then the edges crossing e are in $E(s \to G_4, t \to R_4, R_3 \to R_4)$. Each of the three subsets contributes at most one edge to κ ; thus, $|\kappa| \leq 3$. Therefore, $\kappa \cap E(x \to R_4) = \emptyset$.
 - If $e \in E(R_3 \to R_4)$, then the edges crossing e are either adjacent to s or t, or in $E(R_3 \to R_4)$. Since $twist(R_3 \to R_4) \le 1$, we have that $|\kappa| \le 3$, contradicting our assumption.

This completes the proof for invariants of Case 1.

Case 2. Now assume $(s, x) \in E$ and $(x, t) \in E$; see Fig. 3c. Again, G is decomposed into two edge-disjoint single-source subgraphs sharing a single vertex x; denote the graph containing (s, x) by G^g and the graph containing (x, t) by G^r . By the induction hypothesis, the two graphs admit orders σ^g and σ^r satisfying the invariant. Next we show how to combine the orders into a single one for G.

Let
$$\sigma^g = [s, G_3, x, G_4]$$
 and $\sigma^r = [x, R_3, t, R_4]$. Then we set

$$\sigma = [s, G_3, x, G_4, R_3, t, R_4]$$

and observe that in the order $H_3 = [G_3, x, G_4, R_3]$, $H_4 = R_4$; see Fig. 4b. As in Case 1, σ is a linear extension of V, and we verify the conditions of the invariant.

I.1. Observe that $E(s \to H_4) = \emptyset$ and $E(H_3 \to H_4) = E(x \to R_4, R_3 \to R_4)$. Thus, $twist(s \to H_4, H_3 \to H_4) = twist(x \to R_4, R_3 \to R_4) \le 1$ where the inequality follows from the induction hypothesis I.1 applied to G^T .

I.2. Consider the maximum twist κ in G under vertex order $\sigma = [s, H_3, t, H_4]$, and suppose for contradiction that $|\kappa| \geq 4$. First we rule out the case when κ does not contain an edge from $E(H_3 \to H_3, H_4 \to H_4)$ (that is, when all edges of κ are either in $E(H_3 \to H_4)$, or adjacent to s or t). In that case, κ contains at most one edge adjacent to s and at most one edge adjacent to s, and all the remaining edges are from $E(H_3 \to H_4)$. By I.1 it holds that $twist(H_3 \to H_4) \leq 1$, which implies that $|\kappa| \leq 3$.

Therefore, we may assume that at least one edge $e \in \kappa$ is from $E(H_3 \to H_3, H_4 \to H_4)$. If $e \in E(H_4 \to H_4)$, then all edges of κ are incident to a vertex in R_4 (since only such edges cross e); in particular all edges of κ are in G^r . This is impossible by the induction hypothesis I.2 applied to G^r . Thus we may assume that e belongs to $E(H_3 \to H_3)$.

Since $e \in E(H_3 \to H_3)$, both endpoints of e are in $G_3 \cup \{x\} \cup G_4 \cup R_3$. Notice that if the two endpoints are in the same part (e.g., G_3 , G_4 , or R_3), then all edges of κ have endpoints in that part (since they all cross e), and specifically all edges of κ are either in G^g or G^r , which implies that $|\kappa| \leq 3$ by the induction hypothesis. Hence, we assume that e belongs to $E(G_3 \to G_4, G_3 \to x, x \to G_4, x \to R_3)$ and that none of the edges of κ are in the same part R_i or G_i of V.

- If $e \in E(G_3 \to x, x \to G_4)$, then the only edges crossing e are in $E(s \to G_3, s \to G_4, G_3 \to G_4)$, that is, they all belong to G^g . In that case $|\kappa| \leq 3$ by the hypothesis I.2 applied to G^g . Hence, $\kappa \cap E(G_3 \to x, x \to G_4) = \emptyset$.
- If $e \in E(G_3 \to G_4)$, then the edges crossing e are either incident to s, or incident to x, or in $E(G_3 \to G_4)$. However, $twist(G_3 \to G_4) \le 1$, which implies that $|\kappa| \le 3$. Therefore, $\kappa \cap E(G_3 \to G_4) = \emptyset$.
- If $e \in E(x \to R_3)$, then the edges crossing e are in $E(s \to G_4, G_3 \to G_4, R_3 \to t, R_3 \to R_4)$. Each of the four subsets contributes at most one edge to κ , and edges of $E(s \to G_4, G_3 \to G_4)$ do not cross edges of $E(R_3 \to t, R_3 \to R_4)$, contradicting our assumption that $|\kappa| \geq 4$.

This completes the proof of Lemma 1.

Lemma 2. Every single-source outerplanar DAG admits a 4-stack layout.

Proof. The proof follows the same recursive approach as the proof of Lemma 1; in particular, the vertex order is the same. Next we describe relevant differences.

For a given outerplanar DAG, G = (V, E) with a unique source $s \in V$, we maintain the following invariant. For a base edge (s,t) of G, there exists a layout in 4 stacks, S_1, S_2, S_3, S_4 , with a vertex order consisting of four parts, $\sigma = [s, H_3, t, H_4]$, such that the following holds (see Fig. 12):

- **I.1** $E(s \rightarrow H_3, s \rightarrow H_4) \subseteq S_1;$
- **I.2** $E(H_3 \rightarrow t, t \rightarrow H_4) \subseteq S_2;$
- **I.3** $E(H_3 \rightarrow H_4) \subseteq S_3$.

Now we show that the invariants can be maintained in the two cases.

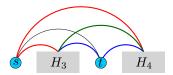


Fig. 12: The invariant used in the proof of Lemma 2

Case 1. Assume $(s, x) \in E$ and $(t, x) \in E$, and that the edges of the two subgraphs are assigned to stacks recursively; see Fig. 13a. By renaming stacks, we may assume that

```
- E(s \rightarrow G_3, s \rightarrow G_4) \subseteq S_1 (red),

- E(G_3 \rightarrow x, x \rightarrow G_4) \subseteq S_2 (blue),

- E(G_3 \rightarrow G_4) \subseteq S_3 (green),

- E(t \rightarrow R_3, t \rightarrow R_4) \subseteq S_2,

- E(R_3 \rightarrow x, x \rightarrow R_4) \subseteq S_4 (purple),

- E(R_3 \rightarrow R_4) \subseteq S_3.
```

For the remaining three edges we use the following stack assignment: $(s, x) \in S_1$, $(t, x) \in S_2$, $(s, t) \in S_2$. One can verify that edges in the same stack do not cross each other, and that the invariants, I.1, I.2, I.3, are maintained.

Case 2. Assume $(s, x) \in E$ and $(x, t) \in E$, and that the edges of the two subgraphs are assigned to stacks recursively; see Fig. 13b. By renaming stacks, we may assume that

```
-E(s \rightarrow G_3, s \rightarrow G_4) \subseteq \mathcal{S}_1 \text{ (red)}, 
-E(G_3 \rightarrow x, x \rightarrow G_4) \subseteq \mathcal{S}_2 \text{ (blue)}, 
-E(G_3 \rightarrow G_4) \subseteq \mathcal{S}_3 \text{ (green)}, 
-E(x \rightarrow R_3, x \rightarrow R_4) \subseteq \mathcal{S}_4 \text{ (purple)}, 
-E(R_3 \rightarrow t, t \rightarrow R_4) \subseteq \mathcal{S}_2, 
-E(R_3 \rightarrow R_4) \subseteq \mathcal{S}_3.
```

For the remaining three edges we use the following stack assignment: $(s, x) \in \mathcal{S}_1$, $(x,t) \in \mathcal{S}_2$, $(s,t) \in \mathcal{S}_2$. One can verify that edges in the same stack do not cross each other, and that invariants I.1 and I.2 are maintained. In order to show that I.3 is maintained, we make an observation that follows directly from the recursive construction:

Observation 1 For the order $\sigma = [s, H_3, t, H_4]$, we have either $E(s \rightarrow H_4) = \emptyset$ or $H_3 = \emptyset$.

Applying the observation for graph G^r , we get that either $E(x \to R_4) = \emptyset$ or $E(R_3 \to R_4) = \emptyset$. In both cases we have that edges $E(H_3 \to H_4)$ are in one stack, S_3 or S_4 , implying that I.3 holds. This completes the proof of Lemma 2.

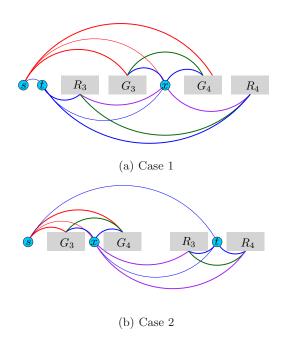


Fig. 13: Stack assignment in the proof of Lemma 2

A.2 Monotone oDAGs

Here we provide a complete proof of Lemma 3.

Lemma 3. Every monotone outerplanar DAG admits an order whose twist size is at most 4.

Proof. We prove the claim by induction on the size of the given oDAG, G = (V, E), by using the following invariants (see Fig. 5): For a base edge $(s, t) \in E$, there exists an order σ of V consisting of six parts, $\sigma = [H_1, s, H_2, H_3, t, H_4]$, such that the following holds:

- **I.1** $E(H_1 \to H_3) = E(H_2 \to H_3) = E(H_2 \to H_4) = \emptyset;$
- **1.2** $twist(H_1 \cup \{s\} \rightarrow \{t\} \cup H_4) \leq 1;$
- **I.3** $twist(H_1 \to H_2 \cup \{t\} \cup H_4) \le 2;$
- **I.4** $twist(H_1 \cup \{s\} \cup H_3 \rightarrow H_4) \leq 2$.
- **I.5** $twist(E) \leq 4$;

Now we prove that the described invariants can be maintained. If G consists of a single edge, then the base of the induction clearly holds. For the inductive case, we consider the base edge (s,t) of G and choose the unique common neighbor of s and t, denoted $x \in V$. Since G is monotone and acyclic, there are two ways the edges between x and s,t are directed: either $(s,x) \in E, (t,x) \in E$ (operation O.2) or $(x,s) \in E, (x,t) \in E$ (operation O.3). Observe that, since a

(monotone) outerplanar DAG remains (monotone) outerplanar after reversing all edge directions and the described invariants are symmetric with respect to parts H_1, H_2 and parts H_3, H_4 , it is sufficient to study only one of the two cases. Therefore in what follows we investigate the former case, while the latter case follows from the symmetry.

Assume $(s, x) \in E$ and $(t, x) \in E$. It is easy to see that G is decomposed into two edge-disjoint monotone oDAGs sharing a vertex $x \in V$; denote the graph containing (s, x) by G^g and the graph containing (t, x) by G^r . By the induction hypothesis, the two graphs admit orders σ^g and σ^r satisfying the described invariant. Next we show how to combine the orders into a single one for G.

Let $\sigma^g = [G_1, s, G_2, G_3, x, G_4]$ and $\sigma^r = [R_1, t, R_2, R_3, x, R_4]$. Then we set

$$\sigma = [G_1, s, G_2, R_1, t, R_2, G_3, R_3, x, R_4, G_4]$$

and note that $H_1 = G_1$, $H_2 = G_2$, $H_3 = R_1$, $H_4 = [R_2, G_3, R_3, x, R_4, G_4]$ in σ ; see Fig. 6. It is easy to see that σ is a linear extension of V, and we verify the conditions of the invariant.

- I.1. The condition follows directly from the construction; see Fig. 6.
- I.2. Consider $twist(H_1 \to H_4, s \to H_4, H_1 \to t, s \to t)$. Observe that $E(H_1 \to t) = E(G_1 \to t) = \emptyset$, $E(H_1 \to H_4) = E(G_1 \to G_4, G_1 \to x)$, and $E(s \to H_4) = E(s \to x, s \to G_4)$. Hence, $twist(H_1 \to H_4, s \to H_4, H_1 \to t, s \to t) = twist(G_1 \to G_4, s \to G_4, G_1 \to x, s \to x, s \to t) \leq 1$, where the inequality follows from the induction hypothesis I.2 applied to G^g and the fact that edge (s, t) does not cross any other edge of the edge set.
- I.3. Consider $twist(H_1 \to H_2, H_1 \to H_4, H_1 \to t)$. Here $E(H_1 \to t) = \emptyset$, $E(H_1 \to H_2) = E(G_1 \to G_2)$, and $E(H_1 \to H_4) = E(G_1 \to G_4, G_1 \to x)$. Therefore,

$$twist(H_1 \to H_2, H_1 \to H_4, H_1 \to t) = twist(G_1 \to G_2, G_1 \to G_4, G_1 \to x) \le 2$$

where the inequality follows from the hypothesis I.3 applied to G^g .

- I.4. Consider $twist(H_1 \to H_4, H_3 \to H_4, s \to H_4)$, and let κ be the maximum twist formed by the edges. Next we argue that $|\kappa| \leq 2$.
 - Observe that κ may contain edges from seven subsets: $E(G_1 \to G_4)$, $E(G_1 \to x)$, $E(s \to G_4)$, $E(s \to x)$, $E(R_1 \to R_4)$, $E(R_1 \to x)$, and $E(R_1 \to R_2)$. First consider the case when κ contains an edge of $E(R_1 \to R_2)$. Such an edge can only cross edges from $E(R_1 \to R_4, R_1 \to x)$. However we have that $twist(R_1 \to R_2, R_1 \to R_4, R_1 \to x) \le 2$ by I.3 applied to G^r , which implies that $|\kappa| \le 2$ in the case. Thus, we may assume that κ contains no edge of $E(R_1 \to R_2)$.

The remaining six edge sets are partitioned into $E(G_1 \to G_4, G_1 \to x, s \to G_4, s \to x)$ and $E(R_1 \to R_4, R_1 \to x)$. The size of the maximum twist in each of the two subsets is at most one by I.2 applied to G^g and for G^r , respectively. Therefore, $|\kappa| \leq 2$ as claimed.

I.5. Consider the maximum twist κ in G under vertex order $\sigma = [H_1, s, H_2, H_3, t, H_4]$. First we rule out the case when κ does not contain an edge from $E(H_i \to H_i)$ for some $1 \le i \le 4$ (that is, when all edges of κ are either in $E(H_i \to H_j)$ for some $i \neq j$, or adjacent to s or t). In that case, κ contains at most one edge adjacent to s and at most one edge adjacent to t. The remaining edges are from $E(H_1 \to H_2, H_3 \to H_4, H_1 \to H_4)$. Observe that on one hand, an edge from $E(H_1 \to H_2)$ cannot form a twist with an edge from $E(H_3 \to H_4)$. On the other hand, we showed that $twist(H_1 \to H_2, H_1 \to H_4) \leq 2$ and $twist(H_3 \to H_4, H_1 \to H_4) \leq 2$, which implies that $|\kappa| \leq 4$.

Therefore, we may assume that at least one edge of κ , denoted $e \in \kappa$, is from $E(H_i \to H_i)$ for some $1 \le i \le 4$. If i = 1, i = 2, or i = 3, then all edges of κ are adjacent to a vertex from G_1 , G_2 , or G_1 , respectively; see Fig. 6. This is impossible by the induction hypothesis I.5 applied to G^g and G^r . Thus we assume that G_1 belongs to G_2 and that all other edges of G_3 have an endpoint in G_4 .

Since $e \in E(H_4 \to H_4)$, both endpoints of e are in $R_2 \cup G_3 \cup R_3 \cup \{x\} \cup R_4 \cup G_4$. Notice that if the two endpoints are both in the same part (e.g., R_i or G_i for some i), then all edges of κ have endpoints in that part (since they all cross e), and specifically all edges of κ are either in G^g or G^r , which implies $|\kappa| \leq 4$ by the induction hypothesis I.5. Hence, we assume that e belongs to $E(G_3 \to G_4, G_3 \to x, x \to G_4, R_3 \to R_4, R_3 \to x, x \to R_4)$ and that none of the edges of κ are in the same part G_i or R_i of V.

- If $e \in E(R_3 \to R_4, R_3 \to x, x \to R_4)$, then all the edges of κ crossing e are either adjacent to x or belong to $E(R_1 \to R_4, R_3 \to R_4, t \to R_4)$. Since $twist(R_1 \to R_4, R_3 \to R_4, t \to R_4) \leq 2$ by I.4 applied to G^r and there is at most one edge in κ adjacent to x, we have that $|\kappa| \leq 3$ in this case. Hence, we may assume that $\kappa \cap E(R_3 \to R_4, R_3 \to x, x \to R_4) = \emptyset$.
- If $e \in E(G_3 \to G_4, G_3 \to x, x \to G_4)$, then all the edges of κ crossing e are either adjacent to x or belong to $E(G_1 \to G_4, G_3 \to G_4, s \to G_4) \cup E(R_1 \to R_4, t \to R_4)$. The bound $|\kappa| \leq 4$ follows from observations that $twist(G_1 \to G_4, G_3 \to G_4, s \to G_4) \leq 2$ (by I.4 applied to G^g), $twist(R_1 \to R_4, t \to R_4) \leq 1$ (by I.2 applied to G^r), and that there is at most one edge in κ adjacent to x.

This completes the proof of Lemma 3.

A.3 Outerpath DAGs

Here we provide a complete proof of Lemma 4.

Lemma 4. Every outerpath DAG admits an order whose twist size is at most 4.

Proof. We prove the claim by induction on the size of the given outerpath DAG, G = (V, E), by using the following invariants: For a base edge $(s, t) \in E$, there exists a vertex order consisting of six parts, $\sigma = [H_1, s, H_2, H_3, t, H_4]$, such that the following holds:

```
I.1 H_2 = \emptyset or H_3 = \emptyset, that is, E(H_2 \to H_3) = \emptyset; I.2 twist(H_2 \to t, H_2 \to H_4) \le 1;
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I.3 twist(H_1 \rightarrow H_3, s \rightarrow H_3) \leq 1;

I.4 twist(H_1 \cup \{s\} \cup H_2 \rightarrow H_3 \cup \{t\} \cup H_4) \leq 2;

I.5 twist(H_1 \rightarrow H_2, H_2 \rightarrow t, H_2 \rightarrow H_4) \leq 3;

I.6 twist(H_1 \rightarrow H_3, s \rightarrow H_3, H_3 \rightarrow H_4) \leq 3;

I.7 twist(H_1 \rightarrow H_4, s \rightarrow H_4, H_2 \rightarrow H_4, H_3 \rightarrow H_4) \leq 3;

I.8 twist(H_1 \rightarrow H_2, H_1 \rightarrow H_3, H_1 \rightarrow t, H_1 \rightarrow H_4) \leq 3;
```

I.9 $twist(E) \leq 4$.

Now we prove that the described invariants can be maintained. If G consists of a single edge, then the base of the induction clearly holds. In order to prove the inductive case, we consider a base edge $(s,t) \in E$ and choose the unique common neighbor of s and t, denoted $x \in V$. There are three operations that can be applied on (s,t): O.1, O.2, and O.3. By symmetry, it is sufficient to study only O.1 and O.2. Depending on which edges of face $\langle s,t,x\rangle$ are utilized for the construction, we distinguish four cases; see Fig. 7. As in earlier proofs we denote the graphs constructed on (s,x) by G^g and the graph on (t,x) by G^r , and assume the two graphs admit orders σ^g and σ^r satisfying the invariants. Notice however that, since G is an outerpath, only one of G^g , G^r contains more than two vertices.

Case 1a. Assume that $(s,x) \in E$, $(t,x) \in E$, $\sigma^g = [G_1, s, G_2, G_3, x, G_4]$, and $\sigma^r = [t, x]$. We set

$$\sigma = [G_1, s, G_2, t, G_3, x, G_4],$$

where $H_1 = G_1$, $H_2 = G_2$, $H_3 = \emptyset$, $H_4 = [G_3, x, G_4]$ in σ ; see Fig. 8a. Next we verify the conditions of the invariants.

- I.1. Follows directly from the construction.
- I.2. Since $E(H_2 \to t) = \emptyset$, the claim follows from I.1 and I.2 applied to G^g .
- I.3. The claim holds since $H_3 = \emptyset$.
- I.4. Consider the maximum twist, κ , among the specified edges. If $(s,t) \in \kappa$, then the only edges crossing (s,t) are in $E(G_2 \to G_4, G_2 \to x)$. By I.2 applied to G^g , we get the claim. If $(s,t) \notin \kappa$, then all the edges are from the hypothesis I.4 for G^g .
- I.5. Since $E(H_1 \to H_2, H_2 \to H_4, H_2 \to t) = E(G_1 \to G_2, G_2 \to G_4, G_2 \to x)$, the claim follows from the hypothesis I.5 applied to G^g .
- I.6. The claim holds since $H_3 = \emptyset$.
- I.7. The claim follows from I.4 applied to G^g .
- I.8. Since $E(H_1 \to H_2, H_1 \to H_3, H_1 \to t, H_1 \to H_4) = E(G_1 \to G_2, G_1 \to G_3, G_1 \to x, G_1 \to G_4)$, the claim follows from the hypothesis I.8 applied to G^g .
- I.9. Consider the maximum twist, κ , formed by E under σ . We may assume that κ contains (s,t) or (t,x), as otherwise all edges of κ are from G^g ; since the two options are symmetric, we assume $(s,t) \in \kappa$. The only edges crossing (s,t) are in $E(G_1 \to G_2, G_2 \to G_4, G_2 \to x)$. By I.5, we have $twist(G_1 \to G_2, G_2 \to G_4, G_2 \to x) \le 3$, which implies that $|\kappa| \le 4$.

Case 1b. Assume $(s, x) \in E, (t, x) \in E, \sigma^g = [s, x], \text{ and } \sigma^r = [R_1, t, R_2, R_3, x, R_4].$ We set

$$\sigma = [s, R_1, t, R_2, R_3, x, R_4],$$

where $H_1 = H_2 = \emptyset$, $H_3 = R_1$, $H_4 = [R_2, R_3, x, R_4]$ in σ ; see Fig. 8b. Next we verify the conditions of the invariant.

- I.1. Follows directly from the construction.
- I.2. The claim holds since $H_2 = \emptyset$.
- I.3. The claim holds since $E(H_1 \rightarrow H_3, s \rightarrow H_3) = \emptyset$.
- I.4. The claim holds since the relevant set of edges consists of (s,t) and (s,x).
- I.5. The claim holds since $H_2 = \emptyset$.
- I.6. As $E(H_1 \rightarrow H_3, H_3 \rightarrow H_4, s \rightarrow H_3) = E(R_1 \rightarrow R_2, R_1 \rightarrow R_3, R_1 \rightarrow x, R_1 \rightarrow R_4)$, the claim follows from the hypothesis I.8 applied to G^r .
- I.7. Observe that $E(H_1 \to H_4, s \to H_4, H_2 \to H_4, H_3 \to H_4) = E(s \to x, R_1 \to R_2, R_1 \to R_3, R_1 \to x, R_1 \to R_4)$. If (s, x) is in the maximum twist, then it can cross only $E(R_1 \to R_4)$; by I.4 applied to G^r , we get the desired bound. Otherwise, if (s, x) is not in the maximum twist, then the claim follows from I.8 applied to G^r .
- I.8. The claim holds since $H_1 = \emptyset$.
- I.9. Consider the maximum twist, κ , formed by E under σ . We may assume that κ contains (s,t) or (s,x), as otherwise all edges of κ are from G^r . If $(s,x) \in \kappa$, then the edges crossing (s,x) are in $E(R_1 \to R_4, t \to R_4, R_2 \to R_4, R_3 \to R_4)$. By I.7 we get the bound. If $(s,t) \in \kappa$, then the edges crossing (s,x) are in $E(R_1 \to R_4, R_1 \to x, R_1 \to R_2, R_1 \to R_3)$. By I.8 we get the bound.

Case 2a. Assume that $(s,x) \in E$, $(x,t) \in E$, $\sigma^g = [G_1, s, G_2, G_3, x, G_4]$, and $\sigma^r = [x,t]$. We set

$$\sigma = [G_1, s, G_2, G_3, x, G_4, t],$$

where $H_1 = G_1$, $H_2 = [G_2, G_3, x, G_4]$, $H_3 = H_4 = \emptyset$ in σ ; see Fig. 8c. Next we verify the conditions of the invariant.

- I.1. Follows directly from the construction.
- I.2. The claim holds since $H_4 = \emptyset$ and $E(H_2 \to t) = E(x \to t)$.
- I.3. The claim holds since $H_3 = \emptyset$.
- I.4. The claim holds since the relevant set of edges consists of (s,t) and (x,t).
- I.5. Observe that $E(H_1 \to H_2, H_2 \to H_4, H_2 \to t) = E(G_1 \to G_2, G_1 \to G_3, G_1 \to x, G_1 \to G_4, x \to t)$. If (x, t) is in the maximum twist, then it can cross only $E(G_1 \to G_4)$; by I.4 applied to G^g , we get the bound. Otherwise, if (x, t) is not in the maximum twist, then the claim follows from I.8 applied to G^g .
- I.6. The claim holds since $H_3 = \emptyset$.
- I.7. The claim holds since $H_4 = \emptyset$.
- I.8. As $E(H_1 \to H_2, H_1 \to H_3, H_1 \to t, H_1 \to H_4) = E(G_1 \to G_2, G_1 \to G_3, G_1 \to x, G_1 \to G_4)$, the claim follows from the hypothesis I.8 applied to G^g .
- I.9. Reduced to I.9 in Case 1b by relabeling vertices and reversing edge directions.

Case 2b. Assume $(s, x) \in E$, $(x, t) \in E$, $\sigma^g = [s, x]$, and $\sigma^r = [R_1, x, R_2, R_3, t, R_4]$. The case is reduced to *Case 2a* by reversing all edge directions; see Fig. 8d. This completes the proof of Lemma 4.

B Complete Proofs for Section 4

Here we provide a complete proof of Theorem 2.

Theorem 2. Every upward planar 3-tree admits an order whose twist size is at most 5.

Proof. We prove the claim by induction on the size of a given upward planar 3-tree, G = (V, E), by using the following invariants (see Fig. 9a): For the outerface $\langle s, m, t \rangle$ of G, there exists an order of V such that it consists of five parts, $\sigma = [s, H_1, m, H_2, t]$, where $H_1, H_2 \subset V$, and the following holds:

I.1
$$twist(H_1 \rightarrow H_2, s \rightarrow H_2, H_1 \rightarrow t, s \rightarrow t) \leq 2;$$

I.2 $twist(E) \leq 5.$

Now we prove that the described invariants can be maintained. If G is a triangle, then the base of the induction clearly holds. In order to prove the inductive case, we consider the outerface, $\langle s, m, t \rangle$, of G and identify the unique vertex, $x \in V$, adjacent to s, m, t. Since G is upward planar, we have $(s, x) \in E$, $(x,t) \in E$; for the direction of the edge between x and m, there are two possible options. Observe that as in the proof of Lemma 3, we can reduce one option to another one by reversing the directions of all edges of G, which preserves upward planarity of the graph. Therefore, it is sufficient to study only one of the options.

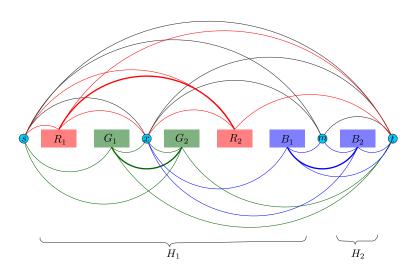


Fig. 14: An inductive step in the proof of Theorem 2

Assume $(x, m) \in E$. It is easy to see that G is decomposed into three upward planar subgraphs bounded by faces $\langle s, x, t \rangle$, $\langle s, x, m \rangle$, and $\langle x, m, t \rangle$; denote the graphs by G^r , G^g , and G^b , respectively; see Fig. 9b. By the induction hypothesis, the three graphs admit orders σ^r , σ^g , σ^b satisfying the described invariants. Next we show how to combine the orders into a single one for G.

Let $\sigma^r = [s, R_1, x, R_2, t], \ \sigma^g = [s, G_1, x, G_2, m], \ \text{and} \ \sigma^b = [x, B_1, m, B_2, t].$ Then we set

$$\sigma = [s, R_1, G_1, x, G_2, R_2, B_1, m, B_2, t]$$

where $H_1 = [R_1, G_1, x, G_2, R_2, B_1]$ and $H_2 = B_2$; see Fig. 14. It is easy to see that σ is a linear extension of V, and we verify the invariant.

I.1 Consider edges $E(H_1 \to H_2, s \to H_2, H_1 \to t, s \to t)$ and denote the maximum twist formed by these edges by κ ; see Fig. 15. We need to show that $|\kappa| \leq 2$. Observe that in this case, $E(s \to H_2) = \emptyset$, while $twist(\{x\} \cup B_1 \to B_2 \cup \{t\}) \leq 2$ by I.1 applied to G^b .

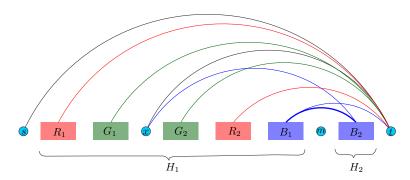


Fig. 15: Maintaining invariant I.1 in the proof of Theorem 2

Consider edges of κ that are adjacent to t; clearly, there is at most one such edge.

- If none of the edges from $E(v \to t), v \in V$ is in κ , then κ is formed by $E(B_1 \to B_2, x \to B_2)$; by induction hypothesis I.1 applied to G^b , $|\kappa| \leq 2$.
- If an edge, $e \in E(B_1 \to t)$ is in κ , then the edges crossing e are from $E(B_1 \to B_2, x \to B_2)$. By induction hypothesis applied to G^b , $|\kappa| \leq 2$.
- Finally, if an edge $e \in E(s \to t, R_1 \to t, x \to t, R_2 \to t)$ is in κ , then all the edges of κ crossing e are from $E(x \to B_2)$, that is, they are adjacent to x. Since only one edge adjacent to a vertex can be in a twist, we have $|\kappa| \leq 2$.
- I.2 Consider the maximum twist κ in G under vertex order $\sigma = [s, H_1, m, H_2, t]$. First we rule out the case when κ does not contain an edge from $E(H_1 \to H_1, H_2 \to H_2)$. In that case, κ contains at most three edges adjacent to s, m, t. By I.1 we have $twist(H_1 \to H_2) \leq 2$, which implies that $|\kappa| \leq 5$ in the considered case.

Therefore, we may assume that at least one edge of κ , denoted $e \in \kappa$, is from $E(H_1 \to H_1, H_2 \to H_2)$. If $e \in E(H_2 \to H_2)$, then all edges of κ are adjacent to a vertex in H_2 ; by the induction hypothesis I.2 applied to G^b , we have $|\kappa| \leq 5$. Thus we may assume that e belongs to $E(H_1 \to H_1)$ and that all other edges of κ have an endpoint in H_1 , as they cross e.

Since $e \in E(H_1 \to H_1)$, both endpoints of e are in $R_1 \cup G_1 \cup \{x\} \cup G_2 \cup R_2 \cup B_1$. Notice that if the two endpoints are both in the same part (e.g., R_i , G_i , or B_i for some i), then all edges of κ have endpoints in that part (since they all cross e), and specifically all edges of κ are either in G^g or G^r or G^b , which implies that $|\kappa| \leq 5$. Hence, we may assume that e belongs to $E(R_1 \to R_2, R_1 \to x, x \to R_2, G_1 \to G_2, G_1 \to x, x \to G_2, x \to B_1)$ and that none of the edges of κ are in the same part G_i , R_i , or B_i of V.

- If $e \in E(G_1 \to G_2, G_1 \to x, x \to G_2)$, then all the edges of κ crossing e are either adjacent to s, x, t, or belong to $E(G_1 \to G_2)$. By I.1 applied to G^g , we have $twist(G_1 \to G_2) \le 2$, which implies that $|\kappa| \le 5$. Hence, we may assume that $\kappa \cap E(G_1 \to G_2, G_1 \to x, x \to G_2) = \emptyset$ if $\kappa > 5$.
- Similarly, if $e \in E(R_1 \to R_2, R_1 \to x, x \to R_2)$, then all the edges of κ crossing e are either adjacent to s, x, t, or belong to $E(R_1 \to R_2)$. By I.1 applied to G^r , we have $twist(R_1 \to R_2) \le 2$, which implies that $|\kappa| \le 5$. Hence, we may assume that $\kappa \cap E(R_1 \to R_2, R_1 \to x, x \to R_2) = \emptyset$.
- Finally, if $e \in E(x \to B_1)$, then all the edges of κ crossing e are either adjacent to s, m, t, or belong to $E(B_1 \to B_2)$. By I.1 applied to G^b , we have $twist(B_1 \to B_2) \leq 2$. At the same time, edges of κ adjacent to s (that is, $E(s \to G_2, s \to R_2)$) do not cross edges from $E(B_1 \to B_2)$. If $\kappa \cap E(B_1 \to B_2) = \emptyset$, then κ contains at most four edges (adjacent to s, m, t, and x). Otherwise, if $\kappa \cap E(B_1 \to B_2) \neq \emptyset$, κ contains one edge adjacent to x, at most one edge adjacent to t, at most one edge adjacent to t, and at most two edges from $E(B_1 \to B_2)$. Therefore, $|\kappa| \leq 5$.

This completes the proof of Theorem 2.