Mutual Witness Proximity Drawings of Isomorphic Trees

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Abstract. A pair $\langle G_0, G_1 \rangle$ of graphs admits a mutual witness proximity drawing $\langle \Gamma_0, \Gamma_1 \rangle$ when: (i) Γ_i represents G_i , and (ii) there is an edge (u, v)in Γ_i if and only if there is no vertex w in Γ_{1-i} that is "too close" to both u and v (i = 0, 1). In this paper, we consider infinitely many definitions of closeness by adopting the β -proximity rule for any $\beta \in [1, \infty]$ and study pairs of isomorphic trees that admit a mutual witness β -proximity drawing. Specifically, we show that every two isomorphic trees admit a mutual witness β -proximity drawing for any $\beta \in [1, \infty]$. The constructive technique can be made "robust": For some tree pairs we can suitably prune linearly many leaves from one of the two trees and still retain their mutual witness β -proximity drawability. Notably, in the special case of isomorphic caterpillars and $\beta = 1$, we construct linearly separable mutual witness Gabriel drawings.

Keywords: Mutual witness proximity drawings, β -proximity, Trees

1 Introduction

Proximity drawings are geometric graphs (i.e., straight-line drawings) such that any two vertices are connected by an edge if and only if they are deemed to be close according to some definition of closeness. Therefore, proximity drawings are such that pairs of non-adjacent vertices are relatively far apart while highly connected subgraphs correspond to groups of vertices that can be naturally clustered together in a visual inspection.

In this paper, we investigate *mutual witness proximity drawings*, which employ the concept of closeness to simultaneously represent pairs of graphs. Specifically, consider a pair of graphs, denoted as $\langle G_0, G_1 \rangle$. The pair admits a mutual witness

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(a) Gabriel drawing (b) Mutual witness relative neighborhood drawing

Fig. 1: Two mutual witness drawings on the same point set.

proximity drawing, denoted as $\langle \Gamma_0, \Gamma_1 \rangle$, under the following conditions: (i) Γ_i represents G_i , and (ii) an edge (u, v) exists in Γ_i if and only if there is no vertex w in Γ_{1-i} that is "too close" to both u and v (where i = 0, 1). Vertex w is called a *witness* and its proximity to u and v impedes the presence of the edge. Clearly, by changing the definition of proximity a pair of graphs may or may not admit a mutual witness proximity drawing.

There is general consensus in the literature to define the closeness of w to both u and v by means of a *proximity region* of u and v, which is a convex region in the plane whose area increases when the distance between u and v increases. For example, the *Gabriel region* [8] of u and v is the disk whose diameter is the line segment \overline{uv} ; the witness w is close to u and v if it is a point of their Gabriel disk. A mutual witness Gabriel drawing of a pair $\langle G_0, G_1 \rangle$ is therefore a pair of drawings Γ_0 of G_0 and Γ_1 of G_1 such that for any two non-adjacent vertices in one drawing their Gabriel disk contains a witness from the other drawing, while for any two adjacent vertices their Gabriel region does not contain any witnesses. Figure 1a shows a mutual witness Gabriel drawing of two caterpillars. As another example, the *relative neighborhood region* [19] of u and v is the intersection of the two disks of radius d(u, v) centered at u and v, respectively. Figure 1b depicts a mutual proximity drawing that adopts the relative neighborhood region: The drawing has the same vertex set but fewer edges than the drawing in Figure 1a.

We want to understand what families of graph pairs admit a mutual witness proximity drawing for a given definition of proximity. Intuitively, the denser the two graphs are, the more likely they admit such a representation: If the graphs are complete, we can draw them sufficiently far apart so that the proximity regions of their edges do not contain any witnesses. On the other hand, when the graphs are sparse there are many non-adjacent vertices requiring the presence of witnesses in their proximity regions, which makes the geometry of the two drawings strongly depend on one another. We specifically study very sparse graphs, namely trees. An outline of our contribution is as follows.

In Section 4, we prove that any pair $\langle G_0, G_1 \rangle$ of isomorphic caterpillars admits a mutual witness Gabriel drawing $\langle \Gamma_0, \Gamma_1 \rangle$ such that Γ_0 and Γ_1 are linearly separable. This is somewhat surprising as caterpillars are very sparse graphs and the linear separability of mutual witness Gabriel drawings was known only for graphs of small diameter, namely at most two [13].

In Section 5, we extend the previous result in two different directions: We consider pairs of general isomorphic trees and we study their drawability for an infinite family of proximity regions called β -regions [12], whose shape depends on a parameter $\beta \in \mathbb{R}$. We show that any pair $\langle G_0, G_1 \rangle$ of isomorphic trees admits a mutual witness proximity drawing for any β -region such that $\beta \in [1, \infty]$. While the two drawings are no longer linearly separable, they have the property that the coordinates of their vertex sets remain the same for any possible value of β . It is worth recalling that the Gabriel disk is the β -region for $\beta = 1$ and that the relative neighborhood region corresponds to the β -region for $\beta = 2$.

In Section 6, we investigate the "robustness" of the construction of Section 5: We show that for some tree pairs, this construction can be modified so that the drawing remains valid even after pruning a suitable set of leaves. While it is known that any two star trees admit a mutual witness Gabriel drawing if and only if the cardinalities of their vertex sets differ by at most two [13], we show that there exist tree pairs which can differ by linearly many leaves and still admit a mutual witness proximity drawing for any β -region such that $\beta \in [1, \infty]$.

Results marked with a (clickable) "*" are proved in the appendix.

2 Related Work

Proximity drawings are a classical research topic in graph drawing; they find application in several areas, including pattern recognition, data mining, machine learning, computational biology, and computational morphology. Proximity drawings have also been used to determine the faithfulness of large graph visualizations. A limited list of references includes [7,10,14,15,16,20].

In the context of designing trained classifiers, mutual witness proximity drawings were first introduced by Ichino and Slansky [9] under the name of *interclass rectangle of influence graphs*. In [9] the proximity region of a pair of vertices, called the *rectangle of influence*, is the smallest axis-aligned rectangle containing the two vertices. This study was then extended to other families of proximity regions, including the Gabriel region, in a sequence of papers by Aronov et al. [1,2,3,4]. Notably, in [4] it is said that once the combinatorial properties of those pairs of graphs that admit a mutual witness Gabriel drawing are understood, "we would have useful tools for the description of the interaction between two point sets". Aronov et al. prove in [3] that any pair of complete graphs admits a mutual witness Gabriel drawing where the two drawings are linearly separable. The linear separability property of mutual witness Gabriel drawings is extended to diameter-2 graphs by Lenhart and Liotta, who also give



Fig. 2: Examples of β -proximity regions for $\beta \geq 1$.

a complete characterization of those complete bipartite graphs that admit a mutual witness Gabriel drawing [13]. Another related contribution of Aronov et al. [1,2,3,4] is to introduce and study *witness proximity drawings*, which can be shortly described as a relaxation of mutual proximity drawings where one of the two drawings has no edges, independently of whether the proximity regions of its vertices do or do not contain any witnesses.

3 Preliminaries

We assume familiarity with basic graph drawing concepts; see e.g. [5,11,17,18].

Let p and q be two distinct points in the plane. We denote by \overline{pq} the straightline segment having p and q as its extreme points. We define β -regions adopting the notation in [6]. A region in the plane is *open* if it is an open set, that is the points on its boundary are not part of the region, and *closed* if all of the points of the boundary are part of the region. Given a pair p, q of points in the plane and a real number $\beta \in [1, \infty]$, the *open* β -*region* of p and q, denoted by $R(p, q, \beta)$, is defined as follows. For $1 \leq \beta < \infty$, $R(p, q, \beta)$ is the intersection of the two open disks of radius $\beta d(p, q)/2$ and centered at the points $(1 - \beta/2)p + (\beta/2)q$ and $(\beta/2)p + (1 - \beta/2)q$. $R(p, q, \infty)$ is the open infinite strip perpendicular to the line segment \overline{pq} and for $\beta \in [1, \infty]$, the closed β -region $R[p, q, \beta]$ is simply the open region $R(p, q, \beta)$ along with its boundary; see Figure 2.

Note that R[p,q,1] is the Gabriel region of p,q and that R(p,q,2) is the relative neighborhood region of p,q. We shall denote as a MW- $[\beta]$ drawing a mutual witness proximity drawing such that for any two vertices p and q the proximity region is $R[p,q,\beta]$. In particular, an MW-[1] drawing is a mutual witness Gabriel drawing. Similarly, a MW- (β) drawing is a mutual witness proximity drawing that uses the open β -region.

As we shall see, some of our constructive arguments produce drawings that are simultaneously MW- (β) and MW- $[\beta]$ drawings; in this case we refer to them simply as MW- β drawings. Note that for any pair p, q of vertices in an MW- β drawing, $R(p, q, \beta)$ contains a witness if p and q are not adjacent, while $R[p, q, \beta]$ contains no witnesses if p and q are adjacent.

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Fig. 3: A winged parallelogram with anchors q_0, q_1 , safe wedges W_0, W_1 , and ports p_0, p_1 .

Let $\langle \Gamma_0, \Gamma_1 \rangle$ be an MW- β drawing of graphs $\langle G_0, G_1 \rangle$ for some value of β . We say that the drawing is *linearly separable* if there exists a line ℓ such that Γ_0 and Γ_1 lie in opposite half-planes with respect to ℓ . The following property rephrases an observation of [13] and will be used in the proof of Theorem 1.

Property 1. Let $\langle \Gamma_0, \Gamma_1 \rangle$ be a linearly separable MW-[1] drawing and let u and v be any two non-adjacent vertices of Γ_i , for i = 0, 1. Then any witness in R[u, v, 1] is also a point in $R[u, v, \infty]$

4 MW-[1] Drawings of Isomorphic Caterpillars

A *caterpillar* is a tree T such that, when removing the leaves of T one is left with a non-empty path called *spine* of T. We call the graph $K_{1,n}$ with $n \ge 1$ a *star*; if n > 1 the non-leaf vertex of a star is the *center* of the star, otherwise (i.e., when the star is an edge) either vertex can be chosen as the center.

In this section, we prove that any two isomorphic caterpillars admit a linearly separable MW-[1] drawing, that is they admit a linearly separable mutual witness Gabriel drawing. As pointed out both in [3] and in [9], the linear separability of mutual witness proximity drawings is a desirable property because it gives useful information about the inter-class structure of two sets of points.

Let $P = \langle a_0, b_0, a_1, b_1 \rangle$ be a parallelogram such that $y(a_0) > y(b_0) > y(b_1) > y(a_1)$ and $x(a_0) = x(b_0) < x(a_1) = x(b_1)$. Let q_0 and q_1 be two points in the interior of P satisfying $y(b_i) = y(q_i)$, $x(q_1) < x(q_0)$, and $x(q_0) - x(b_0) = x(b_1) - x(q_1)$. Let W_i be the wedge with apex b_i not containing any vertex of P other than b_i and defined by two rays ρ_i, ρ'_i such that ρ_i is perpendicular to $\overline{a_i b_{1-i}}$ and ρ'_i is perpendicular to $\overline{q_i b_{1-i}}$. We call W_i safe wedges of P and the q_i anchors. We assume W_i to be an open set. Finally, we identify two ports, the points p_i , where p_i is the point along ρ_i such that $y(p_i) = y(a_i)$. The



Fig. 4: Illustration for Property 2. (P1) Neither a_1 nor any points of $\overline{q_1b_1}$ are points of $R[a_0, z_0, 1]$; (P2) $b_1 \in R(s_0, t_0, 1)$; (P3) $b_1 \in R(a_0, t_0, 1)$.

parallelogram P together with its anchors, safe wedges, and ports is called a *winged parallelogram* $WP(P, q_0, q_1, W_0, W_1, p_0, p_1)$. Figure 3 shows an example of a winged parallelogram. The following property is an immediate consequence of the definition of winged parallelogram; see also Figure 4.

Property 2. Let $WP(P, q_0, q_1, W_0, W_1, p_0, p_1)$ be a winged parallelogram such that the interior angles at points a_i (i = 0, 1) are at most $\frac{\pi}{4}$. Let s_i, t_i, z_i (i = 0, 1) be any three points such that $s_i \in \overline{b_i q_i}, t_i \notin W_i$ with $x(t_0) \ge x(p_0), x(t_1) \le x(p_1)$, and $z_i \in W_i$ with $y(z_i) = y(a_i)$. Then: (P1) neither s_{1-i} nor a_{1-i} are points of $R[a_i, z_i, 1]$; (P2) $b_{1-i} \in R(s_i, t_i, 1)$; (P3) $b_{1-i} \in R[a_i, t_i, 1]$ if t_i on ρ_i and $b_{1-i} \in R(a_i, t_i, 1)$ if t_i is not on ρ_i .

We first show how to draw pairs of isomorphic stars into a winged parallelogram and then generalize the construction to pairs of isomorphic caterpillars.

Lemma 1 (*). Let $\langle T_0, T_1 \rangle$ be a pair of isomorphic stars such that, for $i = 0, 1, T_i$ has root r_i and leaves $v_{i,0}, \ldots, v_{i,k}$. Then $\langle T_0, T_1 \rangle$ admits an MW-[1] drawing $\langle \Gamma_0, \Gamma_1 \rangle$ contained in a winged parallelogram $WP(P, q_0, q_1, W_0, W_1, p_0, p_1)$ such that: (i) r_i is drawn at a_i and the the internal angle of $WP(P, q_0, q_1, W_0, W_1, p_0, p_1)$ at a_i is at most $\frac{\pi}{4}$; (ii) $v_{i,0}$ is drawn at b_i ; and (iii) for $0 < j \le k$, $v_{i,j}$ is drawn at an interior point of the segment $\overline{b_i q_i}$.



Fig. 5: Illustration for the Proof of Lemma 1.

Proof sketch. For i = 0, 1, if T_i has only one leaf, the construction is trivial; see Figure 5a. Otherwise, we draw the leaves of T_i uniformly spaced along a horizontal segment σ_i and then place σ_0 and σ_1 relative to each other so that for every pair of consecutive leaves of T_i , there is a witness for that pair among the leaves of T_{1-i} ; see Figure 5b.

The horizontal line midway between σ_0 and σ_1 will form a separating line for $\langle \Gamma_0, \Gamma_1 \rangle$ once the centers r_i of T_i are placed. The center r_0 of T_0 is then placed vertically above the leftmost leaf of T_0 and the center r_1 of T_1 is placed vertically below the rightmost leaf of T_1 , each center far enough from the separating line so that for i = 0, 1 and $0 \le j \le k$, no proximity region $R[r_i, v_{i,j}, 1]$ contains any witness from T_{1-i} ; see Figure 5c.

In the following we call an MW-[1] drawing $\langle \Gamma_0, \Gamma_1 \rangle$ of two isomorphic stars computed as in the proof of Lemma 1 a *WP-drawing on P* and say that the winged parallelogram *supports* the drawing; see Figure 6. Note that, by construction, the horizontal line *L* having $y(L) = (y(b_0) + y(b_1))/2$ is a separating line for the WP-drawing of two isomorphic stars.

Lemma 2. Let $\langle \Gamma_0, \Gamma_1 \rangle$ be a WP-drawing of two isomorphic stars $\langle T_0, T_1 \rangle$ and let P be the winged parallelogram that supports $\langle \Gamma_0, \Gamma_1 \rangle$. Then, any pair $\langle T'_0, T'_1 \rangle$ of isomorphic stars with at least one leaf and $T'_i \subset T_i$ has a WP-drawing on P.

Proof. Let r_i be the root of T_i and $v_{i,0}, \ldots, v_{i,k}$ be the leaves of T_i . Consider the drawing $\langle \Gamma_0, \Gamma_1 \rangle$ computed in Lemma 1; see Figure 5. We use the same notation as in the proof of Lemma 1. Remove all leaves $v_{i,j}$ that are not in T'_i and reposition the remaining leaves uniformly along σ_i as in the proof of Lemma 1.

By construction, the Gabriel region $R[v_{0,i}, v_{0,j}, 1]$ for every $v_{0,i}, v_{0,j} \in T_i, 1 \le i < j \le k$ still contains the vertex $v_{1,i}$, while the Gabriel region $R[v_{1,i}, v_{1,j}, 1]$

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Fig. 6: A WP-drawing of two isomorphic stars on a parallelogram P

for every $v_{1,i}, v_{1,j} \in T_i, 1 \leq i < j \leq k$ still contains the vertex $v_{1,j}$. Otherwise, if $v_{i,0} \notin T'_i$, then take any leaf $v_{i,j} \in T'_i$, switch its position with $v_{i,0}$ in Γ_i , and then proceed as above.

Theorem 1. Any pair $\langle T_0, T_1 \rangle$ of isomorphic caterpillars admits a linearly separable MW-[1] drawing.

Proof. For i = 0, 1, if each T_i is a path, the pair can easily be realized by two horizontal paths, such that corresponding vertices of T_0 and T_1 have the same *x*-coordinates, all edges have the same length and the *y*-distance between T_0 and T_1 is at most the edge length. So we can assume that the spine of T_i is a path such that at least one spine vertex has degree greater than two.

Let $r_{i,0}, \ldots, r_{i,k}$ be the spine vertices of T_i in the order that they appear along the spine. Decompose T_i into subtrees $T_{i,0}, \ldots, T_{i,k}$, having roots $r_{i,0}, \ldots, r_{i,k}$ respectively. Note that each $\langle T_{0,j}, T_{1,j} \rangle$ is either an isomorphic pair of stars with centers $r_{0,j}$ and $r_{1,j}$, respectively, or it is a pair of isolated vertices.

Let *h* be an index such that $r_{i,h}$ is a vertex of highest degree in T_i . Compute a WP-drawing $\langle \Gamma_{0,h}, \Gamma_{1,h} \rangle$ of $\langle T_{0,h}, T_{1,h} \rangle$ by means of Lemma 1 and let $WP_h = WP(P_h, q_{0,h}, q_{1,h}, W_{0,h}, W_{1,h}, p_{0,h}, p_{1,h})$ be the winged parallelogram that supports the drawing. Let $N = y(r_{0,h})$ and $S = y(r_{1,h})$ and let L_0 and L_1 be the two horizontal lines at heights N and S, respectively. We will construct a MW-[1] drawing of the two caterpillars such that all spine vertices of T_i lie on L_i and such that the horizontal line L at height (N + S)/2 separates T_0 from T_1 .

For any $0 \leq j \leq k, j \neq h$ such that $r_{i,j}$ has at least one leaf, we use Lemma 2 to compute a WP-drawing $\langle \Gamma_{0,j}, \Gamma_{1,j} \rangle$ of $\langle T_{0,j}, T_{1,j} \rangle$ in a winged parallelogram WP_j congruent to WP_h that will be placed so that $r_{i,j}$ lies on L_i . For any $0 \leq j \leq k, j \neq h$ such that $r_{i,j}$ has no children, we will place $r_{i,j}$ on L_i so that the line through $r_{0,j}, r_{1,j}$ is perpendicular to the line through $r_{1,h}, p_{0,h}$.

We now describe how to place each pair $\langle \Gamma_{0,j}, \Gamma_{1,j} \rangle$; note that placing $r_{0,j}$ completely determines the placement of $\langle \Gamma_{0,j}, \Gamma_{1,j} \rangle$. Vertex $r_{0,0}$ can be placed arbitrar-



Fig. 7: Illustration for the Proof of Theorem 1.

ily along L_0 . Assume now that, for some $j \ge 0$, the pairs $\{r_{0,0}, r_{1,0}\}, \ldots, \{r_{0,j}, r_{1,j}\}$ have been placed along L_0 and L_1 . We describe how to place $r_{0,j+1}$. There are three cases; see Figure 7: (1) If $r_{0,j}$ has at least one leaf, place $r_{0,j+1}$ at port $p_{0,j}$. (2) If $r_{0,j}$ has no leaves and $r_{0,j+1}$ has at least one leaf, place $r_{0,j+1}$ so that $r_{1,j}$ is at port $p_{1,j+1}$. (3) If both $r_{0,j}$ and $r_{0,j+1}$ have no leaves, place $r_{0,j+1}$ at the intersection of L_0 with the line through $r_{1,j}$ that is perpendicular to $\overline{r_{0,j}r_{1,j}}$.

This construction is almost an MW-[1] drawing of $\langle T_0, T_1 \rangle$. Consider the mutual witness Gabriel drawing Γ induced by the placement of the vertices of $\langle T_0, T_1 \rangle$ described above. Note that in our constructed drawing: (i) The pairs $\langle T_{0,j}, T_{1,j} \rangle$ are drawn in vertically disjoint strips and by Property 1 form MW-[1] drawings of those pairs. (ii) For any non-spine vertex $u_{0,j} \in T_{0,j}$, and any vertex $u_{0,t} \in T_{0,t}$ ($0 \leq j < t \leq k$), by Property 2 (P2), $b_{1,j} \in R(u_{0,j}, u_{0,t}, 1)$ and so the pair $\{u_{0,j}, u_{0,t}\}$ is not an edge in Γ . (iii) For any spine vertex $r_{0,j} \in T_{0,j}$, and non-spine vertex $u_{0,t} \in T_{0,t}$ ($0 \leq j < t \leq k$), either $r_{0,j}$ has a leaf, and so by Property 2 (P3), $b_{1,j} \in R(r_{0,j}, u_{0,t}, 1)$ or $r_{0,j}$ has no leaves and $r_{1,j} \in R[r_{0,j}, u_{0,t}, 1]$ by the construction described above. Similar statements hold for pairs of vertices in T_1 by the symmetry of the construction.

The drawing Γ is not yet an MW-[1] drawing of $\langle T_0, T_1 \rangle$ because there are no edges in Γ between any pair of consecutive spine vertices of T_i . This problem

can be easily rectified, however. Note that in Γ there are only two types of non-adjacent vertex pairs that only have witnesses on the boundaries of their Gabriel regions (that is, that only have witnesses forming right angles), namely, consecutive leaves in an individual subtree $T_{i,j}$, and consecutive spine vertices in T_i . Let $r_{i,j}$ and $r_{i,j+1}$ be any two consecutive spine vertices of T_i . We can always perturb Γ so that by very slightly moving to the left all vertices of $\langle T_{0,j+1}, T_{1,j+1} \rangle$, we have, by Property 2 (P1), that $R[r_{i,j}, r_{i,j+1}, 1]$ contains no witnesses while for every other pair of non-adjacent vertices their Gabriel regions still contain a witness. Once all spine vertices have been properly connected, the resulting drawing is a linearly separable MW-[1] drawing of $\langle T_0, T_1 \rangle$.

5 MW- β Drawings of Isomorphic Trees

In this section we show that, at the expense of losing linear separability, the result of Theorem 1 can be extended to any two isomorphic trees and to any mutual witness proximity drawing that adopts either the open or the closed β -region for all values of $\beta \in [1, \infty]$. A nice property of our algorithm is that it does not depend on the exact choice of β , i.e., it produces a single drawing that is an MW- β proximity drawing for every $\beta \geq 1$.

Similar to the previous section, we show a construction to recursively draw subtrees inside suitable parallelograms, which are however not winged parallelograms. We start by defining these parallelograms. In the remainder of the section, we shall sometimes assume that our trees are rooted, in which case we denote as (T, r) a tree T with root r.

Let $P = \langle a_0, b_0, a_1, b_1 \rangle$ be a parallelogram where $\overline{a_0a_1}$ is the longer diagonal and no angle is equal to $\frac{\pi}{2}$. We say that P is *nicely oriented* if $y(a_0) > y(b_1) > y(b_0) > y(a_1)$ and $x(a_0) < x(b_0) < x(b_1) < x(a_1)$; see Figure 8a.

Let $\langle (T_0, r_0), (T_1, r_1) \rangle$ be a pair of isomorphic rooted trees with n vertices each. An MW- β parallelogram drawing of $\langle (T_0, r_0), (T_1, r_1) \rangle$ is an MW- β proximity drawing $\langle \Gamma_0, \Gamma_1 \rangle$ contained in a nicely oriented parallelogram $P = \langle a_0, b_0, a_1, b_1 \rangle$ such that, for i = 0, 1, the following holds: (i) point a_i represents the root r_i of T_i ; (ii) if n > 0, point b_i represents a vertex of T_i adjacent to r_i ; (iii) for every other vertex $v_i \in \Gamma_i$ such that v_i is neither the root of T_i nor the vertex at b_i , we have $y(b_1) > y(v_i) > y(b_0)$; (iv) no edge of Γ_i is a vertical segment. Figure 8b shows an example of an MW-1 parallelogram drawing.

Theorem 2. Any two isomorphic trees $\langle T_0, T_1 \rangle$ admit a parallelogram drawing that is an MW- β -drawing for all $\beta \in [1, \infty]$.

Proof. Let r_0 be any vertex of T_0 and let $r_1 \in T_1$ be the isomorphic image of r_0 . We will show by induction on the depth δ of (T_0, r_0) that $\langle (T_0, r_0), (T_1, r_1) \rangle$ admits an MW- β parallelogram drawing of $\langle T_0, T_1 \rangle$ for any $\beta \geq 1$.

If $\delta = 0$ each T_i consists of only its root r_i . Choosing any nicely oriented parallelogram with $\overline{r_0r_1}$ as its long diagonal will result in a valid MW- β drawing. Assume the claim holds for $\delta \leq k$ and suppose $\delta = k + 1$.



Fig. 8: (a) A nicely oriented parallelogram; (b) an MW-1 parallelogram drawing.

Let $\langle (T_{0,0}, r_{0,1}), (T_{1,0}, r_{1,0}) \rangle, \ldots, \langle (T_{0,m}, r_{0,m}), (T_{1,m}, r_{1,m}) \rangle$ be the pairs of isomorphic rooted trees resulting from deleting r_i from T_i . By induction, each $\langle (T_{0,j}, r_{0,j}), (T_{1,j}, r_{1,j}) \rangle$ with $0 \leq j \leq m$ admits a parallelogram drawing which is an MW- β drawing. Let H be any horizontal strip defined by two parallel lines y = s and y = t such that s < t. We uniformly scale and translate the parallelogram drawings of $\langle (T_{0,j}, r_{0,j}), (T_{1,j}, r_{1,j}) \rangle$ such that $y(r_{0,j}) = t$ and $y(r_{1,j}) = s$. Note that this operation does not change any of the β -proximity properties of any of the tree pairs.

Let $P_j = (a_{0,j}, b_{0,j}, a_{1,j}, b_{1,j})$ be the parallelogram that supports the MW- β drawing $\langle (\Gamma_{0,j}), (\Gamma_{1,j}) \rangle$ of $\langle (T_{0,j}, r_{0,j}), (T_{1,j}, r_{1,j}) \rangle$. Let ℓ_j and ℓ'_j be two half-lines such that ℓ_j starts at $r_{0,j}$, is orthogonal to $\overline{r_{0,j}, b_{0,j}}$, and crosses H, and ℓ'_j starts at $r_{1,j}$, is orthogonal to $\overline{r_{1,j}, b_{1,j}}$, and crosses H; see Figure 9. We position P_{j+1} such that (i) ℓ_{j+1} is to the right of ℓ'_j ; (ii) for any edge $e_{1,j} = (u_{1,j}, v_{1,j})$ in $T_{1,j}, r_{0,j+1}$ is to the right of the rightmost intersection point between H and $R[u_{1,j}, v_{1,j}, \infty]$ (since by inductive hypothesis no edge of $\Gamma_{1,j}$ is vertical, the coordinates of such points are finite); and (iii) for any edge $e_{0,j+1} = (u_{0,j+1}, v_{0,j+1}) \in T_{0,j+1}, r_{1,j}$ is to the left of the leftmost intersection point between H and $R[u_{0,j+1}, v_{0,j+1}, \infty]$ (by inductive hypothesis, the coordinates of such points are finite).

Condition (i) guarantees that for any vertices $v_{1,j+1} \in \Gamma_{1,j+1}$ and $v_{1,j} \in \Gamma_{1,j}$, we have $\angle (v_{1,j+1}, r_{0,j+1}, v_{1,j}) > \frac{\pi}{2}$ and thus $r_{0,j+1} \in R(v_{1,j+1}, v_{1,j}, 1)$ and $r_{0,j+1} \in R(v_{1,j+1}, v_{1,j}, \beta)$ for any $\beta \ge 1$. Similarly, for any vertices $v_{0,j+1} \in \Gamma_{0,j+1}$ and $v_{0,j} \in \Gamma_{0,j}$, we have $r_{1,j} \in R(v_{0,j+1}, v_{0,j}, \beta)$ for any $\beta \ge 1$. Conditions (ii) and (iii) guarantee that for any pair of adjacent $v_{i,j}, u_{i,j}$ in $\Gamma_{i,j}$, there is no witness in $R[v_{i,j}, u_{i,j}, \infty]$ and thus no witness in $R[v_{i,j}, u_{i,j}, \beta]$ for any finite $\beta \ge 1$. We now show how to place the roots $r_0 \in T_0$ and $r_1 \in T_1$ to produce an MW- β parallelogram drawing of $\langle (T_0, r_0), (T_1, r_1) \rangle$ for any $\beta \in [1, \infty]$.

Let L_0 be the vertical line through $r_{0,0}$ and let L_1 be the vertical line through $r_{1,m}$; see Figure 10a. We show how to place r_i on L_i such that the *closed* β -region $R[r_i, r_{i,j}, \infty]$ does not contain any witness, while for any other vertex $v \in T_i$, the open β -region $R(r_i, v, 1)$ contains a witness. This implies that $R[r_i, r_{i,j}, \beta]$ does



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Fig. 9: Parallelograms P_j and P_{j+1} placed inside H.



Fig. 10: Placing r_0 in the proof of Theorem 2.

not contain any witnesses for all finite values of β and that $R(r_i, v, \beta)$ contains some witnesses for every $\beta \geq 1$.

We proceed in three steps. In the first step, we identify an interval I'_i of L_i such that for any point $p' \in I'_i$ and for any $r_{i,j}$, $R[p', r_{i,j}, \beta]$ contains no witnesses $(i = 0, 1, 0 \leq j \leq m)$. In the second step, we identify an interval I''_i of L_i such that for each vertex $v \neq r_{i,j}$ $(i = 0, 1, 0 \leq j \leq m)$ in Γ_i and for each point $p'' \in I''_i$, $R(p'', v, \beta)$ contains a witness. In the third step, we identify an interval I''_i of L_i such that for any point $p_0 \in I'''_0$ the segment $\overline{p_0 b_{1,m}}$ does not intersect any parallelogram P_j with $0 \leq j \leq m$. Similarly, for any point $p_1 \in I''_1$, the segment $\overline{p_1, b_{0,0}}$ does not intersect P_j for $0 \leq j \leq m$. As we will see, $I'_i \cap I''_i \cap I''_i$ is a half-infinite strip for i = 0, 1; see Figure 10b. We will describe how to obtain the intervals I'_0, I''_0, I''_0 ; the intervals I'_1, I''_1, I'''_1 can be constructed symmetrically.

We start by defining I'_0 . By construction of the MW- β drawing of the forests $T_{0,0}, \ldots T_{0,m}$ and $T_{1,0}, \ldots T_{1,m}$, there exist horizontal lines h_0, h_1 in the interior of H such that h_i separates $r_{i,0}, \ldots r_{i,m}$ from every other vertex in the forest; see Figure 10a. Let q_0 be the intersection point of h_0 and L_0 . Let z'_0 be the intersection point of L_0 with the line through $r_{0,m}$ perpendicular to $\overline{r_{0,m}q_0}$. Let $I'_0 = \{z \in L_0 : y(z) \ge y(z'_0)\}$. Observe that for any $p_0 \in I'_0$ and any $r_{0,j}$, $R[p_0, r_{0,j}, \infty]$ contains no witnesses.



Fig. 11: Rotating the drawing to obtain an MW- β parallelogram drawing.

We now define I_0'' . For any parallelogram P_j and any vertex $v_{0,j} \in T_{0,j} \setminus \{r_{0,j}\}$, let $z_{0,j}$ be the intersection of L_0 with the line through $b_{1,j}$ perpendicular to $\overline{b_{1,j}v_{0,j}}$. Let z_0'' be the $z_{0,j}$ of maximum y-value over all $z_{0,j}$ $(0 \le j \le m)$. Let $I_0'' = \{z \in L_0 : y(z) \ge y(z_0'')\}$. Observe that for any point $p_0 \in I_0''$ and for any $v_{0,j} \in T_{0,j} \setminus \{r_{0,j}\}$, we have that $\angle (v_{0,j}, b_{1,j}, p_0) \ge \frac{\pi}{2}$ and thus $b_{1,j} \in R[p_0, v_{0,j}, 1]$.

We now define I_0''' . Let α_0 be the acute angle formed by L_0 and the segment $\overline{r_{0,0}b_{0,0}}$. Let α_1 be the acute angle formed by L_1 and the segment $\overline{r_{1,m}b_{1,m}}$ and let $\alpha = \min\{\alpha_0, \alpha_1\}$. Let f_0 be a half-line starting at $r_{1,m}$, having negative slope, and forming an acute angle of $\alpha/2$ with L_1 . Let z_0''' be $f_0 \cap L_0$ and let $I_0''' = \{z \in L_0 : y(z) \ge y(z_0''')\}$.

Let $I_i = I'_i \cap I''_i \cap I''_i$ and let $p_i \in I_i$ be such that $\overline{p_0 r_{1,m}}$ is parallel to $\overline{p_1 r_{0,0}}$. We draw r_i at p_i , which produces an MW- β drawing of $\langle T_0, T_1 \rangle$ in a parallelogram $P = \langle a_0, b_0, a_1, b_1 \rangle = \langle r_0, b_{1,m}, r_1, b_{0,0} \rangle$. This is however not yet a parallelogram drawing, as $y(b_0) = y(b_{0,0}) > y(b_{1,m}) = y(b_1)$ and some edges are vertical.

To complete the proof, we thus show how to rotate P to produce an MW- β parallelogram drawing. Refer to Figure 11a. Let γ_0 be the angle between $\overline{r_{0,0}r_1}$ and $\overline{r_{0,0}b_{0,0}}$ and let γ_1 be the angle between $\overline{r_{1,m}r_0}$ and $\overline{r_{1,m}b_{1,m}}$; Let $\gamma = \min\{\gamma_0, \gamma_1\}$. Let \hat{f}_0 be the ray originating at $r_{0,0}$, forming an angle $\gamma' < \gamma$ with, and lying above, segment $\overline{r_{0,0}r_1}$, so that no edge of the drawing is perpendicular to \hat{f}_0 . Let \hat{f}_1 be the ray originating at $r_{1,m}$ having opposite direction to \hat{f}_0 . Observe that \hat{f}_0 and \hat{f}_1 are parallel and that any vertex of T_i except r_i is in the strip between \hat{f}_0 and \hat{f}_1 . We now rotate P counterclockwise until \hat{f}_0 and \hat{f}_1 become horizontal; see Figure 11b. This produces a parallelogram drawing of $\langle T_0, T_1 \rangle$, since no edge is vertical, $y(r_0) > y(r_{1,m}) > y(r_{0,0}) > y(r_1)$, and $x(r_0) < x(r_{0,0}) < x(r_{1,m}) < x(r_1)$.

6 Pruning Leaves from MW-β Drawings of Isomorphic Trees

In this section, we explore the question of how far from isomorphic two trees might be while still allowing an MW- β drawing. We consider the MW- β drawing $\langle \Gamma_0, \Gamma_1 \rangle$ constructed in the proof of Theorem 2 and ask whether it is possible to prune some leaves from Γ_1 and still have an MW- β drawing of the resulting trees. Precisely, we show that there are cases when we can remove linearly many leaves from Γ_1 and still obtain an MW- β drawing of the resulting tree for any $\beta \in [1, \infty]$. It may be worth recalling that Lenhart and Liotta proved that two stars admit an MW-1 drawing if and only if the cardinalities of their vertex sets differ by at most two [13].

Let (T, r) be a rooted tree and let \mathcal{L} be a set of leaves of T. The vertex v is a *cousin* of a vertex v' if v and v' have a common grandparent but no common parent, i.e., there is a vertex w such that a length-2 directed path w, p, v and a length-2 directed path w, p', v' with $p \neq p'$ exist. We say that $\mathcal{L} \neq \emptyset$ is *sparse* if, for every $v \in \mathcal{L}$, (i) v has at least one sibling, (ii) every sibling v' of v is a leaf with $v' \notin \mathcal{L}$, and (iii) v has a cousin w such that $w \notin \mathcal{L}$ and, for all siblings w' of $w, w' \notin \mathcal{L}$. Note that the existence of a sparse set implies that (T, r) has height at least 2, otherwise there is no vertex that has a cousin.

Theorem 3 (*). Let (T, r) be a rooted tree and let \mathcal{L} be a sparse set of leaves of T. Then the pair $\langle T, T \setminus \mathcal{L} \rangle$ of trees admits an MW- β drawing for all $\beta \in [1, \infty]$.

Corollary 1 (*). For any $m \ge 1$ and n = 7m+1, there exist tree pairs $\langle T_0, T_1 \rangle$ with $|V(T_1)| \le 1 + \frac{5}{6}(|V(T_0)| - 1)$ that admit an MW- β drawing for all $\beta \in [1, \infty]$.

7 Concluding Remarks

In this paper, we studied the mutual witness proximity drawability of pairs of isomorphic trees. We adopted the well-known concept of open/closed β -proximity regions and considered any value of the parameter β such that $\beta \geq 1$. For the special case of $\beta = 1$, the definition of closed β -proximity region coincides with the definition of Gabriel proximity region. We showed in Theorem 1 that any pair of isomorphic caterpillars admits a linearly separable mutual witness Gabriel drawing. We then extended this result in Theorem 2 to any value of $\beta \geq 1$ and to any pair of isomorphic trees, but at the cost of losing linear separability.

It would be interesting to establish whether any two isomorphic trees admit a linearly separable MW- β drawing for $\beta \geq 1$. Also, even for the special case of caterpillars, extending the result of Theorem 1 to values of $\beta > 1$ does not seem immediate. Finally, a characterization of those non-isomorphic pairs of trees that admit a mutual witness β -drawing continues to be elusive. Theorem 3 shows that the trees in the pair may differ by linearly many vertices.

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A Omitted Proofs from Section 4

Lemma 1 (*). Let $\langle T_0, T_1 \rangle$ be a pair of isomorphic stars such that, for $i = 0, 1, T_i$ has root r_i and leaves $v_{i,0}, \ldots, v_{i,k}$. Then $\langle T_0, T_1 \rangle$ admits an MW-[1] drawing $\langle \Gamma_0, \Gamma_1 \rangle$ contained in a winged parallelogram $WP(P, q_0, q_1, W_0, W_1, p_0, p_1)$ such that: (i) r_i is drawn at a_i and the the internal angle of $WP(P, q_0, q_1, W_0, W_1, p_0, p_1)$ at a_i is at most $\frac{\pi}{4}$; (ii) $v_{i,0}$ is drawn at b_i ; and (iii) for $0 < j \le k$, $v_{i,j}$ is drawn at an interior point of the segment $\overline{b_i q_i}$.

Proof. We first consider the case k = 0; see Figure 12a. We place r_0 at position $a_0 = (0, 5)$, $v_{0,0}$ at position $b_0 = (0, 3)$, r_1 at position $a_1 = (2, 0)$, and $v_{1,0}$ at position $b_1 = (2, 2)$. This way, the angle inside $P = (a_0, b_0, a_1, b_1)$ at a_0 is smaller than $\pi/4$. We place the anchor q_0 at (1.1, 3) and q_1 at (0.9, 2) and compute the safe wedges W_0, W_1 and ports p_0, p_1 as above to obtain an MW-[1]-drawing inside a winged parallelogram $WP(P, q_0, q_1, W_0, W_1, p_0, p_1)$ with the desired properties.

Consider now the case that k > 0. We will create an MW-[1] drawing $\langle \Gamma_0, \Gamma_1 \rangle$ that is point symmetric in the origin, i.e., a drawing with $x(r_1) = -x(r_0)$, $y(r_1) = -y(r_0)$, and $x(v_{1,i}) = -x(v_{0,k-i+1})$, $y(v_{1,i}) = -y(v_{0,k-i+1})$ for $1 \le 0 \le k$. By symmetry, we only have to argue that the edges of T_0 are realized correctly.

We first place the leaves $v_{0,i}$, $0 \le i \le k$, at y-coordinate 0.5 and x-coordinate 2i - k + 0.5, such that any pair $v_{0,i}, v_{0,i+1}$ has distance 2 and $\angle (v_{0,i}, v_{1,i}, v_{0,i+1}) = \pi/2$; see Figure 12b. Thus, $v_{1,i}$ lies in $R[v_{0,i}, v_{0,j}, 1]$ for any two vertices with $0 \le i < j \le k$, so no two leaves are adjacent.

Now we place r_0 with $x(r_0) = x(v_{0,0})$; see Figure 12c. We have to make sure that the regions $R[r_0, v_{0,i}, 1]$ contains no witness. Observe that, by definition, any Gabriel region R[u, v, 1] is contained in the disk around u with radius d(u, v). Hence, no point w with d(u, w) > d(u, v) can lie in R[u, v, 1]. Consider the point $p = (x(v_{0,0}), y(v_{1,0})) = (2i - k + 0.5, -0.5)$. By construction, we have $d(r_0, v_{1,i}) < d(r_0, p) = y(r_0) + 0.5$ and we want to make sure that $d(r_0, v_{0,i}) \leq d(r_0, v_{0,k}) < d(r_0, p)$ for each $0 \leq i \leq k$. Consider now the triangle $\triangle(r_0, p, v_{0,k})$. If $\angle(p, v_{0,k}, r_0) = \pi/2$, then by Pythagoras $d(r_0, p) > d(r_0, v_{0,k})$. Let $\alpha = \angle (v_{0,k}, r_0, p)$ and $\beta = \angle (r_0, p, v_{0,k})$ with $\alpha + \beta = \pi/2$. Consider the point $q = (x(v_{0,k}), y(v_{1,k}) = (k - 0.5, -0.5)$ and consider the triangle $\triangle(p, q, v_{0,k})$. We have $\angle(r_0, p, q) = \pi/2$ and $\angle(r_0, p, v_{0,k}) = \beta$, so $\angle(v_{0,k}, p, q) = \alpha$. Since $\angle(p,q,v_{0,k})$, we also have $\angle(q,v_{0,k},p) = \beta$. Hence, the triangles $\triangle(r_0,p,v_{0,k})$ and $\Delta(p, q, v_{0,k})$ are congruent, so we have $\frac{d(r_0, p)}{d(p, v_{0,k})} = \frac{d(p, v_{0,k})}{d(v_{0,k}, q)}$. Since $d(v_{0,k}, q) = 1$ by choice of q, we thus have $d(r_0, p) = d(p, v_{0,k})^2$. By Pythagoras' Theorem, $d(p, v_{0,k})^2 = d(p,q)^2 + d(v_{0,k},q)^2 = 2k^2 + 1$. Thus, $y(r_0) = d(r_0,p) - 0.5 = 2k^2 + 0.5$ ensures that no edges of T_0 has a witness in Γ_1 . Furthermore, note that, by construction, β is larger than $\pi/4$ since $1 = d(v_{0,0}, p) < d(v_{0,0}, v_{0,k})$ as long as k > 0, so α is smaller than $\pi/4$.

We choose the winged parallelogram $WP((a_0, b_0, a_1, b_1), q_0, q_1, W_0, W_1, p_0, p_1)$ as follows. For (a_0, b_0, a_1, b_1) , we choose the positions of $r_0, v_{0,0}, r_1, v_{1,k}$, respectively. We place the point q_0 slightly to the right of $v_{0,k}$ at $(x(v_{0,k}) + \varepsilon, 0.5)$, and the point q_1 slightly to the left of $v_{1,0}$ at $(x(v_{1,0} - \varepsilon, -0.5))$ for some small



Fig. 12: Illustration for the Proof of Lemma 1.

enough ε . The interior angle at points a_0, a_1 is smaller than $\pi/4$ as long as $2k^2 + 1 > 2k + 1$, which is true for k > 0, and all leaves are placed on the desired positions. We choose the safe wedges W_0, W_1 and ports p_0, p_1 as in the definition. For an illustration, see Figure 12d.

B Omitted Proofs from Section 6

Theorem 3 (*). Let (T, r) be a rooted tree and let \mathcal{L} be a sparse set of leaves of T. Then the pair $\langle T, T \setminus \mathcal{L} \rangle$ of trees admits an MW- β drawing for all $\beta \in [1, \infty]$.

We start by a definition and a technical lemma. Let $\langle \Gamma_0, \Gamma_1 \rangle$ be a MW- β parallelogram drawing of two trees in a parallelogram $P = \langle a_0, b_0, a_1, b_1 \rangle$; see Figure 13a. The *strip ratio* $\sigma(\Gamma_0, \Gamma_1)$ of $\langle \Gamma_0, \Gamma_1 \rangle$ is defined as

$$\sigma(\Gamma_0, \Gamma_1) = \frac{|y(b_1) - y(b_0)|}{|y(a_0) - y(a_1)|}.$$

Lemma 3. Let $\langle T_0, T_1 \rangle$ be two isomorphic trees and let $\varepsilon > 0$ be an arbitrarily small real number. There exists a parallelogram drawing of $\langle T_0, T_1 \rangle$ whose strip ratio is $\sigma < \varepsilon$.

Proof. We construct a MW- β parallelogram drawing $\langle \Gamma_0, \Gamma_1 \rangle$ for $\langle T_0, T_1 \rangle$. If $\sigma(\Gamma_0, \Gamma_1) < \varepsilon$, then we are done. Otherwise, we simultaneously move a_0 (and thus r_0) along the ray b_0a_0 upwards and a_1 (and thus r_1) along the ray b_1a_1 downwards until $\sigma(\Gamma_0, \Gamma_1) < \varepsilon$; see Figure 13. Note that this movement corresponds to moving a_0 (r_0) vertically upwards and a_1 (r_1) vertically downwards before the



Fig. 13: Illustration of the strip ratio and the proof of Lemma 3.



Fig. 14: The three types of subtrees in the base case of Theorem 3.

final rotation step in the proof of Theorem 2. Since, for the proof of correctness, it was only important that these two points are far enough above/below the other vertices, the drawing remains an MW- β drawing.

Proof (of Theorem 3). First, note that, for any subtree (T', r') of (T, r) rooted in r' of height at least 2, $\mathcal{L} \cap V(T')$ is sparse for $\langle T', r' \rangle$.

We show by induction on the height $\delta \geq 2$ of (T, r) that an MW- β drawing can always be produced.

Consider first the base case $\delta = 2$. By definition of sparse sets, the children of r cannot be in \mathcal{L} , as they have no cousins. Let $(T_0, r_0) = (T, r)$ and let $(T_{0,0}, r_{0,0}), \ldots, (T_{0,m}, r_{0,m})$ be the subtrees of (T_0, r_0) resulting from deleting r_0 from T_0 . Then each tree $(T_{0,j}, r_{0,j})$ is of one of three types; see Figure 14:

(A) $r_{0,j}$ is a leaf not in \mathcal{L} ,

(B) $(T_{0,j}, r_{0,j})$ has height 1 with exactly one of its leaves $v_{0,j} \in \mathcal{L}$,

(C) $(T_{0,i}, r_{0,i})$ has height 1 with no leaf in \mathcal{L} .

Note that there must be at least one subtree of type (C), but there may be no subtrees of type (A) or (B). We now reorder the children of r_0 such that, from left to right, we first have all subtrees of type (A), then all subtrees of type (B), and then all subtrees of type (C). Within each subtree of $(T_{0,j}, r_{0,j})$ type (B), we order the leaves such $v_{0,j}$ is the rightmost leaf; see Figure 14.

Let (T_1, r_1) be isomorphic to (T_0, r_0) . We first compute a MW- β parallelogram drawing of $\langle (T_0, r_0), (T_1, r_1) \rangle$ in a parallelogram $P = (a_0, b_0, a_1, b_0)$ according to the proof of Theorem 2, but with some small adjustments. Using Lemma 3, we ensure that the rightmost subtree $(T_{i,m}, r_{i,m}), 0 \le i \le 1$, which is of type (C),



Fig. 15: The drawing in the base case of Theorem 3 after removing the leaves of \mathcal{L} . There is not witness in $R[r_0, r_{0,0}, \beta]$ (green disk); $w_1 \in R[r_0, u_{0,i}, \beta]$ (red disk); $r_{1,i} \in R[u_{0,i}, u_{0,j}, \beta]$ (blue disk)

has the largest strip ratio among all subtrees $(T_{i,j}, r_{i,j})$. Let w_1 be the rightmost leaf of $(T_{1,m}, r_{1,m})$. Then, placing the subtrees $(T_{i,j}, r_{i,j})$ in the horizontal strip H as in the proof of Theorem 2, w_1 will be the rightmost and topmost vertex of T_1 in the interior of H.

We place r_0 and r_1 as in the proof of Theorem 2, but with the additional constraint that for every vertex u_0 of (T_0, r_0) in the interior of H, $\angle (u_0, w_1, r_0) \ge \pi/2$, so that w_1 lies in the β -region $R[r_0, u_0, \beta]$. Similar to the proof of Lemma 3, this can be achieved by moving r_0 upwards along the ray b_0r_0 . Since w_1 belongs to a subtree of type (C), $w_1 \notin \mathcal{L}$. Hence, after removing the leaves of \mathcal{L} , all edges between r_0 and any non-adjacent vertex u_0 of T_0 (which lies in the interior of H) still have w_1 as a witness; see Figure 15.

Note that $r_{1,m}$ is placed at point b_1 of the parallelogram, and since $(T_{1,m}, r_{1,m})$ is of type (C), $r_{1,m}$ is not a leaf and thus $r_{1,m} \notin \mathcal{L}$, so removing the leaves of \mathcal{L} from Γ_1 does not destroy the MW- β parallelogram drawing properties.

Furthermore, for any two vertices $u_{0,i}$ in $(T_{0,i}, r_{0,i})$ and $u_{0,j}$ in $(T_{0,j}, r_{0,j})$ with $0 \le i < j \le m$ where $(T_{0,i}, r_{0,i})$ is of type (B), we have that $r_{1,i}$ lies in the β -



Fig. 16: The four types of subtrees in the induction step of Theorem 3.

region $R[u_{0,i}, u_{0,j}, \beta]$, so we can remove $v_{1,i}$ from any subtree of type (B) without destroying the MW- β drawing properties. Hence, we obtain a parallelogram MW- β drawing of $\langle T, T \setminus \mathcal{L} \rangle$. Note that the strip ratio of the drawing can also be lowered by moving r_0 upwards along the ray b_0r_0 and r_1 downwards along the ray b_1r_1 as in the proof of Lemma 3.

Consider now the inductive case of $\delta > 2$. Let $(T_0, r_0) = (T, r)$ and let $(T_{0,0}, r_{0,0}), \ldots, (T_{0,m}, r_{0,m})$ be the subtrees of (T_0, r_0) resulting from deleting r_0 from T_0 . Then each tree $(T_{0,i}, r_{0,j})$ is of one of four types; see Figure 16:

- (A) $r_{0,j}$ is a leaf not in \mathcal{L} ,
- (B) $(T_{0,j}, r_{0,j})$ has height 1 with exactly one of its leaves $v_{0,j} \in \mathcal{L}$,
- (C) $(T_{0,j}, r_{0,j})$ has height 1 with no leaf in \mathcal{L} ,
- (D) $(T_{0,j}, r_{0,j})$ has height at least 2 but smaller than δ .

Note that there must be at least one subtree of type (D), but there might be no subtrees of type (A), (B) or (C). We now reorder the children of r_0 such that, from left to right, we first have all subtrees of type (A), then all subtrees of type (B), then all subtrees of type (C), and then all subtrees of type (D). Within each subtree of $(T_{0,j}, r_{0,j})$ type (B), we order the leaves such $v_{0,j}$ is the rightmost leaf; see Figure 16.

Let $\mathcal{L}' \subseteq \mathcal{L}$ be the set of leaves of \mathcal{L} in the subtrees of type (D). Let (T_1, r_1) be isomorphic to $(T_0 \setminus \mathcal{L}', r_0)$. By induction, every pair of subtrees $\langle (T_{0,j}, r_{0,j}), (T_{1,j}, r_{1,j}) \rangle$ of type (D) has a parallelogram MW- β drawing where the strip ratio can be arbitrarily lowered.

We arrange the parallelogram drawings of the subtrees $\langle (T_{0,j}, r_{0,j}), (T_{1,j}, r_{1,j}) \rangle$ inside a horizontal strip H as in the base case, using Lemma 3 to ensure that the drawing of $\langle (T_{0,m}, r_{0,m}), (T_{1,m}, r_{1,m}) \rangle$, which is of type (D), has the largest strip ratio among all pairs of subtrees $\langle (T_{0,m}, r_{0,m}), (T_{1,m}, r_{1,m}) \rangle$; see Figure 16.

Let w_1 be the topmost (and rightmost) vertex of $(T_{1,m}, r_{1,m})$ inside H. We again move r_0 upwards along the ray b_0r_0 such that, For every vertex u_0 of T_0 in the interior of H, $\angle(u_0, w_1, r) \ge \pi/2$, so that w_1 lies in the β -region $R[r_0, u_0, \beta]$; see Figure 17. Since $w_1 \notin \mathcal{L}$ (otherwise it would not be in $(T_{1,m}, r_{1,m})$, as we already removed the leaves of \mathcal{L}'), after removing the leaves of \mathcal{L} , all edges between r_0 and any non-adjacent vertex u_0 of T_0 (which lies in the interior of H) still have w_1 as a witness. Furthermore, the edges between disjoint subtrees still have witnesses following the same argument as in the base case, and $r_{1,m}$, which is not a leaf and thus not in \mathcal{L} , lies at point b of the parallelogram. Hence, after removing all leaves of \mathcal{L} , we obtain a MW- β parallelogram drawing of $\langle T, T \setminus \mathcal{L} \rangle$.

Corollary 1 (*). For any $m \ge 1$ and n = 7m + 1, there exist tree pairs $\langle T_0, T_1 \rangle$ with $|V(T_1)| \le 1 + \frac{5}{6}(|V(T_0)| - 1)$ that admit an MW- β drawing for all $\beta \in [1, \infty]$.

Proof. We construct an infinite family of trees and sets of leaves as follows. For any m > 0, (T, r) is a tree rooted in r such that removing r yields m subtrees $(T_0, r_0), \ldots, (T_{m-1}, r_{m-1})$. Every subtree $T_j, 0 \le j < m$ consists of the following; see Figure 18.

- (i) The root r_j has 2 children u_j and u'_j ;
- (ii) u_j has one child v_j which is a leaf
- (iii) u'_j has two children w_j and w'_j which are leaves with $w'_j \in \mathcal{L}$.

Then \mathcal{L} is sparse, so $\langle T, T \setminus \mathcal{L} \rangle$ admits an MW- β drawing by Theorem 2. Every subtree T_j has 6 vertices, so |V(T)| = 6m + 1. \mathcal{L} has one leaf per subtrees T_j , so $|\mathcal{L}| = m$ and thus $|V(T \setminus \mathcal{L})| = 5m + 1 = 1 + \frac{5}{6}(|V(T)| - 1)$.



Fig. 17: The drawing in the induction step of Theorem 3 after removing the leaves of \mathcal{L} . There is not witness in $R[r_0, r_{0,0}, \beta]$ (green disk); $w_1 \in R[r_0, u_{0,i}, \beta]$ (red disk); $r_{1,i} \in R[u_{0,i}, u_{0,j}, \beta]$ (blue disk)



Fig. 18: Construction of (T, r) in the proof of Corollary 1.