# Weakly and Strongly Fan-Planar Graphs 

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#### Abstract

We study two notions of fan-planarity introduced by (Cheong et al., GD22), called weak and strong fan-planarity, which separate two non-equivalent definitions of fan-planarity in the literature. We prove that not every weakly fan-planar graph is strongly fan-planar, while the upper bound on the edge density is the same for both families.


Keywords: fan-planarity • density • weak vs. strong

## 1 Introduction

Crossings in graph drawings are known to heavily impede readability [16]17]. Unfortunately, however, minimizing the number of crossings is NP-complete 8 , while many real-world networks turn out to be non-planar. Fortunately, readable drawings of non-planar graphs can be obtained by limiting the topology [15] or the geometry of crossings 10[11. Based on these experimental findings, the research direction of graph drawing beyond planarity has emerged. This line of research is dedicated to the study of so-called beyond planar graph classes that are defined by forbidden edge-crossing patterns. More precisely, a graph belonging to such a class admits a drawing in which the forbidden pattern is absent. Important beyond planar graph classes are $k$-planar graphs, where the forbidden pattern is $k+1$ crossings on the same edge, $k$-quasiplanar graphs, where $k$ mutually crossing edges are prohibited, and RAC-graphs, where edges are not allowed to cross at non-right angles. We refer the interested reader to the survey by Didimo et al. [7 and a recent book [9] on beyond planarity.

In this paper, we study fan-planar graphs, which admit fan-planar drawings. In these drawings, each edge $e$ may only be crossed by a fan of edges, that is a bundle of edges sharing a common endpoint, called anchor of $e$, that all cross $e$ from the same side. Kaufmann and Ueckerdt introduced this graph class in 2014 [12] and described the aforementioned requirement with two forbidden patterns, Pattern (I) and (II) in Fig. 11 the first forbids two edges crossing $e$ to be non-adjacent whereas the second forbids crossings of $e$ by adjacent edges with the common endpoint on different sides of $e$. Since their introduction, fan-planar graphs have received a lot of attention in the scientific community; see [2] for an overview.


Fig. 1: Forbidden configurations.

Recently, Klemz et al. [14] pointed out a missing case in the proof of the edge density upper-bound in the preprint that introduced fan-planar graphs [12. This case was consequently fixed in the journal version [13 by the original authors. However, in the process, they introduced a third forbidden pattern, namely Pattern (III) of Fig. 1, which is quite reminiscent of the previously defined Pattern (II). Namely, in both patterns, the drawing restricted to the edge $e$ and the two edges crossing $e$ has two connected regions, called cells, where one is bounded and the other one is unbounded. The difference between the two configurations is that in Pattern (II), one endpoint of $e$ lies in the bounded cell, while in Pattern (III), both endpoints are contained there. In the new definition of fan-planarity, both endpoints of $e$ must lie in the unbounded cell.

The new forbidden Pattern (III) poses a problem for the existing literature on fan-planarity. Namely, while some previous results still apply when forbidding Pattern (III), other existing results, e.g., the bound of $4 n-12$ on the number of edges of $n$-vertex bipartite "fan-planar" graphs [1], build on the lemmas of the original paper [12] and may be affected by the recent changes to the definition.

Cheong et al. [6] introduced a clear distinction of the two models with the notions of weak and strong fan-planarity. Namely, weak fan-planarity allows Pattern (III) whereas strong fan-planarity forbids it. The original definition of fan-planarity [12 coincides with weak fan-planarity whereas strong fan-planarity matches the definition of the new journal version [13]. Graphs admitting such drawings are called weakly fan-planar and strongly fan-planar, respectively.

The family of graphs that admits drawings where only Pattern (I) is forbidden, called adjacency-crossing graphs, has also been studied by Brandenburg [4]. He showed that there are adjacency-crossing graphs that are not weakly fan-planar, so weakly fan-planar graphs form a proper subset of the adjacency-crossing graphs. Moreover, he shows that for any $n$-vertex adjacency-crossing graph with $m$ edges, one can construct a weakly fan-planar graph with $n$ vertices that also has $m$ edges. Brandenburg concluded from this that an $n$-vertex adjacency-crossing graph has at most $5 n-10$ edges, since that was the bound claimed by [12] for "fan-planar" graphs. Since that bound holds only under strong fan-planarity, this conclusion contains a gap, which the present paper fills.

Our contribution. First, we prove that the family of strongly fan-planar graphs is a proper subset of the weakly fan-planar graphs. Together with Brandenburg's result, this implies that the two inclusions of strongly fan-planar graphs inside
weakly fan-planar graphs and weakly fan-planar graphs inside adjacency-crossing graphs are both proper. We then continue to show that the known upper bound on the edge-density of strongly fan-planar graphs (namely $5 n-10$ for an $n$ vertex graph) carries over to weakly fan-planar graphs. This implies that also Brandenburg's bound [4] is in fact correct. We also prove that the known upper bound of $4 n-12$ on the number of edges of an $n$-vertex bipartite strongly fan-planar graph carries over to bipartite weakly fan-planar graphs.

## 2 Not every weakly fan-planar graph is strongly fan-planar

In this section, we will establish that strongly fan-planar graphs form a proper subset of weakly fan-planar graphs by constructing a graph $G$ with a weakly fan-planar drawing $\Gamma$, where Pattern (III) cannot be avoided in $\Gamma$.

In order to guarantee the existence of at least one Pattern (III) in any valid weakly fan-planar drawing of $G$, we will use the following key idea. We start with a planar graph with a unique embedding. We will then make every edge of this planar graph "uncrossable" by replacing it with a suitable gadget introduced by Binucci et al. [3. Afterwards, we insert into every face of the planar graph a small gadget graph (shown in Fig. 2a), which can only be drawn with a quadrangular outer face if we allow (III). Note that this gadget graph itself is in fact strongly fan-planar, as shown in Fig. 2b, and, hence, does not serve itself as an example of a weakly but not strongly fan-planar graph. In order to achieve our goal, we will leverage the following lemma:

Lemma 1 (Binucci et al. [3]). Let $\mathcal{P}$ be the planarization ${ }^{3}$ of any weakly fan-planar drawing of $K_{7}$. Then, between any pair of vertices of $K_{7}$, there exists a path in $\mathcal{P}$ that contains no real edge of the $K_{7}$.

Moreover, we will use the following definition throughout the paper.
Definition 1. Let $\Gamma$ be a weakly fan-planar drawing of a graph $G$. $\Gamma$ is said to be minimal, if, among all weakly fan-planar drawings of $G$, it contains the smallest possible number of triples of edges that form Pattern (III).

Theorem 1. There exists a weakly fan-planar graph that does not admit a strongly fan-planar drawing.

Proof. Let $G_{0}$ be a 3-connected planar quadrangulation; e.g, one that is obtained by the construction in [5]. Note that by construction, $G_{0}$ is bipartite and has a unique embedding into $\mathbb{R}^{2}$ up to the choice of the outer face and a mirroring [18]. Next, we insert a copy of our gadget graph $H$ shown in Fig. 2a into every face $f$ of $G_{0}$ by identifying the outer cycle of $H$ with the facial cycle of $f$. Denote by $G_{1}$ this supergraph of $G_{0}$. We use the color scheme of Fig. 2a to color all edges of $G_{1}$ - in particular, the edges of $G_{0}$ form a subset of the red edges of $G_{1}$. In the

[^0]next step, we substitute every red edge of $G_{1}$ by a $K_{7}$ and denote the resulting graph by $G$. We claim that $G$ is weakly fan-planar, but not strongly fan-planar.

For the first statement, observe that $K_{7}$ admits a weakly fan-planar drawing, see Fig. 2c, and that our gadget graph $H$ has a weakly fan-planar drawing shown in Fig. 2a Combining both, we obtain a weakly fan-planar drawing of $G$.

Consider now the second statement. Let $\Gamma$ be a weakly fan-planar drawing of $G$ that is minimal and that has the smallest number of crossings among all minimal weakly fan-planar drawings of $G$. We will prove that $\Gamma$ contains at least one Pattern (III), which implies by our choice of $\Gamma$ that every weakly fan-planar drawing of $G$ requires at least one Pattern (III).

Consider a red edge $a b \in G_{1}$ and denote by $a=v_{1}, \ldots, v_{7}=b$ the vertices of the $K_{7}$ which substitute $a b$ in $G$. By Lemma 11, in the drawing $\Gamma$ of this $K_{7}$, there exists a sequence $S$ of crossed edges from $a$ to $b$; see Fig. 2c. By (I), no edge which is not incident to one of $v_{1}, \ldots, v_{7}$ can intersect $S$. By construction, the only edges incident to vertices $v_{2}, \ldots, v_{6}$ are edges of the $K_{7}$. Hence, the only edges that can potentially cross $S$ and interact with the remainder of $G$ are incident to either $a=v_{1}$ or $b=v_{7}$. Suppose for a contradiction that there exists an edge incident to $a$ or $b$ that crosses $S$ in $\Gamma$ such that its other endpoint is not one of the vertices of the $K_{7}$. But then we can easily reroute the edge such that its crossing with $S$ is avoided, see Fig. 2d, a contradiction to our choice of $\Gamma$.

We interpret $\Gamma$ as a drawing $\Gamma^{\prime}$ of $G_{1}$, where the red edges are uncrossed. Since $G_{0}$ has a unique planar embedding into $\mathbb{R}^{2}$ up to the choice of the outer face and a mirroring and since $G_{0}$ consists only of the red edges, in $\Gamma^{\prime}, G_{0}$ is drawn as a planar graph, all faces of which are quadrilaterals. Since the red edges are uncrossed, each quadrilateral face must contain a copy of our gadget $H$. Indeed, since each vertex of $H$ is connected by a path to both $u$ and $u^{\prime}$, it must lie in a face that contains both $u$ and $u^{\prime}$. But by 3 -connectivity of $G_{0}$, this face is unique.

Let $f$ be a bounded quadrilateral face of $G_{0}$. As argued above, $f$ contains a copy of $H$ in its interior in $\Gamma^{\prime}$, i.e. vertices $v, v^{\prime}, w, w^{\prime}, z$ and $z^{\prime}$, refer to Fig. 2a, lie in the interior of $f$. Consider the subgraph $H^{\prime}$ of this $H$ consisting of its red edges. Since the red edges are uncrossed, $H^{\prime}$ is drawn without crossings in $\Gamma$. Since the black edges do not cross $H^{\prime}$, a small case analysis shows that the vertices $u^{\prime}, w$, $z$, and $w^{\prime}$ must lie in the same face of $H^{\prime}$, and in fact the embedding of Fig. 2a is unique up to symmetry with respect to the vertical axis. Thus, the blue edges $(u, v),\left(u, v^{\prime}\right)$ and $\left(w, w^{\prime}\right)$ can only be drawn in the indicated way, but that means that the blue edges form (III), which concludes the proof.

## 3 Density of weakly fan-planar graphs

In this section, we show that the density results for strongly fan-planar graphs also transfer to the weakly fan-planar setting. Let us call a triple $e, e_{\ell}, e_{r}$ of edges in a weakly fan-planar drawing a heart if $e_{\ell}$ and $e_{r}$ share an endpoint $u$, both cross $e$ such that they form Pattern (III), and the part of $e$ between the crossings


Fig. 2: (a) Gadget graph $H$. (b) Gadget graph $H$ with $\left(z, v, v^{\prime}\right)$ chosen as the outer face. Note that there is no pattern (III). (c) A planarization of a fan-planar drawing of $K_{7}$, where the bold edges form a path from $a$ to $b$ that contains no uncrossed edge of the $K_{7}$. (d) An edge incident to $a$ that crosses the $K_{7}$ to avoid this crossing.
with $e_{\ell}$ and $e_{r}$ is not crossed by any edge of the graph, see Fig 3 In the remainder of the paper, we call the intersection points of $e$ and $e_{\ell}$ ( $e_{r}$, resp.) $x_{\ell}$ ( $x_{r}$, resp.).

Lemma 2. Let $\Gamma$ be a weakly fan-planar drawing that is not strongly fan-planar. Then $\Gamma$ contains a heart $\mathcal{H}$.

Proof. By assumption, $\Gamma$ contains three edges $e, e_{\ell}, e_{r}$ that form Pattern (III), where $e_{\ell}$ and $e_{r}$ share endpoint $u$ and cross $e$. Let $E^{\prime}$ be the set of edges that cross $e$. By (I) and (II), any edge $e^{\prime} \in E^{\prime}$ must be incident to $u$, and by (II) it must cross $e$ from the same side as $e_{\ell}$ and $e_{r}$. The edges of $E^{\prime}$ cannot cross each other since they share an endpoint, and each edge $e^{\prime} \in E^{\prime}$ forms Pattern (III) either with $e$ and $e_{r}$, or with $e$ and $e_{\ell}$. Let $E_{\ell} \subset E^{\prime}$ be the set of edges of the first kind, $E_{r}=E^{\prime} \backslash E_{\ell}$ the second kind. If we order $E^{\prime}$ by their crossing point with $e$ along $e$, then we first encounter all elements of $E_{\ell}$, then all elements of $E_{r}$. The last element of $E_{\ell}$ and the first element of $E_{r}$ form a heart with $e$.

We will call the sets $E_{\ell}$ and $E_{r}$ as defined in the previous proof the left valve and the right valve of the heart $\mathcal{H}=e, e_{\ell}, e_{r}$, respectively. We denote by $H$ the


Fig. 3: A heart.


Fig. 4: (a) Illustration of the setting previous to the flip-operation. (b) Transformation from $\Gamma$ to $\Gamma^{\prime}$ by flipping $E_{\ell}$.
edge set containing both valves of $\mathcal{H}$ and the edge $e$, namely $H=E_{\ell} \cup E_{r} \cup e$. In the following, we will define an edge-rerouting operation that will later allow us to reduce the number of hearts in a weakly fan-planar drawing under certain conditions.

Flipping the valve of a heart. Consider a heart $\mathcal{H}$ formed by the edges $e, e_{\ell}, e_{r}$ in a weakly fan-planar drawing $\Gamma$; refer to Fig. 4a for a visualization. In the following, we will define an operation that we call flipping a valve of $\mathcal{H}$ resulting in the drawing $\Gamma^{\prime}$. We describe the flip of $E_{\ell}$, as the other case is symmetric. The general idea is to redraw the edges in $E_{\ell}$ "close" to the ones in the other valve $E_{r}$, in particular, mainly following the curve of $e_{r}$.

Let $e_{1}^{\ell}, e_{2}^{\ell}, \ldots e_{k}^{\ell}$ be the edges of $E_{\ell}$ in the order that they intersect edge $e$ in $\Gamma$ starting at $w$, i.e. $e_{k}^{\ell}=e_{\ell}$. We will draw the curve $\gamma^{i}$ of $e_{i}^{\ell}$ in three parts, denoted as $\gamma_{1}^{i}, \gamma_{2}^{i}, \gamma_{3}^{i}$. We consider the edges in reverse order and start with $e_{k}^{\ell}=e_{\ell}$. The first part $\gamma_{1}^{k}$ of the curve of $e_{k}^{\ell}$ in $\Gamma^{\prime}$ follows the curve of $e_{r}$ slightly outside until $x_{r}$, then $\gamma_{2}^{k}$ follows $e$ until $x_{\ell}$, where the curve intersects $e$ and afterwards it inherits its original curve in $\Gamma$ as the last part $\gamma_{3}^{k}$. So, assume that we have drawn
$e_{i}^{\ell}$ with $2<i \leq k$. The curve of $e_{i-1}^{\ell}$ follows the curve of $e_{i}^{\ell}$ (slightly outside) until $x_{r}$, where it follows $e$ until the intersection point of $e_{i-1}^{l}$ with $e$ in $\Gamma$. Here, the curve intersects $e$ and then again inherits its original curve in $\Gamma$ until it reaches its endpoint different from $u$. After this operation, the edges of $E_{\ell} \cup E_{r}$ do not cross by simplicity and $e$ does not form Pattern (I) to (III) with $E_{\ell} \cup E_{r}$, thus we can make the following observation, illustrated in Fig. 4b,

Observation 1. Let $\Gamma^{\prime}$ be the drawing obtained from fipping a valve of a heart $\mathcal{H}$. Then, $\Gamma^{\prime}[H]$ is a strongly fan-planar drawing.

Moreover, in the entire drawing $\Gamma^{\prime}$ (not limited to $H$ ) resulting from a flip, new crossings can arise only in a restricted part of the flipped edges.

Lemma 3. Let $\mathcal{H}$ be a heart formed by edges $e, e_{\ell}, e_{r}$ in $\Gamma$ and $E_{\ell}=\left\{e_{1}^{\ell}, e_{2}^{\ell}, \ldots e_{k}^{\ell}\right\}$ be the edges of a flipped valve of $\mathcal{H}$ in $\Gamma^{\prime}$. Then any crossing introduced by the flipping operation occurs on the partial curves $\gamma_{1}^{1}, \gamma_{1}^{2}, \ldots \gamma_{1}^{k}$.

Proof. Let us consider a flipped edge $e_{i}^{\ell} \in E_{\ell}$ and its curve $\gamma^{i}$ in $\Gamma^{\prime}$. Clearly, any additional crossing that we introduce may only occur on the first part $\gamma_{1}^{i}$ or second part $\gamma_{2}^{i}$ of $\gamma^{i}$, as the last part $\gamma_{3}^{i}$ is inherited from $\Gamma$. By construction, $\gamma_{2}^{i}$ is crossing free, as the segment of $e$ between $x_{\ell}$ and $x_{r}$ in $\Gamma$ is crossing-free since $e_{\ell}, e_{r}$ and $e$ form a heart and by considering the edges in the reverse order that they intersect $e$ (starting at $w$ ), they do not intersect each other.

For our later proofs of Theorems 2 and 3 on the edge density of (bipartite) weakly fan-planar drawings, we need to show that a certain configuration, as described in the following lemma, cannot occur in a minimal weakly fan-planar drawing.

Lemma 4. Let $e, e_{\ell}, e_{r}$ be a heart in a minimal weakly fan-planar drawing $\Gamma$. Then there is no edge $e^{\prime} \neq e$ in $\Gamma$ that crosses both $e_{\ell}$ and $e_{r}$.

Proof. Assume for a contradiction that there exists an edge $e^{\prime} \neq e$ that intersects both $e_{\ell}$ and $e_{r}$. By (I), $e$ and $e^{\prime}$ share an endpoint, say $w$, see Fig. 5a. This implies by (I) and (II) that every edge which crosses $e_{\ell}$ or $e_{r}$ is incident to $w$. W.l.o.g. assume that $x_{\ell}$ is encountered before $x_{r}$ when traversing $e$ starting at $w$. First, notice that $e, e^{\prime}$ and $e_{r}$ can in turn form (III) themselves, refer to Fig. 5b

Based on this observation, we will define four sets of edges that will be helpful in the remainder. In particular, we construct these sets based on their drawing in $\Gamma$-while some of the edges may be redrawn at a later time, they will always belong to their corresponding sets. Let $E_{\ell}$ and $E_{r}$ be the sets of edges that correspond to the left valve and the right valve of the heart $\mathcal{H}$ induced by $e, e_{\ell}, e_{r}$-in particular, $e_{\ell} \in E_{\ell}$ and $e_{r} \in E_{r}$. Further, let $E_{t}$ be the set of edges that cross both $e_{\ell}$ and $e_{r}$ and do not form (III) with $e_{r}$ and $e$ when traversing $e_{\ell}$ starting at $v$. Complementary, let $E_{b}$ be the set of edges that also cross $e_{\ell}$ and $e_{r}$ and are not contained in $E_{t}$. Further, let $e$ be the first edge in $E_{t}$ that is encountered when traversing $e_{\ell}\left(e_{r}\right)$ starting at $u$. In the following, we will distinguish between two cases. First, if $E_{b}=\emptyset$, we call $\mathcal{H}$ a single heart. Otherwise, that is when $E_{b} \neq \emptyset$, we assume that $e^{\prime}$ is the last edge in $E_{b}$ that is encountered


Fig. 5: (a) A single heart $\mathcal{H}=e, e_{\ell}, e_{r}$. (b) A double heart formed by $\mathcal{H}=e, e_{\ell}, e_{r}$ and $\mathcal{H}^{\prime}=e_{r}, e, e^{\prime}$, the area bounded by $\mathcal{H}^{\prime}$ is indicated in gray.
when traversing $e_{\ell}$ starting at $u$ (and therefore also when traversing $e_{r}$ ). We say that $\mathcal{H}$ forms a double heart with $\mathcal{H}^{\prime}$ defined by the edges $e_{r}, e, e^{\prime}$; see Fig. 6.

The main idea is to apply the flip operation to reduce the number of hearts in the resulting (weakly fan-planar, as to be shown) drawing $\Gamma^{\prime}$ to obtain a contradiction to the minimality of $\Gamma$. For a single heart, one flip will suffice while two flips will be necessary for a double heart. In both cases, we start by flipping the valve $E_{\ell}$ as previously defined to obtain the drawing $\Gamma^{\prime}$. We first assert:
$\Gamma^{\prime}$ is weakly fan-planar. By Observation 1, any new edge triple forming Pattern (I) or (II) in $\Gamma^{\prime}$ must involve at least one edge in the flipped valve $E_{\ell}$ and at least one edge belonging to neither $E_{\ell}$ nor $E_{r}$. Consider a flipped edge $e_{i}^{\ell} \in E_{\ell}$. By Lemma3, any new crossing on the curve $\gamma^{i}$ of $e_{i}^{\ell}$ may only occur on its first part $\gamma_{1}^{i}$ in $\Gamma^{\prime}$. Any edge that intersects $\gamma_{1}^{i}$ in $\Gamma^{\prime}$ also intersects $e_{r}$ and is therefore incident to $w$ by Pattern (I). To show that we do not introduce new edge-triples forming Pattern (I), it remains to show that all edges in $E_{\ell}$ have vertex $w$ as their anchor, even if they only cross $e$ and no other edge in $E_{t}$.

We consider two cases. If $\mathcal{H}$ forms a double heart with $\mathcal{H}^{\prime}$, consider any edge $e_{b} \in E_{b}$. Since $e \in E_{t}$, we observe that each edge $e_{\ell} \in E_{\ell}$ is crossed both by $e$ and $e_{b}$; see Fig. 7a. Thus, its anchor is $w$. Otherwise, $\mathcal{H}$ is a single heart. Consider the region $\mathcal{R}$ defined by $w, x_{\ell}$ and the intersection of $e_{\ell}$ and another edge $e_{t} \in E_{t}$ in the original drawing $\Gamma$, see Fig. 7b for an illustration of this case.

By definition, every edge in $E_{\ell}$ besides $e_{\ell}$ enters $\mathcal{R}$ over $e$. If such an edge is leaving $\mathcal{R}$, then it has to also cross $e_{t}$ by simplicity, but then its anchor is $w$ and by (I) it can only be crossed by edges that are incident to $w$. Suppose now an edge of $E_{\ell}$ ends in $\mathcal{R}$, but its anchor is not $w$. Since the edge intersects $e$, its


Fig. 6: Illustration of edge sets used in (a) a single heart and (b) a double heart.
anchor is therefore $w^{\prime}$. But no edge incident to $w^{\prime}$ can enter $\mathcal{R}$, since it would either cross $e_{\ell}$, whose anchor is $w$ and hence it would coincide with $e=\left(w, w^{\prime}\right)$, it cannot cross $e$ by simplicity and if it crosses $e_{t}$, then it has to be incident to the anchor of $e_{t}$, which is $u$, but then the curve has to intersect the boundary of $\mathcal{R}$ twice which is impossible.

It remains to show that we do not introduce Pattern (II) by the flip-operation. For the sake of contradiction, assume that there is a new edge triple $T$ forming Pattern (II) after the flip. Note that $T$ involves at least one flipped edge $e_{i}^{\ell} \in E_{\ell}$. We will first establish that the anchor of $T$ is either $w$ or $u$. Suppose that $e_{i}^{\ell}$ is not incident to the anchor of $T$, then the anchor of $T$ is $w$ (since all edges in $E_{\ell}$ have $w$ as their anchor). Otherwise, the edge crossing $e_{i}^{\ell}$ in $T$ also crosses $e_{r}$, hence $u$ is the anchor of $T$. In the case that $w$ is the anchor of $T$, observe that all edges in $E_{\ell} \cup E_{r}$ are crossed from the same side by edges incident to $w$ when directed from $u$ to their other endvertex. Hence, $T$ cannot form Pattern (II). If $u$ is the anchor of $T$, it would require a non-empty region between $\gamma_{1}^{i}$ and another partial curve $\gamma_{1}^{j}$ with $j \neq i$ to form Pattern (II), which is a contradiction to our construction.

Thus, we have established that $\Gamma^{\prime}$ is indeed weakly fan-planar and we can now consider the number of Patterns (III). We distinguish two cases.

Case 1: $\mathcal{H}$ forms a single heart. By Observation 1, the drawing $\Gamma^{\prime}[H]$ does not contain any edge triple forming Pattern (III). Hence, any new Pattern (III) in $\Gamma^{\prime}$ involves either one or two flipped edges of $E_{\ell}$. In the latter case, since edges in $E_{\ell}$ do not cross by simplicity, this would require a non-empty region between the first partial curves of two edges in $E_{\ell}$, which is a contradiction to our construction.


Fig. 7: (a) Every edge in $E_{\ell}$ has anchor $w$ if $\mathcal{H}$ is part of a double heart. (b) Any edge of $E_{\ell}$ that enters region $\mathcal{R}$ has $w$ as its anchor, if $\mathcal{H}$ forms a single heart.

Suppose now that one edge of $E_{\ell}$ and two edges $e_{1}, e_{2}$ incident to $w$ form a new Pattern (III). Since the edges in $E_{\ell}$ follow the curve of $e_{r}$ after the flip, $e_{r}$ is necessarily crossed by one of the two edges incident to $w$, say $e_{1}$, in order to contain $u$ in a closed region. But then $e_{1}$ would be already present in $\Gamma$ and belong to the set $E_{b}$, as $e, e_{\ell}, e_{r}$ form Pattern (III), a contradiction to $E_{b}=\emptyset$.

We conclude that no new Pattern (III) is introduced by Observation 1 while all triples in $H$ forming (III) are eliminated, i.e., the overall number is reduced.

Case 2: $\mathcal{H}$ and $\mathcal{H}^{\prime}$ form a double heart. First, observe that all edges in the valves $E_{\ell} \cup E_{r}$ of $\mathcal{H}$ are anchored by $w$ and all edges of the valves $E_{t} \cup E_{b}$ of $\mathcal{H}^{\prime}$ are anchored by $u$. In particular, every edge forming $\mathcal{H}$ and $\mathcal{H}^{\prime}$ is contained in one of the four edge sets $E_{\ell}, E_{r}, E_{t}, E_{b}$. Suppose now that there is a third heart $\mathcal{H}^{\prime \prime}$ distinct from $\mathcal{H}$ that forms a double heart with $\mathcal{H}^{\prime}$. Hence, there are crossings between the valves of $\mathcal{H}^{\prime}$ and $\mathcal{H}^{\prime \prime}$. It follows, that the edges in the valves of $\mathcal{H}^{\prime \prime}$ are necessarily incident to $u$ by (I), but then all crossings between edges of $H^{\prime}$ and $H^{\prime \prime}$ occur from the same side as between edges of $H$ and $H^{\prime}$, as otherwise Pattern (II) would be present in $\Gamma$. It follows that the hearts $\mathcal{H}$ and $\mathcal{H}^{\prime \prime}$ are identical. Thus, we can consider $\mathcal{H}$ and $\mathcal{H}^{\prime}$ symmetrically, since in a double heart one always has the other as its counterpart. In this case, the flip of $E_{\ell}$ can indeed form new Patterns (III). However, as discussed in the previous case, this can only occur when a flipped edge $e_{\ell} \in E_{\ell}$ forms a heart with an edge $e_{t} \in E_{t}$ and $e_{b} \in E_{b}$. Thus, all new Patterns (III) are contained in the single heart $\mathcal{H}^{\prime}$ where $E_{r}$ takes the role of $E_{t}$ in the discussion of the previous case; see Fig. 8a. We can now proceed as in the previous case by flipping $E_{b}$ which eliminates all triples forming Pattern (III) as $E_{\ell}=\emptyset$; see Fig. 8b.

So far, we investigated minimal weakly fan-planar drawings and their properties. One last ingredient is needed to prove Theorem 2. Given a graph $G$ with a minimal weakly fan-planar drawing, we will sometimes reduce the number of Patterns (III) by modifying $G$ into a different, weakly fan-planar graph $G^{\prime}$ with


Fig. 8: Illustration of second flip operation in the double heart case in Lemma 4 .
the same number of vertices and with the same number of edges. The following lemma states when this modification is possible; see Appendix A for the proof:

Lemma 5. Let $e_{\ell}, e_{r}$ and $e=\left(w, w^{\prime}\right)$ be a heart $\mathcal{H}$ in a minimal weakly fanplanar drawing $\Gamma$ of a graph $G=(V, E)$, let $\mathcal{L}$ be the closed curve that consists of the partial curves of the edges $e, e_{\ell}, e_{r}$ up to the crossing points $x_{\ell}$ and $x_{r}$ and let $p_{1}, p_{2}$ be two common neighbors of $w$ and $w^{\prime}$ outside of $\mathcal{L}$. Then, there is a graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ with $\left|V^{\prime}\right|=|V|$ and $\left|E^{\prime}\right|=|E|$ that admits a fan-planar drawing $\Gamma^{\prime}$ with fewer edge triples forming Pattern (III) than $\Gamma$ contains.

We are now ready to prove our main theorem:
Theorem 2. A weakly fan-planar graph $G$ with $n$ vertices has at most $5 n-10$ edges.

Proof. We proceed by induction on the number of edge triples forming Pattern (III) in a minimal weakly fan-planar drawing of $G$. In the base case, this number is zero, so the drawing is strongly fan-planar. Then $G$ is strongly fanplanar, and has at most $5 n-10$ edges by [13]. For the induction step, consider a graph $G$ and let $\Gamma$ be a minimal weakly fan-planar drawing of $G$.

By Lemma 2, $\Gamma$ contains a heart $e=\left(w, w^{\prime}\right), e_{\ell}, e_{r}$, see Fig. 9. If $w$ and $w^{\prime}$ have at least two common neighbors outside of $\mathcal{L}$, we obtain a graph $G^{\prime}$ with the same number of vertices and edges with fewer edge triples forming (III) as stated by Lemma 5 and proceed with $G^{\prime}$.

Otherwise, denote by $G_{1}$ the subgraph of $G$ consisting of those vertices and edges of $G$ that lie (entirely) in the bounded closed region bounded by $\mathcal{L}$. In


Fig. 9: Illustrations for the proof of Theorem 2.
particular, the vertices $u, w, v, v^{\prime}$, and $w^{\prime}$, and the edges $e_{\ell}, e_{r}$, and $e$ all belong to $G_{1}$. Similarly, let $G_{2}$ be the subgraph of $G$ consisting of those vertices and edges of $G$ that lie entirely in the unbounded closed region bounded by $\mathcal{L}$. In particular, vertex $u \in G_{2}$, but none of the edges $e, e_{\ell}, e_{r}$ is in $G_{2}$. Let $\left|V\left[G_{2}\right]\right|=r$ and thus $\left|V\left[G_{1}\right]\right|=n-(r-1)$, as $u$ is part of both $G_{1}$ and $G_{2}$.

Note that the graph $G$ contains edges that are neither in $G_{1}$ nor in $G_{2}$. We will show how to augment $G_{2}$ to $G_{2}^{\prime}$ so that it contains an equal number of extra edges. We will create new weakly fan-planar drawings $\Gamma_{1}^{\prime}$ and $\Gamma_{2}^{\prime}$ for $G_{1}$ and $G_{2}^{\prime}$ that have at least one fewer edge triple forming Pattern (III) each.

We start with $G_{1}$. Let $E_{\ell}$ and $E_{r}$ be the left valve and the right valve of our heart, respectively, such that $e_{\ell} \in E_{\ell}$ and $e_{r} \in E_{r}$ holds. Recall that $G_{1}$ lies entirely in the bounded region $\mathcal{L}$ defined by the partial curves of $e_{\ell}, e_{r}$ and $e$ up to their respective intersection points. In particular, this implies that the partial curve of $e_{\ell}$ between $u$ and $x_{\ell}$ is crossing free. Moreover, the partial curve of $e$ between $x_{\ell}$ and $x_{r}$ is also crossing free as $e_{\ell}, e_{r}$ and $e$ form a heart. Based on this observation, we flip the left valve $E_{\ell}$, obtaining $\Gamma_{1}^{\prime}$; see Fig. 10a.
$\Gamma_{1}^{\prime}$ is weakly fan-planar and contains fewer Pattern (III). To show that $\Gamma_{1}^{\prime}$ is weakly fan-planar and contains at least one Pattern (III) less than the drawing of $G_{1}$ in $\Gamma$, fix an edge $e_{i}^{\ell} \in E_{\ell}$ and let $\gamma^{i}$ be its curve in $\Gamma_{1}^{\prime}$. Recall that by Lemma 3, any new crossings of $e_{i}^{\ell}$ can only occur on the partial curve $\gamma_{1}^{i} \subset \gamma^{i}$. Since no two edges of $E_{\ell}$ intersect in $\Gamma_{1}^{\prime}$ by construction and any other edge that would intersect $\gamma^{i}$ would also cross $e_{r}$ in $\Gamma$ and thus be contained both inside and outside $\mathcal{L}$ by Lemma $4, \gamma_{1}^{i}$ is uncrossed in $\Gamma_{1}^{\prime}$. Hence, all crossings of $e_{i}^{\ell} \in E_{\ell}$ are on the part of $\gamma^{i}$ which was inherited from $\Gamma$; it follows that no new Pattern (I), (II), and (III) is introduced in $\Gamma_{1}^{\prime}$.

Hence, our new weakly fan-planar drawing $\Gamma_{1}^{\prime}$ has $\left|E_{\ell}\right| \times\left|E_{r}\right| \geq 1$ Pattern (III) less than $\Gamma$, and we can apply the inductive assumption to get


Fig. 10: Illustrations for the proof of Theorem 2

$$
\left|E\left(G_{1}\right)\right| \leq 5\left|V\left(G_{1}\right)\right|-10
$$

Now we consider $G_{2}$ and the edges that are neither in $G_{1}$ nor in $G_{2}$.

Edges that are neither in $G_{1}$ nor in $G_{2}$. Consider such an edge $e^{\prime}$. Clearly, $e^{\prime}$ crosses $\mathcal{L}$. This crossing cannot be on $e$ by the heart property, so it must be on $e_{\ell}$ or $e_{r}$. The edge $e^{\prime}$ cannot cross both $e_{\ell}$ and $e_{r}$ by Lemma 4 so $e^{\prime}$ either crosses $e_{\ell}$ and must be incident to $w$ or crosses $e_{r}$ and must then be incident to $w^{\prime}$ by (I) and (II). We claim that $e^{\prime}$ can only cross edges incident to $u$ outside of $\mathcal{L}$; see the light gray edges in Fig. 9. To see this, suppose for a contradiction that $e^{\prime}$ is crossed by an edge $e^{\prime \prime}$ outside of $\mathcal{L}$ which is not incident to $u$. W.l.o.g. assume that $e^{\prime}$ is incident to $w$ and crosses $e_{\ell}$, the other case is symmetric. By ( I ), $e^{\prime \prime}$ is then necessarily incident to $v$, which is contained inside $\mathcal{L}$. However, we already established that only edges such as $e^{\prime}$ which are incident to $w$ or $w^{\prime}$ can leave $\mathcal{L}$, a contradiction to $e^{\prime \prime}$ crossing $e^{\prime}$ outside of $\mathcal{L}$.

Augmenting $G_{2}$. Let $k$ be the number of edges of $G$ neither in $G_{1}$ nor in $G_{2}$. Note that $w$ and $w^{\prime}$ have at most one common neighbor outside of $\mathcal{L}$ (as otherwise we have already applied Lemma 5). Then, we construct a new graph $G_{2}^{\prime}$ from $G_{2}$ by adding edges as follows. Recall that the weakly fan-planar drawing $\Gamma\left[G_{2}\right]$ derived from $\Gamma$ contains an empty region inside $\mathcal{L}$ and contains fewer triples forming Pattern (III), as we removed the heart formed by $e, e_{\ell}, e_{r}$.

We insert a single vertex $v^{*}$ inside this region, and connect it to all neighbors of $w$ and $w^{\prime}$ in $G_{2}$, and to $u$; see Fig. 10b By assumption, $w$ and $w^{\prime}$ share at most one such vertex and hence there are at least $k-1$ neighbors. Recall that any such edge crosses only edges incident to $u$ outside of $\mathcal{L}$ and therefore fan-planarity is


Fig. 11: Illustrations for the proof of Theorem 3
maintained. Hence, we augmented $G_{2}$ to the weakly fan-planar graph $G_{2}^{\prime}$ that contains $r+1$ vertices and onto which we can apply the induction hypothesis.
We can now bound $\mathrm{E}(\mathrm{G})$ as follows:

$$
\begin{aligned}
|E(G)| & =\left|E\left(G_{1}\right)\right|+\left|E\left(G_{2}\right)\right|+k=\left|E\left(G_{1}\right)\right|+\left|E\left(G_{2}^{\prime}\right)\right| \\
& \leq 5(n-r+1)-10+5(r+1)-10=5 n-10
\end{aligned}
$$

which concludes the proof.
For bipartite graphs, we proceed in a similar way.
Theorem 3. An n-vertex bipartite weakly fan-planar graph has at most $4 n-12$ edges.

Proof. Let $\Gamma$ be a minimal weakly fan-planar drawing of $G$. We again proceed by induction on the number of edge triples forming Pattern (III). In the base case, $\Gamma$ is strongly fan-planar and hence $G$ has at most $4 n-12$ edges by [1]. In the induction step, we proceed as in the previous proof with two differences: First, since $G$ is bipartite, $w$ and $w^{\prime}$ cannot have a common neighbor, so we do not need to apply Lemma 5. Second, when constructing $G_{2}^{\prime}$, instead of inserting a single vertex $v^{*}$, we insert two vertices $v_{a}$ and $v_{b}$, and connect $v_{a}$ with the neighbors of $w$ in $G_{2}, v_{b}$ with the neighbors of $w^{\prime}$ in $G_{2}$; thus maintaining bipartiteness; see Fig. 11b, In total, we get

$$
\begin{aligned}
|E(G)| & =\left|E\left(G_{1}\right)\right|+\left|E\left(G_{2}\right)\right|+k=\left|E\left(G_{1}\right)\right|+\left|E\left(G_{2}^{\prime}\right)\right| \\
& \leq 4(n-r+1)-12+4(r+2)-12=4 n-12
\end{aligned}
$$

which concludes the proof.

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Fig. 12: (a) Regions $\mathcal{K}$ and $\mathcal{L}$ and vertices $w_{p}$ and $w_{s}$. (b) Replacement of $e$.

## A Proof of Lemma 5

Lemma 5. Let $e_{\ell}, e_{r}$ and $e=\left(w, w^{\prime}\right)$ be a heart $\mathcal{H}$ in a minimal weakly fanplanar drawing $\Gamma$ of a graph $G=(V, E)$, let $\mathcal{L}$ be the closed curve that consists of the partial curves of the edges $e, e_{\ell}, e_{r}$ up to the crossing points $x_{\ell}$ and $x_{r}$ and let $p_{1}, p_{2}$ be two common neighbors of $w$ and $w^{\prime}$ outside of $\mathcal{L}$. Then, there is a graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ with $\left|V^{\prime}\right|=|V|$ and $\left|E^{\prime}\right|=|E|$ that admits a fan-planar drawing $\Gamma^{\prime}$ with fewer edge triples forming Pattern (III) than $\Gamma$ contains.

Proof. To prove the claim, will remove $e$, which in turn removes all triples of Pattern (III) of $\mathcal{H}$, and add another edge $\left(y_{\mathcal{K}}, y_{\mathcal{L}}\right) \notin E$ to account for the missing edge, maintaining weak fan-planarity in the resulting drawing $\Gamma^{\prime}$ of $G^{\prime}$.

Assume w.l.o.g. that $\mathcal{H}$ is chosen so that $\mathcal{L}$ does not contain any other heart $\mathcal{H}^{\prime}$. We aim to find a sequence of curve-segments in $\Gamma$, along which we can insert the edge $\left(y_{\mathcal{K}}, y_{\mathcal{L}}\right)$ without introducing Pattern (I) to (III). We refer to the curve constructed in this way as $\gamma$. Since the edges of $w$ and $w^{\prime}$ to $p_{1}$ and $p_{2}$ both cross edges incident to $u$ they cannot cross each other. Hence there exists a nesting of $p_{1}$ and $p_{2}$, w.l.o.g. assume that $p_{1}$ is the inner one, i.e., the edge $\left(w, p_{1}\right)$ is encountered before the edge $\left(w, p_{2}\right)$ when following $e_{\ell}$ starting at $v$. In other words, $p_{1}$ is contained inside a region which we denote as $\mathcal{K}$ that is delimited by $e, e_{\ell}, e_{r},\left(w, p_{2}\right)$ and $\left(w^{\prime}, p_{2}\right)$. Further, denote by $w_{p}$ and $w_{s}$ the predecessor and successor of the edge $e$, i.e., the edge $e_{p}=\left(w, w_{p}\right)$ crosses $e_{\ell}$ immediately before $e$ and $e_{s}=\left(w, w_{s}\right)$ immediately after $e$ when following $e_{\ell}$ starting at $u$. Note that, possibly $w_{p}=p_{1}$ and $w_{s}$ NIL; see Fig. 12a.

We first identify a suitable vertex $y_{\mathcal{K}}$ and crossing-free subcurve of $\gamma$ that connects $y_{\mathcal{K}}$ with the segment of $e$ between $x_{\ell}$ and $x_{r}$. Consider $w_{p}$. By Lemma 4 , $\left(w, w_{p}\right)$ cannot cross both $e_{\ell}$ and $e_{r}$, hence $w_{p}$ lies outside of $\mathcal{L}$. Further, since edges of $w$ and $w^{\prime}$ do not intersect, $w_{p}$ is contained inside $\mathcal{K}$. Let $x_{p}$ be the


Fig. 13: Identification of $y_{\mathcal{K}}$.
crossing point of $e_{p}$ and $e_{\ell}$ in $\Gamma$. Observe that as $e_{p}$ is the predecessor of $e$, the curve-segment of $e_{\ell}$ between $x_{\ell}$ and $x_{p}$ is crossing-free, hence we add it to $\gamma$. To identify the endpoint $y$, consider the curve-segment $\gamma_{p}$ of $e_{p}$ between $x_{p}$ and $w_{p}$.

First, assume that $\gamma_{p}$ is crossing-free. In this case, we identify $y_{\mathcal{K}}$ with $w_{p}$ and append $\gamma_{p}$ to $\gamma$; see Fig. 13a.

Second, assume that $\gamma_{p}$ has at least one crossing. Let $e_{1}$ be the first edge crossing $e_{p}$ when traversing $\gamma_{p}$ from $x_{p}$. Let $x_{1}$ be the intersection of $e_{1}$ and $e_{p}$. Observe that the segment from $x_{1}$ to $x_{p}$ on $e_{p}$ is crossing-free and hence can be appended to $\gamma$. Since $\left(x_{\ell}, x_{r}\right)$ is crossing-free by definition, the anchor of $e_{p}$ must be $u$. Hence, $e_{1}$ also intersects the edge $\left(w, p_{2}\right)$, its anchor is $w$, and its endpoint $w_{1}$ lies inside $\mathcal{K}$ (its other endpoint is $u$ ). Finally, we observe that the curve $\gamma_{1}$ of $e_{1}$ between $x_{1}$ and $w_{1}$ is crossing free, since otherwise $\left(w, w_{p}\right)$ is not the predecessor of $e$ at $w$. Thus, we fix $w_{1}$ as $y_{\mathcal{K}}$ and append $\gamma_{1}$ to $\gamma$; see Fig. 13b.

Observe that in both cases $y_{\mathcal{K}}$ lies in $\mathcal{K}$. Also note that $y_{\mathcal{K}}$ can only be adjacent to $w$ and $w^{\prime}$ inside $\mathcal{L}$ as the segment of $e$ between $x_{\ell}$ and $x_{r}$ is uncrossed whereas $w$ and $w^{\prime}$ are the anchors of $e_{\ell}$ and $e_{r}$, respectively. Hence, we can be sure that $\left(y_{\mathcal{K}}, y_{\mathcal{L}}\right)$ does not exist in $G$ yet as long as $y_{\mathcal{L}} \notin\left\{w, w^{\prime}, u\right\}$ and $y_{\mathcal{L}}$ lies in $\mathcal{L}$; see Fig. 12b. Thus, it remains to extend $\gamma$ starting from its intersection with $e$ to a suitable vertex $y_{\mathcal{L}}$ that is contained in $\mathcal{L}$.

Consider the curve $\gamma_{\ell}$ of $e_{\ell}$ from $x_{\ell}$ to $v$. First, if $\gamma_{\ell}$ is crossing-free, we identify $y_{\mathcal{L}}$ with $v$ and add $\gamma_{\ell}$ to $\gamma$; see Fig. 14a.

Otherwise, $\gamma_{\ell}$ contains a crossing. Assume momentarily that $\gamma_{\ell}$ is crossed by an edge incident to $w^{\prime}$. In this scenario, we repeat the argumentation for $w^{\prime}$ for which this case cannot occur at the same time. Thus, we conclude that $e_{s}$ must be present. Denote by $x_{s}$ the crossing between $e_{\ell}$ and $e_{s}$. We add the curve segment between $x_{\ell}$ and $x_{s}$ to $\gamma$.

If the curve of $e_{s}$ between $w_{s}$ and $x_{s}$ is crossing-free, we add it to $\gamma$ and identify $w_{s}$ with $y_{\mathcal{L}}$; see Fig. 14b

Note that in all cases discussed so far, $\gamma$ is crossing-free when removing $e$. Thus, when replacing $e$ with $\left(y_{\mathcal{K}}, y_{\mathcal{L}}\right)$, the number of (I) stays at most the same


Fig. 14: Identification of $y_{\mathcal{L}}$.
while the number of (III) decreases by at least 1 ; see Fig. 12 b (in this case the dotted edge at $w$ does not exist).

Finally, it remains to consider the case where there is an edge $e_{1}^{\prime}$ crossing $e_{s}$. We choose $e_{1}^{\prime}$ such that it is the first edge crossing $e_{s}$ after $x_{s}$ when traversing $e_{s}$ from $x_{s}$ to $w_{s}$. Note that the anchor of $e_{s}$ is either $v$ or $u$. If it is $u$, we observe that $e_{\ell}, e_{s}$ and $e_{1}^{\prime}$ form a heart inside $\mathcal{L}$ (see Fig. 14c); a contradiction to the choice of $\mathcal{H}$. Thus, the anchor of $e_{s}$ must be $v$. In this case, we again choose $y_{\mathcal{L}}=v$ and append the segment between $x_{s}$ and $v$ to $\gamma$. I this case $\gamma$ will cross $\left(w, w_{s}\right)$. However, the anchor of $\left(w, w_{s}\right)$ is $y_{\mathcal{L}}=v$ so the number of (I) is maintained. Moreover, since $\gamma$ intersects $\left(w, w_{s}\right)$ in between $e_{\ell}$ and $e_{1}^{\prime}$ the number of (III) still decreases by at least 1 when replacing $e$ with $\left(y_{\mathcal{K}}, y_{\mathcal{L}}\right)$; see Fig. 12b


[^0]:    ${ }^{3}$ In a planarization $\mathcal{P}$ of a non-planar drawing of a graph $G$, each crossing is replaced with a dummy-vertex that subdivides both edges involved in the crossing. We call an edge of $\mathcal{P}$ that is not incident to any dummy-vertex a real edge of $G$.

