# Generic Model Checking for Modal Fixpoint Logics in COOL-MC

 $\begin{array}{c} {\rm Daniel\; Hausmann^{[0000-0002-0935-8602]1\star},\; Merlin} \\ {\rm Humml^{[0000-0002-2251-8519]2\star\star},\; Simon\; Prucker^{[0009-0000-2317-5565]2},\; Lutz} \\ {\rm Schr\"{o}der^{[0000-0002-3146-5906]2\star\star\star},\; and\; Aaron\; Strahlberger^2} \end{array}$ 

 $^{1}$  University of Gothenburg, Sweden  $^{2}$  Friedrich-Alexander-Universität Erlangen-Nürnberg, Germany

Abstract. We report on COOL-MC, a model checking tool for fix-point logics that is parametric in the branching type of models (non-deterministic, game-based, probabilistic etc.) and in the next-step modalities used in formulae. The tool implements generic model checking algorithms developed in coalgebraic logic that are easily adapted to concrete instance logics. Apart from the standard modal  $\mu$ -calculus, COOL-MC currently supports alternating-time, graded, probabilistic and monotone variants of the  $\mu$ -calculus, but is also effortlessly extensible with new instance logics. The model checking process is realized by polynomial reductions to parity game solving, or, alternatively, by a local model checking algorithm that directly computes the extensions of formulae in a lazy fashion, thereby potentially avoiding the construction of the full parity game. We evaluate COOL-MC on informative benchmark sets.

**Keywords:** Model checking  $\cdot$  parity games  $\cdot$   $\mu$ -calculus  $\cdot$  lazy evaluation

#### 1 Introduction

The  $\mu$ -calculus [24] is one of the most expressive logics for the temporal verification of concurrent systems. Model checking the  $\mu$ -calculus is equivalent to parity game solving, and as such enjoys diversified tool support in the shape of both well-developed parity game solving suites such as PGSolver [11,38] or Oink [7] and dedicated model checking tools such as mCRL2 [3]. While the  $\mu$ -calculus is standardly interpreted over relational transition systems, a wide range of alternative flavours have emerged whose semantics is variously based on concurrent games as in the alternating-time  $\mu$ -calculus [2]; on probabilistic transition systems as in the (two-valued) probabilistic  $\mu$ -calculus [5,29,4]; on counting successors as in the graded  $\mu$ -calculus [25]; or on neighbourhood structures as in the

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monotone  $\mu$ -calculus, the ambient fixpoint logic of game logic [31,33,8]. Model checking tools for such  $\mu$ -calculi are essentially non-existent or limited to fragments (see additional comments under 'related work'). We present the generic model checker COOL-MC, which implements generic model checking algorithms for the coalgebraic  $\mu$ -calculus [5] developed in previous work [22]. The coalgebraic  $\mu$ -calculus is based on the semantic framework of coalgebraic logic, which treats systems generically as coalgebras for a set functor encapsulating the system type, following the paradigm of universal coalgebra [35], and parametrizes the semantics of modalities using so-called *predicate liftings* [32,36]. By fairly simple instantiation to concrete logics, COOL-MC thus serves as the first available model checker for the probabilistic  $\mu$ -calculus, the graded  $\mu$ -calculus, and the full alternating-time  $\mu$ -calculus AMC (model checking tools for alternatingtime temporal logic ATL, a fragment of the AMC, do exist, as discussed further below). Besides presenting the tool itself and discussing implementation issues, we conduct an experimental evaluation of COOL-MC on benchmark series of parity games [11,38] that we adapt to the generalized coalgebraic setting. We thus show that COOL-MC scales even on series of problems designed to be hard in the relational base case.

Related Work As mentioned above, COOL-MC is the only currently available model checker for most of the logics that it supports, other than the standard modal  $\mu$ -calculus (and the main point of its genericity is that support for further logics can be added easily). We refrain from benchmarking COOL-MC against modal  $\mu$ -calculus model checkers (e.g. mCRL2 [3]) as this would essentially amount to comparing the respective backend parity game solvers. Model checking tools for alternating-time temporal logic ATL [2] do exist, such as MOCHA [1], MCMAS [30], and UMC4ATL [23], out of which MCMAS appears to be the fastest one currently available [23]. We do compare COOL-MC to MCMAS on two benchmarks, confirming that MCMAS is faster on ATL. Note however that ATL model checking works along essentially the same lines as for CTL, and as such is much simpler than model checking the alternating-time μ-calculus AMC (e.g., it does not require parity conditions, and unlike AMC model checking it is known to be in PTIME [2]), so it is expected that dedicated ATL model checkers will be faster than an AMC model checker like COOL-MC on ATL. Local solving has been shown to be advantagous in model checking for the relational  $\mu$ -calculus [37] and for standard parity games [12].

COOL-MC uses the basic infrastructure, such as parsers and data structures for formulae, of the Coalgebraic Ontology Logic Solver (COOL/COOL 2) [13,14], a generic reasoner aimed at satisfiability checking rather than model checking. The algorithms we use here [22] improve on either the theoretical complexity or the complexity analysis of previous model checking algorithms for concrete instance logics including the alternating-time [2], graded [10], and monotone [16]  $\mu$ -calculi, as well as of a previous generic model checking algorithm for the coalgebraic  $\mu$ -calculus [18]; see [22] for details.

## 2 Model Checking for the Coalgebraic $\mu$ -Calculus

We briefly recall the syntax and semantics of the the underlying generic logic of COOL-MC, the coalgebraic  $\mu$ -calculus [5], and subsequently sketch two different model checking algorithms implemented for this logic in COOL-MC, a local algorithm that directly computes extensions of formulae, and a more global algorithm that reduces instances of the model checking problem to parity games [22].

Syntax. Formulae of the coalgebraic  $\mu$ -calculus are given by the following grammar, parametrized over a choice of countable sets  $\Lambda$  and V of modalities and (fixpoint) variables, respectively.

$$\varphi, \psi ::= \top \mid \bot \mid \varphi \wedge \psi \mid \varphi \vee \psi \mid \Diamond \varphi \mid X \mid \nu X. \varphi \mid \mu X. \varphi$$

where  $X \in \mathsf{V}$ ; we assume that  $\Lambda$  contains, for each modality  $\heartsuit \in \Lambda$ , also the dual  $\overline{\heartsuit}$  (with  $\overline{\heartsuit} = \heartsuit$ ). The logic generalizes the standard  $\mu$ -calculus by supporting arbitrary monotone modalities  $\heartsuit$  in place of  $\diamondsuit$  and  $\square$ , assuming that the semantics of  $\heartsuit$  can be defined in the framework of coalgebraic logic, recalled below. To ensure monotonicity, the logic does not contain negation as an explicit operator; however, negation of closed formulae can, as usual, be defined via negation normal forms. Given a formula  $\varphi$ , we let  $|\varphi|$  denote its syntactic size. The algorithms we use work on the (Fischer-Ladner) closure  $\mathsf{cl}(\varphi)$  of  $\varphi$ , a succinct graph representation of the respective formula, intuitively obtained from its syntax tree by identifying occurrences of fixpoint variables with their binding fixpoint operators; we have  $|\mathsf{cl}(\varphi)| \leq |\varphi|$  [24]. The alternation-depth  $\mathsf{ad}(\varphi)$  of fixpoint formulae  $\varphi = \eta X. \psi$  is defined in the usual way as the number of dependent alternations between least and greatest fixpoints in  $\varphi$ ; for a detailed account, see [26].

Semantics. The semantics of the coalgebraic  $\mu$ -calculus is parametrized over the choice of a set functor  $\mathcal{F}$  that encapsulates the branching type of systems, e.g. nondeterministic ( $\mathcal{F}X = \mathcal{P}X$ , the powerset of X) or probabilistic ( $\mathcal{F}X = \mathcal{D}X$ , the set of discrete probability distributions on X). Formulae are then evaluated over coalgebras ( $C, \xi : C \to \mathcal{F}C$ ) for  $\mathcal{F}$ , that is, over generalized transition systems consisting of a set C of states and transition function  $\xi$  that associates to each state c a collection  $\xi(c) \in \mathcal{F}C$  of observations and successors, structured according to  $\mathcal{F}$ . For the most basic case, we can pick  $\mathcal{F} = \mathcal{P}$  to be the powerset functor, so that  $\mathcal{F}$ -coalgebras are standard transition systems, with  $\xi(c) \in \mathcal{P}C$  being the set of successor states of c.

The semantics of modalities  $\heartsuit \in \Lambda$  is defined in terms of so-called *predicate liftings*, that is, functions  $\llbracket \heartsuit \rrbracket$  that lift predicates  $D \subseteq C$  on C to predicates  $\llbracket \heartsuit \rrbracket (D) \in \mathcal{P}(\mathcal{F}C)$  on  $\mathcal{F}C$ . A state  $c \in C$  in a coalgebra  $(C, \xi)$  then satisfies a formula  $\heartsuit \psi$  if  $\xi(c) \in \llbracket \heartsuit \rrbracket (\llbracket \psi \rrbracket)$  where  $\llbracket \psi \rrbracket$  is the set of states that satisfy  $\psi$ .

This concept instantiates to the standard modalities  $\Diamond$  and  $\square$  over transition systems (that is, over coalgebras  $(C, \xi : C \to \mathcal{P}C)$  for the functor  $\mathcal{P}$ ) by taking predicate liftings

$$\llbracket \lozenge \rrbracket(D) = \{ E \in \mathcal{P}C \mid E \cap D \neq \emptyset \} \qquad \llbracket \square \rrbracket(D) = \{ E \in \mathcal{P}C \mid E \subseteq D \}.$$

For another example, consider graded modalities of the shape  $\langle n \rangle$  and [n] (for  $n \in \mathbb{N}$ ), expressing that more than n successors or all but at most n successors, respectively, satisfy the argument formula. We interpret such modalities over graded transition systems, in which every transition from one state to another is equipped with a non-negative integer multiplicity; these are coalgebras  $(C, \xi: C \to \mathcal{G}C)$  for the multiset functor  $\mathcal{G}$  that maps a set X to the set  $\mathcal{G}X$  of finite multisets over X, represented as maps  $X \to \mathbb{N}$  with finite support [6]. For  $\theta: C \to \mathbb{N}$  and  $D \subseteq C$ , we put  $\theta(D) = \Sigma_{d \in D} \theta(d)$ , and interpret  $\langle n \rangle$ , [n] as the predicate liftings

$$[\![\langle n\rangle]\!](D) = \{\theta \in \mathcal{G}C \mid \theta(D) > n\} \qquad [\![[n]]\!](D) = \{\theta \in \mathcal{G}C \mid \theta(C \setminus D) \leq n\}.$$

Having defined the semantics of single modal steps, we now extend the semantics to the full logic, introducing the game-based semantics of the coalgebraic  $\mu$ -calculus (which is equivalent to a recursively defined algebraic semantics [39,22,21]). To treat least and greatest fixpoints correctly, this semantics uses parity games, which are infinite-duration games played by two players  $\exists$  and  $\forall$ . A parity game  $G = (V, V_{\exists}, E, \Omega)$  consists of a set V of positions, with positions  $V_{\exists} \subseteq V$  owned by  $\exists$  and the others by  $\forall$ , a move relation  $E \subseteq V \times V$ , and a priority function  $\Omega \colon V \to \mathbb{N}$  that assigns a natural number  $\Omega(v)$  to each position  $v \in V$ . A play is a path in the directed graph (V, E) that is either infinite or ends in a node  $v \in V$  with no outgoing moves. Finite plays  $v_0v_1 \dots v_n$  are won by  $\exists$  if and only if  $v_n \in V_{\forall}$  (i.e. if  $\forall$  is stuck); infinite plays are won by  $\exists$  if and only if the maximal priority that is visited infinitely often is even. A (history-free)  $\exists$ -strategy is a partial function  $s \colon V_{\exists} \rightharpoonup V$  that assigns moves to  $\exists$ -nodes. A play follows a strategy s if for all s if s such that s if s if s and follow s.

For the remainder of the paper, we fix a functor  $\mathcal{F}$ , an  $\mathcal{F}$ -coalgebra  $(C, \xi)$ , a set  $\Lambda$  of modalities with associated monotone predicate liftings, and a formula  $\chi$  (that uses modalities from  $\Lambda$ ); further we let  $\mathsf{cl} = \mathsf{cl}(\chi)$  denote the closure of  $\chi$ , and put  $n := |\mathsf{cl}|$  and  $k := \mathsf{ad}(\chi)$ .

**Definition 1.** The model checking game  $G_{(C,\xi),\chi} = (V, V_{\exists}, E, \Omega)$  is the parity game defined by the following table, where game nodes  $v \in V = V_{\exists} \cup V_{\forall}$  are of the shape  $v = (c, \psi) \in C \times \text{cl}$  or  $v = (D, \psi) \in \mathcal{P}(C) \times \text{cl}$ .

| node                       | owner     | set of allowed moves  |
|----------------------------|-----------|---|
| $(c,\top)$                 | $\forall$ | Ø   |
| $(c, \perp)$               | ∃         | Ø   |
| $(c, \varphi \wedge \psi)$ | $\forall$ | $\{(c,\varphi),(c,\psi)\}$  |
| $(c, \varphi \lor \psi)$   | 3         | $\{(c, \varphi), (c, \psi)\}$                                       |
| $(c, \eta X. \psi)$        | 3         | $\{(c,\psi[\eta X.\psi/X])\}$                                       |
| $(c, \heartsuit \psi)$     | 3         | $\{(D,\psi) \mid \xi(c) \in \llbracket \heartsuit \rrbracket(D) \}$ |
| $(D,\psi)$                 | $\forall$ | $\{(d,\psi)\mid d\in D\}$   |

In order to show satisfaction of  $\heartsuit \psi$  at  $c \in C$ , player  $\exists$  thus has to claim satisfaction of  $\psi$  at a sufficiently large set  $D \subseteq C$  of states; player  $\forall$  in turn can challenge the satisfaction of  $\psi$  at any node  $d \in D$ .

As usual in  $\mu$ -calculi, the priority function  $\Omega$  serves to detect that the outermost fixpoint that is unfolded infinitely often is a greatest fixpoint. It is thus defined ensuring that for nodes  $(c, \varphi)$ ,  $\Omega(c, \varphi)$  is even if  $\varphi = \nu X. \psi$ , odd if  $\varphi = \mu X. \psi$ , and  $\Omega(c, \varphi) = 0$  otherwise, and moreover that larger numbers are assigned to outer fixpoints, using the alternation depth of fixpoints. The formal definition of  $\Omega$  follows the standard method, see e.g. [26].

We say that  $c \in C$  satisfies  $\chi$  (denoted  $(C, \xi), c \models \chi$ ) if and only if player  $\exists$  wins the position  $(c, \chi)$  in  $G_{(C, \xi), \chi}$ . The model checking problem for the coalgebraic  $\mu$ -calculus consists in deciding, for state  $c \in C$  in a coalgebra  $(C, \xi)$ , and formula  $\chi$  of the coalgebraic  $\mu$ -calculus, whether  $(C, \xi), c \models \chi$ .

We point out that  $G_{(C,\xi),\chi}$  is a parity game with k priorities that contains up to  $n \cdot 2^{|C|}$  positions of the form  $(D,\psi)$  for  $D \subseteq C$ . Therefore it is not feasible to perform model checking by explicitly constructing and solving this parity game. In previous work [22,20], we have shown that the model checking problem for the coalgebraic  $\mu$ -calculus is in NP  $\cap$  CO-NP and in QP (under mild assumptions on the complexity of evaluating single modal steps using the predicate liftings), providing two methods to circumvent the explicit construction of the full game:

- 1. Compute the winning region in  $G_{(C,\xi),\chi}$  as a nested fixpoint over the set of positions of the shape  $(c,\psi)$ ; intuitively, this avoids the explicit construction of the intermediate positions of the shape  $(D,\psi)$  by directly computing the extension of subformulae over C. This solution is generic in the sense that it works for any instance of the coalgebraic  $\mu$ -calculus.
- 2. Provide a polynomial-sized game-characterization of the modalities of the concrete logic at hand, enabling a polynomial reduction of the model checking problem to solving parity games. This makes it possible to use parity game solvers, but relies on the logic-specific game characterization of the modalities.

As part of this work, we have implemented and evaluated both methods as an extension of the reasoner COOL, as described next.

# 3 Implementation – Model Checking in COOL-MC

We report on the implementation of model checking for the coalgebraic  $\mu$ -calculus within the framework provided by the COalgebraic Ontology Logic solver (COOL), a coalgebraic reasoner for modal fixpoint logics [13], implemented in OCaml. The satisfiability-checking capacities of COOL have been reported elsewhere [14]. Our tool COOL-MC extends this framework with comprehensive functionality for model checking, along the lines of Section 2. To this end, we use existing infrastructure and data structures of COOL for parsing and representing (the closure of) input formulae  $\chi$  for an extensible selection of logics, induced by the choice of a set functor  $\mathcal{F}$ ; a newly added parser reads input models  $(C, \xi: C \to \mathcal{F}(C))$  in the form of coalgebras for the selected functor (more details on the introduced specification format for coalgebras can be found in the artifact [19]). We thus obtain model checking support for

- the standard modal  $\mu$ -calculus (including its fragment CTL) [24],

- the monotone  $\mu$ -calculus (including its fragment game logic) [31,33,8],
- the alternating-time  $\mu$ -calculus (including its fragment ATL) [2],
- the graded  $\mu$ -calculus [25],
- the probabilistic  $\mu$ -calculus [5,29,4].

By the relation between  $\mu$ -calculus model checking and the solution of games with parity conditions, made more precise in Section 4 below, COOL-MC can also be seen as a generic qualitative solver for (standard, monotone, alternating-time, graded and probabilistic) parity games.

The core model checking functionality is provided by implementations of the two approaches described in Section 2: On the one hand, we implement the direct evaluation of formulae in the form of a generic local model checking algorithm; on the other hand, we also implement a polynomial reduction to standard parity games for each of the logics currently supported. Below, we provide intuitive explanations of the two algorithms, pointing out concrete implementational details only where the implementation is not straight-forward.

Local Model Checking. The local model checking algorithm follows the ideas of [22] by directly encoding the one-step evaluation of formulae  $\psi \in \mathsf{cl}$  by means of functions  $\mathsf{eval}_{\psi} : \mathcal{P}(C \times \mathsf{cl}) \to \mathcal{P}(C \times \mathsf{cl})$ , corresponding to all moves in the model checking game that evaluate  $\psi$  at some state. For instance, we have

$$\begin{split} \operatorname{eval}_{\varphi \vee \psi}(X) &= \{(c, \varphi \vee \psi) \mid (c, \varphi) \in X \text{ or } (c, \psi) \in X\} \\ \operatorname{eval}_{\heartsuit \psi}(X) &= \{(c, \heartsuit \psi) \mid \xi(c) \in [\![ \heartsuit ]\!] (\{d \mid (d, \psi) \in X\})\} \end{split}$$

for  $X\subseteq C\times \operatorname{cl}$ , and similar functions for the remaining operators; intuitively,  $\operatorname{eval}_\psi(X)$  computes the set of positions in the model checking game that have formula component  $\psi$  and are won by player  $\exists$ , assuming that it is already known that  $\exists$  wins all positions in X. Crucially, the evaluation function for modal operators skips the exploration of the intermediate nodes  $(D,\psi)$  in the model checking game by directly evaluating the predicate lifting over the set X. Then we can compute the winning regions in the model checking game as nested fixpoints of the one-step solving function:

assuming w.l.o.g. that k is odd, and denoting by  $\Omega(\psi)$  the priority of all game nodes of the shape  $(c, \psi)$ ; thus the functions below the fixpoints directly correspond to the functions f, g from [22], Definition 5, noting that  $\Omega(\psi) = 0$  whenever  $\psi$  is not a fixpoint formula. We implement this game solving procedure by a higher order function that receives the semantic function for modalities as an argument, and then computes the relevant fixpoints by Kleene fixpoint iteration.

The overall local model checking implementation then builds the model checking game step by step, starting from the initial position  $(c, \varphi)$  and adding nodes to which the respective player can move; crucially, the evaluation functions for modalities allow us to skip all nodes of the form  $(D, \psi)$  during the

exploration of the game arena. At any point during the game construction, the algorithm can attempt to solve the partially constructed game by computing the fixpoints defined above, allowing it (in some cases) to finish *early*, that is, before the whole search space has been explored; this constitutes the *local* nature of the algorithm in the sense that satisfaction of a formula may be proved or refuted without traversing the whole model.

Parity Game Model Checking. Relying on polynomial reductions of modality evaluation to game fragments [22], we implement the generation of model checking parity games in COOL-MC by a higher order function which traverses the input model and formula and translates all connectives into game nodes as described in Section 2, interpreting modal operators using a function it receives as an argument. The parity game thus constructed then can be solved using any parity game solver (including an unoptimized native solver provided by the COOL-MC framework); the current version of COOL-MC uses PGSolver as external parity game solver (support for Oink is planned).

The subgames that evaluate individual modalities in this construction are specific to the logic at hand. Due to space restrictions, we provide sketches of the reductions for two central logics here and refer to [22], Example 15 for full details. For the standard  $\mu$ -calculus, we have modal positions  $(c, \Diamond \psi)$  (or  $(c, \Box \psi)$ ), for which the one-step evaluation games just consist of that single position controlled by player  $\exists$  (or  $\forall$ ), with moves to all positions  $(d, \psi)$  such that  $d \in \xi(c)$ . The evaluation games for, e.g., graded modalities are significantly more involved: For instance, from a position  $(c, \langle n \rangle \psi)$ , the game proceeds in layers, with one layer for each  $d \in C$  to which c has an edge with multiplicity at least 1. In each layer, player  $\exists$  decides whether or not to include d in the set of states that she claims to satisfy  $\psi$ ; all game positions also contain a counter that keeps track of the joint multiplicities of all successors included so far. Player  $\exists$  wins the subgame as soon as this counter exceeds n but loses when the subgame exits the final layer while the counter is still below n. Additionally, player  $\forall$  can choose, for any state d that player  $\exists$  decides to use, to either challenge the satisfaction of  $\psi$  at d by continuing the model checking game a position  $(d, \psi)$ , or accept the choice of d and proceed to the next layer of the local game, increasing the counter by the multiplicity of d as a successor of c.

#### 4 Experimental Evaluation of the Implementation

We experimentally evaluate the performance of our two generic model checking implementations for all logics currently supported. The main interest in COOL-MC lies in its genericity, which enables it to cover a wide range of logics not supported by other tools, so comparison to other tools is mostly omitted for lack of competitors; additional discussion is provided below.

Generalized parity games. As we have seen above, model checking for (coalgebraic)  $\mu$ -calculi reduces to solving parity games. Conversely, parity games can

also be solved by model checking: It is well known that player  $\exists$  wins a node  $v \in V$  in a parity game with k priorities if and only if v satisfies the formula

$$\chi_k := \mu X_k.\,\nu X_{k-1}\dots\mu X_1.\,\nu X_0.\,\bigvee\nolimits_{0 \le i \le k} \varOmega^-(i) \wedge ((V_\exists \wedge \bigtriangledown X_i) \vee (V_\forall \wedge \overline{\bigtriangledown} X_i))$$

where  $\Omega^{-}(i) = \{v \in V \mid \Omega(v) = i\}, \ \nabla = \emptyset \text{ and } \overline{\nabla} = \square \text{ and } k \text{ is w.l.o.g. assumed}$ to be odd. We exploit this characterization to lift benchmarking problems for standard parity games to a coalgebraic level of generality: A parity game is essentially a Kripke structure with propositional atoms for priorities and player ownership, that is, a coalgebraic model based (for transitions) on the powerset functor  $\mathcal{P}$ . We generalize this situation by replacing  $\mathcal{P}$  with other functors  $\mathcal{F}$ , and  $\Diamond, \Box$  with suitable pairs  $\heartsuit, \overline{\heartsuit}$  of dual modalities. In order to win the resulting generalized game, player ∃ then requires a strategy that picks, at each game node v, a set of successors that satisfies  $\heartsuit$  if  $v \in V_{\exists}$  or  $\overline{\heartsuit}$  if  $v \in V_{\forall}$ ; e.g. in the case of standard games player  $\exists$  has to pick a single successor at their nodes  $(\heartsuit = \diamondsuit)$ , while they have to allow all successors at nodes belonging to  $\forall$   $(\overline{\heartsuit} = \square)$ . Furthermore, all plays adhering to such a strategy have to satisfy the parity condition. We then systematically enrich given standard parity games to supply additional functor-specific transition structure in a deterministic way to strike a balance between making the games much harder or much easier than the original game while still making use of the added structure; in our leading examples, we proceed as follows:

- For the monotone  $\mu$ -calculus, we construct monotone parity games; concretely, we build monotone neighbourhood structures N (e.g. [31]; these are coalgebras for the monotone neighbourhood functor [15]), in which two consecutive steps in the original parity game G are merged so that a single step in N corresponds to the evaluation of two-step strategies in G, that is, we define  $\xi(v)$  to be the set  $\{D_1, \ldots, D_m\}$  of (minimal) neighbourhoods  $D_i \subseteq V$  such that the owner of v has a strategy in G to ensure that starting from v and playing two steps, some node from  $D_i$  is reached. Then,  $\Diamond \varphi$  essentially says that  $\exists$  can enforce  $\varphi$  (in two steps), while  $\Box \varphi$  says that  $\exists$  cannot prevent  $\varphi$ .
- For the graded  $\mu$ -calculus (Section 2), we construct graded parity games by equipping moves in G with multiplicities summing up to at least 10 at each node, that is, we assign multiplicity  $(\xi(v))(u) = \lceil 10 \div E(v) \rceil$  to each successor  $u \in E(v)$  of v in G. Then we take  $\heartsuit = \langle 5 \rangle, \heartsuit = [5]$ , so to win in the graded parity game, player  $\exists$  requires a strategy that picks more than five moves, counting multiplicities, at  $\exists$  nodes, and all but at most five moves at  $\forall$  nodes.
- For the two-valued probablistic μ-calculus, we construct qualitive stochastic parity games by imposing a uniform distribution on the moves, thus obtaining probabilistic transition systems, which are coalgebras for the distribution functor that assigns to a set X the set of (discrete) probability distributions on X. Then we take  $\heartsuit = \langle \frac{1}{2} \rangle$ ,  $\overline{\heartsuit} = [\frac{1}{2}]$  where  $\langle p \rangle$  is read "with probability more than p", so player  $\exists$  wins the resulting stochastic parity game if they have a strategy that in each  $\exists$ -move stays within the winning region with probability more than  $\frac{1}{2}$ , and forces  $\forall$  to stay within  $\exists$ 's winning region with probability at least  $\frac{1}{2}$ .

We apply the above constructions to various established parity game benchmarking series, and in each case evaluate the respective variant of the formula  $\chi_k$ , thereby solving the respective monotone, graded, or probabilistic variants of the game. Specifically, we use series of *clique games*, *ladder games*, *Jurdzinski games*, *Towers of Hanoi games*, and *language inclusion games* generated by the parity game solver PGSolver [11,38].

Lazy games. To illustrate the potential advantages of local model checking, we also devise an experiment in which each game from a series of generalized parity games (as detailed above) is prepended with a node owned by player  $\exists$  which has one move that leads to the original game, but also a move to an additional self-looping node with priority 0. The resulting games all have very small solutions that can be found by the local solver, while global solving becomes more and more expensive as the parameters of the game grow.

Modulo game. To evaluate the alternating-time  $\mu$ -calculus [2] instance of the model checking implementation in COOL-MC, we devise a series of games, parameterized by a number of agents and a number m of moves per agent, but with a fixed number of positions  $p_0, \ldots, p_9$  marked by propositional atoms of the same name. At  $p_j$ , the agents concurrently each pick a number from the set  $\{1,\ldots,m\}$ , causing the game to proceed to position  $p_{(h+j) \mod 10}$  where h is the sum of the numbers played. Given a set C of agents, we evaluate the formulae  $\varphi_1 = \bigwedge_{0 \le i < 10} \mu X. p_i \lor [C]X$  and  $\varphi_2 = \nu X. \mu Y. (X \land (p_0 \lor [C]Y) \land (p_5 \lor [C]Y))$  over the modulo game. Formula  $\varphi_1$  says that the coalition C has a joint strategy to reach any given state eventually, while  $\varphi_2$  expresses the Büchi property that C can enforce that both  $p_0$  and  $p_5$  are visited infinitely often.

Evaluation setup. Our main aim in the evaluation is to show that COOL-MC scales even on the benchmark series we use, which are designed to be hard. In the process, we compare the local model checking method with the reduction to parity games (Section 2). To solve the parity games obtained, we use PGSolver's [11,38] implementation of Zielonka's recursive algorithm; we expect that practical performance can be further improved by instead using Oink [7] as a back-end parity game solver, but leave this issue as future work.

For the standard and monotone  $\mu$ -calculi, the reduction to parity games is straightforward, blurring the difference between model checking and parity game solving. For these logics, we thus refrain from a comparison between COOL-MC and other existing model checking tools [3,28], which would essentially boil down to a comparison of the respective backend parity game solvers. On the alternating-time  $\mu$ -calculus (AMC), we do conduct a brief comparison with the model checker MCMAS [30] (further comparison between COOL-MC and MC-MAS can be found in the appendix). We emphasize that the meaningfulness of such a comparison is limited, as on the one hand, MCMAS represents models symbolically while COOL-MC uses an explicit-state representation, and on the other hand, MCMAS only supports alternating-time temporal logic ATL (for which parity-game-based model checking is overkill) while COOL-MC supports

the full AMC. For graded and probabilistic  $\mu$ -calculi, COOL-MC appears to be the only existing model checker, so for these logics we evaluate only the two variants of model checking in COOL-MC; we note that the Probabilistic Symbolic Model Checker (PRISM) [27] uses a specification language based largely on PCTL [17], which is incomparable to the two-valued probabilistic  $\mu$ -calculus [4].

Below, we refer to the different instantiations of COOL-MC by indexing a logic name with either l (for local model checking) or g (for model checking by game reduction); for instance "graded $_g$ " refers to the variant of COOL-MC that reduces model checking for the graded  $\mu$ -calculus to parity game solving.

We measure runtimes as well as the sizes of the graph structures and games constructed, averaging the values measured in our experiments over at least five executions, with a timeout of 60 seconds. All experiments have been executed on a machine with an AMD Ryzen 7 2700 CPU and 32GB of RAM. An artifact containing the source code, evaluation scripts, and benchmarking sets for all experiments described above is available online [19].

Results and interpretation. The runtime results on the generalized parity games experiment are shown in Figs. 1 to 3. The trends for the different logics and variants of generalized games are similar. For readability, we show the measurements for just three logics in each case; additional results can be found in the appendix and in the artifact.

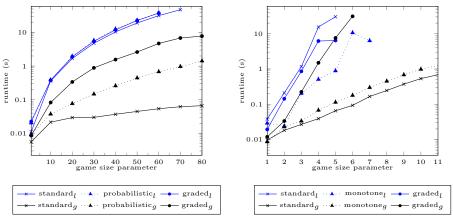


Fig. 1. Ladder games runtimes

Fig. 2. Language inclusion game runtime

It appears that the concrete choice of the logic does not strongly effect the runtimes of the local solvers (the blue plots in Figs. 1 to 3). For game-based solving (the black plots), we observe a considerable impact of the choice of logic on the runtimes, in particular solving the graded and probabilistic parity games through PGSolver takes much longer than for the standard variants. This is in line with expectations: As mentioned in the end of Section 3, the game characterization of the standard (or monotone) modalities  $\lozenge$ ,  $\square$  is straightforward, but the encoding of graded and probabilistic modalities leads to quadratic blow-up in the resulting games. The local solver however, directly evaluates modalities

and thereby avoids this blow-up so that the performance of the local solver is hardly affected by the concrete choice of modalities.

On the other hand, game-based solving typically is faster than local solving. We note that the native fixpoint computation that COOL-MC uses for local solving is completely unoptimized and performs naive Kleene fixpoint iteration, while PGSolver is an optimized tool, and in particular its recursive algorithm shows good performance in practice.

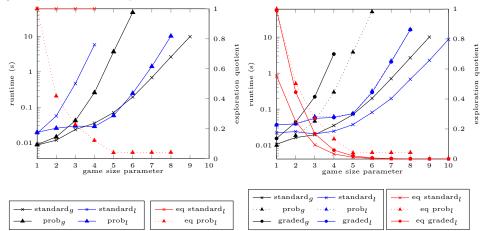


Fig. 3. Towers of Hanoi runtimes

Fig. 4. Lazy Towers of Hanoi runtimes

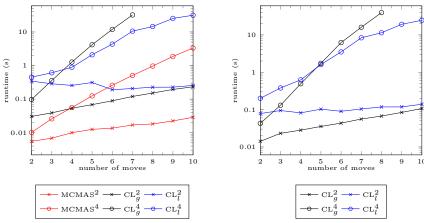
Also, the generalized games used in the benchmarks are constructed from parity games designed to be hard to solve; in particular, we observe that with the notable exceptions of the language inclusion games (Figs. 2 and 5) and the probabilistic variant of the Towers of Hanoi games (Fig. 3), these games typically do not have small solutions so that the local solver cannot play out the strength of on-the-fly model checking.

| Experiment series        | parameter | worlds | full graph | lazy graph | game size |
|--------------------------|-----------|--------|------------|------------|-----------|
| Language incl., monotone | 1         | 3      | 93         | 59         | 126       |
|                          | 7         | 313    | 9,703      | 937        | 13, 146   |
|                          | 30        | †      | †          | †          | 1,099,896 |
| Lazy Hanoi, standard     | 1         | 5      | 103        | 57         | 133       |
|                          | 5         | 245    | 5,143      | 57         | 6,613     |
|                          | 9         | 19,685 | 413,383    | 53         | 531,493   |
|                          | 10        | 59,051 | 1,240,069  | 53         | †         |
| Lazy Hanoi, graded       | 1         | 5      | 103        | 102        | 523       |
|                          | 2         | 11     | 229        | 102        | 2,345     |
|                          | 4         | 83     | 1,741      | 102        | 126, 222  |
|                          | 10        | 59,051 | 1,240,069  | 102        | †         |

Fig. 5. Sizes of (full and lazy) graphs and constructed parity games

This line of argumentation is substantiated by the lazy games experiment conducted on games built from the Towers of Hanoi series, shown in Fig. 4 (the sizes of the constructed graphs and games are listed in Fig. 5). These results

are representative for the lazy modifications of the other parity game series as well. Here, the local solver significantly outperforms the algorithm that first constructs the full game. It appears that the local solver does indeed manage to detect the existence of small winning strategies in these games, thereby avoiding the full exploration of the search space. In each case, the extent to which the local solver explores the full game is shown in Fig. 4 with a red plot that depicts the exploration quotient, i.e. the percentage of the total number of nodes that are actually explored. This effect is observed for all logics currently supported, including the graded and probabilistic variants.



**Fig. 6.** Modulo game runtimes  $(\varphi_1)$ 

**Fig. 7.** Modulo game runtimes  $(\varphi_2)$ 

Figures 6 and 7 show the runtimes for  $\varphi_1$  and  $\varphi_2$  on the modulo game with 2 and 4 agents, respectively. We include runtime plots for MCMAS on  $\varphi_1$ , which is expressible in ATL, while  $\phi_2$  is goes beyond ATL and is thus not handled by MCMAS. As expected, MCMAS is faster on the fragment that it supports; presumably, this is due partly to the fact that ATL allows for dedicated model checking algorithms that avoid parity games and in fact run in polynomial time [2].

#### 5 Conclusions and Future Work

We have presented and evaluated the generic model checker COOL-MC, which implements generic model checking algorithms for the coalgebraic  $\mu$ -calculus [22], and has been instantiated to a range of instance logics. In particular, COOL-MC thus constitutes the first available model checker for the two-valued probabilistic  $\mu$ -calculus [5,29,4], the graded  $\mu$ -calculus [25], and the full alternating-time  $\mu$ -calculus [2] (model checkers for alternating-time temporal logic exist [1,30,23]). The benchmarking results suggest the direct evaluation of modalities in combination with lazy solving as a setup for coalgebraic model checking that scales well in practice. An important issue for future work is to develop and implement symbolic model checking algorithms for the coalgebraic  $\mu$ -calculus.

### **Data-Availability Statement**

All data to reproduce the findings in this paper are available online. The COOL-MC source code used to compile the artifact is available at tag VMCAI-2024 of the COOL git repository [9]. Pre-compiled Linux executables as well as a docker container to reproduce the measurements displayed in the figures and tables of this paper are available online [19].

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#### Appendix

Figures 8 and 9 below show the runtimes for additional experiments on clique games and Jurdzinski games; the results in these experiments show the same trends as the results shown and commented on in the main paper. We also

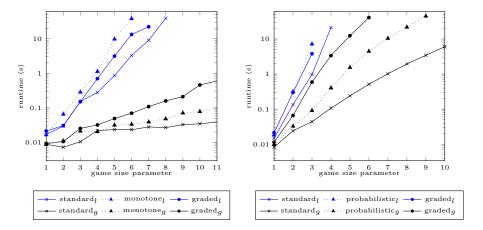


Fig. 8. Clique games runtime

Fig. 9. Jurdzinski games runtime

present an additional benchmark comparing the performance of COOL-MC to the MCMAS model checker. The castle game has been used for benchmarking in previous work on ATL model checking [34,23]. The game is parametrized over the number of castles and the health points all castles start with. Each castle has a corresponding knight that can, in each turn, either be sent out to attack another castle or stay and defend the castle. In each turn, all knights decide concurrently which other castle they want to attack or if they want to stay at their castle and defend. A knight who has attacked in one turn needs to stay and rest in the next turn. A castle that has its knight defending it or resting can block one attack. Each unblocked attack on a castle reduces that castle's number of health points by one. When no health points are left, the castle has lost the game and can no longer attack; this situation is indicated by propositional atoms lost<sub>a</sub>, where a is a knight.

For the castle game we check the following AMC formulas (which are expressible in ATL as used for the MCMAS benchmarks) for satisfaction in the initial state. The formula

$$\nu X$$
.  $\neg \mathsf{lost}_a \wedge [\{a\}]X$ 

expresses that the knight a has a strategy ensuring that her castle never gets destroyed. We check this formula for each  $a \in Ag$ . Moreover, the formula

$$\mu X. \left( \left( \textstyle \bigwedge_{a \in C} \neg \mathsf{lost}_a \right) \wedge \left( \textstyle \bigwedge_{a \in \mathsf{Ag} \backslash C} \mathsf{lost}_a \right) \right) \vee [C] X$$

expresses that the coalition C has a joint strategy to ensure that all other castles are eventually destroyed while none of the allied castles (belonging to C) are destroyed. We check this formula for one coalition of each size.

The castle game has the property that almost none of the joint moves are equivalent, i.e. almost all joint moves lead to a different outcome. Additionally, the castle game can be specified in MCMAS using separated local states of the agents. We chose a straightforward encoding where each agent has a boolean variable ready capturing whether the agent is ready to attack and an integer variable hp holds the current number of health points of the agent. The atoms  $lost_a$  are evaluated to true exactly when the hp variable of a is 0. The main difficulty of this encoding lies in the specification of the Evolution, which encodes the transition function of agents, as shown in Fig. 10: The rules of the game require counting the number of attackers, but MCMAS provides no direct way to count; hence one case has to be generated for each possible number of attackers. Additionally, all the cases have to be disjoint as MCMAS will pick otherwise some matching case non-deterministically. So each of these cases has to list all possible combinations of attackers and non-attackers in a disjunction.

```
Agent ag1
  Lobsvars = { };
  Vars: ready: boolean; hp: 0 .. 2; end Vars
  Actions = { defend, rest, dead, attack_2 };
  Protocol:
    ((hp) >= (1) and ready = true) : { defend, attack_2 };
    ((hp) >= (1) and ready = false) : { rest };
    Other : { dead }; end Protocol
  Evolution:
    hp = (hp) - (1) and ready = false if (Action = attack_2
   and
      ((ag2.Action = attack_1 and (hp) >= (1)) or (hp = 1 and
      ag2.Action = attack_1)));
    hp = (hp) - (0) and ready = true if (!(Action = attack_2)
      ((ag2.Action = attack_1 and (hp) >= (0)) or (hp = 0 and
      ag2.Action = attack_1)));
    ready = false if ((hp) >= (1) and (!(ag2.Action =
   attack_1) and
      Action = attack_2));
    ready = true if ((hp) >= (1) and (!(ag2.Action = attack_1
      !(Action = attack_2))); end Evolution
end Agent
```

Fig. 10. MCMAS encoding of an agent in the two-castle two-health-point game

The encoding of the castle game with n castles and h health points in COOL uses  $(2 \times H)^n$  as state space where  $2 = \{t, f\}$  and  $H = \{x \mid 0 \le x \le h\}$ . In Fig. 11 we see that COOL-MC again can not match the performance of MCMAS due to the same reasons as mentioned in the paper already.

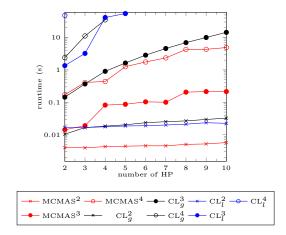


Fig. 11. Castles game runtimes on the ATL formula series