The Frobenius Problem for the Proth Numbers

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Abstract. Let *n* be a positive integer greater than 2. We define the Proth numerical semigroup, $P_k(n)$, generated by $\{k2^{n+i} + 1 \mid i \in \mathbb{N}\}$, where *k* is an odd positive number and $k < 2^n$. In this paper, we introduce the Frobenius problem for the Proth numerical semigroup $P_k(n)$ and give formulas for the embedding dimension of $P_k(n)$. We solve the Frobenius problem for $P_k(n)$ by giving a closed formula for the Frobenius number. Moreover, we show that $P_k(n)$ has an interesting property such as being Wilf.

Keywords: Combinatorial techniques · Frobenius problem · Proth Number · Numerical semigroup · Apéry Set · pseudo-Frobenius number · type · Wilf's conjecture

1 Introduction

The mathematician Ferdinand Frobenius defines the problem that asks to find the largest integer that is not expressible as a non-negative integer linear combination of elements of L, where L is a set of m coprime positive integers.

The Frobenius problem is defined as follows: Given a set $L = \{l_1, l_2, ..., l_m\}$ of coprime positive integers and $l_i \geq 2$, find the largest natural number that is not expressible as a non-negative linear combination of $l_1, l_2, ..., l_m$. It is also known as the money exchange or coin exchange problem in number theory. In literature, the connection between graph theory, theory of computer science and Frobenius problem has been developed (see [10,11,15,14]). This is because the Frobenius problem has attracted mathematicians as well as computer scientists since the 19-th century (see [3], Chapter 1 in [6], Problem C7 in [9], [28]).

For the special case e.g., m = 2, the explicit formula to find the Frobenius number is known, it is $l_1l_2 - l_1 - l_2$ proved in [26]. In addition to that, for the case m = 3, semi-explicit formula is known to find the Frobenius number [17]. Moreover, Rödseth [24], Selmer [25] and Beyer [4] have developed algorithms to solve the Frobenius problem in the case m = 3. In 1996, Ramírez-Alfonsín showed that the Frobenius problem for variable m is NP-hard [16].

The Frobenius problem has been studied for several special cases, e.g., numbers in a geometric sequence, arithmetic sequence, Pythagorean triples, three consecutive squares or cubes [29,30,7,13]. Moreover, the Frobenius problem is

defined on some special structure like Numerical semigroup (see the definition below).

Let \mathbb{N} and \mathbb{Z} be the set of non-negative integers and set of integers, respectively. A subset S of \mathbb{N} containing 0 is a *numerical semigroup* if S is closed under addition and has a finite complement in \mathbb{N} . If S is a numerical semigroup and $S = \langle B \rangle$, then we call B, a system of generators of S. A system of generators B of S is minimal if no proper subset of B generates S. In [18] Rosales et al. proved that every numerical semigroup admits a unique minimal system of generators of S is called the *embedding dimension* of S denoted by e(S).

The Frobenius number of a numerical semigroup $S = \langle \{a_1, a_2, \dots, a_n\} \rangle$ (denoted by F(S)) is the greatest integer that cannot be expressed as a sum $\sum_{i=1}^{n} t_i a_i$,

where $t_1, \ldots, t_n \in \mathbb{N}$ [18,2].

To solve the Frobenius problem for numerical semigroups, several methods were introduced, e.g., see [5,19,18,20]. In particular, in recent articles, the method of computing the Apéry set (see Definition 1) and deduce the Frobenius number using the Apéry set has been presented. In literature, there exists a large list of publications devoted to solve the Frobenius problem for special classes of numerical semigroup, including the Frobenius problem for Fibonacci numerical semigroup [12], Mersenne numerical semigroup [21], Thabit numerical semigroup [22] and repunit numerical semigroup [23]. We note that the study of the Frobenius number for the mentioned numerical semigroups has been inspired by special primes such as Fibonacci, Mersenne, Thabit and repunit primes. In this paper, we introduce Proth numerical semigroup motivated by the Proth number. The main aim of this paper is to study the Proth numerical semigroup and its invariants like embedding dimension, Frobenius number, etc.

In number theory, the *Proth number* (named in honor of the mathematician François Proth) is a natural number of the form $k2^n + 1$, where n and k are positive numbers and $k < 2^n$ is an odd number. We say that a Proth number is a *Proth prime* if it is prime.

A numerical semigroup S is the Proth numerical semigroup if $n \in \mathbb{N}$ such that $S = \langle \{k2^{n+i}+1 \mid i \in \mathbb{N}\} \rangle$, where n and k are positive numbers and $k < 2^n$ is an odd number. We denote by $P_k(n)$ the numerical semigroup $\langle \{k2^{n+i}+1 \mid i \in \mathbb{N}\} \rangle$. It is easy to see that when k = 1 the Proth numerical semigroup is the Cunningham numerical semigroup [27]. Hence, we can assume that $2^r < k < 2^{r+1}$ for some r.

In this paper, we first prove that $e(P_k(n))$ is n + r + 1 where $2^r < k < 2^{r+1}$. Later, we find the Frobenius number of the Proth numerical semigroup. More formally, we prove the following theorem.

Theorem 1. Let n > 2 be a positive integer. Then $F(P_{2^r+1}(n)) = 2s_1 + s_n + s_{n+r} - s_0$, where $s_i = k2^{n+i} + 1$ for $i \in \mathbb{N}$.

Let S be a numerical semigroup. An integer x is a pseudo-Frobenius number of S if $x \in \mathbb{Z} \setminus S$ and $x + s \in S$ for all $s \in S \setminus \{0\}$. The set of pseudo-Frobenius numbers of S is denoted by PF(S), and the cardinality of the set PF(S) is called the *type* of S denoted by t(S) [18,2].

We find the set of pseudo-Frobenius numbers of the Proth numerical semigroup $P_{2r+1}(n)$ and prove that its type is n + r - 1.

In the context of a numerical semigroup, it is reasonable to study the problems that connect the Frobenius number and other invariants of a numerical semigroup. One such problem posed by Wilf (known as Wilf's conjecture) in [31] is as follows: Let S be a numerical semigroup and $\nu(S) = |\{s \in S \mid s \leq F(S)\}|$, is it true that $F(S) + 1 \leq e(S)\nu(S)$, where e(S) is the embedding dimension and F(S) is the Frobenius number of S? Note that the numerical semigroups that satisfy Wilf's conjecture are called Wilf.

The conjecture is still open; in spite of it, an affirmative answer has been given for a few special classes of a numerical semigroup. In this paper, we prove that the Proth numerical semigroup $P_{2^r+1}(n)$ supports Wilf's conjecture.

This paper is an attempt to understand the Frobenius problem and Wilf conjecture for arbitrary embedding dimension through the Proth numerical semigroup. Our approach was inspired by the ideas discussed in [21,22]. However, it is worth noting that our techniques to find the Apéry set of the Proth numerical semigroups differ from the existing ones [21,22].

The reader not familiarized with the study of numerical semigroup and the terminologies like embedding dimension, pseudo-Frobenius numbers, type, etc., can refer to the literature [18,2].

2 The Embedding Dimension

We begin this section by proving that $P_k(n)$ is a numerical semigroup. Later, we prove that the embedding dimension of $P_k(n)$ is n+r+1. Some of the techniques used in this section are introduced earlier see, e.g., [21,22,27].

Lemma 1. (Lemma 2.1 in [18]) Let S be a nonempty subset of \mathbb{N} . Then $\langle S \rangle$ is a numerical semigroup if and only if gcd(S) = 1.

Theorem 2. Let n > 2 be an integer, then $P_k(n)$ is a numerical semigroup.

Proof. It is clear that $P_k(n) \subseteq \mathbb{N}$ is closed under addition and contains zero. Note that from Lemma 1 it is enough to show that $gcd(P_k(n)) = 1$. Let $k2^n + 1$, $k2^{n+1} + 1 \in P_k(n)$. Then $gcd(k2^n + 1, k2^{n+1} + 1) = gcd(k2^n + 1, k2^{n+1} - k2^n) = gcd(k2^n + 1, k2^n) = 1$. Therefore, $P_k(n)$ is a numerical semigroup. \Box

Next we give the minimal system of generators of the Proth numerical semigroup. To this purpose, we need some preliminary results.

Lemma 2. (Lemma 2.1 in [27]) Let S be a numerical semigroup generated by a non-empty set M of positive integers. Then the following conditions are equivalent:

(i) $2m-1 \in S$ for all $m \in M$;

(ii) $2s - 1 \in S$ for all $s \in S \setminus \{0\}$.

Theorem 3. Let n > 2 be an integer, then $P_k(n) = \langle \{k2^{n+i}+1 \mid i = 0, \dots, n+r\} \rangle$.

 $\begin{array}{l} Proof. \mbox{ Let } P = \langle \{k2^{n+i}+1 \mid i \in \{0,1,\ldots,n+r\}\} \rangle. \mbox{ It is clear that } P \subseteq P_k(n). \\ \mbox{ To prove the other direction it is enough to prove that } k2^{n+i}+1 \in P \mbox{ for all } i \in \mathbb{N}. \\ \mbox{ Let } i \in \{0,1,\ldots,n+r-1\}, \mbox{ then } 2(k2^{n+i}+1)-1 = k2^{n+i+1}+1 \in P. \mbox{ For } i = n+r, \\ 2(k2^{n+n+r}+1)-1 = ((k-2^r)2^{n+1}+3)(k2^n+1)+((2^{r+1}-k)2^n-2)(k2^{n+1}+1) \in P. \\ \mbox{ From Lemma 2, we get } 2s-1 \in P \mbox{ for all } s \in P \setminus \{0\}. \mbox{ By induction, we can } \\ \mbox{ deduce that } k2^{n+i}+1 \in P \mbox{ for all } i \geq n+r+1 \mbox{ and hence } P_k(n) = \langle \{k2^{n+i}+1 \mid i = 0,\ldots,n+r\} \rangle. \end{array}$

Note that, Theorem 3 tells us that $\{k2^{n+i}+1 \mid i=0,\ldots,n+r\}$ is a system of generators of $P_k(n)$.

Lemma 3. Let n > 2 be an integer, then $k2^{n+n+r} + 1 \notin \langle \{k2^{n+i} + 1 \mid i \in \{0, 1, ..., n+r-1\} \} \rangle$.

Proof. Assume to the contrary that there exists $a_0, a_1, \ldots, a_{n+r-1} \in \mathbb{N}$ such that

$$k2^{n+n+r} + 1 = \sum_{i=0}^{n+r-1} a_i (k2^{n+i} + 1)$$
$$= k2^n \left(\sum_{i=0}^{n+r-1} 2^i a_i\right) + \sum_{i=0}^{n+r-1} a_i.$$

Hence, $\sum_{i=0}^{n+r-1} a_i = 1 \pmod{k2^n}$ and we get, $\sum_{i=0}^{n+r-1} a_i = tk2^n + 1$ for some $t \in \mathbb{N}$. Observe that $t \neq 0$. Thus, $\sum_{i=0}^{n+r-1} a_i \ge k2^n + 1$. Therefore, $k2^{n+n+r} + 1 = \sum_{i=0}^{n+r-1} a_i (k2^{n+i} + 1) \ge (\sum_{i=0}^{n+r-1} a_i)(k2^n + 1) \ge (k2^n + 1)^2$. Since $2^r < k$ we get

$$\begin{aligned} 2^{r+n} &< 2^n k < 2^n k + 2 \Rightarrow k 2^{r+n+n} < k^2 2^{2n} + 2k 2^n \\ &\Rightarrow k 2^{r+n+n} + 1 < k^2 2^{2n} + 2k 2^n + 1 \\ &\Rightarrow k 2^{n+n+r} + 1 < (k 2^n + 1)^2. \end{aligned}$$

Hence, $k2^{n+n+r} + 1 \ge (k2^n + 1)^2 > k2^{n+n+r} + 1$, which is a contradiction. Therefore, $k2^{n+n+r} + 1 \notin \langle \{k2^{n+i} + 1 \mid i \in \{0, 1, \dots, n+r-1\}\} \rangle$. \Box

Theorem 4. Let n > 2 be an integer and let $P_k(n)$ be the Proth numerical semigroup associated to n, then $e(P_k(n)) = n + r + 1$. Moreover, $\{k2^{n+i} + 1 \mid i \in \{0, 1, ..., n + r\}\}$ is the minimal system of generators of $P_k(n)$.

Proof. By Theorem 3, we know that $\{k2^{n+i}+1 \mid i \in 0, 1, \ldots, n+r\}$ is a system of generator for $P_k(n)$. Suppose that it is not minimal system of generators of $P_k(n)$. Then there exists $l \in \{1, 2, \ldots, n+r-1\}$ such that $k2^{n+l}+1 \in \langle k2^{n+i}+1 \mid i \in \{0, 1, \ldots, l-1\}\rangle$. Let $T = \langle k2^{n+i}+1 \mid i \in \{0, 1, \ldots, l-1\}\rangle$. If $i \in \{0, 1, \ldots, l-2\}$,

then $2(k2^{n+i}+1)-1 = k2^{n+i+1}+1 \in T$ and $2(k2^{n+l-1}+1)-1 = k2^{n+l}+1 \in T$. From Lemma 2, we have $2t-1 \in T$ for all $t \in T \setminus \{0\}$. Hence, by induction we can obtain that $k2^{n+i}+1 \in T$ for all $i \geq l$, which is a contradiction as $k2^{n+n+r}+1 \notin T$ from Lemma 3. Therefore, $\{k2^{n+i}+1 \mid i \in \{0, 1, \ldots, n+r\}\}$ is the minimal system of generators of $P_k(n)$ and $e(P_k(n)) = n+r+1$.

3 The Apéry Set

In this section, we study the notion of Apéry set and give the explicit description of the elements of the Apéry set of the Proth numerical semigroup $P_{2^r+1}(n)$ for all $r \ge 1$. We denote by s_i the element $k2^{n+i} + 1$ for all $i \in \mathbb{N}$. Thus, with this notation, $\{s_0, s_1, ..., s_{n+r}\}$ is the minimal system of generators of $P_k(n)$.

Definition 1. [1,18] Let S be a numerical semigroup and $n \in S \setminus \{0\}$. The Apéry set of S with respect to n is $Ap(S,n) = \{s \in S \mid s - n \notin S\}$.

It is clear from the following lemma that |Ap(S, n)| = n.

Lemma 4. (Lemma 2.4 in [18]) Let S be a numerical semigroup and let n be a nonzero element of S. Then $Ap(S, n) = \{w(0), w(1), \ldots, w(n-1)\}$, where w(i) is the least element of S congruent with i modulo n, for all $i \in \{0, \ldots, n-1\}$.

Our next goal is to describe the elements of $Ap(P_k(n), s_0)$.

Lemma 5. Let n > 2 be an integer. Then:

- (1) if $0 < i \le j < n + r$ then $s_i + 2s_j = 2s_{i-1} + s_{j+1}$;
- (2) if $0 < i \le n+r$ then $s_i+2s_{n+r} = 2s_{i-1}+\alpha s_0+\beta s_1$, where $\alpha = (k-2^r)2^{n+1}+3$ and $\beta = (2^{r+1}-k)2^n-2$.

Proof. (1) If $0 < i \le j < n + r$ then we have

$$s_i + 2s_j = (k2^{n+i} + 1) + 2(k2^{n+j} + 1)$$

= 2(k2^{n+i-1} + 1) + (k2^{n+j+1} + 1) = 2s_{i-1} + s_{j+1}.

(2) If $0 < i \le n + r$ then we get

$$s_{i} + 2s_{n+r} = (k2^{n+i} + 1) + 2(k2^{n+n+r} + 1)$$

= 2(k2ⁿ⁺ⁱ⁻¹ + 1) + k2^{2n+r+1} + 1
= 2s_{i-1} + \alpha(k2^{n} + 1) + \beta(k2^{n+1} + 1) = 2s_{i-1} + \alpha s_{0} + \beta s_{1},

where $\alpha = (k - 2^r)2^{n+1} + 3, \beta = (2^{r+1} - k)2^n - 2.$

Let P(r, n) denotes the set of all n + r-tuple (a_1, \ldots, a_{n+r}) that satisfies the following conditions:

1. for every $i \in \{1, ..., n+r\}$, $a_i \in \{0, 1, 2\}$; 2. if $a_j = 2$ for some j = 2, ..., n+r then $a_i = 0$ for i < j.

Lemma 6. (Lemma 3.3 in [8]) The cardinality of P(r, n) is equal to $2^{n+r+1}-1$.

Lemma 7. Let n > 2 be an integer and let $P_{2^r+1}(n)$ be the Proth numerical semigroup minimally generated by $\{s_0, s_1, \ldots, s_{n+r}\}$. If $s \in \operatorname{Ap}(P_{2^r+1}(n), s_0)$ then there exist $(a_1, \ldots, a_{n+r}) \in P(r, n)$ such that $s = a_1s_1 + \cdots + a_{n+r}s_{n+r}$.

Proof. Let $s \in \operatorname{Ap}(P_{2^r+1}(n), s_0)$. We prove the result of lemma using induction on s. When s = 0 then result follows trivially. Assume that s > 0 and j be the smallest element from $\{0, 1, \ldots, n+r\}$ such that $s - s_j \in P_{2^r+1}(n)$. Since $s \in \operatorname{Ap}(P_{2^r+1}(n), s_0)$ we have $j \neq 0$ and $s - s_j \in \operatorname{Ap}(P_{2^r+1}(n), s_0)$. Now from induction hypothesis there exist $(a_1, \ldots, a_{n+r}) \in P(r, n)$ such that $s - s_j =$ $a_1s_1 + a_2s_2 + \cdots + a_{n+r}s_{n+r}$, hence $s = a_1s_1 + a_2s_2 + \cdots + (a_j + 1)s_j + \cdots +$ $a_{n+r}s_{n+r}$. Note that, to conclude the proof it suffices to prove that $(a_1, \ldots, a_j +$ $1, \ldots, a_{n+r}) \in P(r, n)$.

(1) To prove $(a_1, a_2, ..., a_j + 1, ..., a_{n+r}) \in \{0, 1, 2\}^{n+r}$, it is enough to show that $a_j + 1 \neq 3$. If $a_j + 1 = 3$ then from Lemma 5,

- (i) for j < n+r, we have $s_j + 2s_j = 2s_{j-1} + s_{j+1}$. This implies that, $s - s_{j-1} = a_1s_1 + \dots + s_{j-1} + (a_{j+1} + 1)s_{j+1} + \dots + a_{n+r}s_{n+r}$.
- (ii) for j = n + r, we have, $s_j + 2s_j = 2s_{j-1} + \alpha s_0 + \beta s_1$. This implies that, $s - s_{j-1} = \alpha s_0 + (a_1 + \beta)s_1 + a_2s_2 + \dots + (a_{n+r-1} + 1)s_{n+r-1}$.

In both the cases, we get $s - s_{j-1} \in P_{2^r+1}$, which is a contradiction to the minimality of j. Hence, $a_j + 1 \neq 3$.

(2) From the minimality of j, we obtain that $a_i = 0$ for all $1 \le i < j$. Now assume that there exist l > j such that $a_l = 2$, then again from Lemma 5, we have

(i) for l < n + r, we have $s_j + 2s_l = 2s_{j-1} + s_{l+1}$;

(ii) for l = n + r, we have, $s_j + 2s_l = 2s_{j-1} + \alpha s_0 + \beta s_1$.

Again by the same argument as in (1), we have $s - s_{j-1} \in P_{2^r+1}$, which contradict the minimality of j.

Therefore, $(a_1, \ldots, a_j + 1, \ldots, a_{n+r}) \in P(r, n)$.

It follows from Lemma 7 that $Ap(P_{2r+1}(n), s_0) \subseteq \{a_1s_1 + \dots + a_{n+r}s_{n+r} \mid (a_1, \dots, a_{n+r}) \in P(r, n)\}.$

The next remark tells that the equality in the above expression does not hold in general.

Remark 1. If possible suppose that, $\operatorname{Ap}(P_{2^r+1}(n), s_0) = \{a_1s_1 + \dots + a_{n+r}s_{n+r} \mid (a_1, \dots, a_{n+r}) \in P(r, n)\}$. Then $|\operatorname{Ap}(P_{2^r+1}(n), s_0)| = |\{a_1s_1 + \dots + a_{n+r}s_{n+r} \mid (a_1, \dots, a_{n+r}) \in P(r, n)\}| = 2^{n+r+1} - 1 \neq s_0$.

Thus, it remains to find the elements of the set $\{a_1s_1 + \cdots + a_{n+r}s_{n+r} \mid (a_1, \ldots, a_{n+r}) \in P(r, n)\}$ which belongs to $\operatorname{Ap}(P_{2r+1}(n), s_0)$. To do so, we first define the following sets:

 $F_1 = \{a_1s_1 + \dots + a_{n+r-1}s_{n+r-1} + s_{n+r} \mid a_i \in \{0, 1, 2\} \text{ for } 1 \le i \le n+r-2, a_{n+r-1} \in \{1, 2\} \text{ and if } a_j = 2 \text{ for some } j \text{ then } a_i = 0 \text{ for } i < j\}; \text{ and}$

 $F_{2} = \left(\bigcup_{l=0}^{r-2} E_{l} \cup \{2s_{n+r}\}\right) \setminus \{s_{1} + s_{n} + s_{n+r}, 2s_{1} + s_{n} + s_{n+r}, s_{n} + s_{n+r}\}, \text{ where } E_{l} = \{a_{1}s_{1} + \dots + a_{n+l}s_{n+l} + s_{n+r} \mid a_{i} \in \{0, 1, 2\} \text{ for } 1 \le i \le n+l-1, a_{n+l} \in \{1, 2\} \text{ and if } a_{j} = 2 \text{ then } a_{i} = 0 \text{ for } i < j\}. \text{ Take } F = F_{1} \cup F_{2}.$

Lemma 8. Under the standing hypothesis and notation, the following equalities hold.

- (a) $s_{n+l} + s_{n+r} s_0 = ((2^{n+r} 2^{n+l}) + 2^{n+1} + 4)s_0 + (2^{n+l} 2^n 3)s_1$, for $1 \le l \le r$;
- (b) $s_i + s_n + s_{n+r} s_0 = ((2^r + 1)2^n + 2 (2^i 4))s_0 + (2^i 4)s_1 \text{ for } 2 \le i \le n;$ (c) $s_1 + s_i + s_n + s_{n+r} - s_0 = ((2^r + 1)2^n + 2 - (2^i - 4))s_0 + (2^i - 3)s_1 \text{ for } 2 \le i \le n;$
- (c) $s_1 + s_i + s_n + s_{n+r} s_0 = ((2^r + 1)2^n + 2 (2^i 4))s_0 + (2^i 3)s_1$ for $2 \le i \le n;$

Proof. (a) Let $1 \leq l \leq r$. Consider

$$\begin{aligned} &(2^{n+r}-2^{n+l}+2^{n+1}+4)s_0+(2^{n+l}-2^n-3)s_1\\ &=(2^{n+r}-2^{n+l}+2^{n+1}+4)((2^r+1)2^n+1)+(2^{n+l}-2^n-3)((2^r+1)2^{n+1}+1)\\ &=(2^r+1)2^n(2^{n+r}-2^{n+l}+2\cdot2^n+4+2(2^{n+l}-2^n-3))+2^{n+r}+2^n+1\\ &=(2^r+1)2^n(2^{n+r}+2^{n+l}-2)+2^{n+r}+2^n+1\\ &=(2^r+1)2^n(2^{n+r}+2^{n+l}-1)+1\\ &=(2^r+1)(2^{n+n+r})+1+(2^r+1)2^{n+n+l}+1-(2^r+1)2^n-1\\ &=s_{n+r}+s_{n+l}-s_0.\end{aligned}$$

(b) Let $2 \le i \le n$. Consider

$$\begin{split} & ((2^r+1)2^n+2-(2^i-4))s_0+(2^i-4)s_1 \\ =& ((2^r+1)2^n+2-(2^i-4))((2^r+1)2^n+1)+(2^i-4)((2^r+1)2^{n+1}+1) \\ =& (2^r+1)2^n((2^r+1)2^n+2-2^i+4+2\cdot 2^i-8)+(2^r+1)2^n+2 \\ =& (2^r+1)2^n((2^r+1)2^n+2^i-2)+(2^r+1)2^n+2 \\ =& (2^r+1)2^n((2^r+1)2^n+2^i-1)+2 \\ =& (2^r+1)2^{n+n+r}+1+(2^r+1)2^{n+n}+1+(2^r+1)2^{n+i}+1-((2^r+1)2^n+1) \\ =& s_{n+r}+s_n+s_i-s_0. \end{split}$$

(c) Follows from the proof of part (b).

The following lemmas give the explicit description of the elements in the Apéry set $\operatorname{Ap}(P_{2^r+1}(n), s_0)$.

Lemma 9. Let n > 2 be an integer. Then $F \cap \operatorname{Ap}(P_{2^r+1}(n), s_0) = \phi$.

Proof. Let $a_1s_1 + \cdots + a_{n+r-1}s_{n+r-1} + s_{n+r} \in F_1$. From Lemma 8(a), we have $s_{n+r-1} + s_{n+r} - s_0 \in P_{2r+1}(n)$. Since $a_{n+r-1} \in \{1,2\}$, we have $a_1s_1 + \cdots + a_{n+r-1}s_{n+r-1} + s_{n+r} - s_0 = a_1s_1 + \cdots + (a_{n+r-1}-1)s_{n+r-1} + s_{n+r-1} + s_{n+r} - s_0 \in P_{2r+1}(n)$.

Let $a_1s_1 + \cdots + a_{n+l}s_{n+l} + s_{n+r} \in F_2$ for $1 \le l \le r-2$. From Lemma 8(a), we have $s_{n+l} + s_{n+r} - s_0 \in P_{2^r+1}(n)$. Similar argument as above implies that $a_1s_1 + \cdots + a_{n+l}s_{n+l} + s_{n+r} - s_0 \in P_{2^r+1}(n)$.

Let $a_1s_1 + \cdots + a_ns_n + s_{n+r} \in F_2$ (i.e. l = 0). Note that $a_i \neq 0$ for some $i \in \{2, \ldots, n-1\}$. From Lemma 8(b) and (c), we have $s_i + s_n + s_{n+r} - s_0 \in P_{2^r+1}(n)$ and $s_1 + s_i + s_n + s_{n+r} - s_0 \in P_{2^r+1}(n)$. Since $a_i \neq 0$ for $2 \leq i \leq n-1$, we have $a_1s_1 + \cdots + a_ns_n + s_{n+r} - s_0 \in P_{2^r+1}(n)$.

Finally, consider $2s_{n+r} \in F_2$. From Lemma 8(a), we have $2s_{n+r} - s_0 \in P_{2r+1}(n)$.

Thus, for any element of F say x, we have $x - s_0 \in P_{2^r+1}(n)$ and hence $F \cap \operatorname{Ap}(P_{2^r+1}(n), s_0) = \phi$. \Box

Lemma 10. Under the standing hypothesis and notation, we have $|F| = 2^{n+r} - 2^n - 2$.

Proof. Consider the set $L_{11} = \{a_1s_1 + \dots + a_{n+r-1}s_{n+r-1} + s_{n+r} \mid a_i \in \{0,1\}$ for $1 \le i \le n+r-2$ and $a_{n+r-1} = 1\}$. Clearly, $|L_{11}| = 2^{n+r-2}$. Now we construct a new set L_{12} as follows: Let $a_1s_1 + \dots + a_{n+r-1}s_{n+r-1} + s_{n+r} \in L_{11}$. Take the least index $m \in \{1, 2, \dots, n+r-1\}$ for which $a_m = 1$, add an element $b_1s_1 + \dots + b_{n+r-1}s_{n+r-1} + s_{n+r}$ in L_{12} with $b_m = 2$ and $b_j = a_j$ for all $j \ne m$. Clearly, $|L_{12}| = 2^{n+r-2}$. Note that F_1 is the disjoint union of L_{11} and L_{12} . Hence, $|F_1| = 2^{n+r-1}$.

Consider the set $L_{21} = \{a_1s_1 + \dots + a_{n+l}s_{n+l} + s_{n+r} \mid a_i \in \{0,1\} \text{ for } 1 \leq i \leq n+l-1 \text{ and } a_{n+l} = 1\}$. Clearly, $|L_{21}| = 2^{n+l-1}$. Now we construct a new set L_{22} as follows: Let $a_1s_1 + \dots + a_{n+l}s_{n+l} + s_{n+r} \in L_{21}$. Take the least index m for which $a_m = 1$, add an element $b_1s_1 + \dots + b_{n+l}s_{n+l} + s_{n+r}$ in L_{22} with $b_m = 2$ and $b_j = a_j$ for all $j \neq m$. Clearly, $|L_{22}| = 2^{n+l-1}$. Note that E_l is the disjoint union of L_{21} and L_{22} . Hence, $|E_l| = 2^{n+l}$. Thus we get, $|F_2| = \sum_{l=0}^{r-2} |E_l| + 1 - 3 = \sum_{l=0}^{r-2} 2^{n+l} - 2 = 2^{n+r-1} - 2^n - 2$. Therefore, $|F| = |F_1| + |F_2| = 2^{n+r-1} + 2^{n+r-1} - 2^n - 2 = 2^{n+r} - 2^n - 2$.

Theorem 5. Let n > 2 be an integer. Then

 $Ap(P_{2^r+1}(n), s_0) = \{a_1s_1 + \dots + a_{n+r}s_{n+r} \mid (a_1, \dots, a_{n+r}) \in P(r, n)\} \setminus F.$

Proof. Let $P'(r,n) = \{a_1s_1 + \cdots + a_{n+r}s_{n+r} \mid (a_1, \ldots, a_{n+r}) \in P(r,n)\} \setminus F$. Now from Lemma 7 and Lemma 9, it is clear that $\operatorname{Ap}(P_{2^r+1}(n), s_0) \subseteq P'(r,n))$. Note that from Lemma 6 and Lemma 10, we have

$$|P'(r,n)| = 2^{n+r+1} - 1 - (2^{n+r} - 2^n - 2) = s_0 = |\operatorname{Ap}(P_{2^r+1}(n), s_0)|.$$

Thus,
$$\operatorname{Ap}(P_{2^r+1}(n), s_0) = \{a_1s_1 + \dots + a_{n+r}s_{n+r} \mid (a_1, \dots, a_{n+r}) \in P(r, n)\} \setminus F.$$

4 The Frobenius Problem

In this section, we give the formula for the Frobenius number of the Proth numerical semigroup $P_{2^r+1}(n)$ for all $r \ge 1$. We recall Lemma 4 from Section 3. Let us begin with some preliminary lemmas.

Let us begin with some premimary temmas.

Lemma 11. Let $s \in P_{2^r+1}(n)$ such that $s \not\equiv 0 \pmod{s_0}$, then $s + 1 \in P_{2^r+1}(n)$. Moreover, $w(i+1) \leq w(i) + 1$ for $1 \leq i \leq s_0 - 1$.

Proof. Since $s \in P_{2^r+1}(n)$, there exist $a_0, \ldots, a_{n+r} \in \mathbb{N}$ such that $s = a_0s_0 + \cdots + a_{n+r}s_{n+r}$. If $s \not\equiv 0 \pmod{s_0}$ then there exist $i \in \{1, \ldots, n+r\}$ such that $a_i \neq 0$ and we get, $s+1 = a_0s_0 + \cdots + (a_i-1)s_i + \cdots + a_{n+r}s_{n+r} + s_i + 1$.

Now, $s_i + 1 = k2^{n+i} + 1 + 1 = 2k^{n+i-1} + 2 = 2s_{i-1}$. Hence, $s + 1 = a_0s_0 + \dots + (a_{i-1} + 2)s_{i-1} + (a_i - 1)s_i + \dots + a_{n+r}s_{n+r} \in P_{2^r+1}(n)$.

Moreover, by definition, $w(i) \not\equiv 0 \pmod{s_0}$ for $1 \leq i \leq s_0 - 1$. Thus, $w(i) + 1 \in P_{2^r+1}(n)$. Now, $w(i) + 1 \equiv i + 1 \pmod{s_0}$. As w(i+1) is the least element of $P_{2^r+1}(n)$ which is congruent with i+1 modulo s_0 , we get $w(i+1) \leq w(i) + 1$. \Box

Lemma 12. Let n > 2 be an integer. Then

1. $w(2) = s_1 + s_n + s_{n+r}$; 2. $w(1) = 2s_1 + s_n + s_{n+r}$. Moreover, $w(1) - w(2) = s_1$.

Proof. (1) Consider

$$s_1 + s_n + s_{n+r} - 2 = (2^r + 1)2^{n+1} + 1 + (2^r + 1)2^{n+n} + 1 + (2^r + 1)2^{n+n+r} - 1$$

= 2 \cdot (2^r + 1)2^n + (2^r + 1)2^{2n}(2^r + 1) + 1
= (2^r + 1) \cdot 2^n + 1)^2 = s_0^2.

Therefore, $s_1 + s_n + s_{n+r} \equiv 2 \pmod{s_0}$. From Lemma 5 we have, $s_1 + s_n + s_{n+r} \in Ap(P_{2^r+1}(n), s_0)$. Thus, $w(2) = s_1 + s_n + s_{n+r}$. (2) Note that from (1) we have $s_1 + s_n + s_{n+r} - 2 = s_0^2$. Now

$$2s_1 + s_n + s_{n+r} - 1 = s_1 + s_n + s_{n+r} + 2(2^r + 1)2^n + 1 - 1$$
$$= s_1 + s_n + s_{n+r} - 2 + 2s_0 = s_0^2 + 2s_0.$$

Therefore, $2s_1+s_n+s_{n+r} \equiv 1 \pmod{s_0}$. Again From Lemma 5 we have, $2s_1+s_n+s_{n+r} \in \operatorname{Ap}(P_{2r+1}(n), s_0)$. Thus, $w(1) = 2s_1+s_n+s_{n+r}$. Clearly, $w(1)-w(2) = s_1$.

The next Lemma is due to Selmer [25] gives us the relation among the Frobenius number and Apéry Set.

Lemma 13. ([25], Proposition 5 in [2]) Let S be a numerical semigroup and let n be a non-zero element of S. Then $F(S) = \max(Ap(S, n)) - n$.

Lemma 14. Under the standing notation, we have

$$w(1) = \max(\operatorname{Ap}(P_{2^r+1}(n), s_0)).$$

Proof. From Lemma 11, $w(i+1) \le w(i) + 1$, for $1 \le i \le s_0 - 1$. Thus, we get $w(3) \le w(2) + 1, w(4) \le w(3) + 1 \le w(2) + 2$. In general, for $3 \le j \le s_0 - 1$, we have $w(j) \le w(2) + (j-2)$. Since $w(1) - w(2) = s_1$, we get $w(j) \le w(1) - s_1 + (j-2) = w(1) - (s_1 - (j-2)) < w(1)$ as $s_1 - (j-2) > 0$. Therefore, $w(1) \ge w(i)$ for $0 \le i \le s_0 - 1$ and $w(1) = \max(\operatorname{Ap}(P_{2r+1}(n), s_0))$.

Thus, from Lemma 13 and 14 we obtain the following formula for the Frobenius number of $P_{2^r+1}(n)$.

Theorem 6. Let n > 2 be a positive integer. Then $F(P_{2^r+1}(n)) = 2s_1 + s_n + s_{n+r} - s_0$.

Next we define the genus of a numerical semigroup.

Definition 2. Let S be a numerical semigroup then the set $\mathbb{N} \setminus S$ is called *set* of gaps of S and its cardinality is said to be genus of S denoted by g(S).

Remark 2. It is well known that (see Lemma 3 in [2]), $g(S) \geq \frac{F(S)+1}{2}$.

Corollary 1. Let n > 2 be a positive integer. Then, $g(P_{2^r+1}(n)) \ge k(2^{n+1} + 2^{2n-1} + 2^{2n+r-1} - 2^{n-1}) + 2$.

5 Pseudo-Frobenius Numbers and Type

Our purpose in this section is to give the pseudo-Frobenius set and the formula for the type of the Proth numerical semigroup $P_{2^r+1}(n)$ for all $r \ge 1$. Let us recall the definition of pseudo-Frobenius numbers.

Let S be a numerical semigroup. An integer x is a pseudo-Frobenius number of S if $x \in \mathbb{Z} \setminus S$ and $x + s \in S$ for all $s \in S \setminus \{0\}$.

Consider the following relation on the set of integers \mathbb{Z} : $a \leq_S b$ if $b - a \in S$. Note that this relation is an order relation i.e., it is reflexive, transitive and antisymmetric (see [18]). The next lemma characterizes pseudo-Frobenius numbers in terms of the Apéry set using the relation defined above.

Lemma 15. (Proposition 2.20 in [18]) Let S be a numerical semigroup and let n be a nonzero element of S. Then

$$PF(S) = \{ w - n \mid w \in maximals_{\leq S}(Ap(S, n)) \}.$$

Remark 3. [22] If $w, w' \in \operatorname{Ap}(S, x)$, then $w' - w \in S$ if and only if $w' - w \in \operatorname{Ap}(S, x)$. Hence $maximal_{\leq s}(\operatorname{Ap}(S, x)) = \{w \in \operatorname{Ap}(S, x) \mid w' - w \notin \operatorname{Ap}(S, x) \setminus \{0\} \text{ for all } w' \in \operatorname{Ap}(S, x)\}.$

Let n > 2 be an integer. We define the set X as follows: $X = \{(a_1, \ldots, a_{n+r}) \mid a_1s_1 + \cdots + a_{n+r}s_{n+r} \in F\}$. Let us consider $M(n) = P(r, n) \setminus X$. It is clear that maximal elements in M(n) (with respect to the product order) are

•
$$(2, 1, \ldots, 1, 1, 0), \ldots, (0, \ldots, 0, \overset{\downarrow}{2}, 1, \ldots, 1, 0), \ldots, (0, \ldots, 0, 2, 0);$$

•
$$(2, 1, \dots, \overset{n-1}{\underset{n-1}{\downarrow}}, 0, \dots, 0, 1), \dots, (0, \dots, 0, 2, \overset{n-1}{\underset{n-1}{\downarrow}}, 0, \dots, 0, 1);$$

• $(0, \ldots, 0, \overset{*}{2}, 0, \ldots, 0, 1), (2, 0, \ldots, 0, \overset{*}{1}, 0, \ldots, 0, 1).$

As a consequence of Theorem 5, we get the following lemma.

Lemma 16. Under the standing notation, we have

 $\begin{aligned} \max imal_{\leq P_{2r+1}(n)}(\operatorname{Ap}(P_{2r+1}(n), s_0)) &= \max imal_{\leq P_{2r+1}(n)}\{\{2s_i + s_{i+1} + \dots + s_{n+r-1} \mid 1 \leq i \leq n+r-1\} \cup \{2s_j + s_{j+1} + \dots + s_{n-1} + s_{n+r} \mid 1 \leq j \leq n-2\} \cup \{2s_{n-1} + s_{n+r}, 2s_1 + s_n + s_{n+r}\}\}.\end{aligned}$

We are now already to give the main result of this section.

Theorem 7. Let n > 2 be an integer and let $P_{2^r+1}(n)$ be the Proth numerical semigroup associated to n. Then $maximal_{\leq P_{2^r+1}(n)}(\operatorname{Ap}(P_{2^r+1}(n), s_0)) = \{2s_i + s_{i+1} + \dots + s_{n+r-1} \mid 1 \leq i \leq r\} \cup \{2s_j + s_{j+1} + \dots + s_{n-1} + s_{n+r} \mid 1 \leq j \leq n-2\} \cup \{2s_1 + s_n + s_{n+r}\}.$

Proof. Let $i \in \{r+1, ..., n+r-1\}$, then

$$2s_i + s_{i+1} + \dots + s_{n-1} + s_{n+r} - (2s_{r+i} + s_{r+i+1} + \dots + s_n + s_{n+r-1})$$

= $k2^{n+i} + k2^{n+i}(2^r - 1) + r + k2^{2n+r} + 2 - (k2^{2n}(2^r - 1) + r + k2^{n+r+i} + 1)$
= $(k2^{2n} + 1) = s_n$.

Also, $2s_1 + s_n + s_{n+r} - (2s_{n-1} + s_{n+r}) = 2s_1 + k2^n + 1 - 2(k2^{n-1} + 1) = s_2$.

Hence, we get $2s_{r+i} + s_{r+i+1} + \dots + s_n + s_{n+r-1} \leq_{P_{2^r+1}(n)} 2s_i + s_{i+1} + \dots + s_{n-1} + s_{n+r}$ for $i \in \{r+1, \dots, n+r-1\}$ and $2s_{n-1} + s_{n+r} \leq_{P_{2^r+1}(n)} 2s_1 + s_n + s_{n+r}$. From Lemma 16 we obtain that $maximal_{\leq P_{2^r+1}(n)}(\operatorname{Ap}(P_{2^r+1}(n), s_0)) = maximal_{\leq P_{2^r+1}(n)}\{\{2s_i + s_{i+1} + \dots + s_{n+r-1} \mid 1 \leq i \leq r\} \cup \{2s_j + s_{j+1} + \dots + s_{n-1} + s_{n+r} \mid 1 \leq j \leq n-2\} \cup \{2s_1 + s_n + s_{n+r}\}\}.$

Consider a set $L_1 = \{p_i = 2s_i + s_{i+1} + \dots + s_{n+r-1} \mid 1 \le i \le r\}$ and $L_2 = \{q_j = 2s_j + s_{j+1} + \dots + s_{n-1} + s_{n+r} \mid 1 \le j \le n-2\}$. Take $L = L_1 \cup L_2 \cup \{2s_1 + s_n + s_{n+1}\}$. We show that $L = maximal_{\le P_{2r+1}(n)}(\operatorname{Ap}(P_{2r+1}(n), s_0))$.

Thus, to conclude the proof, it is enough to show that, for any $x, y \in L$, $x \not\leq_{P_{2^r+1}(n)} y$.

Let $p_i, p_{i+1} \in L_1$, then

$$p_{i+1} - p_i = 2s_{i+1} + s_{i+2} + \dots + s_{n+r-1} - (2s_i + s_{i+1} + \dots + s_{n+r-1})$$

= $-2s_i + s_{i+1} = -1.$

Thus, the difference between any two element of L_1 is smaller than $r < s_0$. Which implies that $p_i \not\leq_{P_{2r+1}(n)} p_j$ for any $1 \leq i, j \leq r$ and $i \neq j$.

Similarly, one can check that for $q_i, q_{i+1} \in L_2, q_{i+1} - q_i = -1$ and $q_i \not\leq_{P_{2^r+1}(n)} q_j$ for any $1 \leq i, j \leq n-2$ and $i \neq j$.

Let $p_i \in L_1$ and $q_j \in L_2$. Note that, $q_1 - p_1 = s_{n+r} - (s_n + \dots + s_{n+r-1}) = k2^{2n} + 1 - r$. Now consider $q_j - p_i = q_1 - (j-1) - (p_1 - (i-1)) = q_1 - p_1 - (j-i) = k2^{2n} + 1 - r - j + i$.

Suppose that $k2^{2n}+1-r-j+i \in P_{2^r+1}(n)$, then there exists $\lambda_0, \lambda_1, ..., \lambda_{n+r} \in \mathbb{N}$ such that

$$k2^{2n} + 1 - r - j + i = \lambda_0 s_0 + \lambda_1 s_1 + \dots + \lambda_{n+r} s_{n+r} = (\lambda_0 + \dots + \lambda_{n+r}) + k2^n (\lambda_0 + 2\lambda_1 + \dots + 2^{n+r} \lambda_{n+r}).$$

We get, $(\lambda_0 + \cdots + \lambda_{n+r}) = 1 - r - j + i \leq 0$ which is a contradiction as $\lambda_i \in \mathbb{N}$. Thus, $q_j - p_i \notin P_{2r+1}(n)$ and hence $p_i \not\leq_{P_{2r+1}(n)} q_j$ for $1 \leq i \leq r, 1 \leq j \leq n-2$. Now consider,

$$2s_1 + s_n + s_{n+r} - p_i = 2s_1 + s_n + s_{n+r} - (p_1 - (i-1))$$

= $-s_2 - \dots - s_{n-1} - s_{n+1} - \dots - s_{n+r-1} + s_{n+r} + i - 1$
= $(k2^{n+2} - (n-3)) + k2^{2n} - r + 1 + (i-1)$
= $k2^n(4+2^n) - n - r + 3 + i.$

If possible suppose that $k2^{2n} + k2^{n+2} - n - r + 3 + i \in P_{2r+1}(n)$, then there exists $\lambda_0, \lambda_1, ..., \lambda_{n+r} \in \mathbb{N}$ such that

$$k2^{n}(4+2^{n}) - n - r + 3 + i = \lambda_{0}s_{0} + \lambda_{1}s_{1} + \dots + \lambda_{n+r}s_{n+r}$$

= $(\lambda_{0} + \dots + \lambda_{n+r}) + k2^{n}(2^{0}\lambda_{0} + \dots + 2^{n+r}\lambda_{n+r}).$

We get, $(\lambda_0 + \cdots + \lambda_{n+r}) = -(n+r-3-i) \leq 0$, which is a contradiction as $\lambda_i \in \mathbb{N}$. Therefore, $p_i \not\leq_{P_{2r+1}(n)} 2s_1 + s_n + s_{n+r}$ for $1 \leq i \leq r$.

Similarly, it is clear that $2s_1 + s_n + s_{n+r} - q_j = k2^{n+2} + (j-n+2) \notin P_{2r+1}(n)$. Therefore, $q_j \not\leq_{P_{2r+1}(n)} 2s_1 + s_n + s_{n+r}$ for $1 \leq j \leq n-2$.

Hence, difference between any two elements of L do not belongs to $P_{2^r+1}(n)$. Thus, from Remark 3, we have $L = maximal_{\leq P_{2^r+1}(n)}(\operatorname{Ap}(P_{2^r+1}(n), s_0))$. \Box

By applying Lemma 15 and Theorem 7 we obtained the following theorem.

Theorem 8. Let n > 2 be an integer and let $P_{2^r+1}(n)$ be the Proth numerical semigroup. Then

 $PF(P_{2^r+1}(n)) = \{2s_i + s_{i+1} + \dots + s_{n+r-1} - s_0 \mid 1 \le i \le r\} \cup \{2s_j + s_{j+1} + \dots + s_{n-1} + s_{n+r} - s_0 \mid 1 \le j \le n-2\} \cup \{2s_1 + s_n + s_{n+r} - s_0\}$ and $t(P_{2^r+1}(n)) = |PF(P_{2^r+1}(n))| = r + n - 1.$

6 Wilf's Conjecture

In this section, we prove that the Proth numerical semigroup $P_{2^r+1}(n)$ supports Wilf's conjecture. Let us begin with the statement of Wilf's conjecture.

Conjecture 1. [31] Let S be a numerical semigroup, and $\nu(S) = |\{s \in S \mid s \leq F(S)\}|$, then

$$F(S) + 1 \le e(S)\nu(S),$$

where e(S) is the embedding dimension of S and F(S) is the Frobenius number of S.

Lemma 17. (Corollary 5 in [2]) Let S be a numerical semigroup. We have $F(S) + 1 \leq (t(S) + 1)\nu(S)$.

From the previous lemma we obtain the following theorem.

Theorem 9. The Proth numerical semigroup $P_{2^r+1}(n)$ satisfies Wilf's conjecture.

Proof. Recall that $e(P_{2^r+1}(n)) = n + r + 1$ and from Lemma 17

$$F(P_{2^{r}+1}(n)) + 1 \le (t(P_{2^{r}+1}(n)) + 1) \nu(P_{2^{r}+1}(n))$$

= $(n+r) \nu(P_{2^{r}+1}(n))$
< $(n+r+1) \nu(P_{2^{r}+1}(n))$
= $e(P_{2^{r}+1}(n)) \nu(P_{2^{r}+1}(n).$

7 Conclusion

In this work, we obtained the formula for the embedding dimension of the Proth numerical semigroup $P_k(n)$. As a main result, we solved the Frobenius problem for $P_{2^r+1}(n)$. Moreover, we also attained the pseudo-Frobenius set and the type of $P_{2^r+1}(n)$. We concluded the paper by examining that $P_{2^r+1}(n)$ supports Wilf's conjecture. The following is an immediate open question to investigate: Is there a formula to find the Frobenius number and other invariants of the Proth numerical semigroup $P_k(n)$ for arbitrary k?

References

- 1. Apéry, R.: Sur les branches superlinéaires des courbes algébriques. Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences. 222, 1198-1200 (1946)
- Assi, A., D'Anna, M., García-Sánchez, P.A.: Numerical semigroups and applications. Springer Nature (2020)
- 3. Alfonsín, J.L.: The diophantine Frobenius problem. OUP Oxford (2005)
- 4. Beyer, Ö., Selmer, E.S.: On the linear diophantine problem of Frobenius in three variables (1978)
- 5. Bras-Amorós, M.: Bounds on the number of numerical semigroups of a given genus. Journal of Pure and Applied Algebra. 213(6), 997-1001 (2009)
- Beck, M., Robins, S.: Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra. Undergrad. Texts in Math (2007)
- Gil, B.K., Han, J.W., Kim, T.H., Koo, R.H., Lee, B.W., J., Nam, K.S., Park, H.W., Park, P.S.: Frobenius numbers of Pythagorean triples. International Journal of Number Theory. 11(02), 613-9 (2015)
- 8. Gu, Ze, and Xilin, T.: The Frobenius problem for a class of numerical semigroups. International Journal of Number Theory 13, no. 05 (2017)
- 9. Guy, R.: Unsolved problems in number theory. Springer Science (2004)

- Heap, B.R., Lynn, M.S.: On a linear Diophantine problem of Frobenius: an improved algorithm. Numerische Mathematik. 7(3), 226-31 (1965)
- 11. Hujter, M., Vizvári, B.: The exact solutions to the Frobenius problem with three variables. Ramanujan Math. Soc. 2(2), 117-43 (1987)
- Marín, J. M., Ramírez- Alfonsín, J. L, Revuelta, M. P.: On the Frobenius number of Fibonacci numerical semigroups. Integers. Electronic Journal of Combinatorial Number Theory (2007)
- Lepilov, M., O'Rourke, J., Swanson, I.: Frobenius numbers of numerical semigroups generated by three consecutive squares or cubes. Semigroup Forum. 91, pp. 238-259 (2015)
- Owens, R.W.: An algorithm to solve the Frobenius problem. Mathematics magazine. 76(4), 264-75 (2003)
- 15. Raczunas, M., C. Astowski-Wachtel, P.: A Diophantine problem of Frobenius in terms of the least common multiple. Discrete Math (1996)
- Ramirez-Alfonsin, J.L.: Complexity of the Frobenius problem. Combinatorica. 16(1), pp.143-147 (1996)
- Rosales, J., Robles-Pérez A.: The Frobenius problem for numerical semigroups with embedding dimension equal to three. Mathematics of Computation. 81(279), 1609-17 (2012)
- Rosales, J. C., García-Sánchez, P.A.: Numerical semigroups. New York: Springer (2009)
- Rosales, J. C.: Numerical semigroups with Apéry sets of unique expression. Journal of Algebra. 226, 479-487 (2000)
- Rosales, J. C., García-Sánchez, P. A., García-García, J.I., Jiménez Madrid, J. A.: Fundamental gaps in numerical semigroups. Journal of pure and applied algebra. 189, 301-313 (2004)
- Rosales, J.C., Branco, M., Torrão, D.: The Frobenius problem for Mersenne numerical semigroups. Mathematische Zeitschrift. 286(1), 741-9 (2017)
- Rosales, J.C., Branco, M., Torrão, D.: The Frobenius problem for Thabit numerical semigroups. Journal of Number Theory. 155, 85-99 (2015)
- Rosales, J.C., Branco, M., Torrão, D.: The Frobenius problem for repunit numerical semigroups. The Ramanujan Journal. 40(2), 323-34 (2016)
- 24. Rödseth, Ö.J.: On a linear Diophantine problem of Frobenius (1978)
- 25. Selmer, E. S.: On the linear diophantine problem of Frobenius (1977).
- Sylvester, J.: Mathematical questions with their solutions. Educational times. 41(21), 171-8 (1884)
- 27. Song, K.: The Frobenius problem for numerical semigroups generated by the Thabit numbers of the first, second kind base b and the Cunningham numbers. Bulletin of the Korean Mathematical Society. 57(3), 623-47 (2020)
- Sylvester, J.: On subvariants, i.e. semi-invariants to binary quantics of an unlimited order. American Journal of Mathematics. 5(1), 79-136 (1882)
- Tripathi, A.: On the Frobenius problem for geometric sequences. Integers. 8(1), i43 (2008)
- 30. Tripathi, A.: The Frobenius problem for modified arithmetic progressions. Journal of Integer Sequences. 16(2) (2013)
- Wilf, H.: A circle-of-lights algorithm for the "money-changing problem. The American Mathematical Monthly. 85, 562-565 (1978)