Dimension-Minimality and Primality of Counter Nets⋆⋆⋆

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Abstract. A k-Counter Net (k-CN) is a finite-state automaton equipped with k integer counters that are not allowed to become negative, but do not have explicit zero tests. This language-recognition model can be thought of as labelled vector addition systems with states, some of which are accepting. Certain decision problems for k-CNs become easier, or indeed decidable, when the dimension k is small. Yet, little is known about the effect that the dimension k has on the class of languages recognised by k-CNs. Specifically, it would be useful if we could simplify algorithmic reasoning by reducing the dimension of a given CN.

To this end, we introduce the notion of dimension-primality for k-CN, whereby a k-CN is prime if it recognises a language that cannot be decomposed into a finite intersection of languages recognised by d-CNs, for some $d < k$. We show that primality is undecidable. We also study two related notions: dimension-minimality (where we seek a single language-equivalent d-CN of lower dimension) and language regularity. Additionally, we explore the trade-offs in expressiveness between dimension and non-determinism for CN.

1 Introduction

A k*-dimensional Counter Net* (k-CN) is a finite-state automaton equipped with k integer counters that are not allowed to become negative, but do not have explicit zero tests (see Fig. 1a for an example). This language-recognition model can be thought of as an alphabet-labelled Vector Addition System with States (VASS), some of whose states are accepting [7]. A k-CN $\mathcal{A}$ over alphabet Σ

⋆ S. Almagor was supported by the ISRAEL SCIENCE FOUNDATION (grant No. 989/22), G. Avni was supported by the ISRAEL SCIENCE FOUNDATION (grant No. 1679/21), H. Sinclair-Banks was supported by EPSRC Standard Research Studentship (DTP), grant number EP/T5179X/1.

⋆⋆ The full version can be found on https://arxiv.org/abs/2307.14492

N. Kobayashi and J. Worrell (Eds.): FoSSaCS 2024, LNCS 14575, pp. 229–249, 2024.
https://doi.org/10.1007/978-3-031-57231-9_11

accepts a word $w \in \Sigma^*$ if there is a run of $\mathcal{A}$ on w that ends in an accepting state in which the counters stay non-negative. The *language* of $\mathcal{A}$ is the set $\mathcal{L}(\mathcal{A})$ of words accepted by $\mathcal{A}$.

Counter nets are a natural model of concurrency and are closely related — and equivalent, in some senses — to labelled Petri Nets. These models have received significant attention over the years [6,7,13,14,17,19,27], with specific interest in the one-dimensional case, often referred to as one-counter nets [20,21,1,2]. Unfortunately, most decision problems for k-CNs are notoriously difficult and are often undecidable [1,2]. In particular, k-CNs subsume VASS and Petri nets, for which many problems are known to be Ackermann-complete, for example see the recent breakthrough in the complexity of reachability in VASS [11,25].

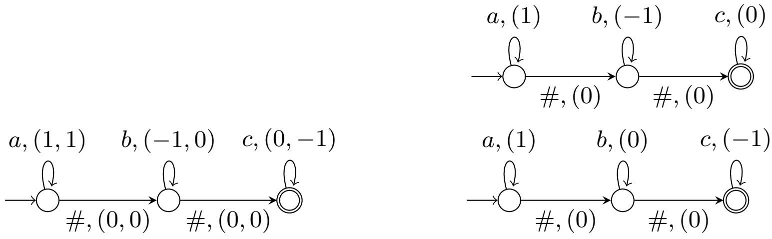
In many cases, the complexity of decision problems for VASS, sometimes with extensions, depends on the dimension, with low dimensions admitting more tractable solutions. [9,8,10,16]. For example, reachability in dimensions one and two is NP-complete [18] and PSPACE-complete [4], respectively, when counter updates are encoded in binary.

A natural question, therefore, is whether we can *decrease* the dimension of a given a k-CN whilst maintaining its language, to facilitate reasoning about it. More generally, the trade-off between expressiveness and the dimension of Counter Nets is poorly understood. We tackle this question in this work by introducing two approaches. The first is straightforward *dimension-minimality*: given a k-CN, does there exist a d-CN $\mathcal{B}$ recognising the same language for some $d < k$?

The second approach is *primality*: given a k-CN, does there exist some $d < k$ and d-CNs $\mathcal{B}_1, \ldots, \mathcal{B}_n$ such that $L(\mathcal{A}) = \bigcap_{i=1}^{n} \mathcal{L}(\mathcal{B}_i)$? That is, we ask whether the language of $\mathcal{A}$ can be decomposed as an intersection of languages recognised by several lower-dimension CNs. We also consider *compositeness*, the dual of primality. Intuitively, in a composite k-CN the usage of the counters can be "split" across several lower-dimension CNs, allowing for properties (such as universality) to be checked on each conjunct separately.

Example 1. We illustrate the model and the definition of compositeness. Consider the 2-CN $\mathcal{A}$ depicted in Fig. 1a, and consider a word $w = a^m\#b^n\#c^k$. We have that $\mathcal{A}$ has an accepting run on w iff $m \geq n$ and $m \geq k$. Indeed, if $m < n$, the first counter drops below 0 while cycling in the second state and so the run is "stuck", and similarly if $m < k$. It is not hard to show that there is no 1-CN that recognizes the language of $\mathcal{A}$. However, Fig. 1b shows two 1-CNs $\mathcal{B}_1$ and $\mathcal{B}_2$ such that $\mathcal{L}(\mathcal{B}) = \mathcal{L}(\mathcal{B}_1) \cap \mathcal{L}(\mathcal{B}_2)$. Indeed, a word $w = a^m\#b^n\#c^k \in \mathcal{L}(\mathcal{B}_1)$ iff $m \geq n$, and $w \in \mathcal{L}(\mathcal{B}_2)$ iff $m \geq k$.

Note that the decomposition in Example 1 is obtained by "splitting" the counters between the two 1-CNs. This raises the question of whether such splittings are always possible. As we show in Proposition 1, for deterministic k-CNs (k-DCNs) this is indeed the case. In general, however, it is not hard to find examples where a k-CN cannot simply be split to an intersection by projecting on each counter. This however, does not rule out that other decompositions are



(a) A composite 2-CN. (b) Two 1-CNs showing compositeness of the 2-CN.

Fig. 1: A composite 2-CN whose language is $\{a^m\#b^n\#c^k \mid m \geq n \wedge m \geq k\}$ and its decomposition into two 1-CNs recognising the languages $\{a^m\#b^n\#c^k \mid m \geq n\}$ and $\{a^m\#b^n\#c^k \mid m \geq k\}$.

possible. Our main result, Theorem 1, gives an example of a prime 2-CN. That is, a 2-CN whose language cannot be expressed as an intersection of 1-CNs.

The notion of primality has been studied for regular languages in [24,23,22], the exact complexity of deciding primality is still open. There, an automaton is composite if it can be written as an intersection of finite automata with fewer *states*. In this work we introduce primality for CNs. We focus on *dimension* as a measure of size, a notion which does not exist for regular languages. Thus, unlike regular languages, the differences between prime and composite CNs is not only in succinctness, but actually in expressiveness, as we later demonstrate.

We parameterise primality and compositeness by the dimension d and the number n of lower-dimension factors. Thus, a k-CN $\mathcal{A}$ is (d,n)*-composite* if it can be written as the intersection above. Then, $\mathcal{A}$ is *composite* if it is (d,n)-composite for some $d < k$ and $n \in \mathbb{N}$. Under this view, dimension-minimality is a special case of compositeness, namely $\mathcal{A}$ is dimension-minimal if it is not $(k-1,1)$-composite. Another particular problem captured by compositeness is *regularity*. Indeed, $\mathcal{L}(\mathcal{A})$ is regular if and only if $\mathcal{A}$ is $(0,1)$-composite, since 0-CNs are just NFAs. Since regularity is already undecidable for 1-CNs [2,28], it follows that deciding whether a k-CN is (d,n)-composite is undecidable. Moreover, it follows that both primality and dimension-minimality are undecidable for 1-CNs.

The undecidability of the above problems is not surprising, as the huge difference in expressive power between 1-CNs and regular languages is well understood. In contrast, even the expressive power difference between 1-CNs and 2-CNs is poorly understood, let alone what effect the dimension has on the expressive power beyond regular languages. Already, 1-VASS and 2-VASS are known to have *flat* equivalents with respect to reachability [26,4], but the complexity differs greatly.

Our goal in this work is to shed light on these differences. In Section 4, we give a concrete example of a prime 2-CN, which turns out to be technically challenging. This example is the heart of our technical contribution, and we emphasise that we *do not* currently have a proved example of a prime 3-CN, let alone for general k-CN (although we conjecture a candidate for such languages). We consider this an interesting open problem, as it highlights the type of pumping machinery that is currently missing from the VASS/CN reasoning arsenal. The technical intricacy in proving our example suggests that generalising it is highly nontrivial. Indeed, proving this claim would require intricate pumping arguments, which are notoriously difficult even for low-dimensional CNs [9].

Using our example, we obtain in Section 5, the undecidability of primality and of dimension-minimality for 2-CNs. To complement this, we show in Theorem 3, that regularity of k-DCNs is decidable. In Section 6, we explore trade-offs in expressiveness of CNs with increasing dimension and with nondeterminism. In particular, we show that there is a strict hierarchy of expressiveness with respect to the dimension. We conclude with a discussion in Section 7. For brevity, some proofs only appear in the full version of the paper.

2 Preliminaries

We denote the non-negative integers $\{0, 1, \ldots\}$ by $\mathbb{N}$. We write vectors in bold, e.g., $\boldsymbol{e} \in \mathbb{Z}^k$, and $\boldsymbol{e}[i]$ is the i-th coordinate. We use $[k] = \{1, \ldots, k\}$ for $k \geq 1$. We use Σ^* to denote the set of all words over an alphabet Σ, and $|w|$ is the length of $w \in \Sigma^*$.

A *k-dimensional Counter Net* (k-CN) $\mathcal{A}$ is a quintuple $\mathcal{A} = \langle \Sigma, Q, Q_0, \delta, F \rangle$ where Σ is a finite alphabet, Q is a finite set of states, $Q_0 \subseteq Q$ is the set of initial states, $\delta \subseteq Q \times \Sigma \times \mathbb{Z}^k \times Q$ is a set of transitions, and $F \subseteq Q$ are the accepting states. A k-CN is *deterministic*, denoted k-DCN, if $|Q_0| = 1$, and for every $p \in Q$ and $\sigma \in \Sigma$ there is at most one transition of the form $(p, \sigma, \boldsymbol{v}, q) \in \delta$. For a transition $(p, \sigma, \boldsymbol{v}, q) \in \delta$, we refer to $\boldsymbol{v} \in \mathbb{Z}^k$ as its *effect*.

An $\mathbb{N}$-configuration (resp. $\mathbb{Z}$-configuration) of a k-CN $\mathcal{A}$ is a pair $(q, \boldsymbol{v}) \in Q \times \mathbb{N}^k$ (resp. $(q, \boldsymbol{v}) \in Q \times \mathbb{Z}^k$) representing the current state and values of the counters. A transition $(p, \sigma, \boldsymbol{e}, q) \in \delta$ is *valid* from $\mathbb{N}$-configuration $(q, \boldsymbol{v})$ if $\boldsymbol{v}+\boldsymbol{e} \in \mathbb{N}^k$, i.e., if all k counters remain non-negative after the transition. A *$\mathbb{Z}$-run* ρ of $\mathcal{A}$ on w is a sequence of $\mathbb{Z}$-configurations $\rho = (q_0, \boldsymbol{v}_0), (q_1, \boldsymbol{v}_1), \ldots, (q_n, \boldsymbol{v}_n)$ such that $(q_i, \sigma_i, \boldsymbol{v}_{i+1} - \boldsymbol{v}_i, q_{i+1}) \in \delta$ for every $0 \leq i \leq n-1$, we may also say that ρ *reads* $w = \sigma_0 \sigma_1 \cdots \sigma_n$. An $\mathbb{N}$-*run* is a $\mathbb{Z}$-run that visits only $\mathbb{N}$-configurations. Note that all the transitions in an $\mathbb{N}$-run are valid. We may omit $\mathbb{N}$ or $\mathbb{Z}$ from the run when it does not matter. For a run $\rho = (q_0, \boldsymbol{v}_0), (q_1, \boldsymbol{v}_1), \ldots, (q_n, \boldsymbol{v}_n)$ of $\mathcal{A}$, we denote $(q_0, \boldsymbol{v}_0) \xrightarrow{\rho} (q_n, \boldsymbol{v}_n)$. We define the *effect* of ρ to be $\mathsf{eff}(\rho) = \boldsymbol{v}_n - \boldsymbol{v}_0$.

An $\mathbb{N}$-run ρ is *accepting* if $q_0 \in Q_0$, $\boldsymbol{v}_0 = \boldsymbol{0}$, and $q_n \in F$. We say that $\mathcal{A}$ *accepts* w if there is an accepting $\mathbb{N}$-run of $\mathcal{A}$ on w. The *language* of $\mathcal{A}$ is $\mathcal{L}(\mathcal{A}) = \{w \in \Sigma^* \mid \mathcal{A} \text{ accepts } w\}$. We say that $\mathcal{A}$ is *unambiguous* if it has at most one accepting run on any given word. Otherwise we say that it is ambiguous.

An infix $\pi = (q_k, \boldsymbol{v}_k), (q_{k+1}, \boldsymbol{v}_{k+1}), \ldots, (q_{k+n}, \boldsymbol{v}_{k+n})$ of a run ρ is a *cycle* if $q_k = q_{k+n}$ and is a *simple cycle* if it does not contain a cycle as a proper infix. When discussing an infix π of a 1-CN – we write that π is > 0, ≥ 0, or < 0 if $\mathsf{eff}(\pi) > 0$, $\mathsf{eff}(\pi) \geq 0$, or $\mathsf{eff}(\pi) < 0$, respectively.

3 Primality and Compositeness

We begin by presenting our main definitions, followed by some introductory properties.

Definition 1 (Compositeness, Primality, and Dimension-Minimality). *Consider a k-CN $\mathcal{A}$, and let $d, n \in \mathbb{N}$. We say that $\mathcal{A}$ is* (d, n)-composite *if there exist d-CNs $\mathcal{B}_1, \ldots, \mathcal{B}_n$ such that $\mathcal{L}(\mathcal{A}) = \bigcap_{i=1}^{n} \mathcal{L}(\mathcal{B}_i)$. If $\mathcal{A}$ is (d, n)-composite for some $d < k$ and $n \in \mathbb{N}$, we say $\mathcal{A}$ is* composite*. Otherwise, $\mathcal{A}$ is* prime*. If $\mathcal{A}$ is not $(k-1, 1)$-composite, we say that $\mathcal{A}$ is* dimension-minimal*. We also extend the definition of primality to languages, and say that a language $\mathcal{L}$ is prime if there is an integer $d > 0$ such that $\mathcal{L} = \mathcal{L}(\mathcal{A})$ for some d-CN $\mathcal{A}$, but there are no $(d-1)$-CNs $\mathcal{B}_1, \ldots \mathcal{B}_n$ such that $\mathcal{L} = \bigcap_{i=1}^{n} \mathcal{L}(\mathcal{B}_i)$.*

Remark 1. Note that the special case where $\mathcal{A}$ is $(0, n)$-composite coincides with the regularity of $\mathcal{L}(\mathcal{A})$, and hence also with being $(0, 1)$-composite.

Observe that in Fig. 1 we in fact show a composite 2-DCN. We now show that every k-DCN is $(1, k)$-composite, by projecting to each of the counters separately. In particular, a k-DCN is prime only when $k = 1$ and it recognises a non-regular language, or when $k = 0$. Formally, consider a k-DCN $\mathcal{D} = \langle \Sigma, Q, Q_0, \delta, F \rangle$ and let $1 \leq i \leq k$. We define the *i-projection* to be the 1-DCN $\mathcal{D}|_i = \langle \Sigma, Q, Q_0, \delta|_i, F \rangle$ where $\delta|_i = \{(p, \sigma, \boldsymbol{v}[i], q) \mid (p, \sigma, \boldsymbol{v}, q) \in \delta\}$.

Proposition 1. *Every k-DCN $\mathcal{D}$ is $(1, k)$-composite, and $\mathcal{L}(\mathcal{D}) = \bigcap_{i=1}^{k} \mathcal{L}(\mathcal{D}|_i)$.*

Proof. Let $w \in \mathcal{L}(\mathcal{D})$ and let ρ be the accepting run of $\mathcal{D}$ on w, then the projection of ρ on counter i induces an accepting run of $\mathcal{D}|_i$ on w, thus $w \in \bigcap_{i=1}^{k} \mathcal{L}(\mathcal{D}|_i)$. Note that this direction does not use the determinism of $\mathcal{D}$.

Conversely, let $w \in \bigcap_{i=1}^{k} \mathcal{L}(\mathcal{D}|_i)$, then each $\mathcal{D}|_i$ has an accepting run ρ_i on w. Since the structure of all the $\mathcal{D}|_i$ is identical to that of $\mathcal{D}$, all the runs ρ_i have identical state sequences, and therefore are also a $\mathbb{Z}$-run of $\mathcal{D}$ on w. Moreover, due to this being a single $\mathbb{N}$-run in each $\mathcal{D}|_i$, it follows that all counter values remain non-negative in the corresponding run of $\mathcal{D}$ on w. Hence, this is an accepting $\mathbb{N}$-run of $\mathcal{D}$ on w, so $w \in \mathcal{L}(\mathcal{D})$. □

Remark 2 (Unambiguous Counter Nets are Composite). The proof of Proposition 1 applies also to *structurally unambiguous* CNs, i.e. CNs whose underlying automaton, disregarding the counters, is unambiguous. Thus, every unambiguous CN is $(1, k)$-composite.

Consider k-CNs $\mathcal{B}_1, \ldots, \mathcal{B}_n$. By taking their product, we can construct a $k \cdot n$-CN $\mathcal{A}$ such that $\mathcal{L}(\mathcal{A}) = \bigcap_{i=1}^{n} \mathcal{L}(\mathcal{B}_i)$. In particular, if each $\mathcal{B}_i$ is a 1-DCN, then $\mathcal{A}$ is an n-DCN. Combining this with Proposition 1, we can deduce the following (proof can be found in the full version).

Proposition 2. *A k-DCN is dimension-minimal if and only if it is not $(1, k-1)$-composite.*

4 A Prime Two-Counter Net

In this section we present our main technical contribution, namely an example of a prime 2-CN. The technical difficulty arises from the need to prove that this example cannot be decomposed as an *intersection* of *nondeterministic* 1-CNs. Since intersection has a "universal flavour", and nondeterminism has an "existential flavour", we have a sort of "quantifier alternation" which is often a source of difficulty.

The importance of this example is threefold. First, it enables us to show that primality is undecidable in Section 5. Second, it offers intuition on what makes a language prime. Third, we suspect that the techniques developed here will be useful in other settings when reasoning about nondeterministic automata, perhaps with counters.

We start by presenting the prime 2-CN, followed by an overview of the proof, before delving into the details.

Example 2. Consider the 2-CN $\mathcal{P}$ over alphabet $\Sigma = \{a, b, c, \#\}$ depicted in Fig. 2. Intuitively, $\mathcal{P}$ starts by reading segments of the form $a^m\#$, where in each segment it nondeterministically chooses whether to increase the first or second counter by m. Then, it reads $b^{m_b}c^{m_c}$ and accepts if the value of the first and second counter is at least m_b and m_c, respectively. Thus, $\mathcal{P}$ accepts a word if its $a^m\#$ segments can be partitioned into two sets I and $\overline{I}$ so that the combined lengths of the segments in I (resp. $\overline{I}$) is at least the length of the b segment (resp. c segment). For example, $a^{10}\#a^{20}\#a^{15}\#b^{15}c^{30} \in \mathcal{L}(\mathcal{P})$, since segments 1 and 2 have length 30, matching c^{30} and segment 3 matches b^{15}. However, $a^{10}\#a^{20}\#a^{15}\#b^{21}c^{21} \notin \mathcal{L}(\mathcal{P})$, since in any partition of $\{10, 20, 15\}$, one set will have sum lower than 21. More precisely, we have the following:

$$\mathcal{L}(\mathcal{P}) = \{a^{m_1}\#a^{m_2}\#\cdots\#a^{m_t}\#b^{m_b}c^{m_c} \mid \exists I \subseteq [t] \text{ s.t. } \sum_{i\in I} m_i \geq m_b \wedge \sum_{i\notin I} m_i \geq m_c\}$$

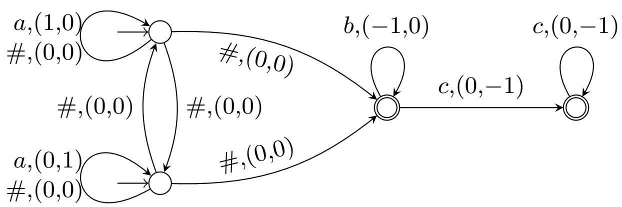


Fig. 2: The prime 2-CN $\mathcal{P}$ for Example 2 and Theorem 1.

Theorem 1. *$\mathcal{P}$ is prime.*

The high-level intuition behind Theorem 1 is that any 1-CN can either guess a subset of segments that covers m_b or m_c, but not both, and in order to make sure the choices between two 1-CNs form a partition, we need to fix the partition in advance. This is only possible if the number of segments is a priori fixed, which is not true (c.f., Remark 3). This intuition, however, is far from a proof.

4.1 Overview of the Proof of Theorem 1

Assume by way of contradiction that $\mathcal{P}$ is not a prime 2-CN. Thus, there exist 1-CNs $\mathcal{V}_1, \dots \mathcal{V}_k$ such that $\mathcal{L}(\mathcal{P}) = \bigcap_{1 \le j \le k} \mathcal{L}(\mathcal{V}_j)$. Throughout the proof, we focus on words of the form $a^{m_1}\#a^{m_2}\# \cdots \#a^{m_{k+1}}\#b^{m_b}c^{m_c}$ for positive integers $\{m_i\}_{i=1}^{k+1}, m_b, m_c$. We index the a^{m_i} segments of these words, so a^{m_i} is the i-th segment. Note that we focus on words with $k+1$ many a segments, one more than the number of $\mathcal{V}_j$ factors in the intersection. It is useful to think about each segment as "paying" for either b or c. Then, a word is accepted if there is a way to choose for each segment whether it pays for b or c, such that there is sufficient budget for both.

Let $i \in [k+1]$ and $j \in [k]$. We say that the i-th segment is *bad* in $\mathcal{V}_j$ if, intuitively, we can pump the length m_i of segment i whilst pumping both m_b and m_c to unbounded lengths, such that the resulting words are accepted by $\mathcal{V}_j$ (see Definition 2 for the formal definition). For example, consider the word $a^{10}\#a^{10}\#a^{10}\#b^{20}c^{10} \in \mathcal{L}(\mathcal{P})$. If the second segment is bad for $\mathcal{V}_j$ then there exist $x, y, z > 0$ such that for every $t, t_b, t_c \in \mathbb{N}$ it holds that the word $a^{10}\#a^{10+tx}\#a^{10}\#b^{20+t_b y}c^{10+t_c z}$ is in $\mathcal{L}(\mathcal{V}_j)$. Observe that such behaviour is undesirable, since for large enough t, t_b, t_c, the resulting word is not in $\mathcal{L}(\mathcal{P})$. Note, however, that the existence of such a bad segment is not a contradiction by itself, since the resulting pumped words might not be accepted by some other 1-CN $\mathcal{V}_{j'}$.

In order to reach a contradiction, we need to show the existence of a segment i that is bad for *every* $\mathcal{V}_j$. Moreover, we must also show that arbitrarily increasing m_i, m_b, m_c can be simultaneously achieved in all the $\mathcal{V}_j$ together (i.e., the above $x, y, z > 0$ are the same for all V_j). This would create a contradiction since all the $\mathcal{V}_j$ accept a word that is not in $\mathcal{L}(\mathcal{P})$. Our goal is therefore to establish a robust and precise definition of a "bad" segment, then find a word w comprising $k+1$ segments where one of the segments is bad for every $\mathcal{V}_j$, and pumping the words in each segment can be done synchronously.

4.2 Pumping Arguments in One-Counter Nets

In this section we establish some pumping results for 1-CN which will be used in the proof of Theorem 1. Throughout this section, we consider a 1-CN $\mathcal{V} = \langle \Sigma, Q, Q_0, \delta, F \rangle$.

Our first lemma states the intuitive fact that without > 0 cycles, the counter value of a run is bounded (proof can be found in the full version).

Lemma 1. *Let (q, n) be a configuration of $\mathcal{V}$, let W be the maximal positive update in $\mathcal{V}$, $\sigma \in \Sigma$, and $N \in \mathbb{N}$. If an $\mathbb{N}$-run ρ of $\mathcal{V}$ on σ^N from configuration (q, n) does not traverse any > 0 cycle, then the maximal possible counter value anywhere along ρ is $n + W|Q|$.*

The next lemma shows that long-enough runs must contain ≥ 0 cycles.

Lemma 2. *Let $\sigma \in \Sigma$ and (q, n) be an $\mathbb{N}$-configuration of $\mathcal{V}$. Then there exists $N \in \mathbb{N}$ such that for all $N' \geq N$, every $\mathbb{N}$-run of $\mathcal{V}$ on $\sigma^{N'}$ from (q, n) traverses a ≥ 0 cycle.*

Proof. Let W be the maximal positive transition update in $\mathcal{V}$, we show that $N = |Q|(n + |Q| \cdot W)$ satisfies the requirements. Assume by way of contradiction that $\mathcal{V}$ can read σ^N via an $\mathbb{N}$-run $\rho = (q_0, n_0 = n) \xrightarrow{\rho} (q_N, n_N)$ that only traverses < 0 cycles.

Since ρ visits $N + 1$ states, then by the Pigeonhole Principle, there exists a state $p \in Q$ that is visited $m \geq (N + 1)/|Q| > N/|Q|$ many times in ρ.

Consider all the indices $0 \leq i_1 < i_2 < \ldots < i_m \leq N$ such that $p = q_{i_1} = \ldots = q_{i_m}$. Each run segment $(q_{i_1}, n_{i_1}) \to (q_{i_2}, n_{i_2}), \ldots, (q_{i_{m-1}}, n_{i_{m-1}}) \to (q_{i_m}, n_{i_m})$ is a cycle in ρ, and therefore must have negative effect. Thus $n_{i_1} > n_{i_2} > \ldots > n_{i_m} \geq 0$, so in particular $n_{i_1} \geq n_{i_m} + m - 1 \geq 0$ (as each cycle has effect at most -1). Moreover, $n_{i_1} < n + |Q| \cdot W$ since the prefix $(q_0, n) \to (q_{i_1}, n_{i_1})$ cannot contain a non-negative cycle. However, since $m > N/|Q| = n + |Q| \cdot W$ and $n_{i_1} \geq n_{i_m} + m - 1 \geq n + |Q| \cdot W$, we get $n + |Q| \cdot W < n + |Q| \cdot W$ which is a contradiction.

□

Next, we show that runs with ≥ 0 and > 0 cycles have "pumpable" infixes.

Lemma 3. *Let $\sigma \in \Sigma$ and consider a > 0 (resp. ≥ 0) cycle $\pi = (q_0, c_0) \xrightarrow{\sigma} (q_1, c_1) \xrightarrow{\sigma} \ldots (q_n = q_0, c_n)$ on σ^n that induces an $\mathbb{N}$-run. Then, there is a sequence of (not necessarily contiguous) indices $0 \leq i_1 \leq \ldots \leq i_k \leq n$ such that $q_{i_1} \xrightarrow{\sigma} q_{i_2} \xrightarrow{\sigma} \cdots q_{i_k}$ is a simple > 0 (resp. ≥ 0) cycle with some effect $e > 0$ (resp. $e \geq 0$). In addition, this simple cycle is "pumpable" from the first occurrence of q_{i_1} in π; namely, for all $m \in \mathbb{N}$ there is a run π_m obtained from π by traversing the cycle m times so that $\mathsf{eff}(\pi_m) = \mathsf{eff}(\pi) + em$.*

Proof. We prove the ≥ 0 case, the > 0 case can be proved mutatis mutandis.

We define $\pi_m = (q_0, c_0) \xrightarrow{\sigma} \ldots (q_{i_1}, c_{i_1}) \xrightarrow{\sigma} \ldots (q_{i_1}, c_{i_1} + em) \xrightarrow{\sigma} \ldots (q_n, c_n + em)$. The proof is now by induction on the length of π.

The base of the induction is a cyclic $\mathbb{N}$-run of length 2. In this case $\pi = (q_0, c_0) \xrightarrow{\sigma} (q_1 = q_0, c_1)$ is itself a ≥ 0 simple cycle that is infinitely pumpable from (q_0, c_0).

We now assume correctness for length n, and discuss $\pi = (q_0, c_0) \xrightarrow{\sigma} (q_1, c_1) \xrightarrow{\sigma} \ldots (q_n = q_0, c_n)$ of length $n + 1$. Let $0 \leq j_1 < j_2 \leq n$ be indices such that $q_{j_1} = q_{j_2}$, for a maximal j_1. Note that the cycle $\tau = (q_{j_1}, c_{j_1}) \xrightarrow{\sigma} \ldots (q_{j_2}, c_{j_2})$ must be simple. If $j_1 = 0$ and $j_2 = n$, then π itself is a simple ≥ 0 cycle, and the pumping argument is straightforward. Otherwise τ is nested. We now split into two cases, based on whether $\mathsf{eff}(\tau) \geq 0$.

1. τ is ≥ 0: then the induction hypothesis applies on τ. We take the guaranteed constants $j_1 \leq i_1 \leq \ldots \leq i_k \leq j_2$, which apply to π as well.

2. τ is < 0: then we remove τ from π to obtain $\pi' = (q_0, c_0) \xrightarrow{\sigma} \dots (q_{j_1}, c_{j_1}) \xrightarrow{\sigma} (q_{j_2+1}, c'_{j_2+1}) \xrightarrow{\sigma} \dots (q_n, c'_n)$, such that $c'_i \geq c_i$ for all $j_2 + 1 \leq i \leq n$. The induction hypothesis applies on π', so let $i_1, \dots, i_k$ be the guaranteed constants. Note that $i_1 \leq j_1$, since the cycle removed when obtaining π' from π is the last occurrence of a repetition of states in π. We therefore know that $q_{i_1} \xrightarrow{\sigma} q_{i_2} \xrightarrow{\sigma} \cdots q_{i_k}$ is a simple ≥ 0 cycle in π' – which applies to π as well. In addition, it is infinitely pumpable from $\mathbb{N}$-configuration (q_{i_1}, c_{i_1}) in π' for $i_1 \leq j_1$. Indeed, since π and π' coincide up to and including (q_{j_1}, c_{j_1}) between π and π' - this cycle is infinitely pumpable in π as well. □

The simple cycle in Lemma 3 has length $k < |Q|$. By pumping it $\frac{|Q|!}{k}$ times we obtain a pumpable cycle of length $|Q|!$, allowing us to conclude with the following.

Corollary 1. *Let ρ be an $\mathbb{N}$-run of $\mathcal{V}$ on σ^n that traverses a ≥ 0 cycle. For every $m \in \mathbb{N}$, we can construct an $\mathbb{N}$-run ρ' of $\mathcal{V}$ on $\sigma^{n+m|Q|!}$ such that $\mathsf{eff}(\rho') \geq \mathsf{eff}(\rho)$ by pumping a ≥ 0 simple cycle in ρ.*

4.3 Good and Bad Segments

We lift the colour scheme[4] of > 0 and ≥ 0 to words and runs as follows. For a word $w = uv$ and a run ρ, we write e.g., uv to denote that ρ traverses a > 0 cycle when reading u, then a ≥ 0 cycle when reading v. Note that this does not preclude other cycles, e.g., there could also be negative cycles in the u part, etc. That is, the colouring is not unique, but represents elements of the run.

Recall our assumption that $\mathcal{L}(\mathcal{P}) = \bigcap_{1 \leq j \leq k} \mathcal{L}(\mathcal{V}_j)$, and for all $j \in [k]$ denote $\mathcal{V}_j = \langle \Sigma, Q_j, I_j, \delta_j, F_j \rangle$. Let $Q_{\max} = \max\{|Q_j|\}_{j=1}^{k}$ and denote $\alpha = Q_{\max}!$. Further recall that we focus on words of the form $a^{m_1}\#a^{m_2}\# \cdots \#a^{m_{k+1}}\#b^{m_b}c^{m_c}$ for integers $\{m_i\}_{i=1}^{k+1}, m_b, m_c \in \mathbb{N}$, and that we refer to the infix a^{m_i} as the i-th segment, for $1 \leq i \leq k + 1$. We proceed to formally define good and bad segments.

Definition 2 (Good and Bad Segments). *The i-th segment is* bad *in $\mathcal{V}_j$ if there exist constants $\{m_i\}_{i=1}^{k+1}, m_b, m_c \in \mathbb{N}$ such that the following hold.*

(a) $\{m_i\}_{i=1}^{k+1}, m_b, m_c$ *are multiples of* α, *and*
(b) *there is an accepting $\mathbb{N}$-run ρ of $\mathcal{V}_j$ on $w = a^{m_1}\#a^{m_2}\# \cdots \#a^{m_{k+1}}\#b^{m_b}c^{m_c}$ that adheres to one of the three forms:*
   (i) $a^{m_1}\#a^{m_2}\# \cdots a^{m_{i-1}}\#a^{m_i}\#a^{m_{i+1}}\# \cdots \#a^{m_{k+1}}\#b^{m_b}c^{m_c}$,
   (ii) $a^{m_1}\#a^{m_2}\# \cdots a^{m_{i-1}}\#a^{m_i}\#a^{m_{i+1}}\# \cdots \#a^{m_{k+1}}\#b^{m_b}c^{m_c}$, *or*
   (iii) $a^{m_1}\#a^{m_2}\# \cdots a^{m_{i-1}}\#a^{m_i}\#a^{m_{i+1}}\# \cdots \#a^{m_{k+1}}\#b^{m_b}c^{m_c}$.

The i-th segment is good *in $\mathcal{V}_j$ if it is not bad in $\mathcal{V}_j$.*

[4] The colours were chosen as accessible for the colourblind. For a greyscale-friendly version, see the full paper.

Lemma 4 formalises the intuition that a bad segment can be pumped simultaneously with both the b and c segments, giving rise to a word accepted by $\mathcal{V}_j$ but rejected by $\mathcal{P}$.

Intuitively, Forms (ii) and (iii) indicate that all segments are bad. Indeed, the i-th segment has a ≥ 0 cycle, so it can be pumped safely, and in Form (ii) both b and c can be pumped using ≥ 0 cycles. Whereas in Form (iii) we can pump b using a > 0 cycle, and can use it to compensate for pumping c, even if the latter requires iterating a negative cycle.

Form (i) is the interesting case, where we use a > 0 cycle in the i-th segment to compensate for pumping both b and c. The requirement that all segments up to the i-th are ≥ 0 is at the core of our proof and is explained in Section 4.4.

Lemma 4. *Suppose the l-th segment is bad in $\mathcal{V}_j$, then there exist $x, y, z \in \mathbb{N}$, that are multiples of α, such that for every $n \in \mathbb{N}$ the following word w is accepted by $\mathcal{V}_j$.*

$$w_n = a^{m_1}\#a^{m_2}\#\cdots\#a^{m_{l-1}}\#a^{m_l+xn}\#a^{m_{l+1}}\#\cdots\#a^{m_{k+1}}\#b^{m_b+yn}c^{m_c+zn}$$

Proof. We can choose $z = \alpha$, then take y to be large enough so that Form (iii) runs can compensate for negative cycles in c^z using > 0 cycles in b^y, whilst not decreasing the counters in Form (ii) runs. We can indeed find such a $y \in \mathbb{N}$ that is a multiple of α, since α is divisible by all lengths of simple cycles. Finally, we choose x so that Form (i) runs can compensate for c^z and b^y using > 0 cycles on a^x in the l-th segment, again whilst not decreasing the counters in Forms (ii) and (iii). $\square$

Recall that our goal is to show that there is a segment $l \in [k+1]$ that is bad in *every* V_j, for $j \in [k]$. In Lemma 5, We show that each V_j has at most one good segment. Therefore, there are at most k good segments in total, leaving at least one segment that is bad in every $\mathcal{V}_j$, as desired.

Lemma 5. *Let $j \in [k]$ and $0 \leq r < s \leq k+1$. Then the r-th or s-th segment is bad in $\mathcal{V}_j$.*

Proof. Since j is fixed, denote $\mathcal{V}_j = \langle \Sigma, Q, Q_0, \delta, F \rangle$. We inductively define constants $\{n_i\}_{i=1}^{k+1}, n_b, n_c \in \mathbb{N}$ as follows. Suppose that n_1 is a large-enough multiple of α so that Lemma 2 guarantees a ≥ 0 cycle in any accepting run of $\mathcal{V}_j$ on a^{n_1} from some $(q_0, 0)$ with $q_0 \in Q_0$. Now, assume that we have defined $n_1, \ldots n_{l-1}$, and consider the word $u = a^{n_1}\#a^{n_2}\#\cdots\#a^{n_{l-1}}\#$. Define $n = |u| \cdot W$ where W is the maximal update of any transition of $\mathcal{V}_j$. Since u consists of $\frac{n}{W}$ letters, $n+1$ is greater than any counter value that can be observed in any run of $\mathcal{V}_j$ on u. We define n_l to be a multiple of α large enough so that Lemma 2 guarantees a ≥ 0 cycle when reading a^{n_l} from any configuration of the form $\{(q, n') \mid q \in Q,\ n' \leq n+1\}$. We set $n_b = n_c = \alpha$, the choice of n_b, n_c is somewhat arbitrary. Finally, we set $w = a^{n_1}\#\cdots\#a^{n_{k+1}}\#b^{n_b}c^{n_c}$.

Now, for every $x \in \mathbb{N}$, we obtain from w a word w_x by pumping $x\alpha$ many a's in the r-th and s-th segments and pumping $x\alpha$ many b's and c's in their segments. That is, let $n'_i = n_i + x\alpha$ for $i \in \{r, s\}$ and $n'_i = n_i$ for $i \notin \{r, s\}$, and let $n'_b = n_b + x\alpha$ and $n'_c = n_c + x\alpha$, then $w_x = a^{n'_1}\# \cdots \# a^{n'_{k+1}} \# b^{n'_b} c^{n'_c}$. Observe that $w_x \in \mathcal{L}(\mathcal{P})$. Indeed, since $n_r \geq n_b = \alpha$ and $n_s \geq n_c = \alpha$ we have that $n_r + x\alpha \geq n_b + x\alpha$ and $n_s + x\alpha \geq n_c + x\alpha$, so the r-th and s-th segments can already pay for the b's and c's, respectively. In particular, $w_x \in \mathcal{L}(\mathcal{V}_j)$ via some accepting $\mathbb{N}$-run ρ_x.

We choose a particular value of x, as follows. Consider x and suppose some accepting $\mathbb{N}$-run ρ_x as above does not traverse a > 0 cycle neither in r-th nor s-th segment. By Lemma 1, the maximal possible counter value of ρ_x after reading

$$a^{n_1}\# \cdots \# a^{n_r + x\alpha} \# \cdots \# a^{n_s + x\alpha} \# \cdots \# a^{n_{k+1}} \#$$

is $M_b = (k + 1 + \sum_{z \in [k+1] \setminus \{r,s\}} n_z) \cdot W + 2|Q| \cdot W$. Crucially, this value does not depend on x. Further, if there is no > 0 cycle in the segment of b's as well, again the maximal counter value of ρ up to the c segment is bounded by $M_c = (k + 2 + \sum_{z \in [k+1] \setminus \{r,s\}} n_z) \cdot W + 3|Q| \cdot W$, that is independent of x and M_b. By Lemma 2, we can now choose x large enough to satisfy that for every accepting $\mathbb{N}$-run ρ_x on w_x:

1. If ρ_x does not traverse any > 0 cycle in the r-th or s-th segments, then ρ_x has a ≥ 0 cycle reading $b^{(n_b + x\alpha)}$ from any configuration in $\{(q, M') \mid q \in Q,\ M' \leq M_b\}$.
2. If ρ_x does not traverse any > 0 cycle in the r-th or s-th segment, nor in the b segment, then ρ_x has a ≥ 0 cycle reading $c^{(n_c + x\alpha)}$ from any configuration in $\{(q, M') | q \in Q,\ M' \leq M_c\}$.

Having fixed x, we claim that for the constants of w_x, one of the r-th or s-th segment is bad in $\mathcal{V}_j$. By construction, Lemma 2 guarantees that ρ_x has ≥ 0 cycles in segments $1, \ldots r - 1$. If ρ_x has a > 0 cycle in segment r, then ρ_x is of Form (i):

$$a^{n_1}\# a^{n_2} \# \cdots \# a^{n_{r-1}} \# a^{n_r + x\alpha} \# \cdots \# a^{n_s + x\alpha} \# \cdots \# a^{n_{k+1}} \# b^{n_b + x\alpha} c^{n_c + x\alpha}$$

and so the r-th segment must be bad in $\mathcal{V}_j$.

Otherwise, if ρ_x does not have a > 0 cycle in the r-th segment, then the construction in Lemma 2 guarantees ≥ 0 cycles in segments indexed $r, r+1, \ldots, s-1$. Indeed, for the r-th segment, we are guaranteed a ≥ 0 cycle reading a^{n_r}, all the more for $a^{n_r + x\alpha}$. As for segments indexed $r + 1, \ldots s - 1$, if ρ_x does not have a > 0 cycle in the r-th segment, then the maximal effect of segment r is $|Q| \cdot W$. However, n_{r+1} was constructed to guarantee a ≥ 0 cycle even in case the effect of segment r is $W n_r \geq W\alpha \geq W|Q|$.

If there is a > 0 cycle in segment s, then ρ_x is again of Form (i):

$$a^{n_1}\# a^{n_2} \# \cdots \# a^{n_{s-1}} \# a^{n_s + x\alpha} \# a^{n_{s+1}} \# \cdots \# a^{n_{k+1}} \# b^{n_b + x\alpha} c^{m_c + x\alpha}$$

and so the s-th segment must be bad in $\mathcal{V}_j$.

Otherwise, using the same arguments as for the r-th segment, we have that segments indexed $s + 1, \ldots, k + 1$ each contain a ≥ 0 cycle. In this case we are left with the b and c segments. The choice of x guarantees a ≥ 0 cycle in the b segment. If ρ_x traverses a > 0 cycle in the b segment, then w_x is of Form (iii).

$$a^{n_1}\#a^{n_2}\#\cdots\#a^{n_{k+1}}\#b^{n_b+x\alpha}c^{n_c+x\alpha}$$

Finally, if there are no > 0 cycles in the b segment, then the choice of x again guarantees a ≥ 0 cycle in the c segment, so w_x is of Form (ii).

$$a^{n_1}\#a^{n_2}\#\cdots\#a^{n_{k+1}}\#b^{n_b+x\alpha}c^{n_c+x\alpha}$$

In the two latter cases, both the r-th and the s-th segments are bad in $\mathcal{V}_j$. $\square$

4.4 Proof of Theorem 1

Given Lemma 5, we now know that each $\mathcal{V}_j$ has at most one good segment. Therefore, all 1-CNs $\mathcal{V}_1, \ldots, \mathcal{V}_k$ together have at most k good segments. Recall that the words we focus on have $k+1$ segments, and therefore there is at least one segment, say the l-th segment, that is bad in every $\mathcal{V}_j$. Note, however, that this segment may correspond to different constants in each $\mathcal{V}_j$. That is, there exists constants $\{m_i^j, m_b^j, m_c^j \mid i \in [k+1], j \in [k]\}$ witnessing that the l-th segment is bad for each $\mathcal{V}_j$. We group the $\mathcal{V}_j$ according to the form of their accepting runs ρ_j (see Definition 2):

(i) $a^{m_1^j}\#a^{m_2^j}\#\cdots\#a^{m_l^j}\#a^{m_{l+1}^j}\#\cdots\#a^{m_{k+1}^j}\#b^{m_b^j}c^{m_c^j}$,
(ii) $a^{m_1^j}\#a^{m_2^j}\#\cdots\#a^{m_l^j}\#a^{m_{l+1}^j}\#\cdots\#a^{m_{k+1}^j}\#b^{m_b^j}c^{m_c^j}$, or
(iii) $a^{m_1^j}\#a^{m_2^j}\#\cdots\#a^{m_l^j}\#a^{m_{l+1}^j}\#\cdots\#a^{m_{k+1}^j}\#b^{m_b^j}c^{m_c^j}$.

We now find constants resulting in a single word for which the l-th segment is bad in every $\mathcal{V}_j$. First, for $i \in [k+1]\setminus\{l\}$, we define $M_i = \max\{m_i^j \mid j \in [k]\}$, note that these values are still multiples of α. Similarly, we define $M_c = \max\{m_c^j \mid j \in [k]\}$. It remains to fix new constants L and B, which we do in phases in the following. The resulting word is then

$$w = a^{M_1}\#\cdots\#a^{M_{l-1}}\#a^L\#a^{M_{l+1}}\#\cdots\#a^{M_{k+1}}\#b^B c^{M_c}.$$

Most steps in the analysis below are based on Lemma 3 and Corollary 1. We first, partially, handle Form (iii) runs. For such $\mathcal{V}_j$, there is an accepting $\mathbb{N}$-run ρ_j on

$$a^{m_1^j}\#\cdots\#a^{m_{l-1}^j}\#a^{m_l^j}\#a^{m_{l+1}^j}\#\cdots\#a^{m_{k+1}^j}\#b^{m_b^j}c^{m_c^j}$$

By pumping ≥ 0 cycles as per Corollary 1 in all segments except l we obtain an accepting $\mathbb{N}$-run ρ'_j on

$$a^{M_1}\#\cdots\#a^{M_{l-1}}\#a^{m_l^j}\#a^{M_{l+1}}\#\cdots\#a^{M_{k+1}}\#b^{m_b^j}c^{m_c^j}.$$

We now pump arbitrary cycles in the c segment to construct a $\mathbb{Z}$-run ρ''_j on

$$a^{M_1}\#\cdots\#a^{M_{l-1}}\#a^{m_l^j}\#a^{M_{l+1}}\#\cdots\#a^{M_{k+1}}\#b^{m_b^j}c^{M_c}.$$

Next, we compensate for possible negative cycles in the c segment by pumping a > 0 cycle in the b segment. Thus, we construct an $\mathbb{N}$-run ρ'''_j on

$$a^{M_1}\#\cdots\#a^{M_{l-1}}\#a^{m_l^j}\#a^{M_{l+1}}\#\cdots\#a^{M_{k+1}}\#b^B c^{M_c},$$

where B is chosen to be large enough such that ρ_j''' is an $\mathbb{N}$-run for all $\mathcal{V}_j$, $j \in [k]$. Note that it remains to fix L.

We now turn to Form (i) with a similar process we start with an accepting $\mathbb{N}$-run ρ_j on

$$a^{m_1^j} \# \cdots \# a^{m_{l-1}^j} \# a^{m_l^j} \# a^{m_{l+1}^j} \# \cdots \# a^{m_{k+1}^j} \# b^{m_b^j} c^{m_c^j}.$$

Pump ≥ 0 cycles in segments indexed $1, \ldots, l-1$ to obtain an accepting $\mathbb{N}$-run ρ_j' on

$$a^{M_1} \# \cdots \# a^{M_{l-1}} \# a^{m_l^j} \# a^{m_{l+1}^j} \# \cdots \# a^{m_{k+1}^j} \# b^{m_b^j} c^{m_c^j}.$$

Now, obtain a $\mathbb{Z}$-run ρ_j'' by pumping arbitrary cycles in the remaining segments, including the b segment.

$$a^{M_1} \# \cdots \# a^{M_{l-1}} \# a^{m_l^j} \# a^{M_{l+1}} \# \cdots \# a^{M_{k+1}} \# b^{B} c^{M_c}$$

Again, compensate for negative cycles by taking L large enough so that pumping > 0 cycles in the l-th segment yields an accepting $\mathbb{N}$-run ρ_j''' on

$$a^{M_1} \# \cdots \# a^{M_{l-1}} \# a^{L} \# a^{M_{l+1}} \# \cdots \# a^{M_{k+1}} \# b^{B} c^{M_c}.$$

We now return to Form (iii) and fix the l-th segment by pumping ≥ 0 cycles to construct an accepting $\mathbb{N}$-run on

$$a^{M_1} \# \cdots \# a^{M_{l-1}} \# a^{L} \# a^{M_{l+1}} \# \cdots \# a^{M_{k+1}} \# b^{B} c^{M_c}.$$

We are left with Form (ii), which are the most straightforward to handle. We simply pump ≥ 0 cycles in all segments to construct an accepting $\mathbb{N}$-run ρ_j' on

$$a^{M_1} \# \cdots \# a^{M_{l-1}} \# a^{L} \# a^{M_{l+1}} \# \cdots \# a^{M_{k+1}} \# b^{B} c^{M_c}.$$

Note that the requirement for all segments before the l-th to be ≥ 0 is crucial, otherwise we won't be able to pump all the cycles in all forms simultaneously.

We now have that w is accepted by every $\mathcal{V}_j$, and the l-th segment is bad for all $\mathcal{V}_j$. By applying Lemma 4 for each of the $\mathcal{V}_j$ and taking global constants to be the products of the respective constants $x, y, z > 0$ for each $\mathcal{V}_j$, we now obtain $X, Y, Z \in \mathbb{N}$, multiples of α, such that for every $n \in \mathbb{N}$ the word

$$w_n = a^{M_1} \# \cdots \# a^{M_{l-1}} \# a^{L+Xn} \# a^{M_{l+1}} \# \cdots \# a^{M_{k+1}} \# b^{B+Yn} c^{M_c+Zn} \in \mathcal{L}(\mathcal{V}_j)$$

is accepted by every $\mathcal{V}_j$, for every $j \in [k]$.

Finally, we choose n large enough to satisfy $\sum_{i \in [k+1] \setminus \{l\}} M_i < \min\{B + Yn, M_c + Zn\}$, so that $w_n \notin \mathcal{L}(\mathcal{P})$. This is possible because, w.l.o.g, the l-th segment can only pay for b, and the remaining segments $[k+1] \setminus \{l\}$ cannot pay for c. This contradicts the assumption that $\mathcal{L}(\mathcal{P}) = \bigcap_{j \in [k]} \mathcal{L}(\mathcal{V}_j)$, concluding the proof of Theorem 1. $\square$

Remark 3 (Unbounded Compositeness). The proof of Theorem 1 shows that if words with $k+1$ segments are allowed, then the language is not $(1, k)$-composite, we use this to establish primality. By intersecting $\mathcal{L}(\mathcal{P})$ with words that allow at most $k+1$ segments, we obtain a language that is not $(1, k)$-composite, but it is not hard to show that it is $(1, 2^{k+1})$-composite. This demonstrates that a 2-CN can be composite, but may require unboundedly many factors.

The intuition behind Theorem 1 is that separate counters are needed to keep track of the elements that "cover" b^{m_b} and c^{m_c}. Extending this idea to k-CN, we require that the a segments are partitioned to k different sets that cover k "targets".

Conjecture 1. The following language is the language of a prime k-CN:

$$L_k = \{a^{m_1}\#a^{m_2}\#\cdots\#a^{m_t}\#b_1^{n_1}\#b_2^{n_2}\cdots\#b_k^{n_k} \mid \exists I_1,\ldots,I_k \subseteq [t]\ \forall i \in [k],\ \sum_{j\in I_i} m_j \geq n_i\ \wedge \forall i \neq j,\ I_i \cap I_j = \emptyset\}$$

While constructing a k-CN for L_k is a simple extension of Example 2, proving that it is indeed prime does not seem to succumb to our techniques, and we leave it as an important open problem (see Section 7).

5 Primality of Counter Nets is Undecidable

In this section we consider the *primality* and *dimension-minimality* decision problems: given a k-CN $\mathcal{A}$, decide whether $\mathcal{A}$ is prime and whether $\mathcal{A}$ is dimension-minimal, respectively.

We use our prime 2-CN from Example 2 and the results of Section 4 to show that both problems are undecidable. Our proof is by reduction from the containment problem[5] for 1-CN: given two 1-CN $\mathcal{A}, \mathcal{B}$ over alphabet Σ, decide whether $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$. This problem was shown to be undecidable in [20].

We begin by describing the reduction that applies to both problems. Consider an instance of 1-CN containment with two 1-CNs $\mathcal{A}$ and $\mathcal{B}$ over the alphabet Σ. We construct a 2-CN $\mathcal{C}$ as follows. Let Λ be the alphabet of the 2-CN from Example 2 and Theorem 1, and let $\$ \notin \Sigma \cup \Lambda$ be a fresh symbol. Intuitively, $\mathcal{C}$ accepts words of the form $u\$v$ when either $u \in \mathcal{L}(\mathcal{A})$ and v is accepted by $\mathcal{P}$ starting from the maximal counter $\mathcal{A}$ ended with on u, or when $u \in \mathcal{L}(\mathcal{B})$ and $v \in \Lambda^*$.

Formally, we convert $\mathcal{A}$ and $\mathcal{B}$ to 2-CNs $\mathcal{A}'$ and $\mathcal{B}'$ by adding a counter and never modifying its value, so a transition (p, σ, v, q) in $\mathcal{A}$ becomes $(p, \sigma, (v, 0), q))$ in $\mathcal{A}'$, for example. We construct a 2-CN $\mathcal{C}$ as follows (see Fig. 3). We take $\mathcal{A}'$, $\mathcal{B}'$, and $\mathcal{P}$, and for every accepting state q of $\mathcal{A}'$ we introduce a transition $(q, \$, \mathbf{0}, p_0)$ where p_0 is an initial state of $\mathcal{P}$. We then add a new accepting state $q_\top$ and add the transitions $(q_\top, \lambda, \mathbf{0}, q_\top)$ for every letter $\lambda \in \Lambda$, in other words $q_\top$ is an accepting sink for Λ. We also add transitions $(s, \$, \mathbf{0}, q_\top)$ from every accepting state s of $\mathcal{B}'$. The initial states are those of $\mathcal{A}'$ and $\mathcal{B}'$, and the accepting states are those of $\mathcal{P}$ and $q_\top$.

Theorem 2. *Primality and dimension-minimality are undecidable, already for 2-CN.*

Proof. We prove the theorem by establishing that $\mathcal{C}$ is not prime if and only if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$, and $\mathcal{C}$ is not dimension-minimal if and only if $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$.

[5] Actually, the complement thereof.

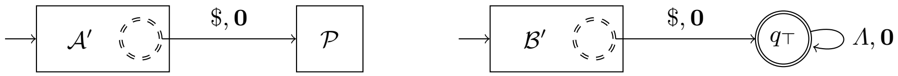


Fig. 3: The reduction from 1-CN non-containment to 2-CN primality and dimension-minimality. The dashed accepting states are those of $\mathcal{A}'$ and $\mathcal{B}'$, and are not accepting in the resulting construction.

Assume that $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$, then the component of $\mathcal{C}$ containing $\mathcal{A}'$ and $\mathcal{P}$ (Fig. 3 left) becomes redundant. Since the component containing $\mathcal{B}'$ and $q_\top$ only makes use of one counter, $\mathcal{C}$ is composite. Formally, we claim that $\mathcal{L}(\mathcal{C}) = \{u\$v \mid u \in \mathcal{L}(\mathcal{B}) \wedge v \in \Lambda^*\}$. Indeed, if $w \in \mathcal{L}(\mathcal{C})$ then $w = u\$v$ so either $u \in \mathcal{L}(\mathcal{A}') = \mathcal{L}(\mathcal{A})$ or $u \in \mathcal{L}(\mathcal{B})$, but since $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{L}(\mathcal{B})$, this is equivalent to $u \in \mathcal{L}(\mathcal{B})$, and in this case there is simply no condition on $v \in \Lambda^*$. Since the second counter is not used in component containing $\mathcal{B}'$ and $q_\top$ (Fig. 3 right), we can construct a 1-CN equivalent to $\mathcal{C}$ by projecting on the first counter and just deleting the component containing $\mathcal{A}'$ and $\mathcal{P}$ completely. It follows that in this case $\mathcal{C}$ is not dimension-minimal, and therefore is not prime either.

For the converse, assume that $\mathcal{L}(\mathcal{A}) \not\subseteq \mathcal{L}(\mathcal{B})$, and let $u \in \mathcal{L}(\mathcal{A}) \setminus \mathcal{L}(\mathcal{B})$. Denote $m = \max\{\mathsf{eff}(\rho) \mid \rho \text{ is an accepting run of } \mathcal{A} \text{ on } u\}$. Thus, for a word $v \in \Lambda^*$ we have that $u\$v \in \mathcal{L}(\mathcal{C})$ if and only if v is accepted in $\mathcal{P}$ with initial counter m. Assume by way of contradiction that $\mathcal{C}$ is not prime, then we can write $\mathcal{L}(\mathcal{C})$ as an intersection of languages of 1-CNs. Loosely speaking, this will create a contradiction as we will be able to argue that $\mathcal{P}$ is not prime. More precisely, take $v = a^{m_1}\#a^{m_2}\#\cdots\#a^{m_{k+1}}\#b^{m_b}c^{m_c}$ for integers $\{m_i\}_{i=1}^{k+1}, m_b, m_c \in \mathbb{N}$ and consider words of the form $u\$v$. Our analysis from Section 4—specifically the arguments used in the proof Lemma 5—on $u\$v$ can show, mutatis mutandis, that the language of $\mathcal{P}$ is not composite regardless of any fixed initial counter value (an analogue of Theorem 1).

We thus have that $\mathcal{C}$ is prime, and in particular $\mathcal{C}$ is dimension-minimal, concluding the correctness of the reduction. □

To contrast the undecidability of primality in nondeterministic CNs, we turn our attention to a decidable fragment of primality, for which we focus on deterministic CNs. Recall that by Proposition 1, a k-DCN is dimension minimal if and only if it is not $(1, k-1)$-composite. Thus, dimension-minimality "captures" primality. We show that regularity, which is equivalent to being $(0, 1)$-composite, is decidable for k-DCNs for every dimension k.

For dimension one, regularity is already known to be decidable in $\mathsf{EXPSPACE}$, even for history-deterministic 1-CNs [5, Theorem 19]. History-determinism is a restricted form of nondeterminism; history-deterministic CNs are less expressive than nondeterministic CNs but more expressive than DCNs. However, already for $k \geq 2$, regularity is undecidable for history-deterministic k-CNs [5, Theorem 20].

Theorem 3. *Regularity of k-DCN is decidable and is in* $\mathsf{EXPSPACE}$.

We provide further details, including a proof of Theorem 3, in the full version. In short, we translate our k-DCN into a regularity preserving Vector Addition System (VAS) and use results on VAS regularity from [3, Theorem 4.5]. We remark that an alternative approach may be taken by adapting the results of [12] on regularity of VASS, although this seems more technically challenging because CNs have accepting states.

6 Expressiveness Trade-Offs between Dimensions and Nondeterminism

Theorem 1 implies that 2-CNs are more expressive than 1-CNs, and that nondeterministic models are more expressive than deterministic ones. In particular, a k-DCN can be decomposed by projection (Proposition 1), and have decidable regularity (Theorem 3). It is therefore interesting to study the interplay between increasing the dimension and introducing nondeterminism. In this section we present two results: first, we show that dimension and nondeterminism, are incomparable notions, in a sense. Second, we show that increasing the dimension strictly increases expressiveness, for both CNs and DCNs. We remark that the latter may seem like an intuitive and simple claim. However, to the best of our knowledge it has never been proved, and moreover, it requires a nontrivial approach to pumping with several counters.

We start by showing that nondeterminism can sometimes compensate for low dimension. Let $k \in \mathbb{N}$ and $\Sigma = \{a_1, \ldots, a_k, b_1, \ldots, b_k, c\}$; consider the language $L_k = \{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k} b_i c^m \mid i \in [k] \wedge n_i \geq m\}$. It is easy to construct a k-DCN as well as a 1-CN for L_k, as depicted by Figs. 4 and 5 for $k = 3$. To construct a 1-CN we guess which b_i will be later read, and verify the guess using the single counter in the $a_i^{n_i}$ part.

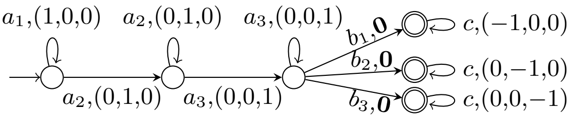


Fig. 4: A 3-DCN for $L_3 = \{a_1^{n_1} a_2^{n_2} a_3^{n_3} b_i c^m \mid i \in [3] \wedge n_i \geq m\}$. Intuitively, the 3-DCN counts the number of occurrences of each letter, and decreases the appropriate counter once the letter b_i selects it.

We now show that $\mathcal{L}_k$'s dimension cannot be minimised whilst maintaining determinism.

Theorem 4. *L_k is not recognisable by a $(k-1)$-DCN.*

Proof. Assume by way of contradiction that there exists a $(k-1)$-DCN $\mathcal{D} = \langle \Sigma, Q, Q_0, \delta, F \rangle$ such that $\mathcal{L}(\mathcal{D}) = L_k$. Let $n > |Q|$ and for every $i \in [k]$ consider

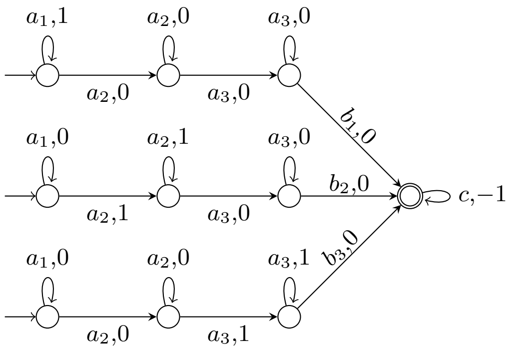


Fig. 5: A 1-CN for $L_3 = \{a_1^{n_1} a_2^{n_2} a_3^{n_3} b_i c^m \mid i \in [3] \wedge n_i \geq m\}$. Intuitively, the CN guesses which b_i will be seen, and counts the respective occurrences of the letter a_i. Then, once b_i is seen, the counter is decreased on c.

the word $w_i = a_1^n a_2^n \cdots a_k^n b_i c^n \in L_k$. Since $\mathcal{D}$ is deterministic and $n > |Q|$, all of the accepting runs on the w_i coincide up to the b_i part and have cycles in each a_i^n segment as well as in the c^n segment (the latter may differ according to i). Let M be the product of the lengths of all these cycles.

First, observe that the cycles in all of the a_i^n segments cannot decrease any counter. Indeed, otherwise by pumping such a cycle for large enough $t > 0$ times, there would not exist an $\mathbb{N}$-run on words with the prefix $a_1^n \cdots a_{i-1}^n a_i^{n+tM}$. This creates a contradiction since, with an appropriate suffix, such words can be accepted.

Thus, all a_i cycles have non-negative effects for all counters. Indeed, for each counter i – associate with i the minimal segment index whose cycle strictly increases i. Since there are $k-1$ counters and k segments this map is not surjective, in other words, there is a segment (without loss of generality, the a_k segment) such that every counter that is increased in the a_k cycle is also increased in a previous segment. Therefore, there exist $s, t > 0$ such that the word

$$a_1^{n+sM} a_2^{n+sM} \cdots a_{k-1}^{s+sM} a_k^n b_k c^{n+tM} \notin L_k$$

is accepted by $\mathcal{D}$, which is a contradiction.

We now turn to show that conversely, dimension can sometimes compensate for nondeterminism. Moreover, we show that there is a strict hierarchy of expressiveness with respect to dimension. Specifically, for $k \in \mathbb{N}$ consider the language $H_k = \{a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k} b_1^{n_1} b_2^{n_2} \cdots b_k^{n_k} \mid \forall 1 \leq i \leq k,\ m_i \geq n_i\}$.

Theorem 5. *H_k is recognisable by a k-DCN, but not by a $(k-1)$-CN.*

Proof (sketch). Constructing a k-DCN for H_k is straightforward, by using the i-th counter to check that $m_i \geq n_i$, for each $i \in [k]$.

We turn to argue that H_k is not recognisable by a $(k-1)$-CN (See the full version for a complete proof). Assume by way of contradiction that $\mathcal{A} =$

$\langle \Sigma, Q, Q_0, \delta, F\rangle$ is a $(k-1)$-CN with $L(\mathcal{A}) = H_k$. We first observe that there exists $m_1 \in \mathbb{N}$ large enough so that every run of $\mathcal{A}$ on $a_1^{m_1}$ must traverse a non-negative cycle, i.e., a cycle whose overall effect is $\boldsymbol{u_1} \in \mathbb{Z}^{k-1}$ such that $\boldsymbol{u_1}[i] \geq 0$ for all $i \in [k-1]$. Indeed, this is immediate by a "uniformly bounded" version of Dickson's lemma [15]; any long-enough "controlled" sequence of vectors in $\mathbb{N}^{k-1}$ must contain an r-increasing chain, for any $r \in \mathbb{N}$.

By repeating this argument we can ultimately find $m_1, \ldots, m_k$ such that any run of $\mathcal{A}$ on $a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k}$ traverses a non-negative cycle in each a_j segment for $j \in [k]$. Consider now the word $w = a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k} b_1^{m_1} b_2^{m_2} \cdots b_k^{m_k} \in H_k$, then there exists an accepting run ρ of $\mathcal{A}$ on w such that for each $j \in [k]$, the run ρ traverses a non-negative cycle in segment a_j, with effect $\boldsymbol{u_j} \in \mathbb{N}^{k-1}$.

Consider the vectors $\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}$. We claim that there exists $\ell \in [k]$ such that the support of $\boldsymbol{u_\ell}$ is *covered* by $\boldsymbol{u_1}, \ldots, \boldsymbol{u_{\ell-1}}$ in the following sense: for every counter $i \in [k-1]$, if $\boldsymbol{u_\ell}[i] > 0$, then there exists $j < \ell$ such that $\boldsymbol{u_j}[i] > 0$. Indeed, this holds since otherwise every $\boldsymbol{u_j}$ must contribute a fresh positive coordinate to the union of supports of the previous vectors, but there are k vectors and only $k-1$ coordinates.

Next, observe that since each $\boldsymbol{u_j}$ is a non-negative cycle taken in ρ, then it can be pumped without decreasing any following counters, and hence induce an accepting run on a pumped word. Intuitively, we now proceed by pumping all the $\boldsymbol{u_j}$ cycles for $j < \ell$ for some large-enough number of times M, which enables us to remove one iteration of the cycle with effect $\boldsymbol{u_\ell}$ while maintaining an accepting run on a word of the form:

$$w' = a_1^{m_1+Md_1} a_2^{m_2+Md_2} \cdots a_{\ell-1}^{m_{\ell-1}+Md_{\ell-1}} a_\ell^{m_\ell-d_\ell} a_{\ell+1}^{m_{\ell+1}} \cdots a_k^{m_k} b_1^{m_1} b_2^{m_2} \cdots b_k^{m_k}.$$

Since $m_\ell > m_\ell - d_\ell$, the b_ℓ segment is longer than the a_ℓ segment. Thus $w' \notin H_k$, this yields a contradiction. □

Apart from showing that nondeterminism cannot always compensate for increased dimension, Theorem 5 also shows that for every dimension k, there are languages recognisable by a $(k+1)$-DCN (and in particular by a $(k+1)$-CN), but not by any k-CN (and in particular not by any k-DCN). Thus, we obtain the following hierarchy.

Corollary 2. *For every $k \in \mathbb{N}$, k-CNs (resp. k-DCNs) are strictly less expressive than $(k+1)$-CNs (resp. $(k+1)$-DCNs).*

7 Discussion

Broadly, this work explores the interplay between the dimension of a CN and its expressive power. This is done by studying the *dimension-minimality* problem, where we ask whether the dimension of a given CN can be decreased while preserving its language, and by the more involved *primality* problem, which allows a decomposition to multiple CNs of lower dimension. We show that both primality and dimension-minimality are undecidable. Moreover, they remain undecidable even when we discard the degenerate dimension 0 case, which corresponds to finite memory, i.e., regular languages. On the other hand, this degenerate case is one where we can show decidability for DCNs.

This work also highlights a technical shortcoming of current understanding of high-dimensional CNs: pumping arguments in the presence of k dimensions and nondeterminism are very involved, and are (to our best efforts) insufficient to prove Conjecture 1. To this end, we present novel pumping arguments in the proof of Theorem 1 and to some extent in the proof of Theorem 5, which make progress towards pumping in the presence of k dimensions and nondeterminism.

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