



Auction-Based Scheduling

Guy Avni¹ , Kaushik Mallik² , and Suman Sadhukhan¹

¹ University of Haifa, Haifa, Israel
gavni@cs.haifa.ac.il,
ssadhukh@campus.haifa.ac.il

² Institute of Science and Technology Austria (ISTA), Klosterneuburg, Austria
kaushik.mallik@ist.ac.at

Abstract. Sequential decision-making tasks often require satisfaction of multiple, partially-contradictory objectives. Existing approaches are monolithic, where a single *policy* fulfills all objectives. We present *auction-based scheduling*, a *decentralized* framework for multi-objective sequential decision making. Each objective is fulfilled using a separate and independent policy. Composition of policies is performed at runtime, where at each step, the policies simultaneously bid from pre-allocated budgets for the privilege of choosing the next action. The framework allows policies to be independently created, modified, and replaced. We study path planning problems on finite graphs with two temporal objectives and present algorithms to synthesize policies together with bidding policies in a decentralized manner. We consider three categories of decentralized synthesis problems, parameterized by the assumptions that the policies make on each other. We identify a class of assumptions called *assume-admissible* for which synthesis is always possible for graphs whose every vertex has at most two outgoing edges.

1 Introduction

Sequential decision-making tasks often require satisfaction of multiple, partially-contradictory objectives. For example, the control *policy* of a traffic light may need to choose signals in a way that the traffic throughput is maximized *while* the maximum waiting time is minimized [34], the control policy operating an unmanned aerial vehicle may need to navigate in a way that the destination is reached *while* no-fly zones are avoided [33], the policy of an operating-system resource manager needs to allocate resources to tasks in a way that deadlocks are avoided *while* fairness is maintained [3].

We propose a decentralized synthesis framework for policies when tasks are given as a conjunction of two objectives Φ_1 and Φ_2 , and the policies need to choose actions from a common action space. The key idea is that Φ_1 and Φ_2 will be accomplished, respectively, using two *action policies* α_1 and α_2 —designed independently, and the composition of α_1 and α_2 at runtime will generate a policy for $\Phi_1 \wedge \Phi_2$. The challenge is that at each time point, one action needs to be chosen, whereas α_1 and α_2 might select conflicting actions. For example,

when developing a plan for a robot, Φ_1 and Φ_2 might specify two target locations, and α_1 and α_2 may select opposite directions in a location.

We propose a novel composition mechanism called *auction-based scheduling*: both policies are allocated bounded monetary *budgets*, and at each point in time, an auction (aka *bidding*) is held, where the policies *bid* from their budgets for the privilege to get scheduled for choosing the action. More formally, we equip each action policy α_i , for $i \in \{1, 2\}$, with a *bidding policy* β_i , which is a function that proposes a bid from the available budget based on the history of the interaction. A *tender* for objective Ψ is a triple $\tau = \langle \alpha, \beta, \mathbb{B} \rangle$, where α is an *action policy*, β is a *bidding policy*, and $\mathbb{B} \in (0, 1)$ is a minimal budget required for the tender to guarantee Ψ . Two tenders τ_1 and τ_2 are *compatible* if $\mathbb{B}_1 + \mathbb{B}_2 < 1$, which is when they can be composed at runtime as follows. Each Tender i , for $i \in \{1, 2\}$, is allocated an initial budget that exceeds $\mathbb{B}_i$, where the sum of budgets equals 1. At each point in time, the tenders simultaneously choose bids using their bidding policies, the higher bidder chooses an action using its action policy and pays the bid to the other tender. Thus, the sum of budgets stays constant at 1. Note that the composition gives rise to a path in the graph. The decentralized synthesis problem asks: Given a graph $\mathcal{G}$ and objectives Φ_1, Φ_2 such that $\Phi_1 \wedge \Phi_2 \neq \text{false}$, for each Φ_i compute τ_i such that no matter which tender it is composed with, the composition generates a path that fulfills Φ_i . The framework is sound-by-construction, namely the composition of compatible tenders satisfies $\Phi_1 \wedge \Phi_2$.

The advantage of auction-based scheduling is modularity at two levels. First, since the designs of policies do not depend on each other, they can be created independently and in parallel, e.g., by different vendors or in a parallel computation. Second, since the policies operate independently, they can be modified and replaced separately. For example, when only the objective Φ_1 changes, there is no need to alter the policy α_2 , and vice versa.

Bidding for the next action encourages the policy with higher scheduling urgency to bid higher, and at the same time, the bounds on budgets ensure *fairness*, namely that no policy is starved. Auction-based scheduling adds new, complementary features to the arsenal of modular approaches in multi-objective decision-making. With the conventional decentralized synthesis approaches, the policies are composed either *concurrently* [39] or in a *turn-based* manner [23]. Concurrent actions are meaningful if each policy needs to act on its own local control variables, e.g., when the local control policies of two robots concurrently move the robot towards their destinations in a shared workspace. In our case, the set of actions is common between policies, and the concurrent interaction is unsuitable. Likewise, turn-based actions are also unsuitable in our setting because it is unclear how to assign turns to policies apriori. We will demonstrate (Ex. 2) that an inappropriate turn-assignment to policies may violate some of the objectives, while auction-based scheduling will succeed to fulfil all of them.

We study auction-based scheduling in the context of path planning on finite directed graphs with pairs of ω -regular objectives on its paths, and present algorithms for the decentralized synthesis problem with increasing levels of assumptions made by the tenders on each other: (a) Strong synthesis, with no as-

sumptions and the most robust solution, (b) assume-admissible synthesis, with the assumption that the other tender is not purely cynical and behaves rationally with respect to its own objective, and (c) assume-guarantee synthesis, with explicit contract-based pre-coordination. We show for graphs whose every vertex has at most two outgoing edges, for every pair of ω -regular objectives Φ_1, Φ_2 , and for all three classes of problems (a), (b), and (c), there exist PTIME decentralized synthesis algorithms that either compute compatible tenders or output that no compatible tenders with the respective assumptions exist; surprisingly, we show that compatible tenders always exist for (b). For general graphs, we show that the problems are in $\text{NP} \cap \text{coNP}$. All our algorithms internally solve *bidding games* using known algorithms from the literature [38,37]. Due to the lack of space, some proofs are omitted, but can be found in the extended version [17].

2 Preliminaries

Let Σ be a finite alphabet. We use Σ^* and Σ^ω to respectively denote the set of finite and infinite words over Σ , and Σ^∞ to denote $\Sigma^* \cup \Sigma^\omega$. Let for two words $u \in \Sigma^*$ and $v \in \Sigma^\infty$, $u \leq v$ denote that u is a prefix of v , i.e., there exists a w such that $v = uw$. Given a language $L \subseteq \Sigma^\infty$, define $\text{pref}(L)$ to be the set of every finite prefix in L , i.e., $\text{pref}(L) := \{u \in \Sigma^* \mid \exists v \in L. u \leq v\}$.

Graphs. We formalize path planning problems on *graphs*. A graph $\mathcal{G}$ is a tuple $\langle V, v^0, E \rangle$ where V is a finite set of vertices, v^0 is a designated initial vertex, and $E \subseteq V \times V$ is a set of directed edges. If $(u, v) \in E$, then v is a *successor* of u . A *binary graph* is a graph whose every vertex has at most two successors. A *path* over $\mathcal{G}$ is a sequence of vertices $v^0 v^1 \dots$ so that every $(v^i, v^{i+1}) \in E$. Unless explicitly mentioned, paths always start at v^0 . We use $\text{Paths}^{\text{fin}}(\mathcal{G})$ and $\text{Paths}^{\text{inf}}(\mathcal{G})$ to denote the sets of finite and infinite paths, respectively.

A *strongly connected component* (SCC) of the graph $\mathcal{G}$ is a set S of vertices, such that there is a path between every pair of vertices of S . An SCC S is called a *bottom strongly connected component* (BSCC) if there does not exist any edge from a vertex in S to a vertex outside of S . The graph $\mathcal{G}$ is itself called strongly connected if V is an SCC.

Objectives. Fix a graph $\mathcal{G}$. An *objective* Φ in $\mathcal{G}$ is a set of infinite paths, i.e., $\Phi \subseteq \text{Paths}^{\text{inf}}(\mathcal{G})$. For an infinite path ρ , we use $\text{Inf}(\rho)$ to denote the set of vertices that ρ visits infinitely often. We focus on the following objectives:

- Reachability:** for $S \subseteq V$, $\text{Reach}_{\mathcal{G}}(S) := \{v^0 v^1 \dots \in \text{Paths}^{\text{inf}}(\mathcal{G}) \mid \exists i \geq 0. v^i \in S\}$,
- Safety:** for $S \subseteq V$, $\text{Safe}_{\mathcal{G}}(S) := \{v^0 v^1 \dots \in \text{Paths}^{\text{inf}}(\mathcal{G}) \mid \forall i \geq 0. v^i \in S\}$,
- Büchi:** for $S \subseteq V$, $\text{Büchi}_{\mathcal{G}}(S) := \{\rho \in \text{Paths}^{\text{inf}}(\mathcal{G}) \mid \text{Inf}(\rho) \cap S \neq \emptyset\}$,
- Parity (max, even):** for $\text{Col}: V \rightarrow [0; k]$ for some $k > 0$, $\text{Parity}_{\mathcal{G}}(\text{Col}) := \{\rho \in \text{Paths}^{\text{inf}}(\mathcal{G}) \mid \max \{i \mid \exists v \in \text{Inf}(\rho). \text{Col}(v) = i\} \text{ is even}\}$,

Given an objective Φ , we will use Φ^c to denote its complement, i.e., $\Phi^c = \text{Paths}^{\text{inf}}(\mathcal{G}) \setminus \Phi$. Observe that $(\text{Reach}_{\mathcal{G}}(S))^c = \text{Safe}_{\mathcal{G}}(V \setminus S)$.

Action policies. Fix a graph $\mathcal{G}$. An *action policy* is a function $\alpha: \text{Paths}^{\text{fin}}(\mathcal{G}) \rightarrow V$, choosing the next vertex to extend any given finite path ρv , where $\langle v, \alpha(\rho v) \rangle \in E$. The action policy α is *memoryless* if for every pair of distinct finite paths $\rho v, \rho' v$ that end in the same vertex v , it holds that $\alpha(\rho v) = \alpha(\rho' v)$; in this case we simply write $\alpha(v)$. An action policy α generates a unique infinite path in $\mathcal{G}$, denoted $\text{out}(\alpha)$, and defined inductively as follows. The initial vertex is v^0 . For every prefix $v^0, \dots, v^i$ of $\text{out}(\alpha)$, for $i \geq 0$, $v^{i+1} = \alpha(v^0, \dots, v^i)$. We say that the policy α *satisfies* a given objective Φ , written $\alpha \models \Phi$ iff $\text{out}(\alpha) \in \Phi$.

3 The Auction-Based Scheduling Framework

Consider a graph $\mathcal{G} = \langle V, v^0, E \rangle$. A pair of objectives $\Phi_1, \Phi_2 \subseteq V^\omega$ in $\mathcal{G}$ are called *overlapping* if they have nonempty intersection (i.e., $\Phi_1 \cap \Phi_2 \neq \emptyset$). The *multi-objective planning problem* asks to synthesize an action policy that satisfies the *global* objective $\Phi_1 \cap \Phi_2$ for overlapping Φ_1, Φ_2 .

We propose a decentralized approach to the problem. Our goal is to design two action policies α_1 and α_2 for Φ_1 and Φ_2 , respectively. We will equip each action policy with a *bidding policy*, which it will use at runtime to bid for choosing the action at each time point. We formalize this below.

Definition 1 (Bidding policies). A bidding policy is a function $\beta: V \times [0, 1] \rightarrow [0, 1]$ with the constraint that $\beta(v, B) \leq B$ for every vertex v and every amount of available budget $B \in [0, 1]$.

We equip a pair of action and bidding policies with a *threshold budget*, which represents the greatest lower bound on the initial budget needed for the policies to guarantee their objective, and we call the resulting triple a *tender*.

Definition 2 (Tenders). A tender for a given graph $\mathcal{G}$ is a triple $\langle \alpha, \beta, \mathbb{B} \rangle$ of an action policy α , a bidding policy β , and a threshold budget $\mathbb{B} \in [0, 1]$. The set of all tenders for $\mathcal{G}$ is denoted $\mathcal{T}^{\mathcal{G}}$. A tender τ satisfies an objective Φ , denoted $\tau \models \Phi$, iff $\alpha \models \Phi$ (i.e., when the tender is operating alone on the graph).

Next, we formalize the composition of two tenders at runtime, which produces an action policy that uses a register of memory to keep track of the available budgets. We introduce some notation. A *configuration* is a pair $\langle v, B_1 \rangle$, where v is a vertex and B_1 is the budget available to the first tender. We normalize the sum of budgets to 1, hence implicitly, the budget available to the second tender is $B_2 = 1 - B_1$. Let $\mathcal{C} = V \times [0, 1]$ be the set of all configurations. For a given sequence of configurations $s = (v^0, B_1^0)(v^1, B_1^1) \dots \in \mathcal{C}^\infty$, let $\text{proj}_V(s)$ denote the path $v^0 v^1 \dots$. A *history* is a finite sequence of configurations $\langle v^0, B_1^0 \rangle \dots \langle v^k, B_1^k \rangle \in \mathcal{C}^*$ with the constraint that $\text{proj}_V(s) \in \text{Paths}^{\text{fin}}(\mathcal{G})$. Let $\mathcal{H}$ be the set of all histories.

Definition 3 (Composition of tenders). Let $\mathcal{G}$ be a graph, and $\tau_1 = \langle \alpha_1, \beta_1, \mathbb{B}_1 \rangle$ and $\tau_2 = \langle \alpha_2, \beta_2, \mathbb{B}_2 \rangle$ be two tenders. The tenders τ_1 and τ_2 are compatible iff $\mathbb{B}_1 + \mathbb{B}_2 < 1$. If compatible, then their composition, denoted $\tau_1 \bowtie \tau_2$, is a function

$\tau_1 \bowtie \tau_2: \mathcal{H} \rightarrow \mathcal{C}$ defined as follows. Given a history $h = \langle v^0, B_1^0 \rangle \dots \langle v^k, B_1^k \rangle \in \mathcal{H}$, let $b_1 := \beta_1(v^k, B_1^k)$ and $b_2 := \beta_2(v^k, 1 - B_1^k)$. Then,

- if $b_1 \geq b_2$, then $\tau_1 \bowtie \tau_2(h) = (\alpha_1(\rho v), B_1 - b_1)$, and
- if $b_1 < b_2$, then $\tau_1 \bowtie \tau_2(h) = (\alpha_2(\rho v), B_1 + b_2)$.

Given an initial configuration $\langle v^0, B_1^0 \rangle$ with $B_1^0 > \mathbb{B}_1$ and $B_2^0 = 1 - B_1^0 > \mathbb{B}_2$, the composition outputs an infinite sequence of configurations, denoted $\text{out}(\tau_1 \bowtie \tau_2)$, where $\text{out}(\tau_1 \bowtie \tau_2) := \langle v^0, B_1^0 \rangle \langle v^1, B_1^1 \rangle \dots \in \mathcal{C}^\omega$ such that for every k , $\langle v^k, B_1^k \rangle = \tau_1 \bowtie \tau_2 (\langle v^0, B_1^0 \rangle \dots \langle v^{k-1}, B_1^{k-1} \rangle)$. We will say $\tau_1 \bowtie \tau_2$ satisfies a given objective Φ , written $\tau_1 \bowtie \tau_2 \models \Phi$, iff $\text{proj}_V(\text{out}(\tau_1 \bowtie \tau_2)) \in \Phi$.

We will often use the index $i \in \{1, 2\}$ to refer to either of the two tenders or their attributes, and will use $-i = 3 - i$ for the “other” one, e.g., τ_i and τ_{-i} . Notice the difference between $\mathbb{B}_i$ and B_i^0 : $\mathbb{B}_i$ is the threshold budget at v^0 which is a constant attribute of τ_i , whereas B_i^0 is the actual budget initially allocated to τ_i whose value can be anything above $\mathbb{B}_i$.

3.1 Classes of decentralized synthesis problem

In this section, we describe three classes of decentralized synthesis problems that we study. Throughout this section, fix a graph $\mathcal{G}$ and a given pair of overlapping objectives Φ_1 and Φ_2 .

Strong decentralized synthesis. Here, tenders make no assumptions on each other, thus the solutions provide the strongest (the most robust) guarantees. Formally, for each $i \in \{1, 2\}$, the goal is to construct τ_i such that for every compatible τ_{-i} , we have $\tau_i \bowtie \tau_{-i} \models \Phi_i$.

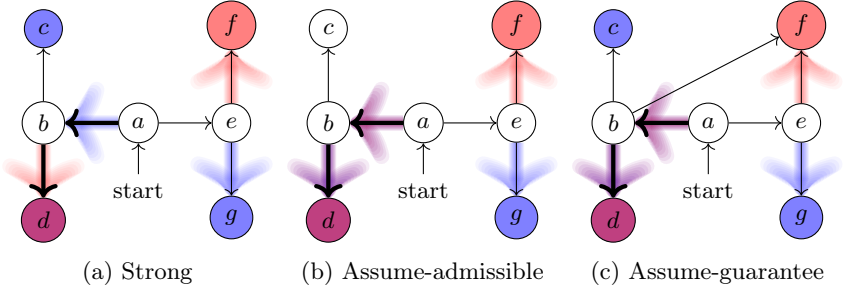


Fig. 1: Graphs with two reachability objectives given by targets: T_{blue} , depicted in blue, T_{red} depicted in red, and $T_{\text{blue}} \cap T_{\text{red}}$ depicted in purple. The action policies of the red and blue tenders choose edges with, respectively, red and blue shadows (shared edges are in purple). If no edges from a vertex have red or blue shadow, then the respective tender is indifferent about the choice made. Thick edges depict the paths taken by the compositions of tenders.

Example 1. Consider the graph depicted in Fig. 1a with a pair of *reachability* objectives having the targets $T_{\text{blue}} = \{c, d, g\}$ and $T_{\text{red}} = \{d, f\}$, respectively. Their intersection $\{d\}$ is depicted in purple. We present a pair of robust tenders τ_{blue} and τ_{red} with $\mathbb{B}_{\text{blue}} = 1/4$ and $\mathbb{B}_{\text{red}} = 1/2$, so that τ_{blue} and τ_{red} are compatible. We will show that τ_{blue} guarantees that no matter which compatible tender it is composed with, eventually T_{blue} is reached, and similarly τ_{red} ensures that T_{red} is reached. Therefore, $\tau_{\text{blue}} \bowtie \tau_{\text{red}}$ ensures that d is reached.

We first describe τ_{blue} . Consider an initial configuration $\langle a, 1/4 + \epsilon \rangle$, for any $\epsilon > 0$. Note that the other tender's budget is $3/4 - \epsilon$. The first action of τ_{blue} is $\langle b, 1/4 \rangle$. There are two possibilities. First, τ_{blue} wins the bidding, then we reach the configuration $\langle b, \epsilon \rangle$, and since both successors of b are in T_{blue} , the objective is satisfied in the next step. Second, τ_{blue} loses the bidding, meaning that the other tender bids at least $1/4$, and in the worst case, we proceed to the configuration $\langle e, 1/2 + \epsilon \rangle$. Next, τ_{blue} chooses $\langle g, 1/2 \rangle$ and necessarily wins as τ_{red} 's budget is only $1/2 - \epsilon$, and we reach $g \in T_{\text{blue}}$. We stress that τ_{blue} can be entirely oblivious about τ_{red} , except for the implicit knowledge of τ_{red} 's budget.

We describe τ_{red} . Consider an initial configuration $\langle a, 1/2 + \epsilon \rangle$, for any $\epsilon > 0$. Initially, τ_{red} bids 0, because it does not have a preference between going left or right. In the worst case, the budget stays $1/2 + \epsilon$ in the next turn. Since both b and e have single successors in T_{red} , thus τ_{red} must win the bidding. It does so by bidding $1/2$, which exceeds the available budget $1/2 - \epsilon$ of τ_{red} . $\triangle$

We now use the same problem as in Ex. 1, and show that the conventional turn-based interaction may fail to fulfill both objectives.

Example 2. Consider again the graph depicted in Fig. 1a with the targets $T_{\text{blue}} = \{c, d, g\}$ and $T_{\text{red}} = \{d, f\}$. Suppose α_{blue} and α_{red} are the two respective action policies, and we arbitrarily decide to make their interaction turn-based, where α_{red} chooses actions in a and α_{blue} chooses actions in b and e . It is clear that no matter which edge α_{red} chooses from a , it cannot guarantee satisfaction of T_{red} , because α_{blue} can take the game to c or g depending on α_{red} 's choice. $\triangle$

Assume-admissible decentralized synthesis. While the guarantees of strong decentralized synthesis are appealing, it often fails as each tender makes the pessimistic assumption that the other tender can behave arbitrarily—even adversarially. We consider *admissibility* [23] as a stronger assumption based on rationality, ensuring compatible tenders to exist even when strong synthesis may fail. We illustrate the idea in the following example.

Example 3. Consider the graph in Fig. 1b, with reachability objectives given by targets $T_{\text{blue}} = \{d, g\}$ and $T_{\text{red}} = \{d, f\}$. We argue that strong decentralized synthesis is not possible. Indeed, using the same reasoning for τ_{red} in Ex. 1, we have $Th_{\text{blue}}(a) = Th_{\text{red}}(a) = 0.5$. On the other hand, observe that when synthesizing τ_{red} , since $c \notin T_{\text{blue}}$, we know that a “rational” τ_{blue} —formally, *admissible* τ_{blue} (see Sec. 6)—will not proceed from b to c , and we can omit the edge. In turn, the threshold in a decreases to $1/4$ for both objectives. Since the sum of thresholds is now less than 1, two compatible tenders can be obtained. $\triangle$

In general, we seek an *admissible-winning tender*, which ensures that its objective is satisfied when composed with any admissible tender. Admissible-winning tenders are modular because they can be reused provided that the set of admissible actions of the other tender remains unchanged. For example, even when vertex g is added to the red target set, the blue tender can be used with no change. Somewhat surprisingly, we show that in graphs in which all vertices have out-degree at most 2, assume-admissible decentralized synthesis is always possible, and a pair of admissible-winning tenders can be found in PTIME.

Assume-guarantee decentralized synthesis. Sometimes, even the admissibility assumption is too weak, and we need more direct synchronization of the tenders. We consider assume-guarantee decentralized synthesis, where each tender needs to respect a pre-specified *contract*, and as a result, their composition satisfies both objectives. We illustrate the idea below.

Example 4. Consider the graph depicted in Fig. 1c, with reachability objectives given by targets $T_{\text{blue}} = \{c, d, g\}$ and $T_{\text{red}} = \{d, f\}$. Here, the strong decentralized synthesis fails due to reasons similar to Ex. 3. The assume-admissible decentralized synthesis fails because from e , both objectives cannot be fulfilled, and from b , no matter which tender wins the bidding can use an admissible edge that violates the other objective (e.g., (b, c) is admissible for τ_{blue} but violates T_{red}). We consider the *contract* $\langle G_{\text{blue}}, G_{\text{red}} \rangle = \langle \mathbf{G} \neg c, \mathbf{G} \neg f \rangle$, which is satisfied when (a) if α_{blue} fulfills G_{blue} , then α_{red} fulfills G_{red} , and (b) if α_{red} fulfills G_{red} , then α_{blue} fulfills G_{blue} . Now whichever tender wins the bidding at b needs to fulfill its guarantee, because it cannot judge from the past interaction if the other tender violates its guarantee. Therefore, from b , the next vertex will be d under the contract, and using the same tenders from Ex. 3, both objectives will be fulfilled.

4 An Aside on Bidding Games on Graphs

All our synthesis algorithms internally solve *bidding games*, which we briefly review here; see the survey [8] for more details. A (two-player) bidding game is played between *Player X* and *Player Y*, and is a tuple $\langle \mathcal{G}, \Phi \rangle$, where $\mathcal{G} = \langle V, E \rangle$ is the (finite, directed) graph and $\Phi \subseteq V^\omega$ is the objective for *Player X*. The game is *zero-sum*, meaning that the objective of *Player Y* is $V^\omega \setminus \Phi$, i.e., the violation of Φ . This differs from auction-based scheduling where objectives overlap; otherwise, the interaction between *Player X* and *Player Y* is the same as the one between tenders. A *strategy* for a player is a pair $\langle \alpha, \beta \rangle$ where α is an action policy and β is a bidding policy. As in the composition of tenders, two strategies and an initial configuration $\langle v, B_1 \rangle$ give rise to an infinite sequence of configurations called a *play*. A strategy is *winning* if no matter which strategy the opponent follows, the play satisfies the player's objective. A central quantity in bidding games is the *threshold budget* in a vertex v , which is intuitively, a necessary and sufficient initial budget for *Player X* to guarantee winning.

Definition 4 (Threshold budgets). Consider a bidding game $\langle \mathcal{G}, \Phi \rangle$ with $\mathcal{G} = \langle V, E \rangle$. The threshold of *Player X* is given by $\text{Th}_{\Phi}^{\mathcal{G}} : V \rightarrow [0, 1]$, where for every $v \in V$, we have $\text{Th}_{\Phi}^{\mathcal{G}}(v) = \inf_B \{ \text{Player X has a winning strategy from } \langle v, B \rangle \}$.

The threshold of *Player Y* is denoted as $Th_{\Phi^c}^G(v)$. The following theorem characterizes the structure of thresholds and states that the two players' thresholds are complementary. The intuition can be found in the full version [17], where we also show how winning strategies can be constructed from thresholds.

Theorem 1 ([38]). *Consider a reachability bidding game $\langle \mathcal{G}, \Phi \rangle$ where Φ is $Reach_{\mathcal{G}}(T)$ where, without loss of generality, T is a given set of sink vertices. For every vertex v , we have $Th_{\Phi}^G(v) = 1 - Th_{\Phi^c}^G(v)$. Moreover, for every sink vertex t , we have $Th_{\Phi}^G(t) = 0$, if $t \in T$, and $Th_{\Phi}^G(t) = 1$ otherwise. For every vertex v , we have $Th_{\Phi}^G(v) = 0.5 \cdot (Th_{\Phi}^G(v^+) + Th_{\Phi}^G(v^-))$, where v^- and v^+ are successors of v , such that for every other successor u , we have $Th_{\Phi}^G(v^-) \leq Th_{\Phi}^G(u) \leq Th_{\Phi}^G(v^+)$. Verifying if $Th_{\Phi}^G(v) > 0.5$ for a given vertex v is in $NP \cap coNP$ in general and is in $PTime$ for binary graphs.*

For infinite-horizon objectives, like parity, it is known that eventually one of the BSCCs will be reached, and inside every BSCC every vertex can be reached by both players infinitely often with every arbitrary initial budget. This implies that for every parity objective, the threshold of every vertex inside every BSCC in a game graph is either 0 or 1, and fulfilling a given parity objective is equivalent to reaching a BSCC whose every vertex has threshold 0. We state this formally.

Theorem 2 ([10]). *Consider a bidding game $\langle \mathcal{G}, \Phi \rangle$ with a parity objective Φ . Let S be a BSCC of $\mathcal{G}$. Every vertex in S has threshold either 0 or 1, and it is 1 iff the highest parity index in S is odd. Moreover, for a vertex v not in a BSCC, we have $Th_{\Phi}^G(v) = Th_{Reach_{\mathcal{G}}(T)}^G(v)$, where T is the union of BSCCs whose vertices have threshold 0.*

5 Strong Decentralized Synthesis

We study the *strong decentralized synthesis* problem, where the goal is to synthesize two compatible *robust tenders*, i.e., tenders that guarantee the fulfillment of their objectives when composed with *any* compatible tender.

Definition 5 (Robust tenders). *Let $\mathcal{G}$ be a graph and Φ_i be an objective in $\mathcal{G}$. A tender τ_i is robust for Φ_i if for every other compatible tender $\tau_{-i} \in \mathcal{T}^{\mathcal{G}}$, we have $\tau_i \bowtie \tau_{-i} \models \Phi_i$.*

Problem 1 (STRONG-SYNT). *Define STRONG-SYNT as the problem whose input is a tuple $\langle \mathcal{G}, \Phi_1, \Phi_2 \rangle$, where $\mathcal{G}$ is a graph and Φ_1 and Φ_2 are overlapping ω -regular objectives in $\mathcal{G}$, and the goal is to decide whether there exists a pair of tenders $\tau_1, \tau_2 \in \mathcal{T}^{\mathcal{G}}$ such that:*

- (I) τ_1 and τ_2 are compatible,
- (II) τ_1 is robust for Φ_1 , and
- (III) τ_2 is robust for Φ_2 .

Since each robust tender τ_i guarantees that Φ_i is satisfied when composed with *any* tender, the composition of two robust tenders satisfies both objectives:

Proposition 1 (Sound composition of robust tenders). *Let τ_1 and τ_2 be two compatible robust tenders for $\langle \mathcal{G}, \Phi_1, \Phi_2 \rangle$. Then $\tau_1 \bowtie \tau_2 \models \Phi_1 \cap \Phi_2$.*

We reduce the strong decentralized synthesis problem to the solution of two independent bidding games, both played on the graph $\mathcal{G}$, one with *Player X*'s objective Φ_1 and the other one with *Player X*'s objective Φ_2 . When the sum of thresholds in v^0 is less than 1, we set the two tenders to be winning *Player X* strategies in the two games with the threshold budgets of the tenders being set as the respective thresholds in v^0 . It follows from the construction that both tenders are robust, and hence their composition will fulfill both objectives (Prop. 1).

Theorem 3 (Strong decentralized synthesis). *Let $\mathcal{G} = \langle V, v^0, E \rangle$ be a graph and Φ_1 and Φ_2 be a pair of overlapping ω -regular objectives. A pair of robust tenders exists iff $Th_{\Phi_1}^{\mathcal{G}}(v^0) + Th_{\Phi_2}^{\mathcal{G}}(v^0) < 1$. Moreover, STRONG-SYNT is in $NP \cap coNP$ in general and is in PTIME for binary graphs.*

Proof. First, assume that $Th_{\Phi_1}^{\mathcal{G}}(v^0) + Th_{\Phi_2}^{\mathcal{G}}(v^0) < 1$. For $i \in \{1, 2\}$, let $\langle \alpha_i, \beta_i \rangle$ denote a winning *Player X* strategy in the bidding game $\langle \mathcal{G}, \Phi_i \rangle$ from every configuration $\langle v^0, B \rangle$ with $B > Th_{\Phi_i}^{\mathcal{G}}(v^0)$. We argue that the render $\tau_1 = \langle \alpha_1, \beta_1, Th_{\Phi_1}^{\mathcal{G}}(v^0) \rangle$ is robust for Φ_1 , and the proof for τ_2 is dual. Indeed, for any compatible tender $\tau'_2 = \langle \alpha'_2, \beta'_2, \mathbb{B}'_2 \rangle$, the pair $\langle \alpha'_2, \beta'_2 \rangle$ corresponds to a *Player Y* strategy in the bidding game $\langle \mathcal{G}, \Phi_1 \rangle$. The resulting play coincides with $out(\tau_1 \bowtie \tau'_2)(\langle v^0, B \rangle)$ and satisfies Φ_1 since the strategy $\langle \alpha_1, \beta_1 \rangle$ is winning.

Second, suppose that $Th_{\Phi_1}^{\mathcal{G}}(v^0) + Th_{\Phi_2}^{\mathcal{G}}(v^0) \geq 1$. For any allocation $\mathbb{B}_1 + \mathbb{B}_2 < 1$, there is an $i \in \{1, 2\}$ such that $\mathbb{B}_i \leq Th_{\Phi_i}^{\mathcal{G}}(v^0)$. Assume WLog that $\mathbb{B}_1 \leq Th_{\Phi_1}^{\mathcal{G}}(v^0)$. Consider a winning *Player Y* strategy $\langle \alpha_2, \beta_2 \rangle$ in the bidding game $\langle \mathcal{G}, \Phi_1 \rangle$ from $\langle v^0, \mathbb{B}_1 \rangle$. The tender $\tau'_2 = \langle \alpha_2, \beta_2, 1 - \mathbb{B}_1 \rangle$ is compatible and $out(\tau_1 \bowtie \tau'_2)(\langle v^0, \mathbb{B}_1 \rangle)$ violates Φ_1 .

Finally, in order to obtain the complexity bounds, we guess memoryless action policies in both games, which are known to exist [37], and verify that they are optimal. Based on the guess, we devise a linear program to compute the thresholds. Finally, we verify that the sum of thresholds in v^0 is less than 1. For binary graphs, there is no need to guess the action policy in order to find thresholds (Thm. 1). $\square$

We identify a setting where strong decentralized synthesis is always possible. The following theorem follows from the result that threshold budgets in strongly-connected Büchi games containing at least one accepting vertex are 0.

Theorem 4 (Strong decentralized synthesis on SCCs). *Consider a strongly-connected graph $\mathcal{G}$ and a pair of non-empty Büchi objectives in $\mathcal{G}$. Then, a pair of robust tenders exists in $\mathcal{G}$.*

We demonstrate the effectiveness of strong synthesis using path planning problems with two reachability objectives. Consider a fixed grid but four different instances of the problem, as shown in Fig. 2. For the first three cases, we successfully obtain pairs of robust tenders whose compositions fulfill both objectives. Moreover, since the blue target remained the same in all cases, we needed to redesign only the red tender, saving us a significant amount of computation.

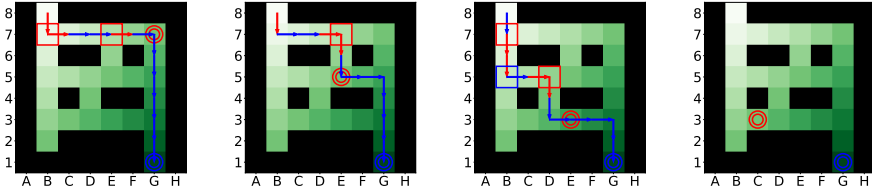


Fig. 2: Robust tenders for path planning with two reachability objectives on a one-way grid, where the black cells are obstacles and *the only permissible moves are from lighter to darker green cells*—and not the other way round. The cell B8 is the initial location. The cells with double circles of colors red (respectively, G7, E5, E3, C3) and blue (G1) are the targets to reach. The path shows the output of the composition of the two tenders, where the red and blue segments are actions which were chosen by the red and blue tenders, respectively. The cells with red and blue squares are locations where the respective tenders win the bidding; in the rest of the cells on the paths, the bidding ended in ties which were resolved randomly. Strong synthesis was successful in the first three instances and failed in the last one. The pairs of thresholds of red and blue targets are, respectively (left to right): (0.75, 0.125), (0.625, 0.125), (0.75, 0.125), (0.875, 0.125).

6 Assume-Admissible Decentralized Synthesis

In assume-admissible decentralized synthesis, each tender assumes that the other tender is *rational* and pursues its own objective. We formalize rationality by adapting the well-known concepts of *dominance* and *admissibility* from game theory [1,22]. Intuitively, τ_i dominates τ'_i if τ_i is always at least as good as τ_i and sometimes strictly better than τ_i ; therefore, there is no reason to use τ'_i . An admissible tender is one that is not dominated by any other tender.

Definition 6 (Dominance, admissibility). *Let $\mathcal{G}$ be a graph and Φ be an objective. We provide definitions for the first tender and the definitions for the second tender are dual. Let $\mathbb{B}_1 < 1$. For two tenders τ_1 and τ'_1 that have the same budget allocation, τ_1 dominates τ'_1 when*

- (a) τ_1 performs as well as τ'_1 when composed with any compatible τ_2 ; formally, for every compatible tender τ_2 , $\tau'_1 \bowtie \tau_2 \models \Phi$ implies $\tau_1 \bowtie \tau_2 \models \Phi$, and
- (b) there is a compatible tender τ_2 for which τ_1 performs better than τ'_1 ; formally, there exists a compatible τ_2 with $\tau_1 \bowtie \tau_2 \models \Phi$, and $\tau'_1 \bowtie \tau_2 \not\models \Phi$.

A tender τ_1 is called Φ -admissible in $\mathcal{G}$ iff it is not dominated by any other tender. We denote the set of Φ -admissible tenders in $\mathcal{G}$ by $\text{Adm}^{\mathcal{G}}(\Phi)$.

Next, we define admissible-winning tenders, which are tenders that fulfill their objectives when composed with *any* admissible tender.

Definition 7 (Admissible-winning tenders). Let $\mathcal{G}$ be a graph and Φ_1, Φ_2 be a pair of overlapping objectives in $\mathcal{G}$. A tender τ_i is called Φ_{-i} -admissible-winning for Φ_i if and only if $\tau_i \in \text{Adm}^{\mathcal{G}}(\Phi_i)$, and for every other tender $\tau_{-i} \in \text{Adm}^{\mathcal{G}}(\Phi_{-i})$ compatible with τ_i , we have $\tau_i \bowtie \tau_{-i} \models \Phi_i$.

When the objectives are clear from the context, we will omit them and will simply write a tender is “admissible tender,” “admissible-winning tender,” etc.

Problem 2 (AA-SYNT). Define AA-SYNT as the problem whose input is a tuple $\langle \mathcal{G}, \Phi_1, \Phi_2 \rangle$, where $\mathcal{G}$ is a graph and Φ_1 and Φ_2 are overlapping ω -regular objectives in $\mathcal{G}$, and the goal is to decide whether there exists a pair of tenders $\tau_1 \in \text{Adm}^{\mathcal{G}}(\Phi_1)$ and $\tau_2 \in \text{Adm}^{\mathcal{G}}(\Phi_2)$ such that:

- (I) τ_1 and τ_2 are compatible,
- (II) τ_1 is Φ_2 -admissible-winning for Φ_1 , and
- (III) τ_2 is Φ_1 -admissible-winning for Φ_2 .

The following proposition follows from the requirement that τ_1 and τ_2 are admissible.

Proposition 2 (Sound composition of admissible-winning tenders). Let τ_1 and τ_2 be tenders that fulfill the requirements stated in Prob. 2. Then, $\tau_1 \bowtie \tau_2 \models \Phi_1 \cap \Phi_2$.

Remark 1. Note that the synthesis procedure for each Φ_{-i} -admissible-winning tender τ_i for Φ_i requires the knowledge of $\text{Adm}^{\mathcal{G}}(\Phi_{-i})$. Assume-admissible decentralized synthesis is modular in the following sense. First, the specific implementation of the tender τ_{-i} with which each τ_i is composed is not known during synthesis. All that is known is the objective Φ_{-i} for which τ_{-i} is synthesized. Second, each τ_i can remain unchanged even when Φ_{-i} changes to Φ'_{-i} , as long as $\text{Adm}^{\mathcal{G}}(\Phi'_{-i}) \subseteq \text{Adm}^{\mathcal{G}}(\Phi_{-i})$.

6.1 Reachability objectives

Throughout this section we focus on overlapping reachability objectives $\Phi_1 = \text{Reach}_{\mathcal{G}}(T_1)$ and $\Phi_2 = \text{Reach}_{\mathcal{G}}(T_2)$ with $T_1, T_2 \subseteq V$ being sets of sink target vertices. This is without loss of generality, as every graph with non-sink target vertices can be converted into a graph with sink target vertices by adding memory (see the full version [17]).

We reduce the decentralized assume-admissible synthesis problem to solving a pair of zero-sum bidding games on a sub-graph of $\mathcal{G}$. Intuitively, an edge $e = \langle u, v \rangle$ is *dominated* for the i -th tender, for $i \in \{1, 2\}$, if it is possible to achieve the objective Φ_i from u but not from v . Clearly, a tender that chooses e is dominated and is thus not admissible (see the proof of the lemma in the full version [17]). Recall that $\text{Th}_{\Phi_i}^{\mathcal{G}}(v)$ denotes the threshold in the zero-sum bidding game played on $\mathcal{G}$ with the *Player X* objective Φ_i , and that $\text{Th}_{\Phi_i}^{\mathcal{G}}(v) = 1$ means there is no path from v to T_i .

Lemma 1 (A necessary condition for admissibility). *For every vertex u having at least two successors v, w with $Th_{\Phi_i}^{\mathcal{G}}(v) < 1$ and $Th_{\Phi_i}^{\mathcal{G}}(w) = 1$, if a Player i tender $\langle \alpha_i, \beta_i, \mathbb{B}_i \rangle$ is in $Adm^{\mathcal{G}}(\Phi_i)$, then $\alpha_i(u) \neq w$, for both $i \in \{1, 2\}$.*

Proof. We argue that choosing w from u is dominated by the action of choosing v from u , no matter what the budget at u is. Firstly, Cond. (a) of Def. 6 trivially holds. Secondly, consider the other tender τ_{-i} which bids zero at u , and later cooperates with τ_i to satisfy Φ_i . Clearly, the τ_i 's action policy that selects v at u will be able to satisfy Φ_i , but the one that selects w will not. $\square$

We obtain the reduced graph by omitting edges that are dominated for both players. For example, in Fig. 1b, the edge $\langle b, c \rangle$ is dominated for both players (see Ex. 3) and in Fig. 1c, no edge is dominated for both players (see Ex. 4).

Definition 8 (Largest admissible sub-graphs for reachability). *The largest admissible sub-graph of $\mathcal{G}$ with respect to two reachability objectives Φ_1 and Φ_2 is $\widehat{\mathcal{G}}_{\Phi_1, \Phi_2} = \langle V', E' \rangle$ with $V' = V \setminus \left\{ v \in V \mid Th_{\Phi_1}^{\mathcal{G}}(v) = 1 \wedge Th_{\Phi_2}^{\mathcal{G}}(v) = 1 \right\}$ and $E' = (V' \times V') \cap E$. We omit Φ_1, Φ_2 from $\widehat{\mathcal{G}}_{\Phi_1, \Phi_2}$ when it is clear from the context.*

For a vertex v in $\mathcal{G}$ and $i \in \{1, 2\}$, recall that $Th_{\mathcal{G}}^{\Phi_i}(v)$ denotes the threshold in $\mathcal{G}$ for objective Φ_i , and $Th_{\widehat{\mathcal{G}}}^{\Phi_i}(v)$ denotes the threshold in the reduced graph. Observe that a winning strategy in $\mathcal{G}$ will never cross a dominated edge. Removing dominated edges restricts the opponent, thus $Th_{\mathcal{G}}^{\Phi_i}(v) \geq Th_{\widehat{\mathcal{G}}}^{\Phi_i}(v)$. The next lemma shows that, surprisingly, the decrease in sum of thresholds is guaranteed to be significant. The proof (see the full version [17]) which holds for non-binary graphs, intuitively follows from observing that in $\widehat{\mathcal{G}}$, necessarily a sink that is a target for one of the players is reached, and since there is an overlap in at least one sink, the sum of thresholds is at most 1.

Lemma 2 (On the sum of thresholds in $\widehat{\mathcal{G}}$). *For every vertex v , we have $Th_{\widehat{\mathcal{G}}}^{\Phi_1}(v) + Th_{\widehat{\mathcal{G}}}^{\Phi_2}(v) \leq 1$. Moreover, if $\mathcal{G}$ is binary then $Th_{\widehat{\mathcal{G}}}^{\Phi_1}(v) + Th_{\widehat{\mathcal{G}}}^{\Phi_2}(v) < 1$.*

Our synthesis procedure proceeds as in strong decentralized synthesis: Find and output a pair of robust tenders in $\widehat{\mathcal{G}}$, which are guaranteed to exist when $\mathcal{G}$ is binary. In order to maintain soundness (see Prop. 2), it is key to show that a robust tender τ_i in $\widehat{\mathcal{G}}$ is admissible in $\mathcal{G}$. The proof of the following lemma is intricate (see the full version [17]). We show that even when one can find τ'_i and τ_{-i} such that $\tau_i \bowtie \tau_{-i} \not\models \Phi_i$ but $\tau'_i \bowtie \tau_{-i} \models \Phi_i$, it is possible to construct τ'_{-i} for which $\tau_i \bowtie \tau'_{-i} \models \Phi_i$ but $\tau'_i \bowtie \tau'_{-i} \not\models \Phi_i$, thus τ'_i does not dominate τ_i . Furthermore, such a tender wins against a set of tenders which *over-approximates* admissible tender for Φ_{-i} .

Lemma 3 (Algorithm for computing admissible-winning tenders). *For $i \in \{1, 2\}$, a robust tender for Φ_i in $\widehat{\mathcal{G}}$ is Φ_{-i} -admissible-winning for Φ_i in $\mathcal{G}$.*

The following theorem is obtained by combining Lemmas 2 and 3.

Theorem 5 (Assume-admissible decentralized synthesis for reachability). *The problem AA-SYNT is a tautology for binary graphs: for every binary graph and two overlapping reachability objectives, there exists a pair of compatible admissible-winning tenders. Moreover, the tenders can be computed in PTIME.*

Remark 2. For general (i.e., non-binary) graphs, AA-SYNT is not a tautology anymore; a counter-example is given in Ex. 4. However, the same PTIME algorithm for computing tenders can still be used to obtain a sound solution; the completeness question is left open for future work.

6.2 Büchi objectives

In this section, we consider binary graphs with a pair of overlapping Büchi objectives. We first demonstrate that, unlike reachability, it is not guaranteed that an assume-admissible decentralized solution exists.

Example 5. Consider the graph depicted in Fig. 3 with the Büchi objectives given by the accepting vertices $S_{\text{red}} = \{b, d\}$ and $S_{\text{blue}} = \{a, c\}$. Note that the objectives are overlapping since the path $(bc)^\omega$ satisfies both. We argue that no pair of compatible admissible-winning tenders exist. Note that a robust (hence dominant) red tender forces reaching d , thus forcing Φ_{red} to be satisfied. Dually, a robust blue tender forces Φ_{blue} in a . It can be shown that $Th_{\Phi_{\text{red}}}^{\mathcal{G}}(b) = 2/3$ and $Th_{\Phi_{\text{blue}}}^{\mathcal{G}}(b) = 1/3$. Thus, for any $\mathbb{B}_{\text{red}}$ and $\mathbb{B}_{\text{blue}}$ with $\mathbb{B}_{\text{red}} + \mathbb{B}_{\text{blue}} < 1$, there is a robust tender that violates the other tender's objective. $\triangle$

We generalize the concept of largest admissible subgraphs to Büchi objectives. It is not hard to show that proceeding into a BSCC with an accepting state is admissible. Indeed, Thm. 2 shows that there is a robust (hence admissible) tender in such BSCCs. On the other hand, proceeding to a BSCC with no accepting vertex is clearly not admissible. The largest admissible subgraph is obtained by repeatedly removing BSCC that are not admissible for both tenders. Formally, for a given action policy α and a given vertex v of $\mathcal{G}$, we will write $\alpha \not\models_v \Phi_1 \cup \Phi_2$ to indicate that the action policy cannot fulfill $\Phi_1 \cup \Phi_2$ from the initial vertex v .

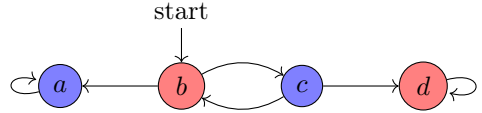


Fig. 3: A graph with no assume-admissible decentralized solution.

Definition 9 (Largest admissible sub-graphs for Büchi). *The largest admissible sub-graph $\hat{\mathcal{G}}_{\mathcal{B}}$ of $\mathcal{G}$ for the Büchi objectives Φ_1, Φ_2 is the graph $\langle V', E' \rangle$ with $V' = V \setminus \{v \in V \mid \forall \text{ action policy } \alpha . \alpha \not\models_v \Phi_1 \cup \Phi_2\}$, and $E' = (V' \times V') \cap E$.*

We describe a reduction to reachability games. For $i \in \{1, 2\}$, let T_i denote the union of BSCCs of $\hat{\mathcal{G}}_{\mathcal{B}}$ in which there is at least one Büchi accepting vertex for Φ_i . We call $\langle T_1, T_2 \rangle$ the *reachability core* of $\langle \Phi_1, \Phi_2 \rangle$ in $\hat{\mathcal{G}}_{\mathcal{B}}$. Let $\Phi'_1 = \text{Reach}_{\hat{\mathcal{G}}_{\mathcal{B}}}(T_1)$ and $\Phi'_2 = \text{Reach}_{\hat{\mathcal{G}}_{\mathcal{B}}}(T_2)$. We proceed as in strong decentralized synthesis: we find

$Th_{\Phi'_1}^{\widehat{\mathcal{G}}_{\mathcal{B}}}(v^0)$ and $Th_{\Phi'_2}^{\widehat{\mathcal{G}}_{\mathcal{B}}}(v^0)$ and return a pair of robust tenders if their sum is strictly less than 1. Note that unlike reachability objectives, in Büchi objectives the sum might be 1 as in Ex. 5. Moreover, as Ex. 5 illustrates, when the sum is 1, no pair of admissible-winning tenders exist. By adapting results from the previous section, we obtain the following.

Theorem 6 (Assume-admissible decentralized synthesis for Büchi).

Let $\mathcal{G}$ be a binary graph and Φ_1, Φ_2 be a pair of overlapping Büchi objectives. Let $\langle T_1, T_2 \rangle$ be the reachability core of $\langle \Phi_1, \Phi_2 \rangle$ in the largest admissible sub-graph of $\mathcal{G}$ (for Φ_1, Φ_2). A pair of admissible-winning tenders exists iff $T_1 \cap T_2 \neq \emptyset$. Moreover, AA-SYNT for Büchi objectives is in PTIME.

Like reachability (see Rem. 2), for Büchi objectives the same algorithm for AA-SYNT for binary graphs can be used to obtain a sound solution for general graphs, and the completeness question is left open for future work.

7 Assume-Guarantee Decentralized Synthesis

We present the assume-guarantee decentralized synthesis problem, the one with the highest degree of synchronization among the tenders, with the benefit of the most applicability. In this synthesis procedure, we assume that we are given a pair of languages $A_1, A_2 \subseteq V^\omega$, called the *assumptions*. Intuitively, each tender τ_i can *assume* A_i is fulfilled by the other tender, and, in return, needs to *guarantee* that A_{-i} is fulfilled, in addition to fulfilling own objective.

Definition 10 (Contract-abiding tenders). *Let $\mathcal{G}$ be a graph, Φ_i be an ω -regular objective, and A_1, A_2 be a pair of ω -regular assumptions in $\mathcal{G}$. We say a tender $\tau_i = \langle \alpha_i, \cdot, \cdot \rangle \in \mathcal{T}^{\mathcal{G}}$ fulfills Φ_i under the contract $\langle A_1, A_2 \rangle$, written $\tau_i \models \langle A_i \triangleright \Phi_i \triangleright A_{-i} \rangle$, iff*

- (a) *for every finite path ρ , if ρ is in $\text{pref}(A_i)$, then $\rho \cdot \alpha_i(\rho) \in \text{pref}(A_1 \cap A_2)$, and*
- (b) *for every other compatible tender $\tau_{-i} \in \mathcal{T}^{\mathcal{G}}$, we have $\tau_i \bowtie \tau_{-i} \models \text{Safe}_{\mathcal{G}}(\text{pref}(A_1 \cap A_2)) \implies (A_{-i} \wedge (A_i \implies \Phi_i))$.*

Here, each tender τ_i only make safety assumption on the other tender (Cond. (a)), namely that the path does not leave the safe set $\text{pref}(A_i)$, and in return, provides full guarantee on A_{-i} (Cond. (b)). Normally, safety assumptions are not enough for fulfilling liveness guarantees and objectives [5]. But in bidding games, within the safe set, the players can use a known bidding tactic [9] to accumulate enough budgets from time to time to reach the liveness goals always eventually. We use A_1, A_2 as ω -regular sets, though we conjecture that safety assumptions suffice. The assume-guarantee distributed synthesis problem asks to compute a pair of tenders that fulfill their objectives under the given contract, as stated below.

Problem 3 (AG-SYNT). *Define AG-SYNT as the problem that takes as input a tuple $\langle \mathcal{G}, \Phi_1, \Phi_2, A_1, A_2 \rangle$, where $\mathcal{G}$ is a graph, Φ_1 and Φ_2 are overlapping objectives in $\mathcal{G}$, and A_1 and A_2 are two ω -regular languages over V with $v^0 \in \text{pref}(A_1 \cap A_2)$, and the goal is to decide whether there exists a pair of tenders $\tau_1, \tau_2 \in \mathcal{T}^{\mathcal{G}}$ such that:*

- (I) τ_1 and τ_2 are compatible,
- (II) $\tau_1 \models \langle A_1 \triangleright \Phi_1 \triangleright A_2 \rangle$, and
- (III) $\tau_2 \models \langle A_2 \triangleright \Phi_2 \triangleright A_1 \rangle$.

When the assumptions allow all behaviors, i.e., $A_1 = A_2 = V^\omega$, then AG-SYNT is equivalent to STRONG-SYNT. On the other hand, when the assumptions allow only each other's objectives, i.e., $A_1 = \Phi_1$ and $A_2 = \Phi_2$, then we obtain a purely cooperative synthesis algorithm. We prove that satisfaction of the contracts by a pair of tenders will imply satisfaction of $\Phi_1 \cap \Phi_2$.

Proposition 3 (Sound composition of contract-abiding tenders). *Let τ_1 and τ_2 be tenders that fulfill the requirements stated in Prob. 3. Then, $\tau_1 \bowtie \tau_2 \models \Phi_1 \cap \Phi_2$.*

Proof. In the following, for a given language $L \in V^\omega$, we write $\text{Safe}_{\mathcal{G}}(\text{pref}(L))$ to denote the set of infinite paths which can always be extended to L , i.e., $\{v^0 v^1 \dots \in \text{Paths}^{\text{inf}}(\mathcal{G}) \mid \forall i \geq 0. v^0 \dots v^i \in \text{pref}(L)\}$.

We claim that both assumptions A_1, A_2 will be fulfilled, from which Cond. (b) of Def. 10 will imply satisfaction of both Φ_1 and Φ_2 by τ_1 and τ_2 , respectively. Let $A = A_1 \cap A_2$, and A can be decomposed into safety and liveness components as $A = \text{Safe}_{\mathcal{G}}(\text{pref}(A)) \cap (\text{Safe}_{\mathcal{G}}(\text{pref}(A)) \implies A)$. We prove the claim on the two components separately. Firstly, the fact that $\tau_1 \bowtie \tau_2$ implements $\text{pref}(A)$ on $\mathcal{G}$ can be proven by induction over the length of the generated path: The base case is given by the assumption $v^0 \in \text{pref}(A_1 \cap A_2)$, and for every finite path ρ , if τ_i wins the bidding and if $\rho \in \text{pref}(A_1 \cap A_2) \subset \text{pref}(A_i)$ then τ_i needs to ensure that the next vertex v' satisfies $\rho v' \in \text{pref}(A_i \cap A_{-i})$ (consequence of Cond. II-III of Prob. 3 and Cond. (a) of Def. 10), thereby implying that the path will always remain inside $\text{pref}(A_1 \cap A_2)$, proving the safety part.

For the liveness part, we use known results from Richman bidding games, which guarantee that in an infinite horizon game, with any arbitrary positive initial budget, players can always eventually visit any vertex that can be reached [37]. This implies that if the invariance $\text{Safe}_{\mathcal{G}}(\text{pref}(A_1 \cap A_2))$ holds, then each tender τ_i can actually fulfill A_{-i} (they are required to do so by Cond. (b) of Def. 10) when composed with any compatible tender in the long run. Therefore, $A_1 \cap A_2$ will be fulfilled. $\square$

In bidding games literature, it is unknown how to compute strategies for objectives which can be violated if a given assumption is violated by the opponent, like in Cond. (a) in Def. 10. The challenge stems from the lack of separation of the set of available actions to the players, preventing us to impose assumptions only on the opponent's behavior. We present a practically motivated sound, but possibly incomplete, solution for the decentralized synthesis problem, by using a stronger way of satisfying the contract, namely by requiring each tender τ_i to use actions so that the generated path remains in $\text{pref}(A_1 \cap A_2)$ all the time. Formally, we say that the tender τ_i *strongly* fulfills Φ_i under the contract $\langle A_1, A_2 \rangle$, written $\tau_i \models_s \langle A_i \triangleright \Phi_i \triangleright A_{-i} \rangle$, if, instead of Cond. (a) of Def. 10, for every finite path ρ , we have $\rho \cdot \alpha_i(\rho) \in \text{pref}(A_1 \cap A_2)$, regardless of whether $\rho \in \text{pref}(A_i)$

or not, and moreover Cond. (b) of Def. 10 is fulfilled. It is easy to show that $\tau_i \models_s \langle A_i \triangleright \Phi_i \triangleright A_{-i} \rangle$ implies $\tau_i \models \langle A_i \triangleright \Phi_i \triangleright A_{-i} \rangle$, so that $\Phi_1 \cap \Phi_2$ will be fulfilled.

Similar to AA-SYNT, we extract a sub-graph $\mathcal{G}'$ of $\mathcal{G}$, called the *largest contract-satisfying sub-graph*, whose every path belongs to $\text{pref}(A_1 \cap A_2)$, and vice versa; we omit the construction, which follows usual automata-theoretic procedure from the literature [4]. For example, in Ex. 4, the largest contract-satisfying sub-graph of the graph in Fig. 1c is the one that only excludes the vertices c and f . It follows that when the tenders strongly fulfill their objectives under the contracts, it is guaranteed that every path always remains in $\mathcal{G}'$.

Theorem 7 (Assume-guarantee decentralized synthesis). *Let $\mathcal{G} = \langle V, v^0, E \rangle$ be a graph, Φ_1 and Φ_2 be a pair of overlapping ω -regular objectives, and A_1 and A_2 be ω -regular assumptions. Let $\mathcal{G}'$ be the largest contract-satisfying sub-graph of $\mathcal{G}$. A pair of robust tenders exist if $\text{Th}_{A_2 \cap \Phi_1}^{\mathcal{G}'}(v^0) + \text{Th}_{A_1 \cap \Phi_2}^{\mathcal{G}'}(v^0) < 1$. Moreover, AG-SYNT is in PTIME.*

8 Related Work

Shielding [35] is a framework in which a runtime monitor called a *shield* enforces an unverified policy π (e.g., generated using reinforcement learning [7]) to satisfy a given specification. A shield operates by observing, at each point in time, the action proposed by π and can alter it, e.g., if safety is violated. The choice of who acts at each point in time, π or the shield, can be seen as a scheduling choice similar to our setting. However, the goals of the two approaches are different: our goal is to design tenders for modular policy synthesis, whereas a shield is meant as a verified “wrapper” for a complex policy. Technically, in auction-based scheduling, the scheduling depends on the auction which is external to the policies, whereas in shielding, it is the shield who chooses whether to override π .

In *distributed reactive synthesis* [42,36,32], the goal is to design a collection of Mealy machines whose communication is dictated by a given *communication architecture*. Distributed synthesis is well studied and we point to a number of works that considered objectives that are a conjunction $\Phi_1 \wedge \Phi_2 \wedge \dots$ of sub-objectives $\Phi_1, \Phi_2, \dots$ [30,21,39,31,18,6]. While there is a conceptual similarity between our synthesis of tenders and the synthesis of Mealy machines, there is a fundamental difference between the approaches. Namely, our composition is based on scheduling, i.e., exactly one policy is scheduled at each point in time, whereas in distributed synthesis, the composition of the Mealy machines is performed in parallel, i.e., they all read and write at each point in time.

Our algorithms build upon the rich literature on bidding games on graphs. The bidding mechanism that we focus on is called *Richman* bidding [38,37,10]. Other bidding mechanisms have been studied: *poorman* [11], *taxman* [13], and *all-pay* [14,15]. Auction-based scheduling can be instantiated with any of these mechanisms and the properties from bidding games transfer immediately (which differ significantly for quantitative objectives). Of particular interest in practice is *discrete bidding*, in which the granularity of the bids is restricted [27,2,16]. To

the best of our knowledge, beyond our work, non-zero-sum bidding games have only been considered in [40]. The solution concept that they consider is *subgame perfect equilibrium* (SPE). While it is suitable to model the interaction between selfish agents, we argue that it is less suitable in decentralized synthesis.

There are many works on designing optimal policies for multi-objective sequential decision making problems for various different system models; see the survey by Roijers et al. [43] and works on multi-objective stochastic games [20,21,24]. To the best of our knowledge, no prior work considers the decomposition of the problem into individual task-dependent policies like us. Auctions to distribute tasks to agents have been considered extensively [29,44,25,28,19,26,41]. Their goal is very different: their agents bid for tasks, that is, a bid represents an agent’s cost (e.g., in terms of resources) for performing a task. The auction then allocates the tasks to agents so as to minimize the individual costs, giving rise to an efficient global policy.

9 Conclusion and Future Work

We present the auction-based scheduling framework. Rather than synthesizing a monolithic policy for a conjunction of objectives $\Phi_1 \wedge \Phi_2$, we synthesize two independent tenders for each of the objectives and compose the tenders at runtime using auction-based scheduling. A key advantage of the framework is modularity; each tender can be synthesized and modified independently. We study three instantiations of decentralized synthesis in planning problems with varying degree of flexibility and practical usability, and develop algorithms based on bidding games. Interestingly, we show that a pair of admissible-winning tenders always exists in binary graphs for reachability objectives and they can be found in PTIME. This positive result illustrates the strength and potential of the auction-based scheduling framework.

There are plenty of directions of future research and we list a handful. First, we consider only qualitative objectives and it is interesting to lift the results to *quantitative* objectives, where one can quantify the fairness achieved by the scheduling mechanism in a fine-grained manner. Moreover, it is appealing to employ the rich literature on *mean-payoff* bidding games. Second, we consider a conjunction of two objectives, and it is interesting to extend the approach to a conjunction of multiple objectives. This will require extending the theory of bidding games to the multi-player setting, which have not yet been studied. Finally, it is particularly interesting to extend the technique of auction-based scheduling beyond path-planning problems, for example, it is interesting to consider decentralized synthesis of controllers that operate in an adversarial or probabilistic environment. Again, the corresponding bidding games need to be studied (so far only sure winning has been considered for bidding games played on MDPs [12]).

Acknowledgements

This work was supported in part by the ERC project ERC-2020-AdG 101020093 and by ISF grant no. 1679/21.

References

1. Adam, B., Amanda, F., Keisler, H.J.: Admissibility in games. In: *Econometrica* (2008)
2. Aghajohari, M., Avni, G., Henzinger, T.A.: Determinacy in discrete-bidding infinite-duration games. *Log. Methods Comput. Sci.* **17**(1) (2021)
3. de Alfaro, L., Faella, M., Majumdar, R., Raman, V.: Code aware resource management. In: *Proceedings of the 5th ACM international conference on Embedded software*. pp. 191–202 (2005)
4. Alpern, B., Schneider, F.B.: Recognizing safety and liveness. *Distributed computing* **2**, 117–126 (1987)
5. Amla, N., Emerson, E.A., Namjoshi, K., Treffer, R.: Assume-guarantee based compositional reasoning for synchronous timing diagrams. In: *International Conference on Tools and Algorithms for the Construction and Analysis of Systems*. pp. 465–479. Springer (2001)
6. Anand, A., Nayak, S.P., Schmuck, A.K.: Contract-based distributed synthesis in two-objective parity games. *arXiv preprint arXiv:2307.06212* (2023)
7. Avni, G., Bloem, R., Chatterjee, K., Henzinger, T.A., Könighofer, B., Pranger, S.: Run-time optimization for learned controllers through quantitative games. In: *Proc. 31st CAV*. pp. 630–649 (2019)
8. Avni, G., Henzinger, T.A.: A survey of bidding games on graphs. In: *Proc. 31st CONCUR. LIPIcs*, vol. 171, pp. 2:1–2:21. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2020)
9. Avni, G., Henzinger, T.A., Chonev, V.: Infinite-duration bidding games. In: *Proc. 28th CONCUR. LIPIcs*, vol. 85, pp. 21:1–21:18 (2017)
10. Avni, G., Henzinger, T.A., Chonev, V.: Infinite-duration bidding games. *J. ACM* **66**(4), 31:1–31:29 (2019)
11. Avni, G., Henzinger, T.A., Ibsen-Jensen, R.: Infinite-duration poorman-bidding games. In: *Proc. 14th WINE. LNCS*, vol. 11316, pp. 21–36. Springer (2018)
12. Avni, G., Henzinger, T.A., Ibsen-Jensen, R., Novotný, P.: Bidding games on markov decision processes. In: *Proc. 13th RP*. pp. 1–12 (2019)
13. Avni, G., Henzinger, T.A., Zikelic, D.: Bidding mechanisms in graph games. *J. Comput. Syst. Sci.* **119**, 133–144 (2021)
14. Avni, G., Ibsen-Jensen, R., Tkadlec, J.: All-pay bidding games on graphs. In: *Proc. 34th AAAI*. pp. 1798–1805. AAAI Press (2020)
15. Avni, G., Jecker, I., Žikelić, Đ.: Infinite-duration all-pay bidding games. In: *Proc. 32nd SODA*. pp. 617–636 (2021)
16. Avni, G., Sadhukhan, S.: Computing threshold budgets in discrete-bidding games. In: *Proc. 42nd FSTTCS. LIPIcs*, vol. 250, pp. 30:1–30:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik (2022)
17. Avni, G., Mallik, K., Sadhukhan, S.: Auction-based scheduling. *CoRR abs/2310.11798* (2023), <https://doi.org/10.48550/arXiv.2310.11798>
18. Bansal, S., De Giacomo, G., Di Stasio, A., Li, Y., Vardi, M.Y., Zhu, S.: Compositional safety ltl synthesis. In: *Working Conference on Verified Software: Theories, Tools, and Experiments*. pp. 1–19. Springer (2022)
19. Basile, F., Chiacchio, P., Di Marino, E.: An auction-based approach to control automated warehouses using smart vehicles. *Control Engineering Practice* **90**, 285–300 (2019)
20. Basset, N., Kwiatkowska, M., Topcu, U., Wilsche, C.: Strategy synthesis for stochastic games with multiple long-run objectives. In: *Tools and Algorithms for*

- the Construction and Analysis of Systems: 21st International Conference, TACAS 2015, Held as Part of the European Joint Conferences on Theory and Practice of Software, ETAPS 2015, London, UK, April 11–18, 2015, Proceedings 21. pp. 256–271. Springer (2015)
21. Basset, N., Kwiatkowska, M., Wiltsche, C.: Compositional strategy synthesis for stochastic games with multiple objectives. *Information and Computation* **261**, 536–587 (2018)
 22. Berwanger, D.: Admissibility in infinite games. In: Thomas, W., Weil, P. (eds.) STACS 2007, 24th Annual Symposium on Theoretical Aspects of Computer Science, Aachen, Germany, February 22–24, 2007, Proceedings. Lecture Notes in Computer Science, vol. 4393, pp. 188–199. Springer (2007)
 23. Brenguier, R., Raskin, J.F., Sankur, O.: Assume-admissible synthesis. arXiv preprint arXiv:1507.00623 (2015)
 24. Chatterjee, K., Piterman, N.: Combinations of qualitative winning for stochastic parity games. arXiv preprint arXiv:1804.03453 (2018)
 25. Chong, C.Y., Kumar, S.P.: Sensor networks: evolution, opportunities, and challenges. *Proceedings of the IEEE* **91**(8), 1247–1256 (2003)
 26. De Ryck, M., Versteyhe, M., Debrouwere, F.: Automated guided vehicle systems, state-of-the-art control algorithms and techniques. *Journal of Manufacturing Systems* **54**, 152–173 (2020)
 27. Develin, M., Payne, S.: Discrete bidding games. *The Electronic Journal of Combinatorics* **17**(1), R85 (2010)
 28. Dias, M.B., Zlot, R., Kalra, N., Stentz, A.: Market-based multirobot coordination: A survey and analysis. *Proceedings of the IEEE* **94**(7), 1257–1270 (2006)
 29. Farber, D.J., Larson, K.C.: The structure of a distributed computing system—software. In: *Proceedings of the Symposium on Computer-communications Networks and Teletraffic*. pp. 539–545 (1972)
 30. Filiot, E., Jin, N., Raskin, J.F.: Compositional algorithms for ltl synthesis. In: *International Symposium on Automated Technology for Verification and Analysis*. pp. 112–127. Springer (2010)
 31. Finkbeiner, B., Passing, N.: Compositional synthesis of modular systems. *Innovations in Systems and Software Engineering* **18**(3), 455–469 (2022)
 32. Finkbeiner, B., Schewe, S.: Uniform distributed synthesis. In: *20th Annual IEEE Symposium on Logic in Computer Science (LICS’05)*. pp. 321–330. IEEE (2005)
 33. Hahn, E.M., Perez, M., Schewe, S., Somenzi, F., Trivedi, A., Wojtczak, D.: Multi-objective omega-regular reinforcement learning. *Formal Aspects of Computing* (2023)
 34. Houli, D., Zhiheng, L., Yi, Z.: Multiobjective reinforcement learning for traffic signal control using vehicular ad hoc network. *EURASIP journal on advances in signal processing* **2010**, 1–7 (2010)
 35. Könighofer, B., Alshiekh, M., Bloem, R., Humphrey, L., Könighofer, R., Topcu, U., Wang, C.: Shield synthesis. *Formal Methods in System Design* **51**(2), 332–361 (2017)
 36. Kupermann, O., Varfi, M.: Synthesizing distributed systems. In: *Proceedings 16th Annual IEEE Symposium on Logic in Computer Science*. pp. 389–398. IEEE (2001)
 37. Lazarus, A.J., Loeb, D.E., Propp, J.G., Stromquist, W.R., Ullman, D.H.: Combinatorial games under auction play. *Games and Economic Behavior* **27**(2), 229–264 (1999)
 38. Lazarus, A.J., Loeb, D.E., Propp, J.G., Ullman, D.: Richman games. *Games of No Chance* **29**, 439–449 (1996)

39. Majumdar, R., Mallik, K., Schmuck, A.K., Zufferey, D.: Assume-guarantee distributed synthesis. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems* **39**(11), 3215–3226 (2020)
40. Meir, R., Kalai, G., Tennenholtz, M.: Bidding games and efficient allocations. *Games and Economic Behavior* **112**, 166–193 (2018)
41. Ouelhadj, D., Petrovic, S.: A survey of dynamic scheduling in manufacturing systems. *Journal of scheduling* **12**, 417–431 (2009)
42. Pnueli, A., Rosner, R.: Distributed reactive systems are hard to synthesize. In: *Proceedings [1990] 31st Annual Symposium on Foundations of Computer Science*. pp. 746–757. IEEE (1990)
43. Roijers, D.M., Vamplew, P., Whiteson, S., Dazeley, R.: A survey of multi-objective sequential decision-making. *Journal of Artificial Intelligence Research* **48**, 67–113 (2013)
44. Smith, R.G.: The contract net protocol: High-level communication and control in a distributed problem solver. *IEEE Transactions on computers* **29**(12), 1104–1113 (1980)

Open Access This chapter is licensed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license and indicate if changes were made.

The images or other third party material in this chapter are included in the chapter's Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the chapter's Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder.

