Some Results On Point Visibility Graphs [1]

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Abstract

In this paper, we present three necessary conditions for recognizing point visibility graphs. We show that this recognition problem lies in PSPACE. We state new properties of point visibility graphs along with some known properties that are important in understanding point visibility graphs. For planar point visibility graphs, we present a complete characterization which leads to a linear time recognition and reconstruction algorithm.

1 Introduction

The visibility graph is a fundamental structure studied in the field of computational geometry and geometric graph theory [5, 9]. Some of the early applications of visibility graphs included computing Euclidean shortest paths in the presence of obstacles [14] and decomposing two-dimensional shapes into clusters [18]. Here, we consider problems from visibility graph theory.

Let $P = \{p_1, p_2, ..., p_n\}$ be a set of points in the plane (see Fig. 1). We say that two points p_i and p_j of P are *mutually visible* if the line segment p_ip_j does not contain or pass through any other point of P. In other words, p_i and p_j are visible if $P \cap p_ip_j = \{p_i, p_j\}$. If two vertices are not visible, they are called an *invisible pair*. For example, in Fig. 1(c), p_1 and p_5 form a visible pair whereas p_1 and p_3 form an invisible pair. If a point $p_k \in P$ lies on the segment p_ip_j connecting two points p_i and p_j in P, we say that p_k blocks the visibility between p_i and p_j, and p_k is called a *blocker* in P. For example in Fig. 1(c), p_5 blocks the visibility between p_1 and p_3 as p_5 lies on the segment p_1p_3. The *visibility graph* (also called the *point visibility graph* (PVG)) G of P is defined by associating a vertex v_i with each point p_i of P such that (v_i, v_j) is an undirected edge of G if and only if p_i and p_j are mutually visible (see Fig. 1(a)). Observe that if no three points of P are collinear, then G is a complete graph as each pair of points in P is visible since there is no blocker in P. Sometimes the visibility graph is drawn directly on the point set, as shown in Figs. 1(b) and 1(c), which is referred to as a *visibility embedding* of G.

Given a point set P, the visibility graph G of P can be computed as follows. For each point p_i of P, the points of P are sorted in angular order around p_i. If two points p_j and p_k are consecutive in the sorted order, check whether p_i, p_j and p_k are collinear points. By traversing the sorted order, all points of P, that are not visible from p_i, can be identified in $O(n \log n)$ time. Hence, G can be computed from P in $O(n^2 \log n)$ time. Using the result of Chazelle et al. [4] or Edelsbrunner et al. [7], the time complexity of the algorithm can be improved to $O(n^2)$ by computing sorted angular orders for all points together in $O(n^2)$ time.

Consider the opposite problem of determining if there is a set of points P whose visibility graph is the given graph G. This problem is called the visibility graph *recognition* problem. Identifying the set of properties satisfied by all visibility graphs is called the visibility graph *characterization* problem. The problem of actually drawing one such set of points P whose visibility graph is the given graph G, is called the visibility graph *reconstruction* problem.

Here we consider the recognition problem: Given a graph G in adjacency matrix form, determine whether G is the visibility graph of a set of points P in the plane [10]. In Sect. 2, we present three necessary

[1] An extended abstract of this paper appeared in the proceedings of the Eighth International Workshop on Algorithms and Computation, pp.163-175, 2014 [11].

conditions for this recognition problem. In the same section, we establish new properties of point visibility graphs, and in addition, we state some known properties with proofs that are important in understanding point visibility graphs. Though the first necessary condition can be tested in $O(n^3)$ time, it is not clear whether the second necessary and third conditions can be tested in polynomial time. On the other hand, we show in Sect. 3 that the recognition problem lies in PSPACE.

If a given graph G is planar, there can be three cases: (i) G has a planar visibility embedding (Fig. 2), (ii) G admits a visibility embedding, but no visibility embedding of G is planar (Fig. 3), and (iii) G does not have any visibility embedding (Fig. 4). Case (i) has been characterized by Eppstein [6] by presenting four infinite families of G and one particular graph. In order to characterize graphs in Case (i) and Case (ii), we show that two infinite families and five particular graphs are required in addition to graphs for Case (i). Using this characterization, we present an $O(n)$ algorithm for recognizing and reconstructing G in Sect. 4. Note that this algorithm does not require any prior embedding of G. Finally, we conclude the paper with a few remarks.

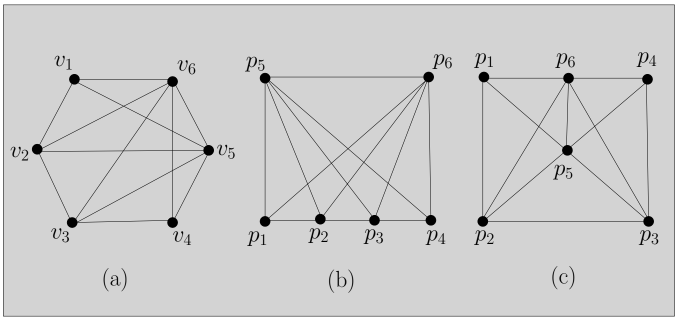

Figure 1: (a) A point visibility graph with (v_1, v_2, v_3, v_4) as a CSP. (b) A visibility embedding of the point visibility graph where (p_1, p_2, p_3, p_4) is a GSP. (c) A visibility embedding of the point visibility graph where (p_1, p_2, p_3, p_4) is not a GSP.

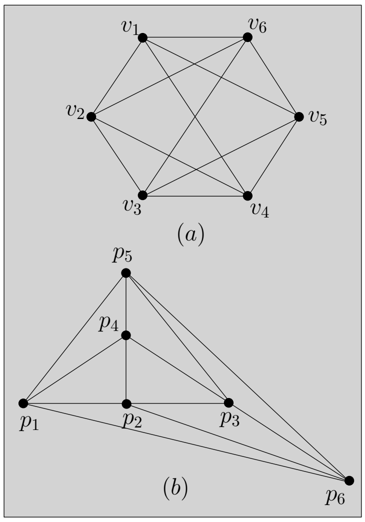

Figure 2: (a) A planar graph G. (b) A planar visibility embedding of G.

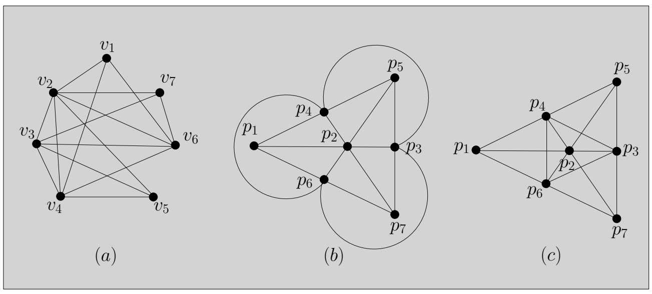

Figure 3: (a) A planar graph G. (b) A planar embedding of G. (c) A non-planar visibility embedding of G

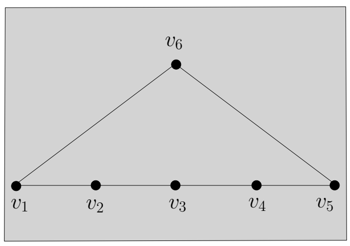

Figure 4: A planar graph G that does not admit a visibility embedding.

2 Properties of point visibility graphs

Consider a subset S of vertices of G such that their corresponding points C in a visibility embedding ξ of G are collinear. The path formed by the points of C is called a *geometric straight path* (GSP). For example, the path (p_1, p_2, p_3, p_4) in Fig. 1(b) is a GSP as the points p_1, p_2, p_3 and p_4 are collinear. Note that there may be another visibility embedding ξ of G as shown in Fig. 1(c), where points p_1, p_2, p_3 and p_4 are not collinear. So, the points forming a GSP in ξ may not form a GSP in every visibility embedding of G. If a GSP is a maximal set of collinear points, it is called a *maximal geometric straight path* (max GSP). A GSP of k collinear points is denoted as *k-GSP*. In the following, we state some properties of PVGs and present three necessary conditions for recognizing G.

Lemma 1. *If G is a PVG but not a path, then for any GSP in any visibility embedding of G, there is a point visible from all the points of the GSP[13].*

Proof. For every GSP, there exists a point p_i whose perpendicular distance to the line containing the GSP is the smallest. So, all points of the GSP are visible from p_i. □

Lemma 2. *If G admits a visibility embedding ξ having a k-GSP, then the number of edges in G is at least $(k-1)+k(n-k)$.*

Proof. Let p_i and p_j be two points of ξ such that p_i is a point of the k-GSP and p_j is not. Consider the segment p_ip_j. If p_i and p_j are mutually visible, then (v_i, v_j) is an edge in G. Otherwise, there exists a blocker p_k on p_ip_j such that (v_j, v_k) is an edge in G. So, p_j has an edge in the direction towards p_i. Therefore, for every such pair p_i and p_j, there is an edge in G. So, $(n-k)k$ such pairs in ξ correspond to $(n-k)k$ edges in G. Moreover, there are $(k-1)$ edges in G corresponding to the k-GSP. Hence, G has at least $(k-1)+k(n-k)$ edges. □

Corollary 1. *If a point p_i in a visibility embedding ξ of G does not belong to a k-GSP in ξ, then its corresponding vertex v_i in G has degree at least k.*

Let H be a path in G such that no edges exist between any two non-consecutive vertices in H. We call H a *combinatorial straight path* (CSP). Observe that in a visibility embedding of G, H may not always correspond to a GSP. In Fig. 1(a), $H = (v_1, v_2, v_3, v_4)$ is a CSP which corresponds to a GSP in Fig. 1(b) but not in Fig. 1(c). Note that a CSP always refers to a path in G, whereas a GSP refers to a path in a visibility embedding of G. A CSP that is a maximal path, is called a *maximal combinatorial straight path* (*max CSP*). A CSP of k-vertices is denoted as *k-CSP*.

Lemma 3. *G is a PVG and bipartite if and only if the entire G is a CSP.*

Proof. If the entire G can be embedded as a GSP, then alternating points in the GSP form the bipartition and the lemma holds. Otherwise, there exists at least one max GSP which does not contain all the points. By Lemma 1, there exists one point p_i adjacent to all points of the GSP. So, p_i must belong to one partition and all points of the GSP (having edges) belong to the other partition. Hence, G cannot be a bipartite graph, a contradiction. The other direction of the proof is trivial. □

Corollary 2. *G is a PVG and triangle-free if and only if the entire G is a CSP.*

Lemma 4. *If G is a PVG, then the size of the maximum clique in G is bounded by twice the minimum degree of G, and the bound is tight.*

Proof. In a visibility embedding of G, draw rays from a point p_i of minimum degree through every visible point of p_i. Observe that any ray may contain several points not visible from p_i. Since any clique can have at most two points from the same ray, the size of the clique is at most twice the number of rays, which gives twice the minimum degree of G. □

Lemma 5. *If G is a PVG and it has more than one max CSP, then the diameter of G is 2 [13].*

Proof. If two vertices v_i and v_j are not adjacent in G, then they belong to a CSP L of length at least two. By Lemma 1, there must be some vertex v_k that is adjacent to every vertex in L. (v_i, v_k, v_j) is the required path of length 2. Therefore, the diameter of G cannot be more than two. □

Corollary 3. *If G is a PVG but not a path, then the BFS tree of G rooted at any vertex v_i of G has at most three levels consisting of v_i in the first level, the neighbours of v_i in G in the second level, and the rest of the vertices of G in the third level.*

Lemma 6. *If G is a PVG but not a path, then the subgraph induced by the neighbours of any vertex v_i, excluding v_i, is connected.*

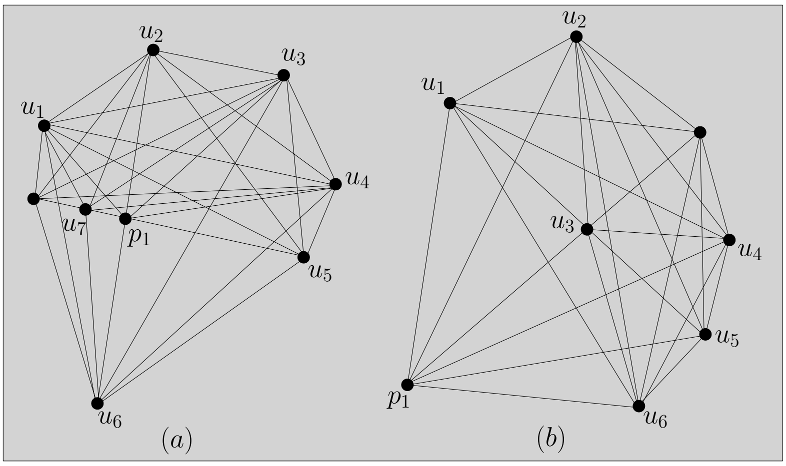

Figure 5: (a) The points $(u_1, u_2, ..., u_7,\ u_1)$ are visible from an internal point p_1. (b) The points $(u_1, u_2, ..., u_6)$ are visible from a convex hull point p_1.

Proof. Consider a visibility embedding of G where G is not a path. Let $(u_1, u_2, ..., u_k, u_1)$ be the visible points of p_i in clockwise angular order. If p_i is not a convex hull point, then $(u_1, u_2), (u_2, u_3), ..., (u_{k-1}, u_k)$, (u_k, u_1) are visible pairs (Fig. 5(a)). If p_i, u_1 and u_k are convex hull points, then $(u_1, u_2), (u_2, u_3), ...$, (u_{k-1}, u_k) are visible pairs (Fig. 5(b)). Since there exists a path between every pair of points in $(u_1, u_2, ..., u_k, u_1)$, the subgraph induced by the neighbours of v_i is connected. □

Necessary Condition 1. *If G is not a CSP, then the BFS tree of G rooted at any vertex can have at most three levels, and the induced subgraph formed by the vertices in the second level must be connected.*

Proof. Follows from Corollary 3 and Lemma 6. □

As defined for point sets, if two vertices v_i and v_j of G are adjacent (or, not adjacent) in G, (v_i, v_j) is referred to as a *visible pair* (respectively, *invisible pair*) of G. Let $(v_1, v_2, ..., v_k)$ be a path in G such that no two non-consecutive vertices are connected by an edge in G (Fig. 6(a)). For any vertex v_j, $2 \leq j \leq k-1$, v_j is called a *vertex-blocker* of (v_{j-1}, v_{j+1}) as (v_{j-1}, v_{j+1}) is not an edge in G and both (v_{j-1}, v_j) and (v_j, v_{j+1}) are edges in G. In the same way, consecutive vertex-blockers on such a path are also called *vertex-blockers*. For example, $v_m * v_{m+1}$ is a vertex-blocker of (v_{m-1}, v_{m+2}) for $2 \leq m \leq k-2$. Note that $*$ represents concatenation of consecutive vertex-blockers.

Consider the graph in Fig. 6(b). Though G satisfies Necessary Condition 1, it is not a PVG because it does not admit a visibility embedding. It can be seen that this graph without the edge (v_2, v_4) admits a visibility embedding (see Fig. 6(a)), where $(v_1, v_2, v_3, v_4, v_5)$ forms a GSP. However, (v_2, v_4) demands visibility between two non-consecutive collinear blockers which cannot be realized in any visibility embedding.

Necessary Condition 2. *There exists an assignment of vertex-blockers to invisible pairs in G such that:*

1. *Every invisible pair is assigned one vertex-blocker.*
2. *If two invisible pairs in G sharing a vertex v_i (say, (v_i, v_j) and (v_i, v_k)), and their assigned vertex-blockers are not disjoint, then all vertices in the two assigned vertex-blockers along with vertices v_i, v_j and v_k must be a CSP in G.*
3. *If two invisible pairs in G are sharing a vertex v_i (say, (v_i, v_j) and (v_i, v_k)), and v_k is assigned as a vertex blocker to (v_i, v_j), then v_j is not assigned as a vertex blocker to (v_i, v_k).*

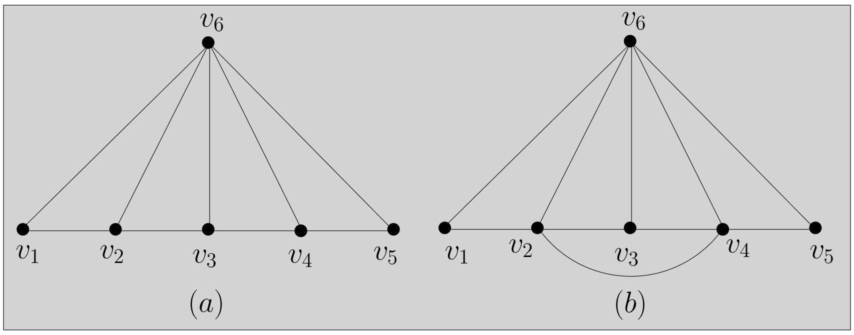

Figure 6: (a) Vertices v_2, v_3, v_4 are vertex-blockers of (v_1, v_3), (v_3, v_4) (v_3, v_5) respectively. Also, $v_2 * v_3 * v_4$ is the vertex-blocker of (v_1, v_5). (b) The graph satisfies Necessary Condition 1 but is not a PVG because of the edge (v_2, v_4).

Proof. In a visibility embedding of G, every segment connecting two points, that are not mutually visible, must pass through another point or a set of collinear points, and they correspond to vertex-blockers in G.

Since (v_i, v_j) and (v_i, v_k) are invisible pairs, the segments (p_i, p_j) and (p_i, p_k) must contain points. If there exists a point p_m on both p_ip_j and p_ip_k, then points p_i, p_m, p_j, p_k must be collinear. So, v_i, v_m, v_j and v_k must belong to a CSP.

Since (v_i, v_j) and (v_i, v_k) are invisible pairs, the segments (p_i, p_j) and (p_i, p_k) must contain points. If the point p_k lies on p_ip_j, then p_j cannot lie on p_ip_k, because it contradicts the order of points on a line. □

Consider the graph G in Fig. 7(a). From its visibility embedding, it is clear that G is a PVG and therefore, satisfies both Necessary Conditions 1 and 2. Let us construct a new graph G' from G by replacing edges v_9v_{10} and $v_{11}v_{12}$ of G by v_9v_{11} and $v_{10}v_{12}$ (see Fig. 7(b)). We have the following lemmas on G'.

Lemma 7. *The graph G' satisfies Necessary Conditions 1 and 2.*

Proof. Observe that the neighbours of any vertex in G' induce a connected subgraph. Also, the diameter of G' is still two. Therefore, G' satisfies Necessary Condition 1.

For showing that G' also satisfies Necessary Condition 2, we consider the assignment of blockers to the mutually invisible pairs of vertices in G' as follows: $(v_0, v_5) \longrightarrow v_1$, $(v_0, v_9) \longrightarrow v_1 * v_5$, $(v_1, v_9) \longrightarrow v_5$, $(v_0, v_6) \longrightarrow v_2$, $(v_0, v_{10}) \longrightarrow v_2 * v_6$, $(v_2, v_{10}) \longrightarrow v_6$, $(v_0, v_7) \longrightarrow v_3$, $(v_0, v_{11}) \longrightarrow v_3 * v_7$, $(v_3, v_{11}) \longrightarrow v_7$, $(v_0, v_8) \longrightarrow v_4$, $(v_0, v_{12}) \longrightarrow v_4 * v_8$, $(v_4, v_{12}) \longrightarrow v_8$, $(v_1, v_3) \longrightarrow v_2$, $(v_1, v_4) \longrightarrow v_2 * v_3$, $(v_2, v_4) \longrightarrow v_3$, $(v_5, v_7) \longrightarrow v_6$, $(v_5, v_8) \longrightarrow v_6 * v_7$, $(v_6, v_8) \longrightarrow v_7$, $(v_9, v_{10}) \longrightarrow v_{11}$, $(v_9, v_{12}) \longrightarrow v_{11} * v_{10}$, $(v_{11}, v_{12}) \longrightarrow v_{10}$. Observe that since the invisible pairs (v_9, v_{11}) and (v_{10}, v_{12}) in G are replaced by (v_9, v_{10}) and (v_{11}, v_{12}) in G', the vertex-blocker assignments have changed accordingly. It can be seen that the above assignment of vertex blockers satisfies Necessary Condition 2.

□

Lemma 8. *The graph G' is not a PVG.*

Proof. Let us assume on the contrary that G has a visibility embedding (say, ξ). Let $p_0, p_1, \ldots, p_{12}$ be the points of ξ corresponding to the vertices $v_0, v_1, \ldots, v_{12}$ respectively. Consider the rays $\overrightarrow{p_0p_1}$, $\overrightarrow{p_0p_2}$, $\overrightarrow{p_0p_3}$ and $\overrightarrow{p_0p_4}$. Since v_0 is not adjacent to any of $v_5, v_6, \ldots, v_{12}$ in G', $p_5, p_6, \ldots, p_{12}$ must lie on these four rays.

Consider the case where p_0 is not a blocker of (p_1, p_4). So, the angle at p_0 between $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_0p_4}$ is not $180°$. Let $w_{1,4}$ denote the wedge formed by $\overrightarrow{p_0p_1}$ and $\overrightarrow{p_0p_4}$ such that the internal angle of $w_{1,4}$ is convex.

Since a blocker of (p_1, p_4) must lie on $\overrightarrow{p_0p_2}$ or $\overrightarrow{p_0p_3}$ (say, $\overrightarrow{p_0p_2}$), $\overrightarrow{p_0p_2}$ divides $w_{1,4}$ into wedges $w_{1,2}$ and $w_{2,4}$ By a similar argument for (p_2, p_4), $\overrightarrow{p_0, p_3}$ passes through $w_{2,4}$. So, the ordering of the rays around p_0 in $w_{1,4}$ is $(\overrightarrow{p_0p_1}, \overrightarrow{p_0p_2}, \overrightarrow{p_0p_3}, \overrightarrow{p_0p_4})$.

Let us locate the positions of p_5, p_6, p_7 and p_8 on $\overrightarrow{p_0p_1}$, $\overrightarrow{p_0p_2}$, $\overrightarrow{p_0p_3}$ and $\overrightarrow{p_0p_4}$. Observe that since each of the vertices v_5, v_6, v_7 and v_8 are adjacent to all of the vertices v_1, v_2, v_3 and v_4, the points p_5, p_6, p_7 and p_8 must be the next points on $\overrightarrow{p_0p_1}$, $\overrightarrow{p_0p_2}$, $\overrightarrow{p_0p_3}$ and $\overrightarrow{p_0p_4}$. In fact, the only two possibilities are $(\overrightarrow{p_0p_1p_5}, \overrightarrow{p_0p_2p_6}, \overrightarrow{p_0p_3p_7}, \overrightarrow{p_0p_4p_8})$ and $(\overrightarrow{p_0p_1p_8}, \overrightarrow{p_0p_2p_7}, \overrightarrow{p_0p_3p_6}, \overrightarrow{p_0p_4p_5})$ that can satisfy the blocking requirements among p_5, p_6, p_7 and p_8.

Let us locate the positions of p_9, p_{10}, p_{11} and p_{12} on $\overrightarrow{p_0p_1}$, $\overrightarrow{p_0p_2}$, $\overrightarrow{p_0p_3}$ and $\overrightarrow{p_0p_4}$. Since v_9 is adjacent to v_2, v_3 and v_4 but not to v_1 in G', p_9 must lie on $\overrightarrow{p_0p_1}$. Similarly, p_{10} must lie on $\overrightarrow{p_0p_2}$. Since p_9 and p_{10} lie on consecutive rays around p_0, the points p_9 and p_{10} must see each other, which is a contradiction.

Consider the other case where p_0 is the blocker of (p_1, p_4). Observe that a point p_i on $\overrightarrow{p_0p_3}$ is required on p_2p_4 to block the visibility between p_2 and p_4. Similarly another point p_j on $\overrightarrow{p_0p_2}$ is required on p_1p_3 to block the visibility between p_1 and p_3. Moreover, p_i and p_j cannot be visible from p_0 unless they are p_2 and p_3. It can be seen that no pair of points p_i and p_j can satisfy these conditions, which is a contradiction. □

The above lemmas show that Necessary Conditions 1 and 2 are not sufficient for recognizing a PVG, which leads to Necessary Condition 3. An assignment of vertex-blockers in G is said to be a *valid assignment* if it satisfies Necessary Conditions 1 and 2. Let $(v_i, v_{i,1}), (v_i, v_{i,2}), \ldots, (v_i, v_{i,d})$ be all visible pairs of v_i in G. For a valid assignment, let $S_{i,j}$ denote the set of vertices of G such that for every vertex $u \in S_{i,j}$, $v_{i,j}$ is a blocker assigned to the invisible pair (v_i, u) in this assignment.

Necessary Condition 3. *If G is not a CSP, then there exists a valid assignment for G such that for every vertex $v_i \in G$, there is an ordering of visible pairs $(v_i, v_{i,1}), (v_i, v_{i,1}), \ldots, (v_i, v_{i,d})$ around v_i such that if $(v_i, v_{i,j})$ is adjacent to $(v_i, v_{i,k})$ in the ordering, then every vertex of $\{v_{i,j}\} \cup S_{i,j}$ is adjacent to every vertex of $\{v_{i,k}\} \cup S_{i,k}$ in G.*

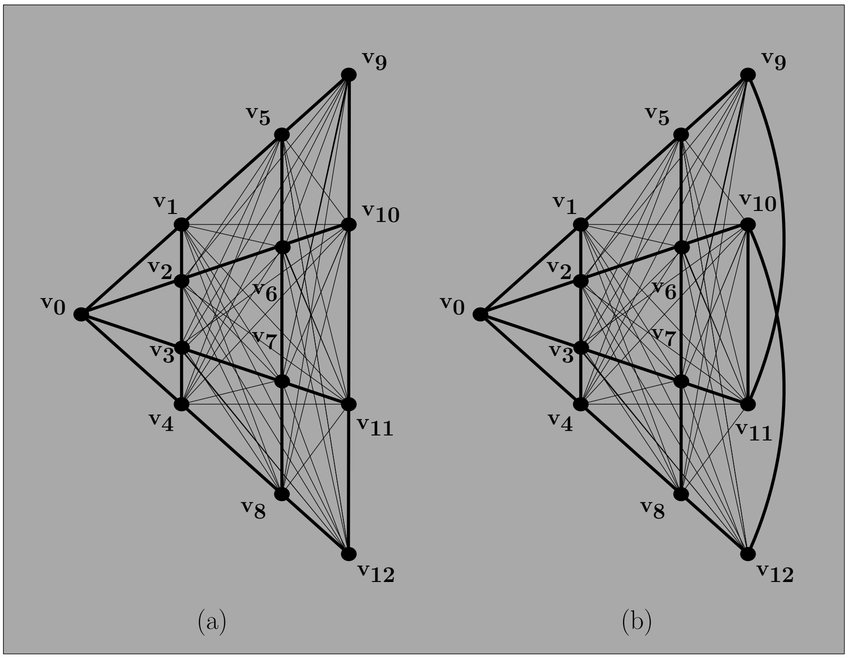

Figure 7: (a) This graph is a PVG drawn in the form of a visibility embedding. (b) This graph is not a PVG but satisfies both Necessary Conditions 1 and 2.

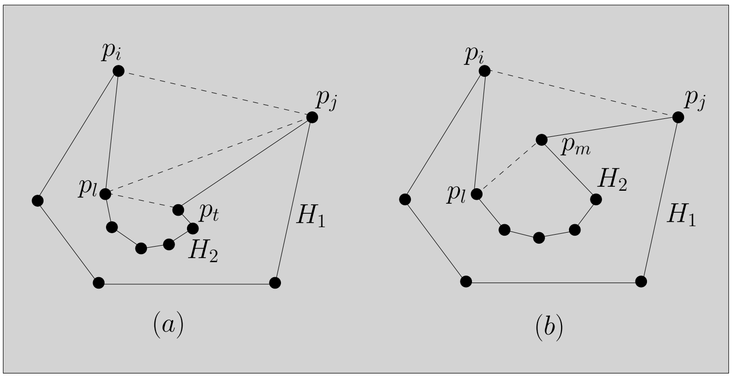

Figure 8: (a) The left tangents of p_i and p_j meet H_2 at the same point p_l. (b) The left tangents of p_i and p_j meet H_2 at points p_l and p_m of the same edge.

Proof. Consider any valid assignment corresponding to a visibility embedding ξ of G. Let $\overrightarrow{p_ip_{i,j}}$ denote the ray drawn from p_i through $p_{i,j}$ in ξ. Consider a clockwise ordering A of $\overrightarrow{p_ip_{i,1}}, \ldots, \overrightarrow{p_ip_{i,d}}$ around p_i in ξ such that the clockwise angle between any two rays in A is convex, except possibly the last and first rays in A. So, every point on a ray in A is visible from every point on its adjacent ray. It can be seen that if any two rays $\overrightarrow{p_ip_{i,j}}$ and $\overrightarrow{p_ip_{i,k}}$ are adjacent in A, then every vertex of $\{v_{i,j}\} \cup S_{i,j}$ is connected by an edge to every vertex of $\{v_{i,k}\} \cup S_{i,k}$ in G. Hence, G satisfies Necessary Condition 3. □

Lemma 9. *If the size of the longest GSP in some visibility embedding of a graph G with n vertices is k, then the degree of each vertex in G is at least* $\lceil \frac{n-1}{k-1} \rceil$ *[16, 15, 17].*

Proof. For any point p_i in a visibility embedding of G, the degree of p_i is the number of points visible from p_i which are in angular order around p_i. Since the longest GSP is of size k, a ray from p_i through any visible point of p_i can contain at most $k-1$ points excluding p_i. So there must be at least $\lceil \frac{n-1}{k-1} \rceil$ such rays, which gives the degree of p_i. □

Theorem 1. *If G is a PVG but not a path, then G has a Hamiltonian cycle.*

Proof. Let $H_1, H_2, ..., H_k$ be the convex layers of points in a visibility embedding of G, where H_1 and H_k are the outermost and innermost layers respectively. Let p_ip_j be an edge of H_1, where p_j is the next clockwise point of p_i on H_1 (Fig. 8(a)). Draw the left tangent of p_i to H_2 meeting H_2 at a point p_l such that the entire H_2 is to the left of the ray starting from p_i through p_l. Similarly, draw the left tangent from p_j to H_2 meeting H_2 at a point p_m. If $p_l = p_m$ then take the next clockwise point of p_l in H_2 and call it p_t. Remove the edges p_ip_j and p_lp_t, and add the edges p_ip_l and p_jp_t (Fig. 8(a)). Consider the other situation where $p_l \neq p_m$. If p_lp_m is an edge, then remove the edges p_ip_j and p_lp_m, and add the edges p_ip_l and p_jp_m (Fig. 8(b)). If p_lp_m is not an edge of H_2, take the next counterclockwise point of p_m on H_2 and call it p_q. Remove the edges p_ip_j and p_qp_m, and add the edges p_ip_q and p_jp_m (Fig. 9(a)).

Thus, H_1 and H_2 are connected forming a cycle $C_{1,2}$. Without the loss of generality, we assume that $p_m \in H_2$ is the next counter-clockwise point of p_j in $C_{1,2}$ (Fig. 9(b)). Starting from p_m, repeat the same construction to connect $C_{1,2}$ with H_3 forming $C_{1,3}$. Repeat till all layers are connected to form a Hamiltonian cycle $C_{1,k}$. Note that if H_k is just a path (Fig. 9(b)), it can be connected trivially to form $C_{1,k}$. □

Corollary 4. *Given G and a visibility embedding of G, a Hamiltonian cycle in G can be constructed in linear time.*

Proof. This is because the combinatorial representation of G contains all its edges, and hence the gift-wrapping algorithm for finding the convex layers of a point set becomes linear in the input size. □

Lemma 10. *Consider a visibility embedding of G. Let A, B and C be three nonempty, disjoint sets of points in it such that* $\forall p_i \in A$ *and* $\forall p_j \in C$*, the GSP between* p_i *and* p_j *contains at least one point from B, and no other point from A or C (Fig. 10(a)). Then* $|B| \geq |A| + |C| - 1$ *[16, 15, 17].*

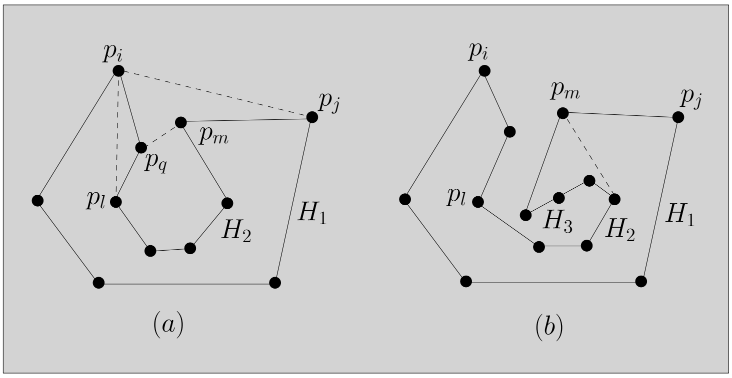

Figure 9: (a) The left tangents of p_i and p_j meet H_2 points p_l and p_m of different edges. (b) The innermost convex layer is a path which is connected to $C_{1,2}$.

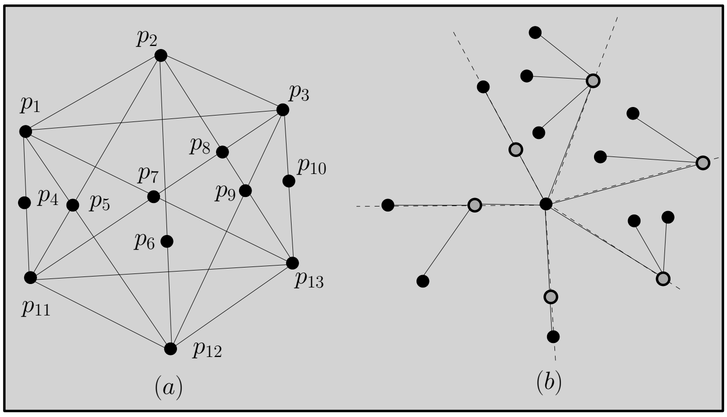

Figure 10: (a) A PVG with A = $\{p_1, p_2, p_3\}$, B=$\{p_4, p_5, p_6, p_7, p_8, p_9, p_{10}\}$ and C=$\{p_{11}, p_{12}$, p$_{13}\}$. (b) Points of A and C connected by edges representing blockers.

Proof. Draw rays from a point $p_i \in A$ through every point of C (Fig. 10(b)). These rays partition the plane into $|C|$ wedges. Since points of C are not visible from p_i, there is at least one blocker lying on each ray between p_i and the point of C on the ray. So, there are at least $|C|$ number of such blockers. Consider the remaining $|A-1|$ points of A lying in different wedges. Consider a wedge bounded by two rays drawn through $p_k \in C$ and $p_l \in C$. Consider the segments from p_k to all points of A in the wedge. Since these segments meet only at p_k, and p_k is not visible from any point of A in the wedge, each of these segments must contain a distinct blocker. So, there are at least $|A|-1$ blockers in all the wedges. Therefore the total number of points in B is at least $|A|+|C|-1$. □

Lemma 11. *Consider a visibility embedding of G. Let A and C be two nonempty and disjoint sets of points such that no point of A is visible from any point of C. Let B be the set of points (or blockers) on the segment p_ip_j, $\forall p_i \in A$ and $\forall p_j \in C$, and blockers in B are allowed to be points of A or C. Then $|B| \geq |A|+|C|-1$ [17].*

Proof. Draw rays from a point $p_i \in A$ through every point of C. These rays partition the plane into at most $|C|$ wedges. Consider a wedge bounded by two rays drawn through $p_k \in C$ and $p_l \in C$. Since these rays may contain other points of A and C, all points between p_i and the farthest point from p_i on a ray, are blockers in B. Observe that all these blockers except one may be from A or C. Thus, excluding p_i, B has at least as many points as from A and C on the ray. Consider the points of A inside the wedge. Draw segments from p_k to all points of A in the wedge. Since these segments may contain multiple points from A, all points on a segment between p_k and the farthest point from p_k are blockers in B. All these points except one may be from A. Thus, B has at least as many points as from A inside the wedge. Therefore the total number of points in B is at least $|A|+|C|-1$. □

3 Computational complexity of the recognition problem

In this section we show that the recognition problem for a PVG lies in PSPACE. Our technique in the proof follows a similar technique used by Everett [8] for showing that the recognition problem for polygonal visibility is in PSPACE. We start with the following theorem of Canny [3].

Theorem 2. *Any sentence in the existential theory of the reals can be decided in PSPACE.*

A sentence in the first order theory of the reals is a formula of the form :

$$\exists x_1 \exists x_2 ... \exists x_n P(x_1, x_2, ..., x_n)$$

where the $x_i's$ are variables ranging over the real numbers and where $P(x_1, x_2, ..., x_n)$ is a predicate built up from $\neg$, $\wedge$, $\vee$, $=$, $<$, $>$, $+$, $\times$, 0, 1 and -1 in the usual way.

Theorem 3. *The recognition problem for point visibility graphs lies in PSPACE.*

Proof. Given a graph $G(V, E)$, we construct a formula in the existential theory of the reals polynomial in size of G which is true if and only if G is a point visibility graph.

Suppose $(v_i, v_j) \notin E$. This means that if G admits a visibility embedding, then there must be a blocker (say, p_k) on the segment joining p_i and p_j. Let the coordinates of the points p_i, p_j and p_k be (x_i, y_i), (x_j, y_j) and (x_k, y_k) respectively. So we have :

$$\exists t_{\in}\mathbb{R}\Big(\big(0 < t\big) \wedge \big(t < 1\big) \wedge \big((x_k - x_i) = t \times (x_j - x_i)\big) \wedge \big((y_k - y_i) = t \times (y_j - y_i)\big)\Big)$$

Now suppose $(v_i, v_j) \in E$. This means that if G admits a visibility embedding, no point in P lies on the segment connecting p_i and p_j to ensure visibility. So, (i) either p_k forms a triangle with p_i and p_j or (ii) p_k lies on the line passing through p_i and p_j but not between p_i and p_j. Determinants of non-collinear points is non-zero. So we have :

$$\exists t \in \mathbb{R}\Big(\big(det(x_i, x_j, x_k, y_i, y_j, y_k) > 0\big) \vee \big(det(x_i, x_j, x_k, y_i, y_j, y_k) < 0\big)\Big) \bigvee \Big(\big(t > 1\big) \vee \big(t < -1\big) \wedge \big((x_k - x_i) = t \times (x_j - x_i)\big) \wedge \big((y_k - y_i) = t \times (y_j - y_i)\big)\Big)$$

For each triple (v_i, v_j, v_k) of vertices in V, we add a $t = t_{i,j,k}$ to the existential part of the formula and

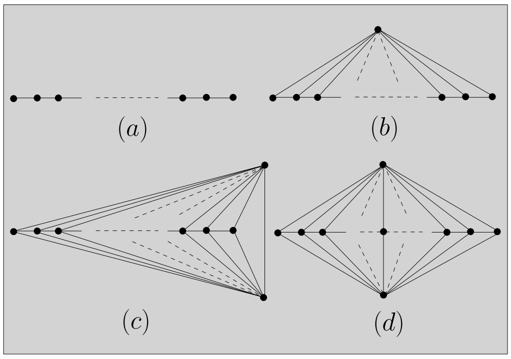

Figure 11: These four infinite families admit planar visibility embedding (Eppstein [6]).

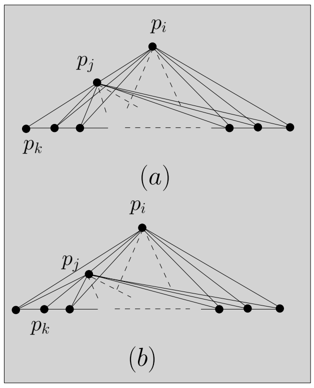

Figure 12: These two infinite families do not admit planar visibility embedding.

the corresponding portion to the predicate. So the formula becomes:

$\exists x_1 \exists y_1 ... \exists x_n \exists y_n \exists t_{1,2,3} .... \exists t_{n-2,n-1,n}\ P(x_1, y_1, ..., x_n, y_n, t_{1,2,3}, ..., t_{n-2,n-1,n})$

which is of size $O(n^3)$. This proves our theorem. □

4 Planar point visibility graphs

In this section, we present a characterization, recognition and reconstruction of planar point visibility graphs. Let G be a given planar graph. We know that the planarity of G can be tested in linear time [2]. If G is planar, a straight line embedding of G can also be constructed in linear time. However, this embedding may not satisfy the required visibility constraints, and therefore, it cannot be a visibility embedding. We know that collinear points play a crucial role in a visibility embedding of G. It is, therefore, important to identify points belonging to a GSP of maximum length. Using this approach, we construct a visibility embedding of a given planar graph G, if it exists. We have the following lemmas on visibility embeddings of G.

Lemma 12. *Assume that G admits a visibility embedding ξ. If ξ has at least one k-GSP for $k \geq 4$, then the number of vertices in G is at most*

$$k + \left\lfloor \frac{2k-5}{k-3} \right\rfloor$$

Proof. By Lemma 2, G can have at least $(k-1) + (n-k)k$ edges. By applying Euler's criterion for planar graphs, we have the following inequality on the number of permissible edges of G.

$$\begin{aligned} (k-1) + (n-k)k &\leq 3(n) - 6 \\ \Rightarrow (k-1) + (n-k)k &\leq 3(k+n-k) - 6 \\ \Rightarrow (k-1) + (n-k)k &\leq 3k + 3(n-k) - 6 \\ \Rightarrow (n-k)(k-3) &\leq 2k-5 \\ \Rightarrow (n-k) &\leq \frac{2k-5}{k-3} \end{aligned} \tag{1}$$

Since $(n-k)$ must be an integer, we have

$$\begin{aligned} (n-k) &\leq \left\lfloor \frac{2k-5}{k-3} \right\rfloor \\ \Rightarrow n &\leq k + \left\lfloor \frac{2k-5}{k-3} \right\rfloor \end{aligned} \tag{2}$$

□

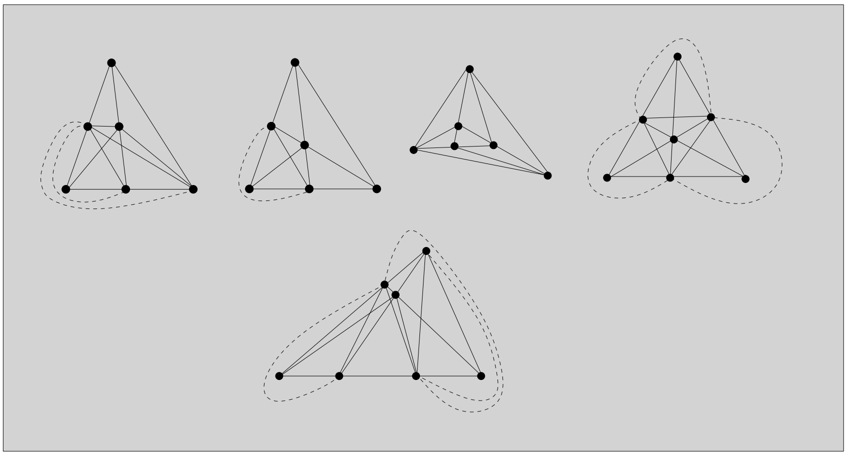

Figure 13: Five planar PVGs that do not belong to any of the six families. Dotted lines show how the edge-crossings in the visibility embedding can be avoided in a planar embedding.

Corollary 5. *There are six infinite families of planar graphs G that admit a visibility embedding ξ with a k-GSP for $k \geq 5$ (Figs. 11 and 12).*

Proof. For $k \geq 5$, $n \leq k+2$. There can be only six infinite families of graphs having at most two points outside a maximum size GSP in ξ (denoted as l) as follows.

1. There is no point lying outside l in ξ (see Fig. 11(a)).
2. There is only one point lying outside l in ξ that is adjacent to all points in l (see Fig. 11(b)).
3. There are two points lying outside l in ξ that are adjacent to all other points in ξ (see Fig. 11(c)).
4. There are two points lying outside l in ξ that are not adjacent to each other but adjacent to all points of l in ξ (see Fig. 11(d)).
5. There are two points p_i and p_j lying outside l in ξ such that p_i and p_j are adjacent to all other points in ξ except an endpoint p_k of l as p_j is a blocker on p_ip_k (see Fig. 12(a)).
6. Same as the previous case, except p_k is now an intermediate point of l in ξ (see Fig. 12(b)).

□

Let us identify those graphs that do not belong to these six infinite families. We show in the following that such graphs can have a maximum of eight vertices.

Lemma 13. *Assume that G admits a visibility embedding ξ. If ξ has at least one 4-GSP, then the number of vertices in G is at most seven.*

Proof. Putting $k = 4$ in the formula of Lemma 12, we get $n \leq 7$. □

Lemma 14. *Assume that G admits a visibility embedding ξ. If G has at least one 3-CSP but no 4-CSP, then G has at most eight vertices.*

Proof. Since G has no 4-CSP, and G is not a clique, there is a 3-GSP in ξ. Starting from the 3-GSP, points are added one at a time to construct ξ. Since no subsequent point can be added on the line passing through points of the 3-GSP to prevent forming a 4-GSP, adding the fourth and fifth points gives at least three edges each in ξ. As ξ does not have a 4-CSP, there can be at most one blocker between an invisible pair of points in ξ. So, for the subsequent points, at least $\lceil \frac{i-1}{2} \rceil$ edges are added for the *ith* point. Since G is planar, by Euler's condition we must have: $8 + \sum_{i=6}^{n} \left\lceil \frac{i-1}{2} \right\rceil \leq 3n - 6$. This inequality is valid only up to $n = 8$. □

Lemma 15. *There are five distinct planar graphs G that admit visibility embeddings but do not belong to the six infinite families (Fig. 13).*

Theorem 4. *Planar point visibility graphs can be characterized by six infinite families of graphs and five particular graphs.*

Proof. Five particular graphs can be identified by enumerating all points of eight vertices as shown in Fig. 13. For the details of the enumeration, see the appendix. □

Theorem 5. *Planar point visibility graphs can be recognized in linear time.*

Proof. Following Theorem 4, G is tested initially whether it is isomorphic to any of the six particular graphs for $n \leq 8$. Then, the maximum CSP is identified before its adjacency is tested with the remaining one or two vertices of G. The entire testing can be carried out in linear time. □

Corollary 6. *Planar point visibility graphs can be reconstructed in linear time.*

Proof. Theorem 5 gives the relative positions and collinearity of points in the visibility embedding of G. Since each point can be drawn with integer coordinates of size $O(logn)$ bits, G can be reconstructed in linear time. □

5 Concluding remarks

We have presented three necessary conditions for recognizing point visibility graphs. Though the first necessary condition can be tested in $O(n^3)$ time, it is not clear how vertex-blockers can be assigned to every invisible pair in G in polynomial time satisfying the second necessary condition. Observe that these assignments in a visibility embedding give the ordering of collinear points along any ray starting from any point through its visible points. These rays together form an arrangement of rays in the plane. It is open whether such an arrangement can be constructed satisfying assigned vertex-blockers in polynomial time. The third necessary condition gives the ordering of these rays around each point. It is also not clear whether the third necessary condition can be tested in polynomial time. Overall, we feel that the three necessary conditions may be sufficient.

Let us consider the complexity issues of the problems of Vertex Cover, Independent Set and Maximum Clique in a point visibility graph. Let G be a graph of n vertices, not necessarily a PVG. We construct another graph G' such that (i) G is an induced subgraph of G', and (ii) G' is a PVG. Let C be a convex polygon drawn along with all its diagonals, where every vertex v_i of G corresponds to a vertex p_i of C. For every edge $(v_i, v_j) \notin G$, introduce a blocker p_t on the edge (p_i, p_j) such that p_t is visible to all points of C and all blockers added so far. Add edges from p_t to all vertices of C and blockers in C. The graph corresponding to this embedding is called G'. So, G' and its embedding can be constructed in polynomial time. Let the sizes of the minimum vertex cover, maximum independent set and maximum clique in G be k_1, k_2 and k_3 respectively. If x is the number of blockers added to C, then the sizes of the minimum vertex cover, maximum independent set and maximum clique in G' are $k_1 + x$, k_2 and $k_3 + x$ respectively. Hence, the problems remain NP-Hard.

Theorem 6. *The problems of Vertex Cover, Independent Set and Maximum Clique remain NP-hard on point visibility graphs.*

Acknowledgements

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Appendix

By enumeration, we identify all five particular graphs (see Fig. 13) that do not belong to the six infinite families (see Figs. 11 and 12), as stated in Theorem 4. We know from Lemmas 13 and 14 that $n \leq 8$. We have the following cases.

Case 1. *There is a 3-GSP but no 4-GSP in some visibility embedding ξ of G.*

If $n \leq 5$, G belongs to one of the infinite families having at most two points outside the 3-GSP.

Consider $n = 6$. Let p_1, p_2 and p_3 be collinear points representing a 3-GSP (denoted as l). If there is no other 3-GSP in ξ, then all edges except (v_1, v_3) are present in G. So, G is not planar as it has K_5 as a subgraph. If there is another 3-GSP (say, l') in ξ, which is disjoint from l, then G is not planar as it has $K_{3,3}$ as a subgraph. So, we consider the situation when l and l' share a point in ξ. There can be three such distinct embeddings of five points as shown in Fig. 14. Before the sixth point p_6 is added in

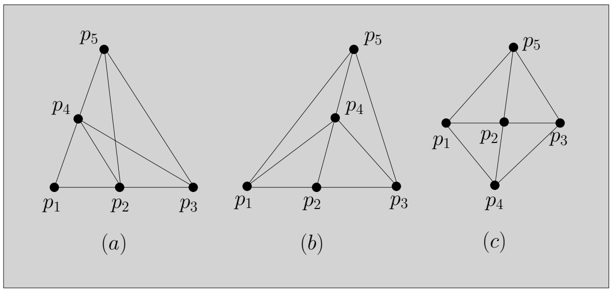

Figure 14: Visibility embeddings of five points containing two overlapping 3-GSPs.

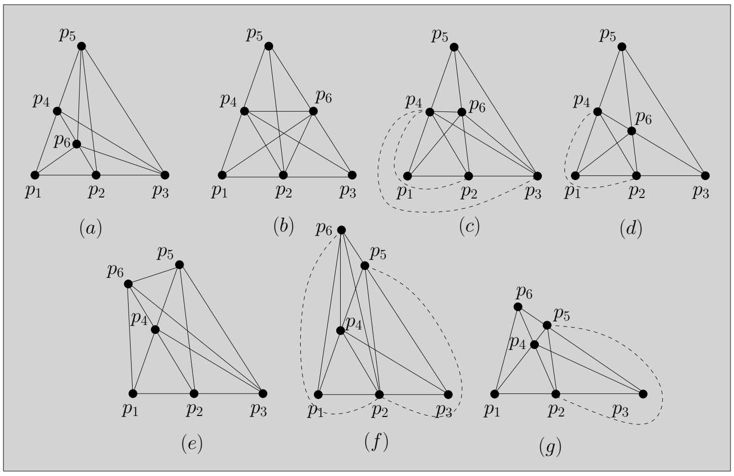

Figure 15: Visibility embeddings of six points after p_6 is added to the embedding in Fig. 14(a). Dotted lines show how the edge-crossings in the visibility embedding can be avoided in a planar embedding.

the embeddings, we need the following lemma.

Lemma 16. *Any planar point visibility graph H of six vertices, with no 4-GSP, has at least three 3-CSPs.*

Proof. We know that if H does not have an edge between two vertices, then it corresponds to a 3-CSP. Since H has at most 12 edges due to Euler's condition, and a complete graph on six vertices has 15 edges, there are at least 3 edges not present in H. Therefore H has at least three 3-CSPs. □

Let us add p_6 to the embedding shown in Fig. 14(a) in such a way that the new embeddings have three 3-GSPs satisfying Lemma 16. So, p_6 must lie on the lines passing through exactly two points, forming a new 3-GSP. Removing symmetric embeddings, we have the following choices of positioning p_6 in the new 3-GSP: $\overline{p_4p_6p_2}$ (Fig. 15(a)), $\overline{p_5p_6p_3}$ (Fig. 15(b)), $\overline{p_5p_6p_2}$ (Fig. 15(c)), $\overline{p_5p_6p_2}$ and $\overline{p_4p_6p_3}$ (Fig. 15(d)), $\overline{p_6p_4p_2}$ (Fig. 15(e)), $\overline{p_6p_5p_3}$ (Fig. 15(f)), $\overline{p_6p_4p_2}$ and $\overline{p_6p_5p_3}$ (Fig. 15(g)). It can be seen that embeddings in Figs. 15(a), 15(b) and 15(e) correspond to non-planar graphs, and embeddings in Figs. 15(c), 15(d), 15(f) and 15(g) correspond to planar graphs. Graphs corresponding to embeddings in Figs. 15(c) and 15(d), are isomorphic to graphs corresponding to embeddings in Figs. 15(f) and 15(g) respectively. Hence, only two non-isomorphic planar graphs arise after adding p_6 to the visibility embedding in Fig. 14(a).

As before, let us add p_6 to the embedding shown in Fig. 14(b). Removing symmetric embeddings, we have the following choices of positioning p_6 in the new 3-GSP: $\overline{p_1p_6p_5}$ (Fig. 16(a)), $\overline{p_1p_5p_6}$ (Fig. 16(b)),

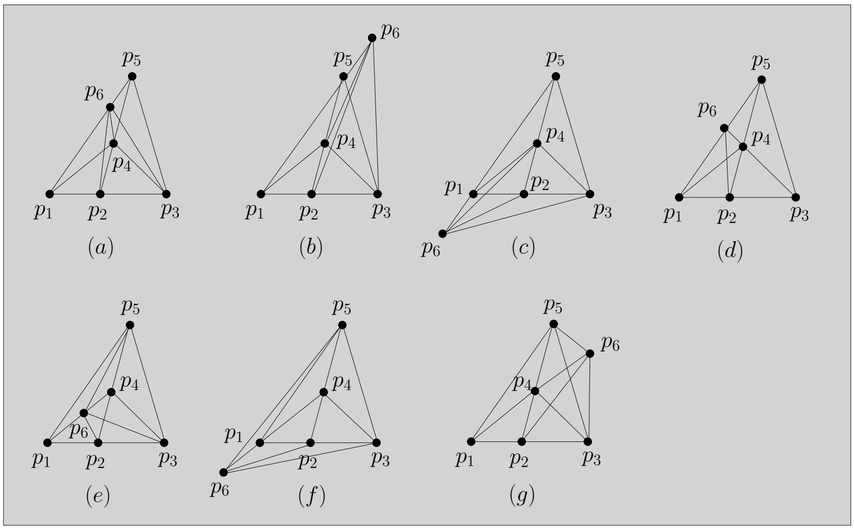

Figure 16: Visibility embeddings of six points after p_6 is added to the embedding in Fig. 14(b).

$\overline{p_6p_1p_5}$ (Fig. 16(c)), $\overline{p_1p_6p_5}$ and $\overline{p_3p_4p_6}$ (Fig. 16(d)), $\overline{p_1p_6p_4}$ (Fig. 16(e)), $\overline{p_6p_1p_4}$ (Fig. 16(f)) and $\overline{p_1p_4p_6}$ (Fig. 16(g)) The embeddings in all the figures except Figure 16(f) have two 3-GSPs that overlap at their end-points, which they are already considered in Fig. 15. Since the embedding in Fig. 16(f) is planar, this is the only new planar graph that arises after adding p_6 to the visibility embedding in Fig. 14(b).

As before, let us add p_6 to the embedding shown in Fig. 14(c). Removing symmetric embeddings, we have the following choices of positioning p_6 in the new 3-GSP: $\overline{p_1p_6p_5}$ (Fig. 17(a)) and $\overline{p_1p_5p_6}$ (Fig. 17(b)). But these two embeddings are already present in Fig. 15. So, no new planar graphs arise after adding p_6 to the embedding visibility in Fig. 14(c). Thus, three particular planar point-visibility graphs of six vertices are identified (see Figs. 15(c), 15(d) and 16(f)). Consider $n = 7$. In the following lemma, we show that there is exactly one particular graph of seven vertices that admits a planar embedding (Fig. 18).

Lemma 17. *Let H be a planar point visibility graph on seven vertices such that it has a 3-GSP but no 4-GSP in every visibility embedding ξ of H. Then ξ has exactly six 3-GSPs.*

Proof. Since H has at most 15 edges due to Euler's condition, and a complete graph on seven vertices has 21 edges, there are at least six invisible pairs in H. So, H has at least six 3-GSPs in ξ, On the other hand, if ξ has seven 3-GSPs, then there are seven invisible pairs in H. So, H can have maximum of 14 edges. But then, every line in ξ must pass through exactly three points, contradicting Sylvester-Gallai Theorem [1]. □

Corollary 7. *If p_7 is added to the embeddings of particular graphs of six vertices in Figs. 15(c), 15(d) and 16(f), then only one embedding gives rise to a planar embedding as shown in Fig. 18.*

Consider $n = 8$. In the following lemma, we show that there is no particular graph on eight vertices.

Lemma 18. *There is no particular planar point visibility graph on eight vertices that has a 3-CSP but no 4-CSP.*

Proof. We know that if G does not have an edge between two vertices, then it corresponds to a 3-CSP. Since G has at most 18 edges due to Euler's condition, and a complete graph on eight vertices has 28 edges, there are at least ten edges not present in G. Therefore G must have at least ten edge disjoint 3-CSPs. But ten edge disjoint 3-CSPs require 20 edges. Since G can have at most 18 edges, such a G cannot exist.

□

Case 2. *There is a 4-GSP but no 5-GSP in every visibility embedding of G.*

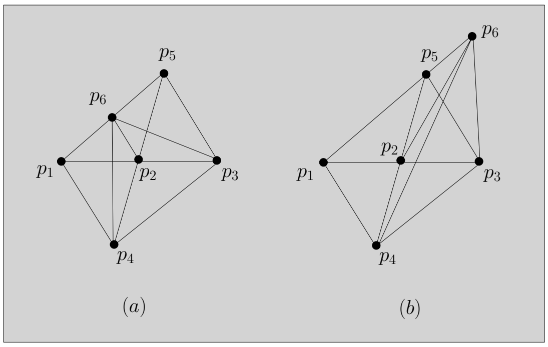

Figure 17: Visibility embeddings of six points after p_6 is added to the embedding in Fig. 14(c).

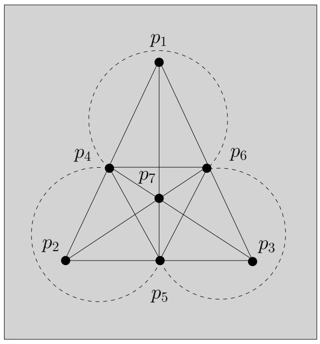

Figure 18: Unique visibility embedding of planar point visibility graph on seven vertices, with a 3-GSP but no 4-GSP. Dotted lines show how the edge-crossings in the visibility embedding can be avoided in a planar embedding.

If $n \leq 6$, G belongs to one of the infinite families having at most two points outside the 4-GSP.

Since G cannot have more than 7 vertices by Lemma 13, we consider only $n = 7$.

Consider any visibility embedding ξ of G. Let p_1, p_2, p_3 and p_4 be collinear points representing a 4-GSP (denoted as l). If the remaining three points p_5, p_6 and p_7 form a 3-GSP disjoint from l, then G is not planar as it has $K_{3,3}$ as a subgraph. If p_5, p_6 and p_7 are mutually visible, and they also see all points of l, then G is not planar as it has $K_{3,3}$ as a subgraph. If p_5, p_6 and p_7 are on opposite sides of l, then, again G is not planar as it has $K_{3,3}$ as a subgraph. So, in every embedding, all points p_5, p_6 and p_7 are on the same side of l. Therefore, an endpoint of every 3-GSP in ξ is a point of l. We have the following lemma.

Lemma 19. *If every visibility embedding of a planar point visibility graph H has a 4-GSP but no 5-GSP, then every visibility embedding of H has at least three 3-GSPs edge disjoint from the 4-GSP.*

Proof. Since H has at most 15 edges due to Euler's condition, and a complete graph on seven vertices has 21 edges, there are at least six invisible pairs in H. Three of these invisible pairs correspond to the 4-GSP. So, the remaining three invisible pairs must correspond to three 3-GSPs edge disjoint-from the 4-GSP. $\square$

Due to the above Lemma, we must ensure that three new 3-GSPs are formed in ξ, by adding p_5, p_6 and p_7. We add p_5 and p_6 to construct the first new 3-GSP as shown in Fig. 19, excluding symmetric cases. Then p_7 is added to these embeddings forming two more 3-GSPs. This can be realized only by placing p_7 at intersection points of pairs of lines containing exactly two points on each line.

Let us add p_7 to the embedding shown in Fig. 19(a). Removing symmetric embeddings, we have the following choices of positioning p_7 in the two new 3-GSPs: $\overline{p_2p_7p_6}$ and $\overline{p_3p_7p_5}$ (Fig. 20(a)), $\overline{p_2p_7p_6}$ and $\overline{p_4p_7p_5}$ (Fig. 20(b)), $\overline{p_3p_7p_6}$ and $\overline{p_4p_7p_5}$ (Fig. 20(c)), $\overline{p_2p_5p_7}$ and $\overline{p_3p_6p_7}$ (Fig. 20(d)), $\overline{p_2p_5p_7}$ and $\overline{p_4p_6p_7}$ (Fig. 20(e)), $\overline{p_3p_5p_7}$ and $\overline{p_4p_6p_7}$ (Fig. 20(f)). It can be seen that embeddings in Figs. 20(a), 20(c), 20(d) and 20(e) correspond to non-planar graphs, and embeddings in Figs. 20(b) and 20(f) correspond to planar graphs isomorphic to each other. Hence, only one particular planar graph arises after adding p_7 to the visibility embedding in Fig. 19(a).

As before, let us add p_7 to the embedding shown in Fig. 19(b). Removing symmetric embeddings, we have the following choices of positioning p_7 in the two new 3-GSPs: $\overline{p_1p_7p_6}$ and $\overline{p_3p_5p_7}$ (Fig. 21(a)), $\overline{p_1p_7p_6}$ and $\overline{p_4p_5p_7}$ (Fig. 21(b)), $\overline{p_1p_5p_7}$ and $\overline{p_3p_7p_6}$ (Fig. 21(c)), $\overline{p_1p_5p_7}$ and $\overline{p_4p_7p_6}$ (Fig. 21(d)), $\overline{p_3p_5p_7}$ and $\overline{p_4p_6p_7}$ (Fig. 21(e)), and $\overline{p_3p_7p_6}$ and $\overline{p_4p_7p_5}$ (Fig. 21(f)). It can be seen that embeddings in Figs. 21(a), 21(c), 21(d), 21(e) and 21(f) correspond to non-planar graphs, and the embedding in Fig. 21(b) corresponds to a planar graph.

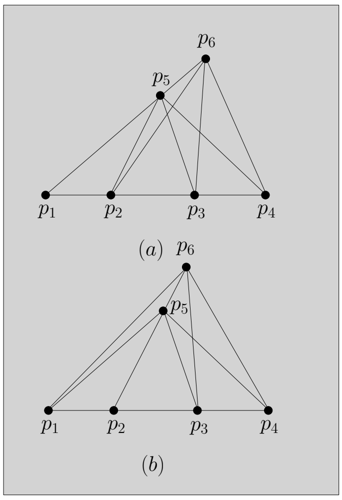

Figure 19: Visibility embeddings of six points containing overlapping but edge disjoint 3-GSP and 4-GSP.

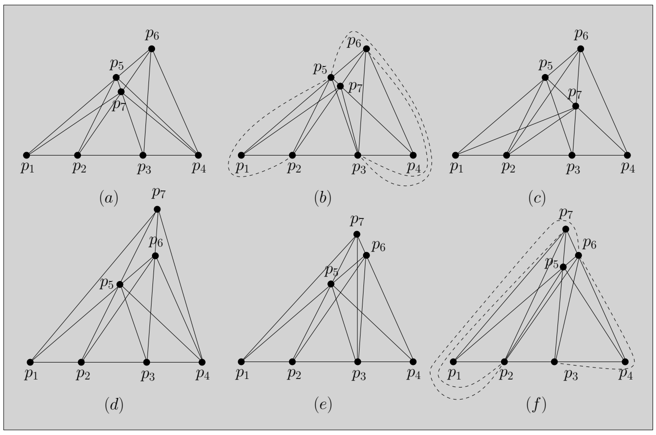

Figure 20: Visibility embeddings of seven points after p_7 is added to the embedding in Fig. 19(a). Dotted lines show how the edge-crossings in the visibility embedding can be avoided in a planar embedding.

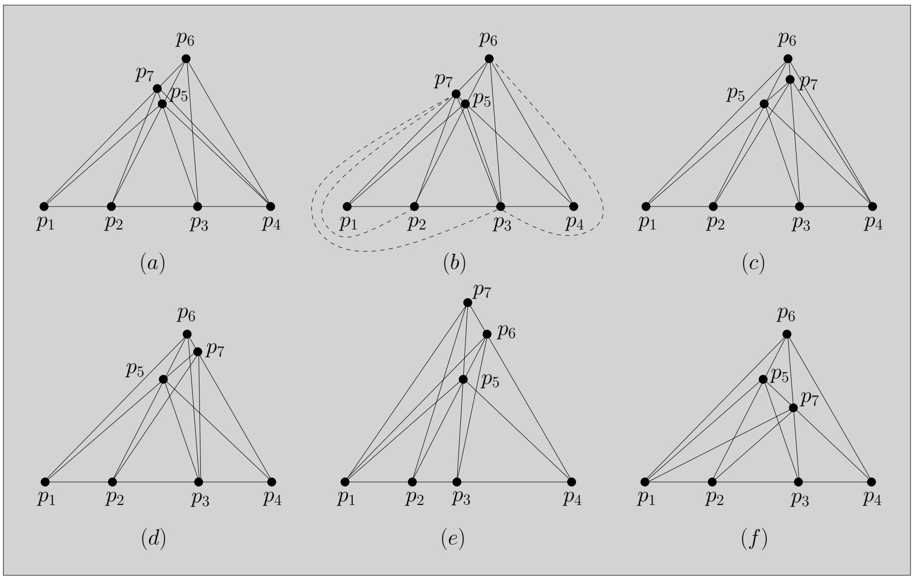

Figure 21: Visibility embeddings of seven points after p_7 is added to the embedding in Fig. 19(b). Dotted lines show how the edge-crossings in the visibility embedding can be avoided in a planar embedding.

But this embedding is already present in Fig. 20. So, no new planar graph arises after adding p_7 to the visibility embedding in Fig. 19(b). Thus, one particular planar point-visibility graph of seven vertices is identified (see Fig. 20(b)).