Complexity of a Problem Concerning Reset Words for Eulerian Binary Automata[☆]

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Abstract

A word is called a reset word for a deterministic finite automaton if it maps all the states of the automaton to a unique state. Deciding about the existence of a reset word of a given maximum length for a given automaton is known to be an NP-complete problem. We prove that it remains NP-complete even if restricted to Eulerian automata with binary alphabets, as it has been conjectured by Martyugin (2011).

1. Introduction and Preliminaries

A *deterministic finite automaton* is a triple $A = (Q, X, \delta)$, where Q and X are finite sets and δ is an arbitrary mapping $Q \times X \to Q$. Elements of Q are called *states*, X is the *alphabet*. The *transition function* δ can be naturally extended to $Q \times X^\star \to Q$, still denoted by δ. We extend it also by defining

$$\delta(S, w) = \{\delta(s, w) \mid s \in S, w \in X^\star\}$$

for each $S \subseteq Q$. If the automaton is fixed, we write

$$r \xrightarrow{w} s$$

instead of $\delta(r, w) = s$.

For a given automaton $A = (Q, X, \delta)$, we call $w \in X^\star$ a *reset word* if

$$|\delta(Q, w)| = 1.$$

If such a word exists, we call the automaton *synchronizing*. Note that each word having a reset word as a factor is also a reset word.

A need for finding reset words appears in several fields of mathematics and engineering. Classical applications (see [11]) include model-based testing,

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robotic manipulation, and symbolic dynamics, but there are important connections also with information theory [10] and with formal models of biomolecular processes [1].

The *Černý Conjecture*, a longstanding open problem, claims that each synchronizing automaton has a reset word of length $(|Q|-1)^2$. Though it still remains open, there are many weaker results in this field, see e.g. [8, 4] for recent ones[1].

Various computational problems arise from the study of synchronization:

- *Given an automaton, decide if it is synchronizing.* Relatively simple algorithm, which could be traced back to [2], works in polynomial time.
- *Given a synchronizing automaton and a number d, decide if d is the length of shortest reset words.* This has been shown to be both NP-hard [3] and coNP-hard. More precisely, it is DP-complete [7].
- *Given a synchronizing automaton and a number d, decide if there exists a reset word of length d.* This problem is of our interest. Lying in NP, it is not so computationally hard as the previous problem. However, it is proven to be NP-complete [3]. Following the notation of [6], we call it Syn. Assuming that $\mathcal{M}$ is a class of automata and membership in $\mathcal{M}$ is polynomially decidable, we define a restricted problem:

  Syn($\mathcal{M}$)
  Input: synchronizing automaton $A = ([n], X, \delta) \in \mathcal{M}$, $d \in \mathbb{N}$
  Output: does A have a reset word of length d?

An automaton $A = (Q, X, \delta)$ is *Eulerian* if

$$\sum_{x \in X} |\{r \in Q \mid \delta(r, x) = q\}| = |X|$$

for each $q \in Q$. Informally, there should be exactly $|X|$ transitions incoming to each state. An automaton is *binary* if $|X| = 2$. The classes of Eulerian and binary automata are denoted by $\mathcal{EU}$ and $\mathcal{AL}_2$ respectively.

Previous results about various restrictions of Syn can be found in [3, 5, 6]. Some of these problems turned out to be polynomially solvable, others are NP-complete. In [6] Martyugin conjectured that Syn($\mathcal{EU} \cap \mathcal{AL}_2$) is NP-complete. This conjecture is confirmed in the rest of the present paper.

2. Main Result

2.1. Proof Outline

We prove the NP-completeness of Syn($\mathcal{EU} \cap \mathcal{AL}_2$) by a polynomial reduction from 3-SAT. So, for arbitrary propositional formula ϕ in 3-CNF we construct

[1] The result published by Trahtman [9] in 2011 has turned out to be proved incorrectly.

an Eulerian binary automaton A and a number d such that

$$\phi \text{ is satisfiable} \Leftrightarrow A \text{ has a reset word of length } d. \tag{1}$$

For the rest of the paper we fix a formula

$$\phi = \bigwedge_{i=1}^{m} \bigvee_{\lambda \in C_i} \lambda$$

on n variables where each C_i is a three-element set of literals, i.e. subset of

$$L_\phi = \{x_1, \ldots, x_n, \neg x_1, \ldots, \neg x_n\}.$$

We index the literals $\lambda \in L_\Phi$ by the following mapping κ:

λ	x_1	x_2	$\ldots$	x_n	$\neg x_1$	$\neg x_2$	$\ldots$	$\neg x_n$
$\kappa(\lambda)$	0	1	$\ldots$	$n-1$	n	$n+1$	$\ldots$	$2n-1$

Let $A = (Q, X, \delta)$, $X = \{a, b\}$. Because the structure of the automaton A will be very heterogeneous, we use an unusual method of description. The basic principles of the method are:

- We describe the automaton A via a labeled directed multigraph G, representing the automaton in a standard way: edges of G are labeled by single letters a and b and carry the structure of the function δ. Paths in G are thus *labeled* by words from $\{a, b\}^\star$.
- There is a collection of labeled directed multigraphs called *templates*. The graph G is one of them. Another template is `SINGLE`, which consists of one vertex and no edges.
- Each template `T`$\neq$`SINGLE` is expressed in a fixed way as a disjoint union through a set $\mathrm{PARTS}_{\mathtt{T}}$ of its proper subgraphs (the *parts* of `T`), extended by a set of additional edges (the *links* of `T`). Each $H \in \mathrm{PARTS}_{\mathtt{T}}$ is isomorphic to some template `U`. We say that H *is of type* `U`.
- Let q be a vertex of a template `T`, lying in a subgraph $H \in \mathrm{PARTS}_{\mathtt{T}}$ which is of type `U` via a vertex mapping $\rho : H \to \mathtt{U}$. The *local address* $\mathrm{adr}_{\mathtt{T}}(q)$ is a finite string of identifiers separated by „|". It is defined inductively by

$$\mathrm{adr}_{\mathtt{T}}(q) = \begin{cases} H \mid \mathrm{adr}_{\mathtt{U}}(\rho(q)) & \text{if } \mathtt{U} \neq \mathtt{SINGLE} \\ H & \text{if } \mathtt{U} = \mathtt{SINGLE}. \end{cases}$$

  The string $\mathrm{adr}_G(q)$ is used as a regular vertex identifier.

Having a word $w \in X^\star$, we denote a t-th letter of w by w_t and define the set $S_t = \delta(Q, w_1 \ldots w_t)$ of *active states at time* t. Whenever we depict a graph, a solid arrow stands for the label a and a dotted arrow stands for the label b.

2.2. Description of the Graph G

Let us define all the templates and informally comment on their purpose. Figure 1 defines the template `ABS`, which does not depend on the formula ϕ.

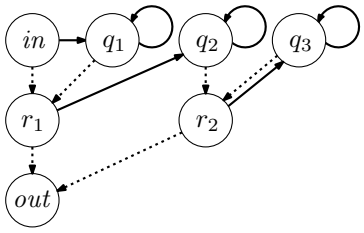


Figure 1: Template `ABS`

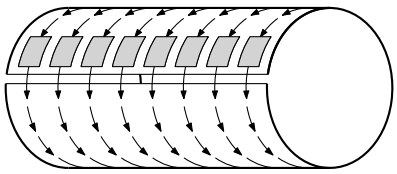


Figure 2: A barrier of `ABS` parts

The state *out* of a part of type `ABS` is always inactive after application of a word of length at least 2 which does not contain b^2 as a factor. This allows us to ensure the existence of a relatively short reset word. Actually, large areas of the graph (namely the `CLAUSE(...)` parts) have roughly the shape depicted in Figure 2, a cylindrical structure with a horizontal barrier of `ABS` parts. If we use a sufficiently long word with no occurrence of b^2, the edges outgoing from the `ABS` parts are never used and almost all states become inactive.

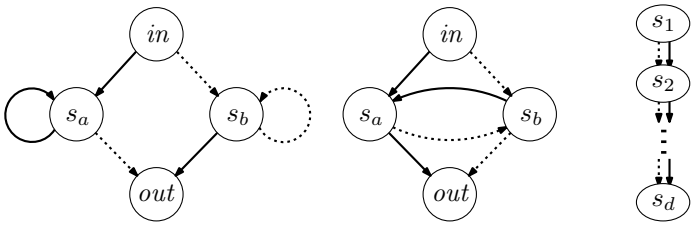


Figure 3: Templates `CCA`, `CCI` and `PIPE(`d`)` respectively

Figure 3 defines simple templates `CCA`, `CCI` and `PIPE(`d`)` for each $d \geq 1$. The activity of an *out* state depends on the last two letters applied. In the case of `CCA` it is inactive if (and typically only if) the two letters were equal. In the case of `CCI` it works oppositely, equal letters correspond to active *out* state. One of the key ideas of the entire construction is the following. Let there be a subgraph of the form

$$\begin{gathered} \text{part of type } \texttt{PIPE(}d\texttt{)} \\ \downarrow a, b \\ \text{part of type } \texttt{CCA} \text{ or } \texttt{CCI} \\ \downarrow a, b \\ \text{part of type } \texttt{PIPE(}d\texttt{)}. \end{gathered} \tag{2}$$

Before the synchronization process starts, all the states are active. As soon as the second letter of an input word is applied, the activity of the *out* state

starts to depend on the last two letters and the pipe below keeps a record of its previous activity. We say that a part H of type `PIPE(`d`)` *records* a sequence $B_1 \dots B_d \in \{\mathbf{0}, \mathbf{1}\}^d$ *at time* t, if it holds that

$$B_k = \mathbf{1} \Leftrightarrow H|s_k \notin S_t.$$

In order to continue with defining templates, let us define a set M_ϕ containing all the literals from L_ϕ and some auxiliary symbols:

$$M_\phi = L_\phi \cup \{y_1, \dots, y_n\} \cup \{z_1, \dots, z_n\} \cup \{q, q', r, r'\}.$$

We index the $4n + 4$ members $\nu \in M_\phi$ by the following mapping μ:

ν	q	r	y_1	x_1	y_2	x_2	$\dots$	y_n	x_n
$\mu(\nu)$	1	2	3	4	5	6		$2n{+}1$	$2n{+}2$

ν	q'	r'	z_1	$\neg x_1$	z_2	$\neg x_2$	$\dots$	z_n	$\neg x_n$
$\mu(\nu)$	$2n{+}3$	$2n{+}4$	$2n{+}5$	$2n{+}6$	$2n{+}7$	$2n{+}8$	$\dots$	$4n{+}3$	$4n{+}4$

The inverse mapping is denoted by μ'. For each $\lambda \in L_\phi$ we define templates `INC(`λ`)` and `NOTINC(`λ`)`, both consisting of $12n + 12$ `SINGLE` parts identified by elements of $\{1, 2, 3\} \times M_\phi$. As depicted by Figure 4a, the links of `INC(`λ`)` are:

$$(1,\nu) \xrightarrow{a} \begin{cases} (2,\lambda) & \text{if } \nu = \lambda \text{ or } \nu = r \\ (2,\nu) & \text{otherwise} \end{cases} \qquad (1,\nu) \xrightarrow{b} \begin{cases} (2,r) & \text{if } \nu = \lambda \text{ or } \nu = r \\ (2,\nu) & \text{otherwise} \end{cases}$$

$$(2,\nu) \xrightarrow{a} \begin{cases} (3,q) & \text{if } \nu = r \text{ or } \nu = q \\ (3,\nu) & \text{otherwise} \end{cases} \qquad (2,\nu) \xrightarrow{b} \begin{cases} (3,r) & \text{if } \nu = r \text{ or } \nu = q \\ (3,\nu) & \text{otherwise} \end{cases}$$

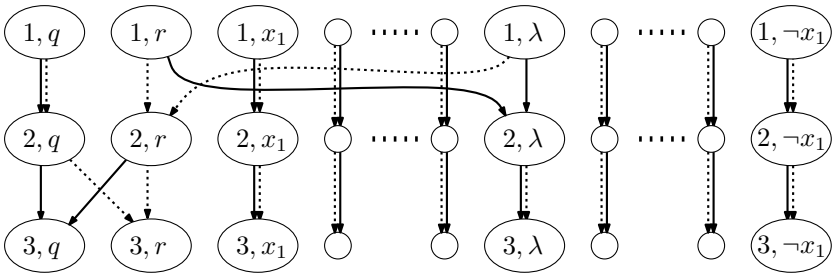
Note that we use the same identifier for an one-vertex subgraph and for its vertex. As it is clear from Figure 4b, the links of `NOTINC(`λ`)` are:

$$(1,\nu) \xrightarrow{a} (2,\lambda) \qquad (1,\nu) \xrightarrow{b} (2,r)$$

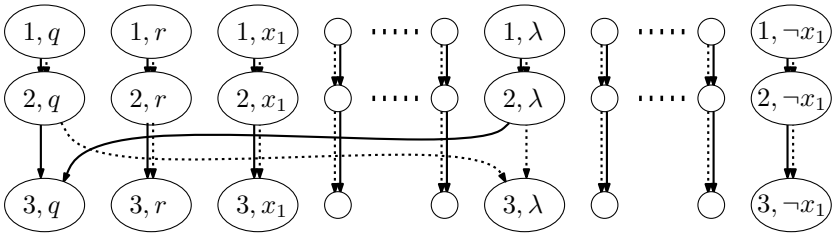
$$(2,\nu) \xrightarrow{a} \begin{cases} (3,q) & \text{if } \nu = q \text{ or } \nu = \lambda \\ (3,\nu) & \text{otherwise} \end{cases} \qquad (2,\nu) \xrightarrow{b} \begin{cases} (3,\lambda) & \text{if } \nu = q \text{ or } \nu = \lambda \\ (3,\nu) & \text{otherwise} \end{cases}$$

The key property of such templates comes to light when we need to apply some two-letter word in order to make the state $(3, \lambda)$ inactive assuming $(1, r)$ inactive. If also $(1, \lambda)$ is initially inactive, we can use the word a^2 in both templates. If it is active (which corresponds to the idea of unsatisfied literal λ), we discover the difference between the two templates: The word a^2 works if the type is `NOTINC(`λ`)`, but fails in the case of `INC(`λ`)`. Such *failure* corresponds to the idea of unsatisfied literal λ occurring in a clause of ϕ.

For each clause (each $i \in \{1, \dots, m\}$) we define a template `TESTER(`i`)`. It consists of $2n$ serially linked parts, namely $level_\lambda$ for each $\lambda \in L_\phi$, each of type `INC(`λ`)` or `NOTINC(`λ`)`. The particular type of each $level_\lambda$ depends on the clause



(a) INC(λ)



(b) NOTINC(λ)

Figure 4: Templates INC(λ) and NOTINC(λ)

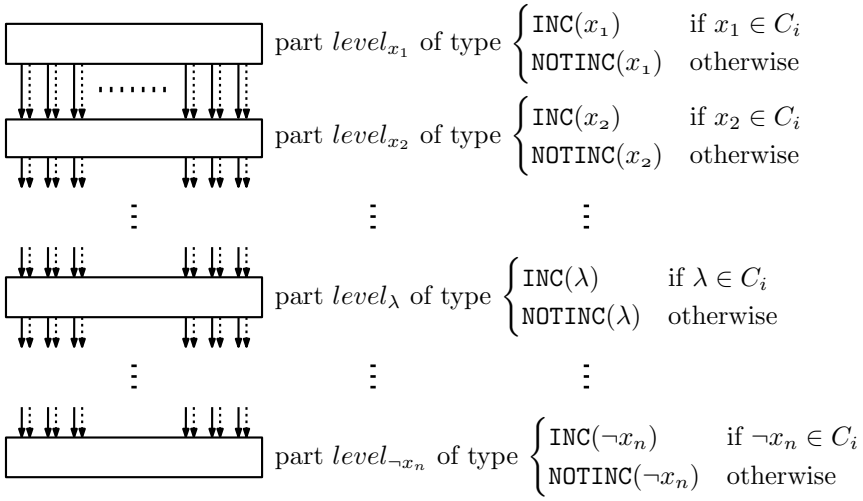


Figure 5: Template TESTER

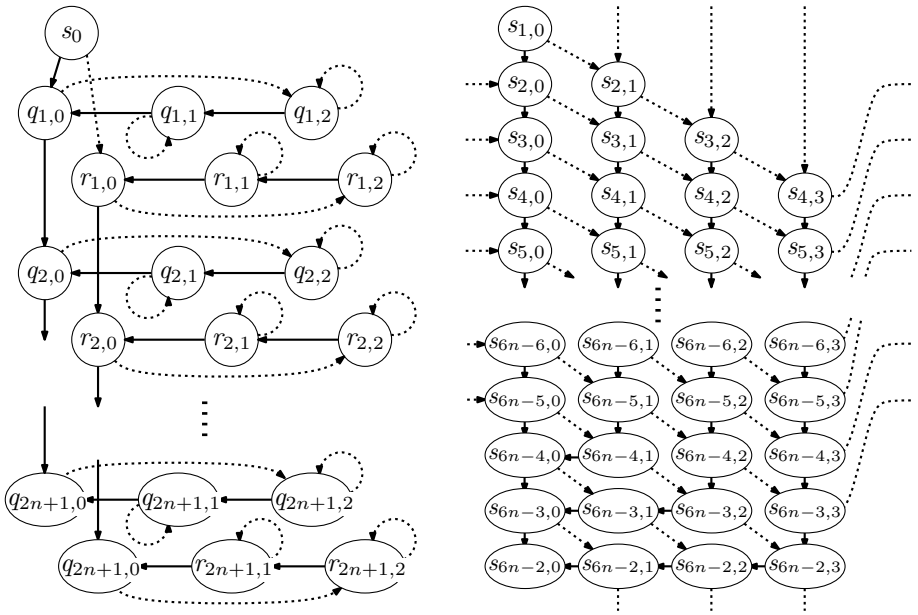


Figure 6: Templates `FORCER` and `LIMITER` respectively

C_i as seen in Figure 5, so exactly three of them are always of type `INC(...)`. If the corresponding clause is unsatisfied, each of its three literals is unsatisfied, which causes three failures within the levels. Three failures imply at least three occurrences of b, which turns up to be too much for a reset word of certain length to exist. Clearly we still need some additional mechanisms to realize this vague vision.

Figure 6 defines templates `FORCER` and `LIMITER`. The idea of template `FORCER` is simple. Imagine a situation when $q_{1,0}$ or $r_{1,0}$ is active and we need to deactivate the entire forcer by a word of length at most $2n+3$. Any use of b would cause an unbearable delay, so if such a word exists, it starts by a^{2n+2}.

The idea of `LIMITER` is similar, but we tolerate some occurrences of b here, namely two of them. This works if we assume $s_{1,0}$ active and it is necessary to deactivate the entire limiter by a word of length at most $6n+1$.

We also need a template `PIPES(`d,k`)` for each $d, k \geq 1$. It consists just of k parallel pipes of length d. Namely there is a `SINGLE` part $s_{d',k'}$ for each $d' \leq d$, $k' \leq k$ and all the edges are of the form $s_{d',k'} \longrightarrow s_{d'+1,k'}$.

The most complex templates are `CLAUSE(`i`)` for each $i \in \{1, \ldots, m\}$. Denote

$$\begin{aligned} \alpha_i &= (i-1)\,(12n-2)\,, \\ \beta_i &= (m-i)\,(12n-2)\,. \end{aligned}$$

As shown in Figure 7, `CLAUSE(`i`)` consists of the following parts:

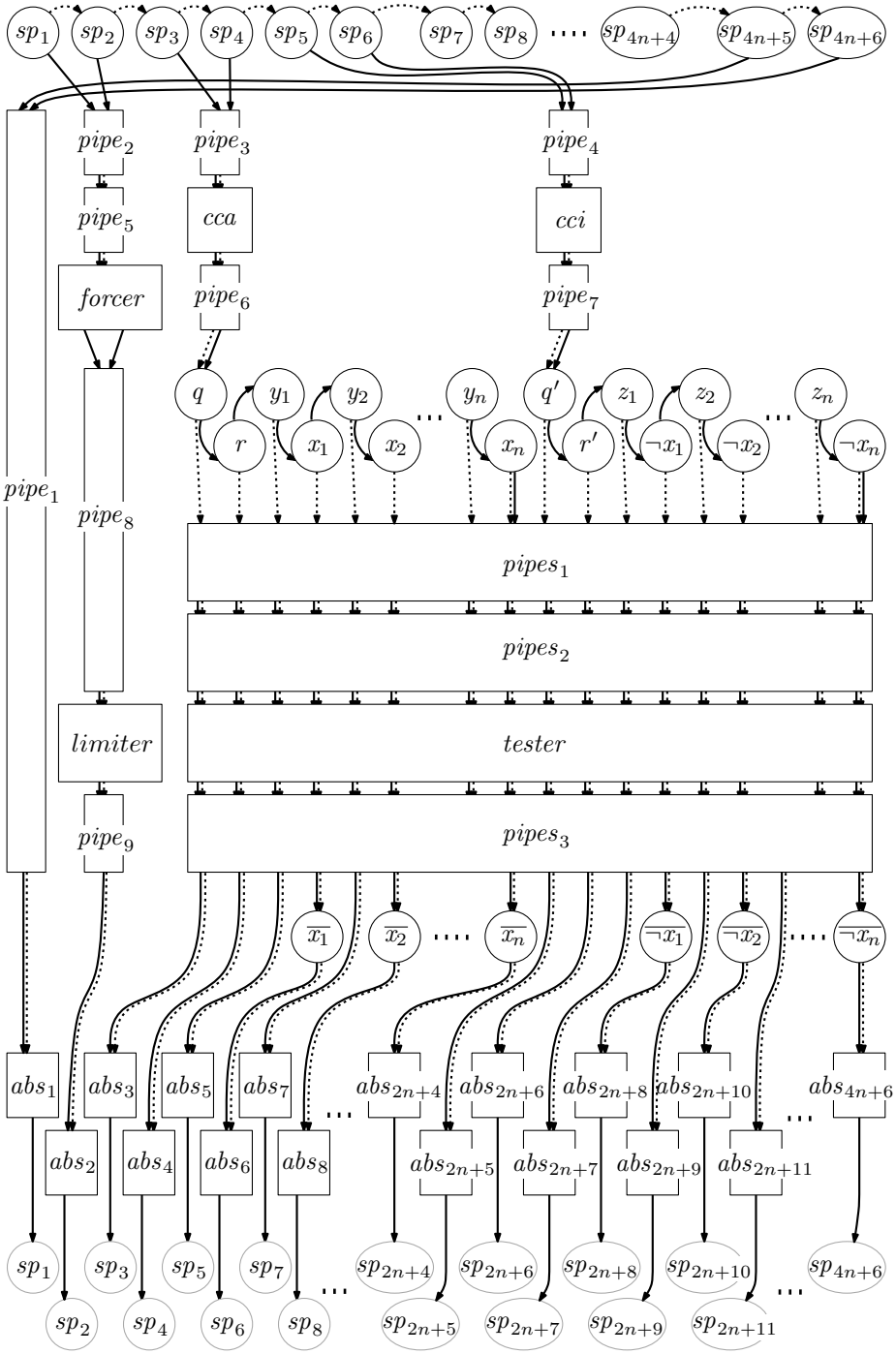


Figure 7: Template `CLAUSE(`i`)`

- Parts $sp_1, \ldots, sp_{4n+6}$ of type `SINGLE`.
- Parts $abs_1, \ldots, abs_{4n+6}$ of type `ABS`. The entire template has a shape similar to Figure 2, including the barrier of `ABS` parts.
- Parts $pipe_2$, $pipe_3$, $pipe_4$ of types `PIPE(`$2n-1$`)` and $pipe_6$, $pipe_7$ of types `PIPE(`$2n+2$`)`.
- Parts *cca* and *cci* of types `CCA` and `CCI` respectively. Together with the pipes above they realize the idea described in (2). As they form two constellations which work simultaneously, the parts $pipe_6$ and $pipe_7$ typically record mutually inverse sequences. We interpret them as an assignment of the variables $x_1, \ldots, x_n$. Such assignment is then processed by the tester.
- A part ν of type `SINGLE` for each $\nu \in M_\phi$.
- A part *tester* of type `TESTER(`i`)`.
- A part $\overline{\lambda}$ of type `SINGLE` for each $\lambda \in L_\phi$. While describing the templates `INC(`λ`)` and `NOTINC(`λ`)` we claimed that in certain case there arises a need to make the state $(3, \lambda)$ inactive. This happens when the border of inactive area moves down through the tester levels. The point is that any word of length $6n$ deactivates the entire tester, but we need to ensure that some tester columns, namely the $\kappa(\lambda)$-th for each $\lambda \in L_\phi$, are deactivated one step earlier. If some of them is still active just before the deactivation of tester finishes, the state $\overline{\lambda}$ becomes active, which slows down the synchronization process.
- Parts $pipes_1$, $pipes_2$ and $pipes_3$ of types `PIPES(`$\alpha_i, 4n+4$`)`, `PIPES(`$6n-2, 4n+4$`)` and `PIPES(`$\beta_i, 4n+4$`)` respectively. There are multiple clauses in ϕ, but multiple testers cannot work in parallel. That is why each of them is padded by a passive `PIPES(...)` part of size depending on particular i. If $\alpha_i = 0$ or $\beta_i = 0$, the corresponding `PIPES` part is not present in cl_i.
- Parts $pipe_1$, $pipe_5$, $pipe_8$, $pipe_9$ of types `PIPE(`$12mn + 4n - 2m + 6$`)`, `PIPE(4)`, `PIPE(`$\alpha_i + 6n - 1$`)`, `PIPE(`β_i`)` respectively.
- The part *forcer* of type `FORCER`. This part guarantees that only the letter a is used in certain segment of the word w. This is necessary for the data produced by *cca* and *cci* to safely leave the parts $pipe_3$, $pipe_4$ and line up in the states of the form ν for $\nu \in M_\phi$, from where they are shifted to the tester.
- The part *limiter* of type `LIMITER`. This part guarantees that the letter b occurs at most twice when the border of inactive area passes through the tester. Because each unsatisfied literal from the clause requests an occurrence of b, only a satisfied clause meets all the conditions for a reset word of certain length to exist.

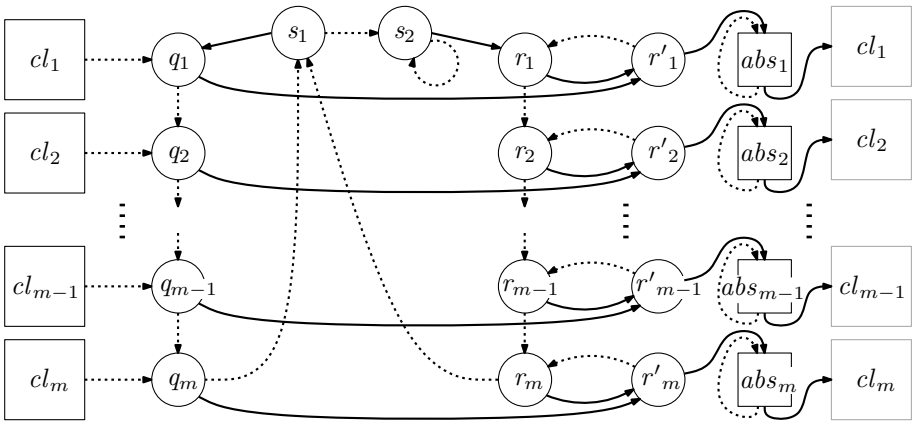


Figure 8: The graph G

Links of $\texttt{CLAUSE}(i)$, which are not clear from Figure 7 are

$$\nu \xrightarrow{a} \begin{cases} pipes_1|s_{1,\mu(\nu)} & \text{if } \nu = \neg x_n \\ \mu'(\mu(\nu)+1) & \text{otherwise} \end{cases} \qquad \nu \xrightarrow{b} pipes_1|s_{1,\mu(\nu)}$$

for each $\nu \in M_\phi$ and

$$pipes_3|s_{\beta_i,k} \xrightarrow{a,b} \begin{cases} \overline{\mu'(k)} & \text{if } \mu'(k) \in L_\phi \\ abs_{k+2}|in & \text{otherwise} \end{cases} \qquad \overline{\lambda} \xrightarrow{a,b} abs_{\mu(\lambda)+2}|in$$

for each $k \in \{1, \ldots, 4n+4\}$, $\lambda \in L_\phi$.

We are ready to form the whole graph G, see Figure 8. For each $i, k \in \{1, \ldots m\}$ there are parts cl_k, abs_k of types $\texttt{CLAUSE}(i)$ and $\texttt{ABS}$ respectively and parts q_k, r_k, r'_k, s_1, s_2 of type $\texttt{SINGLE}$. The edge incoming to a cl_i part ends in $cl_i|sp_1$, the outgoing one starts in $cl_i|sp_{4n+6}$. When no states outside $\texttt{ABS}$ parts are active within each $\texttt{CLAUSE}(\ldots)$ part and no out, r_1 nor r_2 state is active in any $\texttt{ABS}$ part, the word b^2ab^{4n+m+7} takes all active states to s_2 and completes the synchronization. Graph G does not fully represent the automaton A yet because there are

- $8mn + 4m$ vertices with only one outgoing edge, namely $cl_i|abs_k|out$ and $cl_i|sp_l$ for each $i \in \{1, \ldots, m\}, k \in \{1, \ldots, 4n+6\}, l \in \{7, \ldots, 4n+4\}$,
- $8mn+4m$ vertices with only one incoming edge: $cl_i|\nu$ and $cl_i|pipes_1|(1, \nu')$ for each $i \in \{1, \ldots, m\}, \nu \in M_\phi \setminus \{q, q'\}, \nu' \in M_\phi \setminus \{x_n, \neg x_n\}$.

But we do not need to specify the missing edges exactly, let us just say that they somehow connect the relevant states and the automaton A is complete. Let us set

$$d = 12mn + 8n - m + 18$$

and prove that the equivalence (1) holds.

2.3. From an Assignment to a Word

First let us suppose that there is an assignment $\xi_1, \ldots, \xi_n \in \{\mathbf{0}, \mathbf{1}\}$ of the variables $x_1, \ldots, x_n$ (respectively) satisfying the formula ϕ and prove that the automaton A has a reset word w of length d. For each $j \in \{1, \ldots, n\}$ we denote

$$\sigma_j = \begin{cases} a & \text{if } \xi_j = \mathbf{1} \\ b & \text{if } \xi_j = \mathbf{0} \end{cases}$$

and for each $i \in \{1, \ldots, m\}$ we choose a satisfied literal $\overline{\lambda}_i$ from C_i. We set

$$w = a^2 (\sigma_n a) (\sigma_{n-1} a) \ldots (\sigma_1 a)\, aba^{2n+3} b \left(a^{6n-2} v_1\right) \ldots \left(a^{6n-2} v_m\right) b^2 a b^{4n+m+7},$$

where for each $i \in \{1, \ldots, m\}$ we use the word

$$v_i = u_{i,x_1} \ldots u_{i,x_n} u_{i,\neg x_1} \ldots u_{i,\neg x_n},$$

denoting

$$u_{i,\lambda} = \begin{cases} a^3 & \text{if } \lambda = \overline{\lambda}_i \text{ or } \lambda \notin C_i \\ ba^2 & \text{if } \lambda \neq \overline{\lambda}_i \text{ and } \lambda \in C_i \end{cases}$$

for each $\lambda \in L_\phi$. We see that $|v_i| = 6n$ and therefore

$$|w| = 4n + 8 + m\,(12n - 2) + 4n + m + 10 = 12mn + 8n - m + 18 = d.$$

Let us denote

$$\gamma = 12mn + 4n - 2m + 9$$

and

$$\overline{S}_t = Q \backslash S_t$$

for each $t \leq d$. Because the first occurrence of b^2 in w starts by the γ-th letter, we have:

Lemma 2.1. *Each state of a form $cl_{\ldots}|abs_{\ldots}|out$ or $abs_{\ldots}|out$ lies in $\overline{S}_2 \cap \cdots \cap \overline{S}_\gamma$.*

Let us fix an arbitrary $i \in \{1, \ldots, m\}$ and describe a growing area of inactive states within cl_i. We use the following method of verifying inactivity of states: Having a state $s \in Q$ and $t, k \geq 1$ such that any path of length k ending in s uses a member of $\overline{S}_{t-k} \cap \cdots \cap \overline{S}_{t-1}$, we easily deduce that $s \in \overline{S}_t$. In such case let us just say that k *witnesses* that $s \in \overline{S}_t$. The following claims follow directly from the definition of w. Note that Claim 7 relies on the fact that b occurs only twice in v_i.

Lemma 2.2.

1.	$\{cl_i\|sp_1, \ldots, cl_i\|sp_{4n+6}\}$	$\subseteq$	$\overline{S}_2 \cap \cdots \cap \overline{S}_\gamma$
2.	$cl_i\|pipe_2 \cup cl_i\|pipe_3 \cup cl_i\|pipe_4$	$\subseteq$	$\overline{S}_{2n+1} \cap \cdots \cap \overline{S}_\gamma$
3.	$cl_i\|cca \cup cl_i\|cci \cup cl_i\|pipe_5$	$\subseteq$	$\overline{S}_{2n+5} \cap \cdots \cap \overline{S}_\gamma$
4.	$cl_i\|pipe_6 \cup cl_i\|pipe_7 \cup cl_i\|forcer$	$\subseteq$	$\overline{S}_{4n+7} \cap \cdots \cap \overline{S}_\gamma$
5.	$\{cl_i\|\nu : \nu \in M_\phi\}$	$\subseteq$	$\overline{S}_{4n+8} \cap \cdots \cap \overline{S}_\gamma$
6.	$cl_i\|pipes_1 \cup cl_i\|pipes_2 \cup cl_i\|pipe_8$	$\subseteq$	$\overline{S}_{10n+\alpha_i+6} \cap \cdots \cap \overline{S}_\gamma$
7.	$cl_i\|limiter \cup cl_i\|tester$	$\subseteq$	$\overline{S}_{16n+\alpha_i+6} \cap \cdots \cap \overline{S}_\gamma$
8.	$cl_i\|pipe_1 \cup cl_i\|pipe_9 \cup cl_i\|pipes_3$	$\subseteq$	$\overline{S}_{\gamma-1} \cap \overline{S}_\gamma$

Proof.

1. *Claim:* $\{cl_i|sp_1, \ldots, cl_i|sp_{4n+6}\} \subseteq \overline{S}_2 \cap \cdots \cap \overline{S}_\gamma$.
   We have $w_1 w_2 = a^2$ and there is no path labeled by a^2 ending in any $cl_i|sp_{\ldots}$ state, so such states lie in $\overline{S}_2$. For each $t = 3, \ldots, \gamma$ we can inductively use $k = 1$ to witness the memberships in $\overline{S}_t$. In the induction step we use Lemma 2.1, which excludes the *out* states of the `ABS` parts from each corresponding $\overline{S}_{t-1}$.
2. *Claim:* $cl_i|pipe_2 \cup cl_i|pipe_3 \cup cl_i|pipe_4 \subseteq \overline{S}_{2n+1} \cap \cdots \cap \overline{S}_\gamma$.
   All the memberships are witnessed by $k = 2n - 1$, because any path of the length $2n - 1$ ending in such state must use a $cl_i|sp_{\ldots}$ state and such states lie in $\overline{S}_2 \cap \cdots \cap \overline{S}_\gamma$ by the previous claim.
3. *Claim:* $cl_i|cca \cup cl_i|cci \cup cl_i|pipe_5 \subseteq \overline{S}_{2n+5} \cap \cdots \cap \overline{S}_\gamma$.
   We have $w_{2n+2} \ldots w_{2n+5} = a^2ba$, which clearly maps each state of $cl_i|cca$, $cl_i|cci$ or $cl_i|pipe_5$ out of those parts. Each path of length 4 leading into the parts from outside starts in $\overline{S}_{2n+1}$, so it follows that all the states lie in $\overline{S}_{2n+5}$. To prove the rest we inductively use the witness $k = 1$.
4. *Claim:* $cl_i|pipe_6 \cup cl_i|pipe_7 \cup cl_i|forcer \subseteq \overline{S}_{4n+7} \cap \cdots \cap \overline{S}_\gamma$.
   In the cases of $cl_i|pipe_6$ *and* $cl_i|pipe_7$ we just use the witness $k = 2n + 2$. In the case of $cl_i|forcer$ we proceed the same way as in the previous claim. We have $w_{2n+6} \ldots w_{4n+7} = a^{2n+2}$. Because also $w_{2n+5} = a$, only the states $q_{\ldots,0}$ can be active within the part $cl_i|forcer$ in time $2n + 6$. The word $w_{2n+7} \ldots w_{4n+7}$ maps all such states out of $cl_i|forcer$. Each path of length $2n + 2$ leading into $cl_i|forcer$ from outside starts in $\overline{S}_{2n+5}$, so it follows that all states from $cl_i|forcer$ lie in $\overline{S}_{4n+7}$. To handle $t = 4n + 8, \ldots, \gamma$ we inductively use the witness $k = 1$.
5. *Claim:* $\{cl_i|\nu : \nu \in M_\phi\} \subseteq \overline{S}_{4n+8} \cap \cdots \cap \overline{S}_\gamma$.
   In the cases of $cl_i|q$ and $cl_i|q'$ we use the witness 1. We have $w_{4n+8} = b$ and the only edges labeled by b incoming to remaining states could be some of the $8mn + 4m$ unspecified edges of G. But we have $w_{4n+6} w_{4n+7} = a^2$, so each *out* state of any `ABS` part lies in $\overline{S}_{4n+7}$ and thus no unspecified edge starts in a state outside $\overline{S}_{4n+7}$.

6. *Claim:* $cl_i|pipes_1 \cup cl_i|pipes_2 \cup cl_i|pipe_8 \subseteq \overline{S}_{10n+\alpha_i+6} \cap \cdots \cap \overline{S}_\gamma$.
   We use witnesses $k = \alpha_i$ for $cl_i|pipes_1$, $k = 6n-2$ for $cl_i|pipes_2$ and $k = \alpha_i + 6n - 1$ for $cl_i|pipe_8$.
7. *Claim:* $cl_i|limiter \cup cl_i|tester \subseteq \overline{S}_{16n+\alpha_i+6} \cap \cdots \cap \overline{S}_\gamma$.
   Because
   $$w_{4n+\alpha_i+9} \ldots w_{10n+\alpha_i+6} = a^{6n-2},$$
   there are only states of the form $cl_i|limiter|s_{\ldots,0}$ in the intersection of $cl_i|limiter$ and $S_{10n+\alpha_i+6}$. Together with the fact that there are only two occurrences of b in v_i it confirms that the case of $cl_i|limiter$ holds. The case of $cl_i|tester$ is easily witnessed by $k = 6n$.
8. *Claim:* $cl_i|pipe_1 \cup cl_i|pipe_9 \cup cl_i|pipes_3 \subseteq \overline{S}_{\gamma-1} \cap \overline{S}_\gamma$.
   We use witnesses $k = 12mn + 4n - 2m + 6$ for $cl_i|pipe_1$ and $k = \beta_i$ for $cl_i|pipe_9$, $cl_i|pipes_3$.

□

For each $\lambda \in L_\phi$ we ensure by the word $u_{i,\lambda}$ that the $\kappa(\lambda)$-th tester column is deactivated in advance, namely at time $t = 16n + \alpha_i + 5$. The advance allows the following key claim to hold true.

Lemma 2.3. $\left\{ cl_i|\overline{\lambda} \,:\, \lambda \in L_\phi \right\} \subseteq \overline{S}_{\gamma-1} \cap \overline{S}_\gamma$.

Proof. For each such λ we choose

$$k = 6n - 3\kappa(\lambda) + \beta_i + 1$$

as a witness of $cl_i|\overline{\lambda} \in \overline{S}_{\gamma-1}$. There is only one state where a path of length k ending in $\overline{\lambda}$ starts: the state

$$s = cl_i|tester|level_\lambda|\,(3, \lambda)\,.$$

It holds that

$$s \in \overline{S}_{10n+\alpha_i+3\kappa(\lambda)+6} \cap \cdots \cap \overline{S}_\gamma,$$

as is easily witnessed by $k' = 3\kappa(\lambda)$ using Claim 6 of Lemma 2.2. But we are going to show also that

$$s \in \overline{S}_{10n+\alpha_i+3\kappa(\lambda)+5}, \tag{3}$$

which will imply that k is a true witness of $\overline{\lambda} \in \overline{S}_{\gamma-1}$, because

$$(\gamma - 1) - k = 10n + \alpha_i + 3\kappa(\lambda) + 5.$$

So let us prove the membership (3). We need to observe, using the definition of w, that:

- At time $2n + 5$ the part $pipe_6$ records the sequence

  $$\mathbf{0}, \mathbf{1}, \xi_1, \xi_1, \xi_2, \xi_2, \ldots, \xi_n, \xi_n$$

and the part $pipe_7$ records the sequence of inverted values. Because

$$w_{2n+6} \dots w_{4n+7} = a^{2n+2},$$

at time $4n+7$ the states q, r' are active, the states q', r are inactive and for each $j \in \{1, \dots, n\}$ it holds that

$$x_j \in \overline{S}_{4n+7} \Leftrightarrow y_j \in \overline{S}_{4n+7} \Leftrightarrow \neg x_j \in S_{4n+7} \Leftrightarrow z_j \in S_{4n+7} \Leftrightarrow \xi_j = \mathbf{1}.$$

Because $w_{4n+8} = b$, at time $10n+\alpha_i+6$ we find the whole structure above shifted to the first row of $cl_i|tester$, so particularly for $\lambda \in L_\phi$:

$$cl_i|tester|level_{x_1}|\,(1,\lambda) \in \overline{S}_{10n+\alpha_i+6} \Leftrightarrow \lambda \text{ is satisfied by } \xi_1, \dots, \xi_n.$$

- From a simple induction on tester levels it follows that

$$cl_i|tester|level_\lambda|\,(1,r) \in \overline{S}_{10n+\alpha_i+3\kappa(\lambda)+3}.$$

Note that

$$w_{10n+\alpha_i+3\kappa(\lambda)+4} w_{10n+\alpha_i+3\kappa(\lambda)+5} w_{10n+\alpha_i+3\kappa(\lambda)+6} = u_{i,\lambda}$$

and distinguish the following cases:

- If $\lambda = \overline{\lambda}_i$, we have $\lambda \in C_i$, the part $cl_i|tester|level_\lambda$ is of type `INC(λ)` and $u_{i,\lambda} = a^3$. We also know that λ is satisfied, so

$$cl_i|tester|level_{x_1}|\,(1,\lambda) \in \overline{S}_{10n+\alpha_i+6}.$$

  The state above is the only state, from which any path of length $3\kappa(\lambda)-3$ leads to $cl_i|tester|level_\lambda|\,(1,\lambda)$, so we deduce that

$$cl_i|tester|level_\lambda|\,(1,\lambda) \in \overline{S}_{10n+\alpha_i+3\kappa(\lambda)+3}.$$

  We see that each path labeled by a^2 ending in $cl_i|tester|level_\lambda|\,(3,\lambda)$ starts in $cl_i|tester|level_\lambda|\,(1,\lambda)$ or in $cl_i|tester|level_\lambda|\,(1,r)$, but each of the two states lies in $\overline{S}_{10n+\alpha_i+3\kappa(\lambda)+3}$. So the membership (3) holds.

- If $\lambda \notin C$, the part $cl_i|tester|level_\lambda$ is of type `NOTINC(λ)` and $u_{i,\lambda} = a^3$. Particularly $w_{10n+\alpha_i+3\kappa(\lambda)+5} = a$ but no edge labeled by a comes to $cl_i|tester|level_\lambda|\,(3,\lambda)$ and the membership (3) follows trivially.

- If $\lambda \neq \overline{\lambda}_i$ and $\lambda \in C_i$, the part $cl_i|tester|level_\lambda$ is of type `INC(λ)` and $u_{i,\lambda} = ba^2$. Particularly

$$w_{10n+\alpha_i+3\kappa(\lambda)+4} w_{10n+\alpha_i+3\kappa(\lambda)+5} = ba,$$

  but no path labeled by ba comes to $cl_i|tester|level_\lambda|\,(3,\lambda)$, so we reach the same conclusion as in the previous case.

We have proven that $cl_i|\overline{\lambda}$ lies in $\overline{S}_{\gamma-1}$. From Claim 8 of Lemma 2.2 it follows directly that it lies also in $\overline{S}_\gamma$. □

We see that within cl_i only states from the `ABS` parts can lie in $S_{\gamma-1}$. Since $w_{\gamma-2} w_{\gamma-1} = a^2$, no state r_1, r_2 or *out* from any `ABS` part lies in $S_{\gamma-1}$. Now we easily check that all the states possibly present in $S_{\gamma-1}$ are mapped to s_2 by the word $w_\gamma \dots w_d = b^2 a b^{4n+m+7}$.

2.4. From a Word to an Assignment.

Since now we suppose that there is a reset word w of length

$$d = 12mn + 8n - m + 18.$$

The following lemma is not hard to verify.

Lemma 2.4.

1. *Up to labeling there is a unique pair of paths, both of a length $l \leq d-2$, leading from $cl_1|pipe_1|s_1$ and $cl_2|pipe_1|s_1$ to a common end. They are of length $d-2$ and meet in s_2.*
2. *The word w starts by a^2.*

Proof.

1. The leading segments of both paths are similar since they stay within the parts cl_1 and cl_2:

$$pipe_1|s_1 \xrightarrow{a,b} \ldots \xrightarrow{a,b} pipe_1|s_{12mn+4n-2m+6} \xrightarrow{a,b} abs_1|in \xrightarrow{b} \\ \xrightarrow{b} abs_1|r_1 \xrightarrow{b} abs_1|out \xrightarrow{a} sp_1 \xrightarrow{b} \ldots \xrightarrow{b} sp_{4n+6}.$$

   Once the paths leave the parts cl_1 and cl_2, the shortest way to merge is the following:

$$\begin{array}{l} cl_1|sp_{4n+6} \xrightarrow{b} q_1 \xrightarrow{b} q_2 \xrightarrow{b} \quad \ldots \quad \xrightarrow{b} q_{m-1} \xrightarrow{b} q_m \xrightarrow{b} s_1 \xrightarrow{b} \\ cl_2|sp_{4n+6} \xrightarrow{b} q_2 \xrightarrow{b} q_3 \xrightarrow{b} \quad \ldots \quad \xrightarrow{b} q_m \quad \xrightarrow{b} s_1 \xrightarrow{b} s_2 \xrightarrow{b} \end{array} s_2$$

   Having the description above it is easy to verify that the length is $d-2$ and there is no way to make the paths shorter.
2. Suppose that $w_1w_2 \neq a^2$. Any of the three possible values of w_1w_2 implies that

$$\left\{ cl_i|sp_3, \ldots, cl_i|sp_{4n+6} \right\} \subseteq S_2$$

   for each i. It cannot hold that $w = w_1w_2b^{d-2}$, because in such case all $cl_{\ldots}|cca|s_b$ states would be active in any time $t \geq 3$. So the word w has a prefix $w_1w_2b^ka$ for some $k \geq 0$. If $k \leq 4n+3$, it holds that $cl_i|sp_{4n+6} \in S_{k+2}$ and therefore $cl_i|pipe_1|s_1 \in S_{k+3}$, which contradicts the first claim. Let $k \geq 4n+4$. Some state of a form $cl_i|forcer|q_{1,\ldots}$ or $cl_i|forcer|r_{1,\ldots}$ lies in S_{k+2} for each i. This holds particularly for $i = 1$ and $i = 2$, but there is no pair of paths of length at most

$$d - (4n+4) \geq d - k$$

   leading from such two states to a common end.

□

The second claim implies that $cl_i|pipe_1|s_1 \in S_2$ for each $i \in \{1, \ldots, m\}$, so it follows that

$$\delta(Q, w) = \{s_2\}.$$

Let us denote

$$\overline{d} = 12mn + 4n - 2m + 11$$

and

$$\overline{w} = w_1 \ldots w_{\overline{d}}.$$

The following lemma holds because no edges labeled by a are available for final segments of the paths described in the first claim of Lemma 2.4.

Lemma 2.5.

1. *The word w can be written as $w = \overline{w}b^{4n+m+7}$ for some word $\overline{w}$.*
2. *For any $t \geq \overline{d}$, no state from any $cl_{\ldots}$ part lie in S_t, except for the $sp_{\ldots}$ states.*

Proof.

1. Let us write $w = w_1 w_2 w'$. From Lemma 2.4 it follows that

   $$\delta(cl_1|pipe_1|s_1, w') = \delta(cl_2|pipe_1|s_1, w')$$

   and w' have to label some of the paths determined up to labeling in Lemma 2.4(1). The final $4n+m+7$ edges of the paths lead from $cl_1|sp_1$ and $cl_2|sp_1$ to s_2. All the transitions used here are necessarily labeled by b.
2. The claim is easy to observe, since the first claim implies that S_t is a subset of

   $$S' = \left\{s \in Q \mid (\exists d \in \mathbb{N})\, \delta\left(s, b^d\right) = s_2\right\}.$$

□

The next lemma is based on properties of the parts $cl_{\ldots}|forcer$ but to prove that no more a follows the enforced factor a^{2n+1} we also need to observe that each $cl_{\ldots}|cca|out$ or each $cl_{\ldots}|cci|out$ lies in S_{2n+4}.

Lemma 2.6. *The word $\overline{w}$ starts by $\overline{u}a^{2n+1}b$ for some $\overline{u}$ of length $2n + 6$.*

Proof. At first we prove that $\overline{w}$ starts by $\overline{u}a^{2n+1}$. Lemma 2.4(2) implies that $cl_1|pipe_2|s_1 \in S_2$, so obviously some of the states $cl_1|forcer|q_{1,0}$ and $cl_1|forcer|r_{1,0}$ lies in S_{2n+6}. If $w_{2n+6+k} = b$ for some $k \in \{1, \ldots, 2n+1\}$, it holds that $cl_i|forcer|q_{k,2}$ or $cl_i|forcer|r_{k,2}$ lies in S_{2n+6+k}. From such state no path of length at most $2n + 3 - k$ leads to $cl_i|pipe_8|s_1$ and therefore no path of length at most

$$(2n + 3 - k) + (\alpha_i + 6n - 1) + (6n - 2) + \beta_i + 3 = \overline{d} - (2n + 6 + k)$$

leads into S', which contradicts Lemma 2.5(2). It remains to show that there is b after the prefix $\overline{u}a^{2n+1}$. Lemma 2.4(2) implies that both $cl_1|cca|in$ and $cl_1|cci|in$ lie in S_{2n+1}, from which it is not hard to deduce that $cl_1|cca|out$ or $cl_1|cci|out$ lies in S_{2n+4} and therefore $cl_1|q$ or $cl_1|r$ lies in S_{4n+7}. Any path of length $\overline{d} - (4n + 7)$ leading from $cl_1|q$ or $cl_1|r$ into $\overline{S}$ starts by an edge labeled by b. □

Now we are able to write the word $\overline{w}$ as

$$\overline{w} = \overline{u}a^{2n+1}b\,(\overline{v}_1 v'_1 c_1)\ldots(\overline{v}_m v'_m c_m)\,w_{\overline{d}-2}w_{\overline{d}-1}w_{\overline{d}},$$

where $|\overline{v}_k| = 6n-2$, $|v'_k| = 6n-1$ and $|c_k| = 1$ for each k and denote $d_i = 10n+\alpha_i+6$. At time $2n+5$ the parts $cl_{\ldots}|pipe_6$ and $cl_{\ldots}|pipe_7$ record mutually inverse sequences. Because there is the factor a^{2n+1} after $\overline{u}$, at time d_i we find the information pushed to the first rows of testers:

Lemma 2.7. *For each $i \in \{1,\ldots,m\}$, $j \in \{1,\ldots,n\}$ it holds that*

$$\begin{aligned} cl_i|tester|level_{x_1}|\,(1,x_j) \in S_{d_i} \quad &\Leftrightarrow \\ cl_i|tester|level_{x_1}|\,(1,\neg x_j) \notin S_{d_i} \quad &\Leftrightarrow \quad w_{2n-2j+2} \neq w_{2n-2j+3}. \end{aligned}$$

Proof. From the definition of `CCA` and `CCI` it follows that at time $2n+5$ the parts $pipe_6$ and $pipe_7$ record the sequences $B_{(2n+3)}\ldots B_{(2)}$ and $B'_{(2n+3)}\ldots B'_{(2)}$ respectively, where

$$B_{(k)} = \begin{cases} \mathbf{1} & \text{if } w_k = w_{k+1} \\ \mathbf{0} & \text{otherwise} \end{cases} \qquad B'_{(k)} = \begin{cases} \mathbf{0} & \text{if } w_k = w_{k+1} \\ \mathbf{1} & \text{otherwise.} \end{cases}$$

Whatever the letter w_{2n+6} is, Lemma 2.6 implies that

$$cl_i|x_j \in S_{4n+7} \Leftrightarrow cl_i|\neg x_j \notin S_{4n+7} \Leftrightarrow w_{2n-2j+2} \neq w_{2n-2j+3},$$

from which the claim follows easily using Lemma 2.6 again. □

Let us define the assignment $\xi_1,\ldots,\xi_n \in \{\mathbf{0},\mathbf{1}\}$. By Lemma 2.7 the definition is correct and does not depend on i:

$$\xi_j = \begin{cases} \mathbf{1} & \text{if } cl_i|tester|level_{x_1}|\,(1,x_j) \notin S_{d_i} \\ \mathbf{0} & \text{if } cl_i|tester|level_{x_1}|\,(1,\neg x_j) \notin S_{d_i}. \end{cases}$$

The following lemma holds due to $cl_{\ldots}|limiter$ parts.

Lemma 2.8. *For each $i \in \{1,\ldots,m\}$ there are at most two occurrences of b in the word v'_i.*

Proof. It is easy to see that $cl_i|limiter|s_{1,0} \in S_{10n+\alpha_i+6}$ and to note that

$$v'_i = w_{10n+\alpha_i+7}\ldots w_{16n+\alpha_i+5}.$$

Within the part $cl_i|limiter$ no state except for $s_{6n-2,0}$ can lie in $S_{16n+\alpha_i+5}$, because from such states there is no path of length at most

$$\overline{d} - (16n+\alpha_i+5) = \beta_i + 4$$

leading into S'.

The shortest paths from $s_{1,0}$ to $s_{6n-2,0}$ have length $6n-3$ and each path from $s_{1,0}$ into S' uses the state $s_{6n-2,0}$. So there is a path P leading from $s_{1,0}$ to $s_{6n-2,0}$ labeled by a prefix of v'. We distinguish the following cases:

- If P is of length $6n-3$, we just note that such path is unique and labeled by a^{6n-3}. No b occurs in v' except for the last two positions.
- If P is of length $6n-2$, it uses an edge of the form $s_{k,0} \xrightarrow{b} s_{k+1,1}$. Such edges preserve the distance to s_{6n-2}, so the rest of P must be a shortest path from $s_{k+1,1}$ to $s_{6n-2,0}$. Such paths are unique and labeled by a^{6n-2-k}. Any other b can occur only at the last position.
- If P is of length $6n-1$, it is labeled by whole v'. Because any edge labeled by b preserves or increases the distance to s_{6n-2}, the path P can use at most two of them.

□

Now we choose any $i \in \{1, \ldots, m\}$ and prove that the assignment $\xi_1, \ldots, \xi_n$ satisfies the clause $\bigvee_{\lambda \in C_i} \lambda$. Let $p \in \{0, 1, 2, 3\}$ denote the number of unsatisfied literals in C_i.

As we claimed before, all tester columns corresponding to any $\lambda \in L_\phi$ have to be deactivated earlier than other columns. Namely, if $cl_i|tester|level_{x_1}|\,(1, \lambda)$ is active at time d_i, which happens if and only if λ is not satisfied by $\xi_1, \ldots, \xi_n$, the word $v'_i c_i$ must not map it to $cl_i|pipes_3|s_{1,\mu(\lambda)}$. If $cl_i|tester|level_\lambda$ is of type $\texttt{INC}(\lambda)$, the only way to ensure this is to use the letter b when the border of inactive area lies at the first row of $cl_i|tester|level_\lambda$. Thus each unsatisfied $\lambda \in C_i$ implies an occurrence of b in corresponding segment of v'_i:

Lemma 2.9. *There are at least p occurrences of the letter b in the word v'_i.*

Proof. Let $\lambda_1, \ldots, \lambda_p$ be the unsatisfied literals of C_i. From Lemma 2.7 it follows easily that

$$cl_i|tester|level_{\lambda_k}|\,(1, \lambda_k) \in S_{d_i + 3\kappa(\lambda_k)}$$

for each $k \in \{1, \ldots, p\}$. The part $cl_i|tester|level_{\lambda_k}$ is of type $\texttt{INC}(\lambda_k)$, which implies that any path of the length

$$\left(\overline{d} - 3\right) - (d_i + 3\kappa(\lambda_k))$$

starting by a takes $cl_i|tester|level_{\lambda_k}|\,(1, \lambda_k)$ to the state $cl_i|\overline{\lambda}$, which lies outside $S_{\overline{d}-3}$, as it is implied by Lemma 2.5(2). We deduce that $w_{d_i + 3\kappa(\lambda_k)+1} = b$. □

By Lemma 2.8 there are at most two occurrences of b in v'_i, so we get $p \leq 2$ and there is at least one satisfied literal in C_i.

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