# Primal implication as encryption 

Vladimir N. Krupski

August 10, 2018


#### Abstract

We propose a "cryptographic" interpretation for the propositional connectives of primal infon logic introduced by Y. Gurevich and I. Neeman and prove the corresponding soundness and completeness results. Primal implication $\varphi \rightarrow_{p} \psi$ corresponds to the encryption of $\psi$ with a secret key $\varphi$, primal disjunction $\varphi \vee_{p} \psi$ is a group key and $\perp$ reflects some backdoor constructions such as full superuser permissions or a universal decryption key. For the logic of $\perp$ as a universal key (it was never considered before) we prove that the derivability problem has linear time complexity. We also show that the universal key can be emulated using primal disjunction.


## 1 Introduction

Primal Infon Logic ([1], [2] , [3], [4], [5]) formalizes the concept of infon, i.e. a message as a piece of information. The corresponding derivability statement $\Gamma \vdash \varphi$ means that the principal can get (by herself, without any communication) the information $\varphi$ provided she already has all infons $\psi \in \Gamma$.

Primal implication $\left(\rightarrow_{p}\right)$ that is used in Primal Infon Logic to represent the conditional information is a restricted form of intuitionistic implication defined by the following inference rules:

$$
\frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \rightarrow_{p} \psi}\left(\rightarrow_{p} I\right) \quad, \quad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi \rightarrow_{p} \psi}{\Gamma \vdash \psi}\left(\rightarrow_{p} E\right) .
$$

These rules admit cryptographic interpretation of primal implication $\varphi \rightarrow_{p}$ $\psi$ as some kind of digital envelop: it is an infon, containing the information $\psi$
encrypted by a symmetric key (generated from) $\varphi$. Indeed, the introduction rule $\left(\rightarrow_{p} I\right)$ allows to encrypt any available message by any key. Similarly, the elimination rule $\left(\rightarrow_{p} E\right)$ allows to extract the information from the ciphertext provided the key is also available. So the infon logic incorporated into communication protocols ([1], [2]) is a natural tool for manipulating with commitment schemes (see [7]) without detailed analysis of the scheme itself.

Example. (cf. [8]). Alice and Bob live in different places and communicate via a telephone line or by e-mail. They wish to play the following game distantly. Each of them picks a bit, randomly or somehow else. If the bits coincide then Alice wins; otherwise Bob wins. Both of them decide to play fair but don't believe in the fairness of the opponent. To play fair means that they honestly declare their choice of a bit, independently of what the other player said. So they use cryptography.

We discuss the symmetric version of the coin flipping protocol from [8] in order to make the policies of both players the same. Consider the policy of one player, say Alice. Her initial state can be represented by the context

$$
\Gamma=\left\{\text { A said } m_{a}, \text { A said } k_{a}, \text { A IsTrustedOn } m_{a}, \text { A IsTrustedOn } k_{a}\right\},
$$

where infons $m_{a}$ and $k_{a}$ represent the chosen bit and the key Alice intends to use for encryption. Her choice is recorded by infons $A$ said $m_{a}$ and $A$ said $k_{a}$ where $A$ said is the quotation modality governed by the modal logic $\mathbf{K}]^{11}$ Alice simply says, to herself, the infons $m_{a}$ and $k_{a}$.

The remaining two members of $\Gamma$ reflect the decision to play fair. The infon $X$ IsTrustedOn $y$ abbreviates $(X$ said $y) \rightarrow_{p} y$. It provides the ability to obtain the actual value of $y$ from the declaration $X$ said $y$, so Alice can deduce the actual $m_{a}$ and $k_{a}$ she has spoken about.

The commit phase. Alice derives $m_{a}$ and $k_{a} \rightarrow_{p} m_{a}$ from her context by rules $\left(\rightarrow_{p} E\right),\left(\rightarrow_{p} I\right)$ and sends the infon $k_{a} \rightarrow_{p} m_{a}$ to Bob. Bob acts similarly, so Alice will receive a message from him and her context will be extended to

$$
\Gamma^{\prime}=\Gamma \cup\left\{B \operatorname{said}\left(k_{b} \rightarrow_{p} m_{b}\right)\right\} .
$$

The reveal phase. After updating the context Alice obtains $k_{a}$ by rule $\left(\rightarrow_{p} E\right)$ and sends it to Bob. He does the same, so Alice's context will be

$$
\Gamma^{\prime \prime}=\Gamma^{\prime} \cup\left\{B \text { said } k_{b}\right\}
$$

${ }^{1}$ The only modal inference rule that is used in this paper is $X$ said $\varphi, X \operatorname{said}\left(\varphi \rightarrow_{p}\right.$ $\psi) \vdash X$ said $\psi$. It is admissible in $\mathbf{K}$. For more details about modalities in the infon logic see [4, 5].

Now by reasoning in $\mathbf{K}$ Alice deduces $B$ said $m_{b}$. She also has $A$ said $m_{a}$, so it is clear to her who wins. Alice simply compares these infons with the patterns $B$ said $0, B$ said 1 and $A$ said $0, A$ said 1 respectively.

The standard analysis of the protocol shows that Bob will come to the same conclusion. Moreover, Alice can be sure that she is not cheated provided she successively follows her policy up to the end ${ }^{2}$ The same with Bob.

Note that infon logic is used here as a part of the protocol. It is one of the tools that provide the correctness. But it does not prove the correctness. In order to formalize and prove the correctness of protocols one should use much more powerful formal systems.

We make our observation precise by defining interpretations of purely propositional part of infon logic in "cryptographic" infon algebras and proving the corresponding soundness and completeness theorems.

In Section 2 this is done for the system $\mathbf{P}$ which is the $\left\{T, \wedge, \rightarrow_{p}\right\}$ fragment of infon logic. We also show that the quasi-boolean semantics for $\mathbf{P}$ (see [4]) is essentially a special case of our semantics.

In Section 3 we show that $\perp$ can be used to reflect some backdoor constructions. Two variants are considered: system $\mathbf{P}[\perp]$ from [4] with the usual elimination rule for $\perp$ and a new system $\mathbf{P}\left[\perp_{w}\right]$ with a weak form of elimination rule for $\perp$. The first one treats $\perp$ as a root password, and the second one - as a universal key for decryption. For almost all propositional primal infon logics the derivability problem has linear time complexity. We prove the same complexity bound for $\mathbf{P}\left[\perp_{w}\right]$ in Section 4 .

Finally we consider a system $\mathbf{P}\left[\vee_{p}\right]$ which is the modal-free fragment of Basic Propositional Primal Infon Logic PPIL from [5]. The primal disjunction $\vee_{p}$ in $\mathbf{P}\left[\mathrm{V}_{p}\right]$ has usual introduction rules and no elimination rules. We treat it as a group-key constructor and provide a linear time reduction of $\mathbf{P}\left[\perp_{w}\right]$ to $\mathbf{P}\left[\mathrm{V}_{p}\right]$. It thus gives another proof of linear time complexity bound for $\mathbf{P}\left[\perp_{w}\right]$.

[^0]
## 2 Semantics for $\left\{\top, \wedge, \rightarrow_{p}\right\}$-fragment

Let $\Sigma$ be a finite alphabet, say $\Sigma=\{0,1\}$. Let us fix a total pairing function $\pi:\left(\Sigma^{*}\right)^{2} \rightarrow \Sigma^{*}$ with projections $l, r: \Sigma^{*} \rightarrow \Sigma^{*}$, where $\Sigma^{*}$ is the set of all binary strings,

$$
\begin{equation*}
l(\pi(x, y))=x, \quad r(\pi(x, y))=y \tag{1}
\end{equation*}
$$

and two functions enc, dec: $\left(\Sigma^{*}\right)^{2} \rightarrow \Sigma^{*}$ such that enc is total and

$$
\begin{equation*}
\operatorname{dec}(x, \operatorname{enc}(x, y))=y \tag{2}
\end{equation*}
$$

String enc $(x, y)$ will be treated as a ciphertext containing string $y$ encrypted with key $x$. Function dec is the decryption method that exploits the same key. In this text we do not restrict ourselves to encryptions that are strong in some sense. For example, enc $(x, y)$ may be the concatenation of strings $x$ and $y$. Then dec on arguments $x, y$ simply removes the prefix $x$ from $y$. The totality of functions $l, r, d e c$ is not supposed, but the left-hand parts of (1) and (2) must be defined for all $x, y \in \Sigma^{*}$.

We also fix some set $E \subset \Sigma^{*}, E \neq \emptyset$. It will represent the information known by everyone, for example, facts like $0<1$ and $2 \cdot 2=4$. The structure $\mathcal{A}=\left\langle\Sigma^{*}, \pi, l, r, e n c, d e c, E\right\rangle$ will be referred as an infon algebra $3^{3}$

Definition 2.1 A set $M \subseteq \Sigma^{*}$ will be called closed if $E \subseteq M$ and $M$ satisfies the following closure conditions:

1. $a, b \in M \Leftrightarrow \pi(a, b) \in M$,
2. $a, \operatorname{enc}(a, b) \in M \Rightarrow b \in M$,
3. $a \in \Sigma^{*}, b \in M \Rightarrow \operatorname{enc}(a, b) \in M$.

A closed set $M$ represents the information that is potentially available to an agent in a local state, i.e. between two consecutive communication steps of a protocol. The information is represented by texts. $M$ contains all public and some private texts. The agent can combine several texts in a single multi-part document using $\pi$ function as well as to extract its parts by means of $l$ and $r$. She has access to the encryption tool enc, so she can

[^1]convert a plaintext into a ciphertext. The backward conversion (by dec) is also available provided she has the encryption key.

Note that in the closure condition 3 we do not require that $a \in M$. The agent will never need to decrypt the ciphertext enc $(a, b)$ encrypted by herself because she already has the plaintext $b$. The key $a$ can be generated by some trusted third party and sent to those who really need it. This is the case when the encryption is used to provide secure communications between agents when only the connections to the third party are secure (and the authentication is reliable). On the other hand, some protocols may require the agent to distribute keys by herself. Then she can use a key that is known to her or get it from the third party. In the latter case $a$ will be available in her new local state that will be updated by the communication with the third party.

The natural deduction calculus for primal infon $\operatorname{logic} \mathbf{P}$ is considered in [4]. The corresponding derivability relation $\Gamma \vdash \varphi$ is defined by the following rules:

$$
\begin{aligned}
\overline{\vdash \top} \quad & \frac{\Gamma \vdash \varphi}{\varphi \vdash \varphi}(\text { Weakening }) \quad \frac{\Gamma \vdash \varphi_{1} \quad \Gamma, \varphi_{1} \vdash \varphi_{2}}{\Gamma \vdash, \Delta \vdash \varphi}(\text { Cut }) \\
& \frac{\Gamma \vdash \varphi_{1} \quad \Gamma \vdash \varphi_{2}}{\Gamma \vdash \varphi_{1} \wedge \varphi_{2}}(\wedge I) \quad \frac{\Gamma \vdash \varphi_{1} \wedge \varphi_{2}}{\Gamma \vdash \varphi_{i}}\left(\wedge E_{i}\right) \quad(i=1,2) \\
& \frac{\Gamma \vdash \varphi_{2}}{\Gamma \vdash \varphi_{1} \rightarrow_{p} \varphi_{2}}\left(\rightarrow_{p} I\right) \quad \frac{\Gamma \vdash \varphi_{1} \Gamma \vdash \varphi_{1} \rightarrow_{p} \varphi_{2}}{\Gamma \vdash \varphi_{2}}\left(\rightarrow_{p} E\right) .
\end{aligned}
$$

Here $\varphi, \varphi_{1}, \varphi_{2}$ are infons, i.e. the expressions constructed from the set At of atomic infons by the grammar

$$
\varphi::=\top|A t|(\varphi \wedge \varphi) \mid\left(\varphi \rightarrow_{p} \varphi\right),
$$

and $\Gamma, \Delta$ are sets of infons.
As usual, a derivation of $\varphi$ from a set of assumptions $\Gamma$ is a sequence of infons $\varphi_{1}, \ldots, \varphi_{n}$ where $\varphi_{n}=\varphi$ and each $\varphi_{k}$ is either a member of $\Gamma \cup\{\top\}$ or is obtained from some members of $\left\{\varphi_{j} \mid j<k\right\}$ by one of the rules

$$
\frac{\varphi_{1} \varphi_{2}}{\varphi_{1} \wedge \varphi_{2}} \quad \frac{\varphi_{1} \wedge \varphi_{2}}{\varphi_{i}} \quad \frac{\varphi_{2}}{\varphi_{1} \rightarrow_{p} \varphi_{2}} \quad \frac{\varphi_{1} \varphi_{1} \rightarrow_{p} \varphi_{2}}{\varphi_{2}}
$$

It is easy to see that $\Gamma \vdash \varphi$ iff there exists a derivation of $\varphi$ from $\Gamma$. So rules like (Weakening) or (Cut) from the definition of derivability relation are never used in a derivation itself.

Definition 2.2 An interpretation (of the infon language) is a pair $I=\langle\mathcal{A}, v\rangle$ where $\mathcal{A}=\left\langle\Sigma^{*}, \pi, l, r, e n c, \operatorname{dec}, E\right\rangle$ is an infon algebra and $v: \operatorname{At} \cup\{\top\} \rightarrow \Sigma^{*}$ is a total evaluation that assigns binary strings to atomic infons and to constant $T, v(T) \in E$. We assume that $v$ is extended as follows:

$$
\begin{gathered}
v\left(\varphi_{1} \wedge \varphi_{2}\right)=\pi\left(v\left(\varphi_{1}\right), v\left(\varphi_{2}\right)\right), \quad v\left(\varphi_{1} \rightarrow_{p} \varphi_{2}\right)=\operatorname{enc}\left(v\left(\varphi_{1}\right), v\left(\varphi_{2}\right)\right), \\
v(\Gamma)=\{v(\varphi) \mid \varphi \in \Gamma\} .
\end{gathered}
$$

A model is a pair $\langle I, M\rangle$ where $I$ is an interpretation and $M \subseteq \Sigma^{*}$ is a closed set.

In the paper [4] it is established that $\mathbf{P}$ is sound and complete with respect to quasi-boolean semantics. A quasi-boolean model is a validity relation $\models$ that enjoys the following properties:

- $\models \mathrm{T}$,
- $\models \varphi_{1} \wedge \varphi_{2} \Leftrightarrow \models \varphi_{1}$ and $\models \varphi_{2}$,
- $\models \varphi_{2} \Rightarrow \models \varphi_{1} \rightarrow_{p} \varphi_{2}$,
- $\models \varphi_{1} \rightarrow_{p} \varphi_{2} \Rightarrow \not \vDash \varphi_{1}$ or $\models \varphi_{2}$.

An infon $\varphi$ is derivable in the infon $\operatorname{logic} \mathbf{P}$ from the context $\Gamma$ iff $\models \Gamma$ implies $\models \varphi$ for all quasi-boolean models $\models$.

It can be seen that the definition of a quasi-boolean model is essentially a special case of Definition [2.2. Indeed, suppose that atomic infons are words in the unary alphabet $\{\mid\}$. Then all infons turn out to be words in some finite alphabet $\Sigma_{0}$. Consider a translation $\ulcorner\urcorner:. \Sigma_{0}^{*} \rightarrow\{0,1\}^{*}$ that maps all elements of $\Sigma_{0}$ into distinct binary strings of the same length, $\ulcorner\Lambda\urcorner=\Lambda$ for the empty word $\Lambda$ and $\left\ulcorner a_{1} \ldots a_{n}\right\urcorner=\left\ulcorner a_{1}\right\urcorner \ldots\left\ulcorner a_{n}\right\urcorner$ for $a_{1} \ldots, a_{n} \in \Sigma_{0}$.

The corresponding infon algebra $\mathcal{A}$ and the evaluation $v$ can be defined as follows: $v(a)=\ulcorner a\urcorner$ for $a \in A t \cup\{T\}$,

$$
\begin{equation*}
\pi(x, y)=\ulcorner( \urcorner x\ulcorner\wedge\urcorner y\ulcorner )\urcorner, \quad \operatorname{enc}(x, y)=\left\ulcorner( \urcorner x\left\ulcorner\rightarrow_{p}\right\urcorner y\ulcorner )\right\urcorner, \quad E=\{\ulcorner\top\urcorner\} . \tag{3}
\end{equation*}
$$

Projections and the decryption function can be found from (1) and (2). Note that for this interpretation the equality $v(\varphi)=\ulcorner\varphi\urcorner$ holds for every infon $\varphi$.

Consider a quasi-boolean model $\models$. Let $M$ be the closure of the set $M_{0}=\{\ulcorner\varphi\urcorner \mid \models \varphi\}$, i.e. the least closed extension of $M_{0}$.

Lemma $2.3 \models \varphi$ iff $v(\varphi) \in M$.
Proof. It is sufficient to prove that the set $M \backslash M_{0}$ does not contain words of the form $v(\varphi)$. Any element $b \in M \backslash M_{0}$ can be obtained from some elements of $M_{0}$ by a finite sequence of steps $1,2,3$ that correspond to closure conditions:

1. $x, y \mapsto\ulcorner( \urcorner x\ulcorner\wedge\urcorner y\ulcorner )\urcorner ; \quad\ulcorner( \urcorner x\ulcorner\wedge\urcorner y\ulcorner )\urcorner \mapsto x ; \quad\ulcorner( \urcorner x\ulcorner\wedge\urcorner y\ulcorner )\urcorner \mapsto y$;
2. $x,\left\ulcorner( \urcorner x\left\ulcorner\rightarrow_{p}\right\urcorner y\ulcorner )\right\urcorner \mapsto y$;
3. $y \mapsto\left\ulcorner( \urcorner x\left\ulcorner\rightarrow_{p}\right\urcorner y)\urcorner\right.$.

The history of this process is a derivation of $b$ from $M_{0}$ with $1,2,3$ treated as inference rules. Let $b=v(\varphi)$ and $b_{1}, \ldots, b_{n}=b$ be the derivation. Consider the (partial) top-down syntactic analysis of strings $b_{1}, \ldots, b_{n}$ using patterns

$$
\left\ulcorner( \urcorner \cdot\ulcorner\wedge\urcorner \cdot)\urcorner, \quad\left\ulcorner( \urcorner \cdot\left\ulcorner\rightarrow_{p}\right\urcorner \cdot\ulcorner )\right\urcorner, \quad\ulcorner\| \ldots \mid\urcorner .\right.
$$

We replace all substrings that remain unparsed by $v(a)$ where $a=\| \ldots \mid$ is some fresh atomic infon. The resulting sequence $c_{1}, \ldots, c_{n}$ is also a derivation of $b$ from $M_{0}$ because any string of the from $v(\psi)$ has no unparsed substrings. All its members have the form $c_{i}=v\left(\varphi_{i}\right)$ for some infons $\varphi_{i}$. Moreover, $\varphi_{1}, \ldots, \varphi_{n}$ is a derivation of $\varphi=\varphi_{n}$ in $\mathbf{P}$ from the set of hypotheses $\Gamma=\left\{\varphi_{j} \mid c_{j} \in M_{0}\right\}$. But $\models \Gamma$ and $\mathbf{P}$ is sound with respect to quasi-boolean models, so $\models \varphi$ and $b=v(\varphi) \in M_{0}$. Contradiction.

Theorem $2.4 \Gamma \vdash \varphi$ in $\mathbf{P}$ iff $v(\varphi) \in M$ for every model $\langle I, M\rangle$ with $v(\Gamma) \subseteq$ $M$.

Proof. The theorem states that the infon $\operatorname{logic} \mathbf{P}$ is sound and complete with respect to the class of models introduced by Definition 2.2. The soundness can be proven by straightforward induction on the derivation of $\varphi$ from $\Gamma$. The completeness follows from Lemma 2.3 and the completeness result for quasi-boolean models (see [4]).

A set $\{v(\psi) \mid \psi \in T\} \subseteq \Sigma^{*}$ will be called deductively closed if $T \vdash \psi$ implies $\psi \in T$ for all infons $\psi$, i.e. $T$ is deductively closed in $\mathbf{P}$. In the proof
of Lemma 2.3 we actually establish that the particular interpretation $\langle\mathcal{A}, v\rangle$ is conservative in the following sense: the closure $M$ of any deductively closed set $M_{0} \subseteq \Sigma^{*}$ does not contain "new" strings of the form $v(\psi) \notin M_{0}$. It is also injective: $v\left(\varphi_{1}\right)=v\left(\varphi_{2}\right)$ implies $\varphi_{1}=\varphi_{2}$. An interpretation that enjoys these two properties will be called plain.

Lemma 2.5 There exists a plain interpretation.
The completeness part of Theorem 2.4 can be strengthened.
Theorem 2.6 Let the interpretation $I=\langle\mathcal{A}, v\rangle$ be plain. For any context $\Gamma$ there exists a model $\langle I, M\rangle$ with $v(\Gamma) \subseteq M$ such that $\Gamma \nvdash \varphi \operatorname{implies} v(\varphi) \notin M$ for all infons $\varphi$.

Proof. Let $M$ be the closure of the set $M_{0}=\{v(\psi) \mid \Gamma \vdash \psi\}$. Then $v(\Gamma) \subseteq M$. The set $M_{0}$ is deductively closed, so $M \backslash M_{0}$ does not contain strings of the form $v(\psi)$. Suppose $\Gamma \nvdash \varphi$. Then $v(\varphi) \notin M_{0}$ because the interpretation is injective. Thus $v(\varphi) \notin M$.

## 3 Constant $\perp$ and backdoors

## $\perp$ as superuser permissions

Infon $\operatorname{logic} \mathbf{P}[\perp]$ is the extension of $\mathbf{P}$ by additional constant $\perp$ that satisfies the elimination rule

$$
\frac{\Gamma \vdash \perp}{\Gamma \vdash \varphi}(\perp E) .
$$

The corresponding changes in Definition 2.2 are as follows. We add to the alphabet a new letter $\mathbf{f} \notin \Sigma$ and set $\Sigma_{\perp}=\Sigma \cup\{\mathbf{f}\}, v(\perp)=\mathbf{f}$. Functions $\pi, l$, $r$, enc, dec act on words from $\Sigma_{\perp}^{*}$ but still satisfy the conditions (1), (2). We suppose them to preserve $\Sigma^{*}$ : the value should be a binary string provided all arguments are. We also suppose that $v(T) \in E \subseteq \Sigma^{*}$ and $v(\varphi) \in \Sigma^{*}$ for $\varphi \in A t$ and add new closure condition to Definition 2.1.
4. $\mathbf{f} \in M, a \in \Sigma_{\perp}^{*} \Rightarrow a \in M$.

Models for $\mathbf{P}[\perp]$ are all pairs $\langle I, M\rangle$ where $I$ is an interpretation and $M$ is a closed set, both in the updated sense. The definition of plain interpretation is just the same.

Constant $\perp$ is some kind of root password that grants the superuser permissions to its owner. The owner has the direct access to all the information available in the system without any communication or decryption. At the same time $\perp$ can be incorporated into some messages that will be used in communication.

## $\perp$ as universal key

The root password provides the direct access to all the information in the system including private information of any agent that was never sent to anybody else. It is also natural to consider a restricted form of superuser permissions that protect the privacy of agents but provide the ability to decrypt any available ciphertext. It can be simulated by infon $\operatorname{logic} \mathbf{P}\left[\perp_{w}\right]$ with constant $\perp$ treated as a universal key. The corresponding inference rule is a weak form of $(\perp E)$ rule,

$$
\frac{\Gamma \vdash \perp \quad \Gamma \vdash \varphi \rightarrow_{p} \psi}{\Gamma \vdash \psi}\left(\perp E_{w}\right),
$$

that has an additional premise $\Gamma \vdash \varphi \rightarrow_{p} \psi$. So the owner of $\perp$ can get an infon only if she already has the same information as a ciphertext. The rule $\left(\perp E_{w}\right)$ is really weaker than $(\perp E)$ because $\psi \rightarrow_{p} \psi$ is not derivable in $\mathbf{P}$.

All definitions concerning models for $\mathbf{P}\left[\perp_{w}\right]$ are similar to the case of $\mathbf{P}[\perp]$ with closure condition 4 replaced by
$4^{\prime} . \mathbf{f}, \operatorname{enc}(a, b) \in M \Rightarrow b \in M$.
Essentially we extend the signature of infon algebras by additional (partial) operation $\operatorname{crack}(x, y)$ that satisfies the equality

$$
\begin{equation*}
\operatorname{crack}(\mathbf{f}, \operatorname{enc}(a, b))=b \tag{4}
\end{equation*}
$$

and allow any agent to use it, so her local state satisfies the closure condition $4^{\prime}$.

Lemma 3.1 There exist plain interpretations for $\mathbf{P}[\perp]$ and for $\mathbf{P}\left[\perp_{w}\right]$.
Proof. We extend the example of plain interpretation for $\left\{\top, \wedge, \rightarrow_{p}\right\}$ fragment from Section 2 (see (3)). Set $\ulcorner\perp\urcorner=\mathbf{f}$ and extend the interpretation in accordance with (3). The resulting interpretation is plain in the sense of
$\mathbf{P}[\perp]$. Indeed, it is injective because $\mathbf{f} \notin \Sigma$. It is also conservative. In order to prove this we use the construction from Lemma 2.3.

Let the set $M_{0}=\{v(\psi) \mid \psi \in T\} \subseteq(\Sigma \cup\{\mathbf{f}\})^{*}$ be deductively closed and $M$ be its closure. Suppose $v(\varphi) \in M \backslash M_{0}$ for some infon $\varphi$. Then $b_{n}=v(\varphi)$ has a derivation $b_{1}, \ldots, b_{n}$ from $M_{0}$ in the calculus with closure conditions considered as inference rules:

1. $x, y \mapsto\ulcorner( \urcorner x\ulcorner\wedge\urcorner y\ulcorner )\urcorner ; \quad\ulcorner( \urcorner x\ulcorner\wedge\urcorner y\ulcorner )\urcorner \mapsto x ; \quad\ulcorner( \urcorner x\ulcorner\wedge\urcorner y\ulcorner )\urcorner \mapsto y$;
2. $x,\left\ulcorner( \urcorner x\left\ulcorner\rightarrow_{p}\right\urcorner y\ulcorner )\right\urcorner \mapsto y$;
3. $y \mapsto\left\ulcorner( \urcorner x\left\ulcorner\rightarrow_{p}\right\urcorner y)\urcorner\right.$;
4. $\mathbf{f} \mapsto x$.

Consider the (partial) top-down syntactic analysis of strings $b_{1}, \ldots, b_{n}$ using patterns

$$
\left\ulcorner( \urcorner \cdot\ulcorner\wedge\urcorner \cdot)\urcorner, \quad\left\ulcorner( \urcorner \cdot\left\ulcorner\rightarrow_{p}\right\urcorner \cdot\ulcorner )\right\urcorner, \quad\ulcorner\| \ldots \mid\urcorner, \quad \mathbf{f} .\right.
$$

Replace all substrings that remain unparsed by $v(a)$ where $a=\| \ldots \mid$ is some fresh atomic infon. The resulting sequence $c_{1}, \ldots, c_{n}$ is also a derivation of $v(\varphi)$ from $M_{0}$ because any string of the from $v(\psi)$ has no unparsed substrings. All its members have the form $c_{i}=v\left(\varphi_{i}\right)$ for some infons $\varphi_{i}$ and $\varphi_{1}, \ldots, \varphi_{n}$ is a derivation of $\varphi=\varphi_{n}$ in $\mathbf{P}[\perp]$ from the set of hypotheses $T$. But $T$ is deductively closed, so $\varphi \in M_{0}$. Contradiction.

Now set

$$
\operatorname{crack}(x, y):= \begin{cases}b, & \text { if } x=\mathbf{f} \text { and } y=\left\ulcorner( \urcorner a\left\ulcorner\rightarrow_{p}\right\urcorner b\ulcorner )\right\urcorner, \\ \text { undefined, } & \text { otherwise. }\end{cases}
$$

It satisfies the condition (4), so the interpretation for $\mathbf{P}\left[\perp_{w}\right]$ is defined. One can prove in a similar way that the interpretation is plain (w.r.t. $\mathbf{P}\left[\perp_{w}\right]$ ).

The completeness results from Section 2 hold for $\operatorname{logics} \mathbf{P}[\perp]$ and $\mathbf{P}\left[\perp_{w}\right]$ too. The proofs are essentially the same with one difference: the quasiboolean semantics from [4] does not cover the case of $\mathbf{P}\left[\perp_{w}\right]$. Let $\mathbf{L}$ be one of the logics $\mathbf{P}[\perp]$ or $\mathbf{P}\left[\perp_{w}\right]$.

Theorem 3.2 $\Gamma \vdash \varphi$ in $\mathbf{L}$ iff $v(\varphi) \in M$ for every model $\langle I, M\rangle$ of $\mathbf{L}$ with $v(\Gamma) \subseteq M$.

Proof. The soundness part can be proven by straightforward induction on the derivation of $\varphi$ from $\Gamma$. The completeness follows from Lemma 3.1 and Theorem 3.3,

Theorem 3.3 Let I be a plain interpretation of $\mathbf{L}$. For any context $\Gamma$ there exists a model $\langle I, M\rangle$ of $\mathbf{L}$ with $v(\Gamma) \subseteq M$ such that $\Gamma \nvdash \varphi \operatorname{implies} v(\varphi) \notin M$ for all infons $\varphi$.

Proof. Similar to Theorem 2.6.

## 4 Decision algorithm for $\mathrm{P}\left[\perp_{w}\right]$

The derivability problems for infon logics $\mathbf{P}$ and $\mathbf{P}[\perp]$ are linear time decidable ([3], [4, [5]). We provide a decision algorithm for $\mathbf{P}\left[\perp_{w}\right]$ with the same complexity bound.

Definition 4.1 (Positive atoms.) In what follows we assume that the language of $\mathbf{P}$ also contains $\perp$, but it is an ordinary member of $A t$ without any specific inference rule for it. Let

$$
\begin{aligned}
& A t^{+}(\varphi)=\{\varphi\} \text { for } \varphi \in A t \cup\{\top, \perp\} \\
& A t^{+}(\varphi \wedge \psi)=A t^{+}(\varphi) \cup A t^{+}(\psi) \\
& A t^{+}\left(\varphi \rightarrow_{p} \psi\right)=A t^{+}(\psi)
\end{aligned}
$$

For a context $\Gamma \operatorname{set} A t^{+}(\Gamma)=\bigcup_{\varphi \in \Gamma} A t^{+}(\varphi)$.
Lemma 4.2 Let $\Gamma \vdash \perp$ in $\mathbf{P}\left[\perp_{w}\right]$. Then $\Gamma \vdash \varphi$ in $\mathbf{P}\left[\perp_{w}\right]$ iff $A t^{+}(\varphi) \subseteq$ $A t^{+}(\Gamma)$.

Proof. Suppose $\Gamma \vdash \varphi$. The inclusion $A t^{+}(\varphi) \subseteq A t^{+}(\Gamma)$ can be proved by straightforward induction on the derivation of $\varphi$ from $\Gamma$.

Now suppose that $\Gamma \vdash \perp$ and $A t^{+}(\varphi) \subseteq A t^{+}(\Gamma)$. By rules $\left(\wedge E_{i}\right)$ and $\left(\perp E_{w}\right)$ we prove that $\Gamma \vdash \psi$ for every infon $\psi \in A t^{+}(\Gamma)$. Then we derive $\Gamma \vdash \varphi$ by rules $(\wedge I),\left(\rightarrow_{p} I\right)$.

Lemma 4.3 If $\Gamma \nvdash \perp$ in $\mathbf{P}$ and $\Gamma \vdash \varphi$ in $\mathbf{P}\left[\perp_{w}\right]$ then $\Gamma \vdash \varphi$ in $\mathbf{P}$.
Proof. $\Gamma \nvdash \perp$ in $\mathbf{P}$ implies that $\Gamma \nvdash \perp$ in $\mathbf{P}\left[\perp_{w}\right]$ because the shortest derivation of $\perp$ from $\Gamma$ cannot use the $\left(\perp E_{w}\right)$ rule. So any derivation in $\mathbf{P}\left[\perp_{w}\right]$ from $\Gamma$ cannot use this rule.

The decision algorithm for $\mathbf{P}\left[\perp_{w}\right]$ consists of the following three steps:

1. Test whether $\Gamma \vdash \varphi$ in $\mathbf{P}$. If yes, then $\Gamma \vdash \varphi$ in $\mathbf{P}\left[\perp_{w}\right]$ too. Else go to step 2.
2. Test whether $\Gamma \nvdash \perp$ in $\mathbf{P}$. If yes, then $\Gamma \nvdash \varphi$ in $\mathbf{P}\left[\perp_{w}\right]$ by Lemma 4.3. Else go to step 3.
3. We have $\Gamma \vdash \perp$ in $\mathbf{P}$, so it is also true in $\mathbf{P}\left[\perp_{w}\right]$. Test the condition $A t^{+}(\varphi) \subseteq A t^{+}(\Gamma)$. If it is fulfilled then $\Gamma \vdash \varphi$ in $\mathbf{P}\left[\perp_{w}\right]$; otherwise $\Gamma \nvdash \varphi$ in $\mathbf{P}\left[\perp_{w}\right]$ (Lemma 4.2).

Linear time complexity bounds for steps 1,2 follow from the linear bound for $\mathbf{P}$. In order to prove the same bound for step 3 we use the preprocessing stage of the linear time decision algorithm from [5]. It deals with sequents $\Gamma \vdash \varphi$ in a language that extends the language of $\mathbf{P}\left[\perp_{w}\right]$. The preprocessing stage is purely syntactic, so it does not depend on the logic involved and can be used for $\mathbf{P}\left[\perp_{w}\right]$ as well.

The algorithm constructs the parse tree for the sequent. Two nodes are called homonyms if they represent two occurrences of the same infon. For every homonymy class, the algorithm chooses a single element of it, the homonymy leader, and labels all nodes with pointers that provide a constant time access from a node to its homonymy leader. All this can be done in linear time (see [5]).

Now it takes a single walk through the parse tree to mark by a special flag all homonymy leaders that correspond to infons $\psi \in A t^{+}(\Gamma)$. One more walk is required to test whether all homonymy leaders that correspond to $\psi \in A t^{+}(\varphi)$ already have this flag. Thus we have a linear time test for the inclusion $A t^{+}(\varphi) \subseteq A t^{+}(\Gamma)$.

Theorem 4.4 The derivability problem for infon logic $\mathbf{P}\left[\perp_{w}\right]$ is linear time decidable.

## 5 Primal disjunction and backdoor emulation

Primal infon logic with disjunction $\mathbf{P}[\mathrm{V}]$ was studied in [4]. It is defined by all rules of $\mathbf{P}$ and usual introduction and elimination rules for disjunction. $\mathbf{P}[\mathrm{V}]$ can emulate the classical propositional logic, so the derivability problem for it is co-NP-complete.

Here we consider the logic $\mathbf{P}\left[\mathrm{V}_{p}\right]$, an efficient variant of $\mathbf{P}[\mathrm{V}]$. It was mentioned in [4] and later was incorporated into Basic Propositional Primal Infon Logic PPIL [5] as its purely propositional fragment without modalities. In $\mathbf{P}\left[\mathrm{V}_{p}\right]$ the standard disjunction is replaced by a "primal" disjunction $\mathrm{V}_{p}$ with introduction rules

$$
\frac{\Gamma \vdash \varphi_{i}}{\Gamma \vdash \varphi_{1} \vee_{p} \varphi_{2}}\left(\vee_{p} I_{i}\right) \quad(i=1,2)
$$

and without elimination rules. It results in a linear-time complexity bound for $\mathbf{P}\left[\mathrm{V}_{p}\right.$ ] (and for PPIL too, see [4], [5]).

When the primal implication is treated as encryption, the primal disjunction can be used as a method to construct group keys. An infon of the form

$$
\begin{equation*}
\left(\varphi_{1} \vee_{p} \varphi_{2}\right) \rightarrow_{p} \psi \tag{5}
\end{equation*}
$$

represents a ciphertext that can be decrypted by anyone who has at least one of the keys $\varphi_{1}$ or $\varphi_{2}$. In $\mathbf{P}$ the same effect can be produced by the infon

$$
\begin{equation*}
\left(\varphi_{1} \rightarrow_{p} \psi\right) \wedge\left(\varphi_{2} \rightarrow_{p} \psi\right) \tag{6}
\end{equation*}
$$

but it requires two copies of $\psi$ to be encrypted. Moreover, a principal A who does not know both keys $\varphi_{1}$ and $\varphi_{2}$ fails to distinguish between (6) and $\left(\varphi_{1} \rightarrow_{p} \psi_{1}\right) \wedge\left(\varphi_{2} \rightarrow_{p} \psi_{2}\right)$. If A receives (6) from some third party and forwards it to some principals B and C, she will never be sure that B and C will get the same plaintext after decryption. Group keys eliminate the length growth and ambiguity.

An infon algebra for $\mathbf{P}\left[\vee_{p}\right]$ has an additional total operation $g r:\left(\Sigma^{*}\right)^{2} \rightarrow$ $\Sigma^{*}$ for evaluation of primal disjunction: $v\left(\varphi \vee_{p} \psi\right)=\operatorname{gr}(v(\varphi), v(\psi))$. The corresponding closure condition in Definition 2.1 will be
5. If $a \in M, b \in \Sigma^{*}$ or $b \in M, a \in \Sigma^{*}$ then $\operatorname{gr}(a, b) \in M$.

All the results of Section 3 (Lemma 3.1, Theorems 3.2, 3.3) hold for $\mathbf{P}\left[\mathrm{V}_{p}\right]$ too. The proofs are essentially the same.
$\mathbf{P}\left[\perp_{w}\right]$ is linear-time reducible to $\mathbf{P}\left[\vee_{p}\right]$, so $\mathbf{P}\left[\mathrm{V}_{p}\right]$ and PPIL can emulate the backdoor based on a universal key. The reduction also gives another proof for Theorem 4.4.

Remember that in the language of $\mathbf{P}\left[\mathrm{V}_{p}\right]$ symbol $\perp$ denotes some regular atomic infon. Consider the following translation:

$$
\begin{aligned}
& q^{*}=q \text { for } q \in A t \cup\{\top, \perp\}, \\
& (\varphi \wedge \psi)^{*}=\varphi^{*} \wedge \psi^{*} \\
& \left(\varphi \rightarrow_{p} \psi\right)^{*}=\left(\perp \vee_{p} \varphi^{*}\right) \rightarrow_{p} \psi^{*}, \\
& \Gamma^{*}=\left\{\varphi^{*} \mid \varphi \in \Gamma\right\}
\end{aligned}
$$

The transformation of $\Gamma, \varphi$ into $\Gamma^{*}, \varphi^{*}$ can be implemented in linear time.
Theorem 5.1 $\Gamma \vdash \varphi$ in $\mathbf{P}\left[\perp_{w}\right]$ iff $\Gamma^{*} \vdash \varphi^{*}$ in $\mathbf{P}\left[\vee_{p}\right]$.
Proof. Part "only if" can be proved by straightforward induction on the derivation of $\varphi$ from assumptions $\Gamma$ in $\mathbf{P}\left[\perp_{w}\right]$. For any inference rule of $\mathbf{P}\left[\perp_{w}\right]$, its translation is derivable in $\mathbf{P}\left[\mathrm{V}_{p}\right]$. For example, consider the elimination rules for $\rightarrow_{p}$ and $\perp$ :

$$
\frac{\frac{\varphi^{*}}{\perp \vee_{p} \varphi^{*}} \perp \vee_{p} \varphi^{*} \rightarrow_{p} \psi^{*}}{\psi^{*}}, \quad \frac{\frac{\perp}{\perp \vee_{p} \varphi^{*}} \quad \perp \vee_{p} \varphi^{*} \rightarrow_{p} \psi^{*}}{\psi^{*}} .
$$

Part "if". Let $\Gamma^{*} \vdash \varphi^{*}$ in $\mathbf{P}\left[\vee_{p}\right]$. Note that $\mathbf{P}\left[\vee_{p}\right]$ is the modal-free fragment of PPIL and the shortest derivation of $\varphi^{*}$ from assumptions $\Gamma^{*}$ in PPIL is also a derivation in $\mathbf{P}\left[\mathrm{V}_{p}\right]$. Let $D$ be this derivation.

It is proved in [5] that any shortest derivation is local. For the case of $\mathbf{P}\left[\mathrm{V}_{p}\right]$ it means that all formulas from $D$ are subformulas of $\Gamma^{*}, \varphi^{*}$. In particular, $\vee_{p}$ occurs in $D$ only in subformulas of the form $\perp \vee_{p} \theta^{*}$.

Case 1. Suppose that the $\left(\vee_{p} I_{1}\right)$ rule is never used in $D$. Remove part " $\perp \vee_{p}$ " from every subformula of the form $\perp \vee_{p} \psi$ that occurs in $D$. The result will be a derivation of $\varphi$ from assumptions $\Gamma$ in $\mathbf{P}$. So $\Gamma \vdash \varphi$ in $\mathbf{P}\left[\perp_{w}\right]$ too.

Case 2. Suppose that the $\left(\vee_{p} I_{1}\right)$ rule is used in $D$. It has the form

$$
\begin{equation*}
\frac{\perp}{\perp \vee_{p} \theta^{*}}, \tag{7}
\end{equation*}
$$

so $D$ also contains a derivation of $\perp$. The corresponding subderivation is the shortest one and does not use the $\left(\vee_{p} I_{1}\right)$ rule. By applying the transformation from Case 1 we prove that $\Gamma \vdash \perp$ in $\mathbf{P}$ and $\perp \in A t^{+}(\Gamma)$.

We extend Definition 4.1 with new item

$$
A t^{+}\left(\psi_{1} \vee_{p} \psi_{2}\right)=A t^{+}\left(\psi_{1}\right) \cup A t^{+}\left(\psi_{2}\right)
$$

so $A t^{+}(\psi)$ is defined for every $\psi$ in the language of $\mathbf{P}\left[\vee_{p}\right]$. Moreover, $A t^{+}\left(\varphi^{*}\right)=$ $A t^{+}(\varphi)$ and $A t^{+}\left(\Gamma^{*}\right)=A t^{+}(\Gamma)$. We claim that $A t^{+}\left(\varphi^{*}\right) \subseteq A t^{+}\left(\Gamma^{*}\right)$.

Indeed, consider $D$ as a proof tree and its node $\psi$ with $A t^{+}(\psi) \nsubseteq A t^{+}\left(\Gamma^{*}\right)$ whereas $A t^{+}\left(\psi^{\prime}\right) \subseteq A t^{+}\left(\Gamma^{*}\right)$ holds for all predecessors $\psi^{\prime}$. The only rule that can produce this effect is (77), so $\psi=\perp \vee_{p} \theta^{*}$ for some $\theta$ where all occurrences of "new" atoms $q \in A t^{+}(\psi) \backslash A t^{+}\left(\Gamma^{*}\right)$ are inside $\theta^{*}$.

Consider the path from the node $\psi$ to the root node $\varphi^{*}$ and the trace of $\psi$ along it. There is no elimination rule for $\vee_{p}$, so $\psi$ cannot be broken into pieces. All occurrences of positive atoms in $\theta^{*}$ will be positive in all formulas along the trace. But $\vee_{p}$ occurs in $\varphi^{*}$ only in the premise of primal implication, so the trace does not reach the root node. Thus, at some step the formula containing $\psi$ will be eliminated and "new" atoms from $\theta^{*}$ will never appear in $A t^{+}\left(\varphi^{*}\right)$ :


We have established that $A t^{+}(\varphi) \subseteq A t^{+}(\Gamma)$. But $\Gamma \vdash \perp$ in $\mathbf{P}$ and in $\mathbf{P}\left[\perp_{w}\right]$, so $\Gamma \vdash \varphi$ in $\mathbf{P}\left[\perp_{w}\right]$ by Lemma 4.2.

Comment. It is also possible to reduce $\mathbf{P}\left[\perp_{w}\right]$ to $\mathbf{P}$. The corresponding reduction is two-step translation. One should convert $\varphi$ into $\varphi^{*}$ and then replace all subformulas of the form (5) in it with (6). Unfortunately, the second step results in the exponential growth of the length of a formula.

## Acknowledgements

I would like to thank Yuri Gurevich, Andreas Blass and Lev Beklemishev for valuable discussion, comments and suggestions.

The research described in this paper was partially supported by Microsoft project DKAL and Russian Foundation for Basic Research (grant 11-0100281).

## References

[1] Y. Gurevich and I. Neeman. DKAL: Distributed-Knowledge Authorization Language. In Proc. of CSF 2008, pages 149-162. IEEE Computer Society, 2008.
[2] Y. Gurevich and I. Neeman. DKAL 2 - A Simplified and Improved Authorization Language. Technical Report MSR-TR-2009-11, Microsoft Research, February 2009.
[3] Y. Gurevich and I. Neeman. Logic of infons: the propositional case. ACM Transactions on Computational Logic, 12(2), 2011.
[4] L. Beklemishev and Y. Gurevich. Propositional primal logic with disjunction. J. of Logic and Computation 22 (2012), 26 pages.
[5] C. Cotrini and Y. Gurevich. Basic primal infon logic. Microsoft Research Technical Report MSR-TR-2012-88, Microsoft Research, August 2012.
[6] A. Troelstra and H. Schwichtenberg. Basic proof theory, Cambridge Tracts in Theoretical Computer Science, 43, Cambridge University Press, Cambridge, 1996.
[7] Oded Goldreich. Foundations of Cryptography: Volume 1, Basic Tools, Cambridge University Press, Cambridge, 2001.
[8] M. Blum. Coin Flipping by Telephone. Proceedings of CRYPTO 1981, pp. 11-15


[^0]:    ${ }^{2}$ Here we suppose that the encryption method is practically strong and unambiguous. It is impossible for a player who does not know the encryption key to restore the plaintext from a ciphertext. It is also impossible for him to generate two key-message pairs with different messages and the same ciphertext.

[^1]:    ${ }^{3}$ We use this term differently from [2] where infon algebras are semi-lattices with information order " $x$ is at least as informative as $y$ ".

