# Constraint Satisfaction with Counting Quantifiers 2

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#### Abstract

We study *constraint satisfaction problems* (CSPs) in the presence of counting quantifiers  $\exists^{\geq j}$ , asserting the existence of *j* distinct witnesses for the variable in question. As a continuation of our previous (CSR 2012) paper [11], we focus on the complexity of undirected graph templates. As our main contribution, we settle the two principal open questions proposed in [11]. Firstly, we complete the classification of clique templates by proving a full trichotomy for all possible combinations of counting quantifiers and clique sizes, placing each case either in P, NP-complete or Pspace-complete. This involves resolution of the cases in which we have the single quantifier  $\exists^{\geq j}$  on the clique  $\mathbb{K}_{2j}$ . Secondly, we confirm a conjecture from [11], which proposes a full dichotomy for  $\exists$  and  $\exists^{\geq 2}$  on all finite undirected graphs.

The main thrust of this second result is the solution of the complexity for the infinite path which we prove is a polynomial-time solvable problem. By adapting the algorithm for the infinite path we are then able to solve the problem for finite paths, and then trees and forests. Thus as a corollary to this work, combining with the other cases from [11], we obtain a full dichotomy for  $\exists$  and  $\exists^{\geq 2}$  quantifiers on finite graphs, each such problem being either in P or NP-hard. Finally, we persevere with the work of [11] in exploring cases in which there is dichotomy between P and Pspace-complete, in contrast with situations in which the intermediate NP-completeness may appear.

*Key words:* constraint satisfaction problem, counting quantifiers, polynomial time, Pspace-hard, computational complexity, graph homomorphism, graph colouring, graph theory, cliques, forests

#### 1. Introduction

The constraint satisfaction problem  $CSP(\mathbb{B})$ , much studied in artificial intelligence, is known to admit several equivalent formulations, two of the best known of which are the query evaluation of primitive positive (pp) sentences – those involving only existential quantification and conjunction – on  $\mathbb{B}$ , and the homomorphism problem to  $\mathbb{B}$  (see, e.g., [10]). The problem  $CSP(\mathbb{B})$  is NP-complete in general, and a great deal of effort has been expended in classifying its complexity for certain restricted cases. Notably it is conjectured [8, 4] that for all fixed  $\mathbb{B}$ , the problem  $CSP(\mathbb{B})$  is in P or NP-complete. While this has not been settled in general, a number of partial results are known – e.g. over structures of size at most three [13, 3] and over smooth digraphs [9, 1]. A popular generalization of the CSP involves considering the query evaluation problem for *positive Horn* logic – involving only the two quantifiers,  $\exists$  and  $\forall$ , together with conjunction. The resulting *quantified constraint satisfaction problems* QCSP( $\mathbb{B}$ ) allow for a broader class, used in artificial intelligence to capture non-monotonic reasoning, whose complexities rise to Pspacecompleteness.

In this paper, we continue the project begun in [11] to study counting quantifiers of the form  $\exists^{\geq j}$ , which allow one to assert the existence of at least j elements such that the ensuing property holds. Thus on a structure  $\mathbb{B}$  with domain of size n, the quantifiers  $\exists^{\geq 1}$  and  $\exists^{\geq n}$  are precisely  $\exists$  and  $\forall$ , respectively.

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We study variants of  $CSP(\mathbb{B})$  in which the input sentence to be evaluated on  $\mathbb{B}$  (of size |B|) remains positive conjunctive in its quantifier-free part, but is quantified by various counting quantifiers.

For  $X \subseteq \{1, ..., |B|\}$ ,  $X \neq \emptyset$ , the X-CSP(B) takes as input a sentence given by a conjunction of atoms quantified by quantifiers of the form  $\exists^{\geq j}$  for  $j \in X$ . It then asks whether this sentence is true on  $\mathbb{B}$ .

In [11], it was shown that X-CSP( $\mathbb{B}$ ) exhibits trichotomy as  $\mathbb{B}$  ranges over undirected, irreflexive cycles, with each problem being in either L, NP-complete or Pspace-complete. The following classification was given for cliques.

**Theorem 1.** [11] *For*  $n \in \mathbb{N}$  *and*  $X \subseteq \{1, ..., n\}$ *:* 

(i) X-CSP( $\mathbb{K}_n$ ) is in L if  $n \leq 2$  or  $X \cap \{1, \dots, \lfloor n/2 \rfloor\} = \emptyset$ . (ii) X-CSP( $\mathbb{K}_n$ ) is NP-complete if n > 2 and  $X = \{1\}$ .

- (iii) X-CSP( $\mathbb{K}_n$ ) is Pspace-complete if n > 2 and either  $j \in X$  for 1 < j < n/2 or  $\{1, j\} \subseteq X$  for  $j \in X$  $\{ \lceil n/2 \rceil, \ldots, n \}.$

Precisely the cases  $\{j\}$ -CSP $(\mathbb{K}_{2i})$  are left open here. Of course,  $\{1\}$ -CSP $(\mathbb{K}_2)$  is graph 2-colorability and is in L, but for j > 1 the situation was very unclear, and the referees noted specifically this lacuna.

In this paper we settle this question, and find the surprising situation that  $\{2\}$ -CSP( $\mathbb{K}_4$ ) is in P while  $\{j\}$ -CSP $(\mathbb{K}_{2i})$  is Pspace-complete for  $j \geq 3$ . The algorithm for the case  $\{2\}$ -CSP $(\mathbb{K}_4)$  is specialized and non-trivial, and consists in iteratively constructing a collection of forcing triples where we proceed to look for a contradiction.

As a second focus of the paper, we continue the study of  $\{1, 2\}$ -CSP( $\mathbb{H}$ ). In particular, we focus on finite undirected graphs for which a dichotomy was proposed in [11]. As a fundamental step towards this, we first investigate the complexity of  $\{1,2\}$ -CSP( $\mathbb{P}_{\infty}$ ), where  $\mathbb{P}_{\infty}$  denotes the infinite undirected path. We find tractability here in describing a particular unique obstruction, which takes the form of a special walk, whose presence or absence yields the answer to the problem. Again the algorithm is specialized and non-trivial, and in carefully augmenting it, we construct another polynomial-time algorithm, this time for all finite paths.

**Theorem 2.**  $\{1,2\}$ -*CSP*( $\mathbb{P}_n$ ) is in *P*, for each  $n \in \mathbb{N}$ .

A corollary of this is the following key result.

**Corollary 3.**  $\{1,2\}$ -*CSP*( $\mathbb{H}$ ) *is in P, for each forest*  $\mathbb{H}$ .

Combined with the results from [9, 11], this allows us to observe a dichotomy for  $\{1, 2\}$ -CSP( $\mathbb{H}$ ) as  $\mathbb{H}$ ranges over undirected graphs, each problem being either in P or NP-hard, in turn settling a conjecture proposed in [11].

**Corollary 4.** *Let H be a graph.* 

- (i)  $\{1,2\}$ -*CSP*(*H*) is in *P* if *H* is a forest or is bipartite with a 4-cycle,
- (ii)  $\{1,2\}$ -CSP(H) is NP-hard in all other cases.

In [11], the main preoccupation was in the distinction between P and NP-hard. Here we concentrate our observations to show situations in which we have sharp dichotomies between P and Pspace-complete, as well as cases in which NP-completeness manifests. This allows us to generalize the above as follows.

**Theorem 5.** *Let H be a bipartite graph.* 

- (i)  $\{1,2\}$ -CSP(H) is in P if H is a forest or contains a 4-cycle,
- (ii) {1,2}-CSP(H) is Pspace-complete in all other cases.

Taken together, our work seems to indicate a rich and largely uncharted complexity landscape that these types of problems constitute. The associated combinatorics to this landscape appears quite complex and the absence of simple algebraic approach is telling. We will return to the question of algebra in the final remarks of the paper.

The paper is structured as follows. In section 2, we describe a characterization and a polynomial time algorithm for {2}-CSP( $\mathbb{K}_4$ ). In section 3, we show Pspace-hardness for {n}-CSP( $\mathbb{K}_{2n}$ ) for  $n \ge 3$ . In section 4, we characterize {1,2}-CSP for the infinite path  $\mathbb{P}_{\infty}$  and describe the resulting polynomial algorithm. Then, in section 5, we generalize this to finite paths and prove Theorem 2 and associated corollaries, in sections 6 and 7. Subsequently, in section 8, we discuss the P/Pspace-complete dichotomy of bipartite graphs, under {1,2}-CSP, as well as situations in which the intermediate NP-completeness arises, in sections 9 and 10. We conclude the paper in section 11 by giving some final thoughts.

### 1.1. Preliminaries

Our proofs use the game characterization and structural interpretation from [11]. For completeness, we summarize it here. This is as follows.

Given an input  $\Psi$  for X-CSP(**B**), we define the following game  $\mathscr{G}(\Psi, \mathbb{B})$ :

**Definition 1.** Let  $\Psi := Q_1 x_1 Q_2 x_2 \dots Q_m x_m \psi(x_1, x_2, \dots, x_m)$ . Working from the outside in, coming to a quantified variable  $\exists^{\geq j} x$ , the <u>Prover</u> (female) picks a subset  $B_x$  of j elements of B as witnesses for x, and an <u>Adversary</u> (male) chooses one of these, say  $b_x$ , to be the value of x, denoted by f(x).

Prover wins if *f* is a homomorphism to  $\mathbb{B}$ , i.e., if  $\mathbb{B} \models \psi(f(x_1), f(x_2), \dots, f(x_m))$ .

**Lemma 6.** Prover has a winning strategy in the game  $\mathscr{G}(\Psi, \mathbb{B})$  iff  $\mathbb{B} \models \Psi$ .

**Definition 2.** Let *H* be a graph. For an instance  $\Psi$  of *X*-*CSP*(*H*):

- define  $\mathcal{D}_{\psi}$  to be the graph whose vertices are the variables of  $\Psi$  and edges are between variables  $v_i, v_j$  for which  $E(v_i, v_j)$  appears in  $\Psi$ .
- denote  $\prec$  the total order of variables of  $\Psi$  as they are quantified in the formula (from left to right).

### 2. Algorithm for $\{2\}$ -CSP( $\mathbb{K}_4$ )

**Theorem 7.**  $\{2\}$ -*CSP*( $\mathbb{K}_4$ ) *is decidable in polynomial time.* 

The template  $\mathbb{K}_4$  has vertices  $\{1, 2, 3, 4\}$  and all possible edges between distinct vertices. Consider the instance  $\Psi$  of  $\{2\}$ -CSP( $\mathbb{K}_4$ ) as a graph  $G = \mathcal{D}_{\psi}$  together with a linear ordering  $\prec$  on V(G) (see Definition 2).

We iteratively construct the following three sets:  $R^+$ ,  $R^-$ , and F. The set F will be a collection of unordered pairs of vertices of G, while  $R^+$  and  $R^-$  will consist of unordered triples of vertices. (For simplicity we write  $xy \in F$  in place of  $\{x, y\} \in F$ , and write  $xyz \in R^+$  or  $R^-$  in place of  $\{x, y, z\} \in R^+$  or  $R^-$ .)

We start by initializing the sets as follows: F = E(G) and  $R^+ = R^- = \emptyset$ . Then we perform the following rules as long as possible:

- (X1) If there are  $x, y, z \in V(G)$  such that  $\{x, y\} < z$  where  $xz, yz \in F$ , then add xyz into  $R^-$ .
- (X2) If there are vertices  $x, y, w, z \in V(G)$  such that  $\{x, y, w\} < z$  with  $wz \in F$  and  $xyz \in R^-$ , then add xyw into  $R^+$ .
- (X3) If there are  $x, y, w, z \in V(G)$  such that  $\{x, y, w\} < z$  with  $wz \in F$  and  $xyz \in R^+$ , then if  $\{x, y\} < w$ , then add xyw into  $R^-$ , else add xw and yw into F.
- (X4) If there are vertices  $x, y, w, z \in V(G)$  such that  $\{x, w\} < y < z$  with  $xyz \in R^+$  and  $wyz \in R^-$ , then add xw into F, and add xyw into  $R^+$ .
- (X5) If there are vertices  $x, y, w, z \in V(G)$  such that  $\{x, y, w\} < z$  where either  $xyz, wyz \in R^+$ , or  $xyz, wyz \in R^-$ , then add xyw into  $R^+$ .
- (X6) If there are vertices  $x, y, q, w, z \in V(G)$  such that  $\{x, y, w\} < q < z$  where either  $xyz, wqz \in R^+$ , or  $xyz, wqz \in R^-$ , then add xyw and xyq into  $R^+$ .
- (X7) If there are vertices  $x, y, q, w, z \in V(G)$  such that  $\{x, y, w\} < q < z$  where either  $xyz \in R^+$  and  $wqz \in R^-$ , or  $xyz \in R^-$  and  $wqz \in R^+$ , then add xyq into  $R^-$ , and if  $\{x, y\} < w$ , also add xyw into  $R^-$ , else add xw and yw into F.



Figure 1: Illustrating rules (X1)-(X4).

**Theorem 8.** *The following are equivalent:* 

- (i)  $\mathbb{K}_4 \models \Psi$
- (ii) Prover has a winning strategy in  $\mathscr{G}(\Psi, \mathbb{K}_4)$ .
- (iii) Prover can play so that in every instance of the game, the resulting mapping  $f : V(G) \rightarrow \{1, 2, 3, 4\}$  satisfies the following properties:
  - (S1) For every  $xy \in F$ , we have:  $f(x) \neq f(y)$ .
  - (S2) For every  $xyz \in \mathbb{R}^+$  such that x < y < z: if  $f(x) \neq f(y)$ , then  $f(z) \in \{f(x), f(y)\}$ .
  - (S3) For every  $xyz \in R^-$  such that x < y < z: if  $f(x) \neq f(y)$ , then  $f(z) \notin \{f(x), f(y)\}$ .
- (iv) there is no triple xyz in  $R^+$  such that x < y < z and (see Fig. 2)
  - $xz \in F$  or  $yz \in F$ ,
  - or  $xwz \in R^-$  for some w < z (possibly w = y),
  - or  $ywz \in R^-$  for some y < w < z.



Figure 2: Five forbidden configurations of Theorem 8.

### 2.1. Proof of Theorem 8

(i)  $\iff$  (ii) is by definition. (iii)  $\Rightarrow$  (ii) is implied by the fact that  $F \supseteq E(G)$ , and that by (iii) Prover can play to satisfy (S1). Thus in every instance of the game the mapping f is a homomorphism of G to  $\mathbb{K}_4 \Rightarrow$  (ii).

(ii) $\Rightarrow$ (iii): Suppose that Prover plays a winning strategy in the game  $\mathscr{G}(\Psi, \mathbb{K}_4)$  but (iii) fails. We show that this is impossible. Namely, we show how Adversary can play to win.

Consider an instance of the game producing a mapping f. We say that (S1) fails at a vertex v if there exists  $a \in V(G)$  with a < v such that  $av \in F$  and f(a) = f(v). We say that (S2) fails at v if there exist  $a, b \in V(G)$  with a < b < v such that  $abv \in R^+$  while  $f(a) \neq f(b)$  and  $f(v) \notin \{f(a), f(b)\}$ . We say that (S3) fails at v if there exist  $a, b \in V(G)$  with a < b < v such that  $abv \in R^-$  while  $f(a) \neq f(b)$  and  $f(v) \notin \{f(a), f(b)\}$ . We say that (S3) fails at v if there exist  $a, b \in V(G)$  with a < b < v such that  $abv \in R^-$  while  $f(a) \neq f(b)$  and  $f(v) \notin \{f(a), f(b)\}$ .

Since (iii) fails, there is an instance of the game producing a mapping f that fails (S1)-(S3) at some vertex v. Among all such instances, pick one for which v is largest possible with respect to the order <. We will show that this is impossible, namely we will produce a (possibly) different instance violating the maximality of this choice. Note that, since we assume that Prover plays a winning strategy, the mapping f is a homomorphism of G to  $\mathbb{K}_4$ .

**Case 1:** Suppose that (S1) fails at v. Then there is a < v such that  $av \in F$  and f(a) = f(v). If  $av \in E(G)$ , then the mapping f is not a homomorphism of G to  $\mathbb{K}_4$ . Thus Adversary wins, which contradicts (ii). So we may assume that  $av \notin E(G)$ . This implies that av was added to F using one of the rules (X3), (X4), (X7).

**Case 1.1:** Suppose that *av* was added to *F* using (X3). Then there exist vertices *x*, *y*, *w*, *z* where  $\{x, y, w\} < z$  and  $\{x, y\} \not\leq w$  such that  $wz \in F$  and  $xyz \in R^+$ , and either  $a \in \{x, y\}$  and v = w, or  $v \in \{x, y\}$  and a = w. In particular, since f(a) = f(v), we deduce that f(x) = f(w) or f(y) = f(w).

We may assume by symmetry that x < y. Recall that  $\{x, y\} \not< w$ . Thus  $\{x, w\} < y$ . Consider the point of the instance of the game producing f when Prover offers values for y. From this point on, we have Adversary play as follows: for y, if f(x) = f(w), choose any value that is different from f(w); if  $f(x) \neq f(w)$ , choose f(y) for y. Let  $\beta$  denote the value chosen for y. Observe that the choice is always possible, since Prover offers for y two distinct values, one of which is f(y). Moreover, the choice guarantees that  $f(x) \neq \beta$ , since either  $f(x) = f(w) \neq \beta$  or  $f(x) \neq f(w) = f(y) = \beta$ . For this, recall that f(x) = f(w) or f(y) = f(w). Then for z, if f(w) is offered by Prover, we let Adversary choose f(w) for z; otherwise Adversary chooses any value different from f(x) and  $\beta$ . Let  $\alpha$  denote the value chosen for z. Again, note that the choice is always possible, in particular in the latter case where Prover offers for z two distinct values, neither of which is f(w), while f(w) = f(x) or  $f(w) = f(y) = \beta$ . For the remaining vertices, we let Adversary play any choices. This produces a (possibly) different instance of the game. It follows that this instance fails (S1) or (S2) at z. Namely, if f(w) was offered for z, then  $\alpha = f(w)$  and (S1) fails at z, since  $wz \in F$ . If f(w) was not offered for z, then  $\alpha \notin \{f(x), \beta\}$ , in which case (S2) fails at z, because  $xyz \in R^+$ . However, since v < z, this contradicts our choice of v.

**Case 1.2:** Suppose that *av* was added to *F* using (X4). Then there exist vertices *x*, *y*, *w*, *z* where  $\{x, w\} < y < z$  such that  $xyz \in R^+$  and  $wyz \in R^-$ , and where  $\{x, w\} = \{a, v\}$ . In particular, we deduce that f(x) = f(w).

We consider the point when Prover offers values for y and have Adversary play as follows: for y, choose any value different from f(x); let  $\beta$  denote this value. Note that  $\beta \neq f(x)$ . Then for z, choose any value, denote it  $\alpha$ , and then play any choices for the remaining vertices. The instance this produces fails (S2) or (S3) at z. Namely, if  $\alpha \notin \{f(x), \beta\}$ , then (S2) fails, since  $xyz \in R^+$ , while if  $\alpha \in \{f(x), \beta\}$ , then (S3) fails, since f(x) = f(w) and  $wyz \in R^-$ . This again contradicts our choice of v, since v < z.

**Case 1.3:** Suppose that *av* was added to *F* using (X7). Then there exist vertices x, y, q, w, z where  $\{x, y, w\} < q < z$  and  $\{x, y\} \not\leq w$  such that  $xyz \in R^+$ ,  $wqz \in R^-$ , or  $xyz \in R^-$ ,  $wqz \in R^+$ , and such that  $a \in \{x, y\}$  and v = w, or a = w and  $v \in \{x, y\}$ . Since f(a) = f(v), we deduce that f(x) = f(w) or f(y) = f(w).

By symmetry, assume x < y. Thus  $\{x, w\} < y < q < z$  because  $\{x, y\} \not< w$ . We have Adversary play from y as follows: for y, if f(x) = f(w), choose any value different from f(w); if  $f(x) \neq f(w)$ , choose f(y) for y. Let  $\beta$  denote the value chosen for y. Note that  $f(x) \neq \beta$  and  $f(w) \in \{f(x), \beta\}$ , since either  $f(x) = f(w) \neq \beta$  or  $f(x) \neq f(w) = f(y) = \beta$ . For this, recall that f(x) = f(w) or f(y) = f(w). Next, for q, we let Adversary choose any value different from f(w), and denote it  $\gamma$ . Note that  $\gamma \neq f(w)$ . Finally, for z, if  $xyz \in R^+$  and  $wqz \in R^-$ , Adversary chooses f(w) if offered by Prover, and if not, he chooses any value different from f(x) and  $\beta$ . Similarly, if  $xyz \in R^-$  and  $wqz \in R^+$ , Adversary chooses f(x) or  $\beta$  if offered by Prover, and otherwise he chooses any value different from f(w) and  $\gamma$ . Let  $\alpha$  denote the value chosen for z. Again, we see that this choice is always possible, since  $f(w) \in \{f(x), \beta\}$ . Adversary plays any choices for the rest. We claim that the instance this produces fails (S2) or (S3) at z. Namely, if  $xyz \in R^+$  and  $wqz \in R^-$ , then (S2) fails at z if  $\alpha \notin \{f(x), \beta\}$ , since  $xyz \in R^+$ , while if  $\alpha \in \{f(x), \beta\}$ , then (S3) fails at z, since in that case we must have  $\alpha = f(w) \neq \gamma$  by the choice of  $\alpha$ , while  $wqz \in R^-$ . Similarly, if  $xyz \in R^-$  and  $wqz \in R^+$ , then either  $\alpha \in \{f(x), \beta\}$  and (S3) fails at z, since  $xyz \in R^-$ , or  $\alpha \notin \{f(x), \beta\}$  in which case  $\alpha \notin \{f(w), \gamma\}$ and (S2) fails, since  $wqz \in R^+$ . Since v < z, this contradicts our choice of v.

From now on, we may assume that (S1) does not fail at v.

**Case 2:** Suppose that (S2) fails at *v*. Then there are vertices *a*, *b* with a < b < v such that  $abv \in R^+$  while  $f(a) \neq f(b)$  and  $f(v) \notin \{f(a), f(b)\}$ . Since the set  $R^+$  is initially empty, the triple *abv* was added to  $R^+$  by one of the rules (X2), (X4), (X5), or (X6).

**Case 2.1:** Suppose that *abv* was added to  $R^+$  using (X2). Then there exist vertices x, y, w, z where  $\{x, y, w\} < z$  such that  $wz \in F$  and  $xyz \in R^-$ , and where  $\{a, b, v\} = \{x, y, w\}$ . We deduce that f(x), f(y), f(w) are three distinct values, since f(a), f(b), f(v) are. We have Adversary play from z as follows: for z, from the two offered values, one will be in  $\{f(x), f(y), f(w)\}$ ; choose this value  $\alpha$ . For the rest, play any choices. It follows that this instance fails (S1) or (S3) at z. Namely, if  $\alpha = f(w)$ , then (S1) fails, since  $wz \in F$ , while if  $\alpha \in \{f(x), f(y)\}$ , then (S3) fails, since  $xyz \in R^-$ . This contradicts our choice of v, since v < z.

**Case 2.2:** Suppose that *abv* was added to  $R^+$  using (X4). Then there are vertices x, y, w, z where  $\{x, w\} < y < z$  such that  $xyz \in R^+$  and  $wyz \in R^-$ , and where  $\{a, b, v\} = \{x, y, w\}$ . Again, we deduce that f(x), f(y), f(w) are pairwise distinct. Adversary plays from z as follows: for z, if f(y) is offered, choose f(y); otherwise, choose any value different from f(x). For the rest, play any choices. Let  $\alpha$  be the value chosen for z. We claim that this instance fails (S2) or (S3) at z. Namely, if  $\alpha = f(y)$ , then (S3) fails, since  $f(w) \neq f(y)$  and  $wyz \in R^-$ , while if  $\alpha \neq f(y)$ , then  $\alpha \notin \{f(x), f(y)\}$ , in which case (S2) fails, since  $f(x) \neq f(y)$  and  $xyz \in R^+$ . This contradicts our choice of v, since v < z.

**Case 2.3:** Suppose that *abv* was added to  $R^+$  using (X5). Then there exist vertices x, y, w, z where  $\{x, y, w\} < z$  such that either  $xyz, wyz \in R^+$  or  $xyz, wyz \in R^-$ , and where  $\{a, b, v\} = \{x, y, w\}$ . Hence, we deduce that f(x), f(y), f(w) are pairwise distinct. Adversary plays from z as follows: for z, if  $xyz, wyz \in R^+$ , choose any value different from f(y); if  $xyz, wyz \in R^-$ , choose any value in  $\{f(x), f(y), f(w)\}$ . For the rest, play any choices. Let  $\alpha$  denote the value chosen for z. We claim that this instance fails (S2) or (S3) at z. Namely, if  $xyz, wyz \in R^+$ , then  $\alpha \neq f(y)$  and also  $\alpha \neq f(x)$  or  $\alpha \neq f(w)$ , since  $f(x) \neq f(w)$ . Thus (S2) fails, since either  $\alpha \notin \{f(x), f(y), f(w)\}$ . Thus if  $\alpha \in \{f(x), f(y)\}$  and  $wyz \in R^-$ , while if  $\alpha = f(w)$ , then (S3) fails, since  $wyz \in R^-$ . This contradicts the choice of v, since v < z.

**Case 2.4:** Suppose that *abv* was added to  $R^+$  using (X6). Then there are vertices x, y, q, w, z where  $\{x, y, w\} < q < z$  such that either  $xyz, wqz \in R^+$  or  $xyz, wqz \in R^-$ , and where either  $\{a, b, v\} = \{x, y, w\}$  or  $\{a, b, v\} = \{x, y, q\}$ . In either case, we have  $f(x) \neq f(y)$ . Adversary plays from q as follows: if  $f(w) \in \{f(x), f(y)\}$ , then choose f(q) for q; otherwise choose any value different from f(w). Let  $\gamma$  denote the value chosen for q. Then for z, if  $xyz, wqz \in R^-$ , choose any value in  $\{f(x), f(y), f(w), \gamma\}$ ; if  $xyz, wqz \in R^+$  and  $f(w) \in \{f(x), f(y)\}$ , choose any value different from f(w); otherwise choose any value different from  $\gamma$ . Let  $\alpha$  denote the value chosen for z. Note that  $\gamma \neq f(w)$ . Indeed, if  $f(w) \notin \{f(x), f(y)\}$ , then  $\gamma \neq f(w)$  by our choice. If  $f(w) \in \{f(x), f(y)\}$ , then  $\{a, b, v\} \neq \{x, y, w\}$ , since f(a), f(b), f(v) are pairwise distinct. Thus  $\{a, b, v\} = \{x, y, q\}$  implying that f(x), f(y), f(q) are pairwise distinct; so  $f(w) \neq \gamma = f(q)$ , since  $f(w) \in \{f(x), f(y)\}$ .

We claim that this instance fails (S2) or (S3) at z. Namely, if  $xyz, wqz \in R^-$ , then  $\alpha \in \{f(x), f(y), f(w), \gamma\}$  and so (S3) either fails because  $\alpha \in \{f(x), f(y)\}$  while  $xyz \in R^-$ , or it fails because  $\alpha \in \{f(w), \gamma\}$  and  $f(w) \neq \gamma$  while  $wqz \in R^-$ . If  $xyz, wqz \in R^+$  and  $f(w) \in \{f(x), f(y)\}$ , then  $\alpha \neq f(w)$  and  $\gamma = f(q) \notin \{f(x), f(y)\}$ ; thus (S2) fails either because  $\alpha \notin \{f(x), f(y)\}$  while  $xyz \in R^+$ , or  $\{\alpha, f(w)\} = \{f(x), f(y)\}$  and  $\gamma \notin \{f(x), f(y)\}$  while  $wqz \in R^+$ . Finally, if  $xyz, wqz \in R^+$  and  $f(w) \notin \{f(x), f(y)\}$ , then  $\alpha \neq \gamma$ , and (S2) fails either because  $\alpha = f(w)$  while  $xyz \in R^+$ , or because  $\alpha \neq f(w)$  and  $f(w) \neq \gamma \neq \alpha$ , while  $wqz \in R^+$ . This contradicts the choice of v, since v < z.

**Case 3:** Suppose that (S3) fails at v. Then there are vertices a, b with a < b < v such that  $abv \in R^-$  while  $f(a) \neq f(b)$  and  $f(v) \in \{f(a), f(b)\}$ . Since the set  $R^-$  is initially empty, the triple abv was added to  $R^-$  by one of the rules (X1), (X3), or (X7).

**Case 3.1:** Suppose that *abv* was added to  $R^-$  using (X1). Then *av*, *bv*  $\in$  *F* and it follows that (S1) fails at *v*. Namely, if f(v) = f(a), then (S1) fails, since  $av \in F$ , while if f(v) = f(b), then (S1) fails, since  $bv \in F$ . However, we assume that (S1) does not fail at *v* (as this leads to Case 1), a contradiction.

**Case 3.2:** Suppose that *abv* was added to  $R^-$  using (X3). Then there exist vertices x, y, w, z where  $\{x, y, w\} < z$  and  $\{x, y\} < w$  such that  $wz \in F$  and  $xyz \in R^+$ , and where  $\{a, b, v\} = \{x, y, w\}$ . Since a < b < v, we have  $\{a, b\} = \{x, y\}$  and v = w. In particular, we deduce that  $f(w) \in \{f(x), f(y)\}$ . Adversary plays from z as follows: for z, if f(w) offered, choose this value; otherwise, choose any value different from f(x) and f(y). Let  $\alpha$  denote the value chosen for z. Note that this choice is always possible, since Prover offers for z two distinct values; if neither is f(w), then at least one of them is distinct from both f(x) and f(y), since  $f(w) \in \{f(x), f(y)\}$ . For the rest, Adversary play any choices. We claim that this instance fails (S1) or (S2) at z. Namely, if  $\alpha = f(w)$ , then (S1) fails at z, since  $wz \in F$ , while if  $\alpha \neq f(w)$ , then  $\alpha \notin \{f(x), f(y)\}$ , in which case (S2) fails, since  $f(x) \neq f(y)$  and  $xyz \in R^+$ . This contradicts the choice of v, since v < z.

**Case 3.3:** Suppose that *abv* was added to  $R^-$  using (X7). Then there are vertices x, y, q, w, z where  $\{x, y, w\} < q < z$  such that either  $xyz \in R^+$  and  $wqz \in R^-$ , or  $xyz \in R^-$  and  $wqz \in R^+$ , and where either  $\{a, b, v\} = \{x, y, q\}$ , or where  $\{x, y\} < w$  and  $\{a, b, v\} = \{x, y, w\}$ . Since a < b < v, we deduce that  $\{a, b\} = \{x, y\}$  and  $v \in \{w, q\}$ . In particular,  $f(x) \neq f(y)$  and either  $f(w) \in \{f(x), f(y)\}$  or  $f(q) \in \{f(x), f(y)\}$ . Adversary plays from q as follows: if  $f(w) \notin \{f(x), f(y)\}$ , then choose f(q) for q; otherwise, choose any value different from f(w). Let  $\gamma$  denote the value chosen for q. Then for z, if  $xyz \in R^+$  and  $wqz \in R^-$ , choose f(w) or  $\gamma$  if offered, else choose any value distinct from f(x) and f(y). Note that this choice is always possible, since in the latter case Prover offers two distinct values, neither of which is f(w),  $\gamma$ , while either  $f(w) \in \{f(x), f(y)\}$  or  $\gamma = f(q) \in \{f(x), f(y)\}$ . Similarly, if  $xyz \in R^-$  and  $wqz \in R^+$ , we have Adversary choose f(x) or f(y) if offered, and else choose any value distinct from f(w) and  $\gamma$ . Again, this choice is always possible, since  $\{f(w), \gamma\} \cap \{f(x), f(y)\} \neq \emptyset$ . Let  $\alpha$  denote the value chosen for z. For the rest, Adversary plays any choices. Note that  $\gamma \neq f(w)$ . Indeed, if  $f(w) \in \{f(x), f(y)\}$ , then  $\gamma \neq f(w)$  by our choice. If  $f(w) \notin \{f(x), f(y)\}$ , then  $f(q) \in \{f(x), f(y)\}$  and  $\gamma = f(q)$ ; thus  $\gamma \neq f(w)$ , since  $\gamma$  is in  $\{f(x), f(y)\}$  while f(w) is not.

We claim that this instance fails (S2) or (S3) at *z*. Namely, if  $xyz \in R^+$  and  $wqz \in R^-$ , then either (S3) fails, since  $\alpha \in \{f(w), \gamma\}$  while  $wqz \in R^-$ , or (S2) fails, since  $\alpha \notin \{f(x), f(y)\}$  while  $xyz \in R^+$ . Similarly, if  $xyz \in R^-$  and  $wqz \in R^+$ , then either (S3) fails, since  $\alpha \in \{f(x), f(y)\}$  while  $xyz \in R^-$ , or (S2) fails, since  $\alpha \notin \{f(w), \gamma\}$  while  $wqz \in R^+$ . This contradicts the choice of *v*, since v < z.

This exhausts all possibilities. Therefore no such instance of the game exists, which proves (ii) $\Rightarrow$ (iii).

(iii)  $\Rightarrow$  (iv): Assume that Prover has a strategy as described in (iii), but (iv) fails, i.e. there exists a triple  $xyz \in R^+$  such that x < y < z and either  $xz \in F$ , or  $yz \in F$ , or  $xwz \in R^-$  for some w < z, or  $ywz \in R^-$  for some y < w < z. We show that this is impossible. Namely, we show that there is a way that Adversary can play to violate the conditions of (iii). As usual, we let f denote the mapping produced during the game.

Suppose first that  $xz \in F$  or  $yz \in F$ . Adversary plays as follows: until the game reaches y, Adversary plays any choices. When the game reaches y, Prover offers two values for y; from the two, Adversary chooses, as the value f(y), any offered value that is different from f(x). Then Adversary again plays any choices until the game reaches z when Prover offers two distinct values for z. If any of the two values is not in  $\{f(x), f(y)\}$ , then Adversary chooses this value to be the value f(z). Otherwise, he chooses f(x) if  $xz \in F$ , and chooses f(y) if  $yz \in F$ . For the rest, Adversary plays any choices. It follows that the mapping f fails to satisfy (S1) or (S2). Namely, if  $f(z) \notin \{f(x), f(y)\}$ , then (S2) fails, since  $f(x) \neq f(y)$  and  $xyz \in R^+$ . If  $f(z) \in \{f(x), f(y)\}$ , then either f(z) = f(x) in case  $xz \in F$  and so (S1) fails, or f(z) = f(y) in case  $yz \in F$ and so (S1) fails again. This contradicts our assumption (iii).

Now, assume that  $ywz \in R^-$  for some y < w < z. Adversary again chooses f(x) and f(y) to be distinct, and then chooses f(w) to be distinct from f(y). When z is reached, Adversary chooses f(y) or f(w) if offered by Prover, and else he chooses any value distinct from f(x). For the rest, Adversary plays any choices. It follows that (S2) or (S3) fails for f. Namely, if  $f(z) \in \{f(y), f(w)\}$ , then (S3) fails, since  $f(y) \neq f(w)$  and  $ywz \in R^-$ . If  $f(z) \notin \{f(y), f(w)\}$ , then  $f(z) \neq f(x)$  and (S2) fails, since  $f(x) \neq f(y)$  and  $xyz \in R^+$ . This contradicts (iii).

Lastly, assume that  $xwz \in R^-$  where w < z (possibly w = y). Adversary chooses f(x) and f(w) to be distinct and also chooses f(y) so that f(x) and f(y) are distinct (possibly y = w). When z is reached, Adversary chooses f(x) or f(w) if offered by Prover, and else he chooses any value distinct from f(y). For the rest, Adversary plays any choices. Again, we have that (S2) or (S3) fails for f. Namely, if  $f(z) \in$  $\{f(x), f(w)\}$ , then (S3) fails, since  $f(x) \neq f(w)$  and  $xwz \in R^-$ . If  $f(z) \notin \{f(x), f(w)\}$ , then also  $f(z) \neq$ f(y) in which case (S2) fails, since  $f(x) \neq f(y)$  and  $f(z) \notin \{f(x), f(y)\}$ , while  $xyz \in R^+$ . This again contradicts (iii).

This concludes the proof of (iii) $\Rightarrow$ (iv).

(iv)  $\Rightarrow$  (iii): Assume (iv). We describe a strategy for Prover that will satisfy (iii). As usual, let  $f(\cdot)$  denote the values chosen by Adversary during the game (partial mapping from V(G) to {1,2,3,4}). When asked to offer values for z, Prover offers values as follows.

- (1) If there exist  $x, y \in V(G)$  where
  - *x* < *y* < *z*
  - $xyz \in R^+$
  - $f(x) \neq f(y)$

then Prover offers  $\{f(x), f(y)\}$ .

- (2) Else if there exist  $x, y \in V(G)$  where
  - *x* < *y* < *z*
  - $xyz \in R^-$
  - $f(x) \neq f(y)$

then Prover offers  $\{1, 2, 3, 4\} \setminus \{f(x), f(y)\}$ .

- (3) Else if there exists  $x \in V(G)$  with x < z and  $xz \in F$ , then Prover offers any two values different from f(x).
- (4) Else Prover offers any two values.

We prove that this strategy satisfies the conditions of (iii). For contradiction, suppose that Adversary can play against this strategy so that the resulting mapping f fails one of the conditions (S1)-(S3).

Consider the first point of the game when the value f(z) was assigned to z causing one of (S1)-(S3) to fail. Recall that we assume (iv). We examine the three possibilities as follows.

**Case 1:** Suppose that (S1) fails when the value is chosen for *z*. Namely, suppose that there is  $a \in V(G)$  with a < z where  $az \in F$  and f(a) = f(z). This means that f(a) was one of the values offered by Prover for *z*. Recall that Prover offered values for *z* in steps (1)-(4) in that order.

**Case 1.1:** Suppose that Prover offered values for *z* in step (1). Then there exist vertices *x*, *y* where x < y < z such that  $xyz \in R^+$  and  $f(x) \neq f(y)$ , and Prover offered  $\{f(x), f(y)\}$  for *z*. Thus  $f(a) \in \{f(x), f(y)\}$ , since f(a) = f(z). It follows that  $a \notin \{x, y\}$ , since otherwise we contradict (iv). If  $\{x, y\} < a$ , then we have  $xya \in R^-$  by (X3). But then (S3) is violated at *a*, since  $f(a) \in \{f(x), f(y)\}$ . Similarly, if  $\{x, y\} \not< a$ , then we have  $xa, ya \in F$  by (X3), and (S1) is violated either at *y* if f(a) = f(y), or else at *a* or *x* if f(a) = f(x). This contradicts our choice of *z*, since  $\{x, y, a\} < z$ .

**Case 1.2:** Suppose that Prover offered values for *z* in step (2). Then there exist vertices *x*, *y* where x < y < z such that  $xyz \in R^-$  and  $f(x) \neq f(y)$ , and Prover offered for *z* the set  $\{1,2,3,4\} \setminus \{f(x), f(y)\}$ . Since f(z) was chosen from this set, we have  $f(z) \notin \{f(x), f(y)\}$ . Recall that f(a) = f(z). Thus also  $f(a) \notin \{f(x), f(y)\}$  and hence  $a \notin \{x, y\}$ . This implies by (X2) that  $xya \in R^+$ . But since  $f(a) \notin \{f(x), f(y)\}$  and  $f(x) \neq f(y)$ , we notice that f(a), f(x), f(y) are pairwise distinct, and hence, (S2) is violated at either *y* or *a*, since  $xya \in R^+$ . This contradicts our choice of *z*, since  $\{y, a\} < z$ .

**Case 1.3:** Suppose that Prover offered values for z in step (3). Then there exists a vertex x where x < z such that  $xz \in F$  and Prover offered for z a set of two distinct values, neither of which was f(x). Since f(z) was chosen from this set, we have  $f(z) \neq f(x)$ . Recall that f(a) = f(z). Thus  $f(a) \neq f(x)$  and  $a \neq x$ . From this we deduce using (X1) that  $xaz \in R^-$ . Consequently, Prover should have offered values in step (2), never reaching step (3), since  $f(a) \neq f(x)$  and  $xaz \in R^-$ . Thus Prover never reached step (3), a contradiction.

**Case 1.4:** Suppose that Prover offered values for z in step (4). Since step (4) was reached, there is no x such that x < z and  $xz \in F$ . Thus it is impossible that (S1) failed when the value for z was chosen, a contradiction.

**Case 2:** Suppose that (S2) fails when the value is chosen for *z*. Namely, suppose that there exist vertices *a*, *b* where a < b < z and  $abz \in R^+$  such that  $f(a) \neq f(b)$  and  $f(z) \notin \{f(a), f(b)\}$ . Note that this implies that Prover offered values for *z* in step (1), since we may always take x = a and y = b to satisfy the conditions of step (1). Thus we only need to consider this possibility. Namely, we have that there exist vertices *x*, *y* where x < y < z such that  $xyz \in R^+$  and  $f(x) \neq f(y)$ , and Prover offered  $\{f(x), f(y)\}$  for *z*. Thus  $f(z) \in \{f(x), f(y)\}$ . Recall that  $f(z) \notin \{f(a), f(b)\}$ . Hence  $\{a, b\} \neq \{x, y\}$ . Moreover, since  $f(a) \neq f(b)$  and  $f(x) \neq f(y)$ , it follows that  $\{f(a), f(b)\} \neq \{f(x), f(y)\}$ , since  $\{f(x), f(y)\}$  contains f(z) while  $\{f(a), f(b)\}$  does not.

Assume first that  $\{x, y\}$  is disjoint from  $\{a, b\}$ . If y < b, then we deduce using (X6) that  $xya, xyb \in R^+$ . This means that (S2) fails at *b* or at one of *y*, *a*, since  $f(x) \neq f(y)$  and  $\{f(a), f(b)\} \neq \{f(x), f(y)\}$ . Similarly, if b < y, we deduce using (X6) that  $abx, aby \in R^+$ , and (S2) fails at *y* or at one of *b*, *x*, since  $f(a) \neq f(b)$  and  $\{f(x), f(y)\} \neq \{f(a), f(b)\}$ . This contradicts our choice of *z*, since  $\{x, y, a, b\} < z$ .

So we may assume that  $\{x, y\}$  intersects  $\{a, b\}$ . If  $y \in \{a, b\}$ , then  $xab \in R^+$  by (X5). Recall that  $\{f(a), f(b)\} \neq \{f(x), f(y)\}$ . Since  $y \in \{a, b\}$ , we deduce that  $f(x) \notin \{f(a), f(b)\}$ . This implies that f(x), f(a), f(b) are pairwise distinct, since also  $f(a) \neq f(b)$ . Thus (S2) fails at b, since  $xab \in R^+$ . Similarly, if  $x \in \{a, b\}$ , then  $aby \in R^+$  by (X5) and we have  $f(y) \notin \{f(a), f(b)\}$ . Hence, f(a), f(b), f(y) are pairwise distinct and so (S2) fails at y or b, since  $aby \in R^+$ . This contradicts our choice of z, since  $\{y, b\} < z$ .

**Case 3:** Suppose that (S3) fails when the value is chosen for *z*. Namely, suppose that there exist vertices *a*, *b* where a < b < z and  $abz \in R^-$  such that  $f(a) \neq f(b)$  and  $f(z) \in \{f(a), f(b)\}$ . Note that this implies that Prover offered values for *z* in either step (1) or step (2), since we may always take x = a and y = b to satisfy the conditions of step (2). Thus we only need to consider the steps (1) and (2).

**Case 3.1:** Suppose that Prover offered values for *z* in step (1). Then there exist vertices *x*, *y* where x < y < z such that  $xyz \in R^+$  and  $f(x) \neq f(y)$ , and Prover offered  $\{f(x), f(y)\}$  for *z*. Thus  $f(z) \in \{f(x), f(y)\}$ . Recall that  $f(z) \in \{f(a), f(b)\}$ . We deduce that  $\{f(x), f(y)\} \cap \{f(a), f(b)\} \neq \emptyset$ .

Assume first that  $\{a, b\}$  and  $\{x, y\}$  are disjoint. If y < b, then we deduce using (X7) that  $xyb \in R^$ and either  $xya \in R^-$  if  $\{x, y\} < a$ , or else  $xa, ya \in F$ . Thus if  $f(b) \in \{f(x), f(y)\}$ , then (S3) fails at b, since  $f(x) \neq f(y)$  and  $xyb \in R^-$ . So we may assume that  $f(b) \notin \{f(x), f(y)\}$  which yields that  $f(a) \in \{f(x), f(y)\}$ , since  $\{f(a), f(b)\} \cap \{f(x), f(y)\} \neq \emptyset$ . Thus if  $\{x, y\} < a$ , we have  $xya \in R^-$  and so (S3) fails at *a*, since  $f(x) \neq f(y)$ . If  $\{x, y\} \not\leq a$ , we have  $xa, ya \in F$  in which case (S1) fails at either *y* or one of *x*, *a*. Similarly, if b < y, we have by (X7) that  $aby \in R^-$  and either  $abx \in R^-$  if  $\{a, b\} < x$ , or  $ax, bx \in F$  if otherwise. Thus either (S3) fails at y if  $f(y) \in \{f(a), f(b)\}$ , or we have  $f(x) \in \{f(a), f(b)\}$  in which case either (S3) fails at x if  $\{a, b\} < x$ , or (S1) fails at a or b or x if  $\{a, b\} < x$ . This contradicts our choice of z, since  $\{x, y, a, b\} < z$ .

Thus we may assume that  $\{a, b\}$  intersects  $\{x, y\}$ . Recall that a < b and x < y. We observe that if  $x \in \{a, b\}$  or y = a, then we contradict (iv), the second or third condition thereof, respectively. Thus it follows that  $x \notin \{a, b\}$  and y = b. From this we deduce using (X4) that  $ax \in F$  and  $axy \in R^+$ . We recall that  $f(x) \neq f(y)$  and  $f(a) \neq f(b)$ . Since y = b, we deduce that  $f(y) \notin \{f(a), f(x)\}$ . Thus either f(a) = f(x)and (S1) fails at one of a, x, since  $ax \in F$ , or we have  $f(a) \neq f(x)$  in which case (S2) fails at y, since  $axy \in R^+$  and  $f(y) \notin \{f(a), f(x)\}$ . This again contradicts our choice of z, since  $\{a, x, y\} < z$ .

**Case 3.2:** Suppose that Prover offered values for z in step (2). Then there exist vertices x, y where x < y < zsuch that  $xyz \in R^-$  and  $f(x) \neq f(y)$ , and Prover offered for z the set  $\{1, 2, 3, 4\} \setminus \{f(x), f(y)\}$ . Since f(z)was chosen from this set, we have  $f(z) \notin \{f(x), f(y)\}$ . Recall that  $f(z) \in \{f(a), f(b)\}$  and  $f(a) \neq f(b)$ . We deduce that  $\{f(a), f(b)\} \neq \{f(x), f(y)\}$  and so  $\{a, b\} \neq \{x, y\}$ . Now we proceed exactly as in Case 2.

If  $\{x, y\}$  is disjoint from  $\{a, b\}$ , we consider two cases: y < b or b < y. If y < b, then  $xya, xyb \in \mathbb{R}^+$ by (X6), while if b < y, we have  $abx, aby \in R^+$ . In either case, we deduce that (S2) fails at one of x, y, a, b, since  $\{f(a), f(b)\} \neq \{f(x), f(y)\}$ . If  $\{x, y\} \cap \{a, b\} \neq \emptyset$ , then we again have two cases:  $y \in \{a, b\}$ or  $x \in \{a, b\}$ . If  $y \in \{a, b\}$ , we have  $xab \in R^+$  by (X5) and we deduce that f(x), f(a), f(b) are pairwise distinct. Thus (S2) fails at *b*, since  $xab \in R^+$ . If  $x \in \{a, b\}$ , then  $aby \in R^+$  by (X5) and f(y), f(a), f(b) are pairwise distinct. Thus (S2) fails at b or y, since  $aby \in R^+$ . This contradicts our choice of z, since  $\{y, b\} < z$ .

This exhausts all possibilities. Thus we conclude that no such vertex z exists which proves that the strategy for Prover described in steps (1)-(4) is indeed a strategy satisfying the conditions of (iii). Therefore  $(iv) \Rightarrow (iii).$ 

This completes the proof of Theorem 8.

With this characterization, we can now prove Theorem 7 as follows.

### 2.2. Proof of Theorem 7

By Theorem 8, it suffices to construct the sets F,  $R^+$ , and  $R^-$ , and check the conditions of item (iv) of the said theorem. This can clearly be accomplished in polynomial time, since each of the three sets contains at most  $n^3$  elements, where n is the number of variables in the input formula, and elements are only added (never removed) from the sets. Thus either a new pair (triple) needs to be added as follows from one of the rules (X1)-(X7), or we can stop and the output the resulting sets. 

# 3. Hardness of $\{n\}$ -CSP( $\mathbb{K}_{2n}$ ) for $n \geq 3$

**Theorem 9.**  $\{n\}$ -*CSP*( $\mathbb{K}_{2n}$ ) is Pspace-complete for all  $n \geq 3$ .

The template  $\mathbb{K}_{2n}$  consists of vertices  $\{1, 2, ..., 2n\}$  and all possible edges between distinct vertices. We shall call these vertices *colours*. We describe a reduction from  $QCSP(\mathbb{K}_n) = \{1, n\} - CSP(\mathbb{K}_n)$  to  $\{n\} - CSP(\mathbb{K}_{2n})$ . Consider an instance of  $QCSP(\mathbb{K}_n)$ , namely a formula  $\Psi$  where

$$\Psi = \exists^{\geq b_1} v_1 \exists^{\geq b_2} v_2 \dots \exists^{\geq b_N} v_N \psi$$

where each  $b_i \in \{1, n\}$ . As usual (see Definition 2), let G denote the graph  $\mathcal{D}_{\psi}$  with vertex set  $\{v_1, \ldots, v_N\}$ and edge set  $\{v_i v_i | E(v_i, v_i) \text{ appears in } \psi\}$ .

We construct an instance  $\Phi$  of  $\{n\}$ -CSP( $\mathbb{K}_{2n}$ ) with the property that  $\Psi$  is a yes-instance of QCSP( $\mathbb{K}_n$ ) if and only if  $\Phi$  is a yes-instance of  $\{n\}$ -CSP( $\mathbb{K}_{2n}$ ).



Figure 3: The edge gadget (here, as an example, *x* is an  $\exists$  vertex while *y* is a  $\forall$  vertex).

In short, we shall model the *n*-colouring using 2n - 1 colours, n - 1 of which will treated as *don't care* colours (vertices coloured using any of such colours will be ignored). We make sure that the colourings where no vertex is assigned a don't-care colour precisely model all colourings that we need to check to verify that  $\Psi$  is a yes-instance.

We describe  $\Phi$  by giving a graph *H* together with a total order of its vertices with the usual interpretation that the vertices are the variables of  $\Phi$ , the total order is the order of quantification of the variables, and the edges of *H* define the conjunction of predicates  $E(\cdot, \cdot)$  which forms the quantifier-free part  $\phi$  of  $\Phi$ .

We start constructing *H* by adding the vertices  $v_1, v_2, ..., v_N$  and no edges. Then we add new vertices  $u_1, u_2, ..., u_n$  and make them pairwise adjacent.

We make each  $v_i$  adjacent to  $u_1$ , and if  $b_i = n$  (i.e. if  $v_i$  was quantified  $\forall$ ), then we also make  $v_i$  adjacent to  $u_2, u_3, \ldots, u_n$ .

We complete *H* by introducing for each edge  $xy \in E(G)$ , a gadget consisting of new vertices w, q, z, a, b, c with edges wa, wb, qb, qc, za, zb, and we connect this gadget to the rest of the graph as follows: we make x adjacent to a, make y adjacent to b, make a adjacent to  $u_1$ , make c adjacent to  $u_1, u_2, u_3$ , and make each of a, b, c adjacent to  $u_4, \ldots, u_n$ . We refer to Figure 3 for an illustration.

The total order of V(H) first lists  $u_1, u_2, \ldots, u_n$ , then  $v_1, v_2, \ldots, v_N$  (exactly in the same order as quantified in  $\Psi$ ), and then lists the remaining vertices of each gadget, in turn, as depicted in Figure 3 (listing w, q, z, a, b, c in this order).

We consider the game  $\mathscr{G}(\Phi, \mathbb{K}_{2n})$  of Prover and Adversary played on  $\Phi$  where Prover and Adversary take turns, for each variable in  $\Phi$  in the order of quantification, respectively providing a set of *n* colours and choosing a colour from the set. Prover wins if this process leads to a proper 2*n*-colouring of *H* (no adjacent vertices receive the same colour), otherwise Prover loses and Adversary wins. The formula  $\Phi$  is a yes-instance if and only if Prover has a winning strategy.

Without loss of generality (up to renaming colours), we may assume that the vertices  $u_1, u_2, \ldots, u_n$  get assigned colours  $n + 1, n + 2, \ldots, 2n$ , respectively, i.e. each  $u_i$  gets colour n + i. (The edges between these vertices make sure that Prover must offer distinct colours while Adversary has no way of forcing a conflict, since there are 2n colours available.)

The claim of Theorem 9 will then follow from the following two lemmas. (details omitted due to space restrictions – see the appendix)

**Lemma 10.** If Adversary is allowed to choose for the vertices x, y in the edge gadget (Figure 3) the same colour from  $\{1, 2, ..., n\}$ , then Adversary wins. If Adversary is allowed to choose n + 1 for x or y, then Adversary also wins.

In all other cases, Prover wins.

**Proof.** If Prover offers n + 1 for x or y, then Adversary can choose this colour (for x or y) and Prover immediately loses, since both x and y are adjacent to  $u_1$  which is assumed to be assigned the colour n + 1. (Prover loses since the colouring is not proper.)

Assume that *x* and *y* are assigned the same colour *i* from  $\{1, 2, ..., n\}$ . We describe a winning strategy for Adversary. Consider the set of *n* colours Prover offers for *w*. Since the colours are distinct and there is *n* of them, at least one of the colours, denote it *k*, is different from *i* and each of n + 1, n + 4, n + 5, ..., 2n. Adversary chooses the colour *k* for *w*.

Then consider the *n* colours Prover offers for *q*. If any of the *n* colours, denote it *j*, is from  $\{1, 2, ..., n\}$ , then Adversary chooses the colour *j* for *q*, which makes Prover lose when considering the vertex *c* where Prover must offer *n* values different from n + 1, n + 2, ..., 2n and from  $j \in \{1, 2, ..., n\}$ , which is impossible. (Note that *c* is adjacent to *q* as well as  $u_1, u_2, ..., u_n$ .)

Therefore we may assume that Prover offers the set  $\{n + 1, n + 2, ..., 2n\}$  for q. Let  $\ell$  be any colour in the set  $\{n + 2, n + 3\} \setminus \{k\}$ . By definition,  $\ell$  is different from k, and clearly also different from i, since  $i \in \{1, 2, ..., n\}$  while  $\ell \in \{n + 2, n + 3\}$ . We make Adversary choose the colour  $\ell$  for q.

Now if Prover offers for z a colour, denote it r, different from i, k and each of n + 1, n + 4, n + 5, ..., 2n, then Adversary chooses this colour and Prover loses at a when she has to provide n colours distinct from i, k, r, n + 1, n + 4, n + 5, ..., 2n, which is impossible. Similarly, Prover loses, this time at b, if she offers for z a colour different from  $i, k, \ell, n + 4, n + 5, ..., 2n$ . Notice that there is no set of n values that excludes both these situations, since  $i, k, \ell, n + 1, n + 4, ..., 2n$  are distinct values. This shows that Adversary wins no matter what Prover does.

Now, for the second part of the claim, assume that *x* and *y* are either given distinct colours different from n + 1, or same colours from  $\{n + 2, n + 3, ..., 2n\}$ . This time Prover wins no matter what Adversary does.

First, assume that *x* and *y* have distinct colours *i* and *j*, respectively where  $i, j \neq n + 1$ . We consider three cases.

**Case 1**: assume that  $j \in \{n + 4, n + 5, ..., 2n\}$ . Then we have Prover offer for the vertices w and q the set  $\{n + 1, n + 2, ..., 2n\}$ . Let k be the colour chosen by Adversary for w, and let  $\ell$  be the colour chosen for q.

If  $\{i, k, n + 1, n + 4, n + 5, ..., 2n\}$  contains *n* distinct elements, we have Prover offer this set for *z*; otherwise Prover offers any set of *n* distinct elements for *z*. Let *r* be the colour chosen for *z*. Now Prover offers for *a* any set of *n* colours disjoint from  $\{i, k, r, n + 1, n + 4, ..., 2n\}$ , which is possible since  $r \in \{i, k, n + 1, n + 4, ..., 2n\}$ . For *b* Prover offers any set of *n* colours disjoint from  $\{j, k, \ell, r, n + 4, ..., 2n\}$ , which is again possible because  $j \in \{n + 4, ..., 2n\}$ . Finally, Prover offers  $\{1, 2, ..., n\}$  for *c*. It is now easy to see that any choice of Adversary yields a proper colouring and so Prover wins, as claimed.

**Case 2**: assume that  $i \in \{n + 4, n + 5, ..., 2n\}$ . We similarly have Prover offer  $\{n + 1, n + 2, ..., 2n\}$  for both w and q, and let k and  $\ell$  be the colours chosen by Adversary for the two vertices. If  $\{j, k, \ell, n + 4, n + 5, ..., 2n\}$  contains n distinct elements, Prover offers this set for z; otherwise Prover offers any set of n distinct elements. Just like in Case 1.1, this now allows us to choose n distinct colours for each of a, b, c so that none of the colours appears on their neighbours. So again, for any choice of Adversary, Prover wins as required.

**Case 3**: assume that  $i, j \notin \{n + 4, n + 5, ..., 2n\}$ . Recall that  $i, j \neq i + 1$  and  $i \neq j$ . Thus we have Prover offer for w the set  $\{n + 1, i, j, n + 4, ..., 2n\}$  and for q the set  $\{n + 1, ..., 2n\}$ . Let k be the colour chosen by Adversary for w, and let  $\ell$  be the colour chosen for q.

Suppose first that  $k \in \{i, n + 1, n + 4, ..., 2n\}$ . If  $\{j, k, \ell, n + 4, ..., 2n\}$  contains *n* distinct elements, Prover offers this set for *z*; otherwise, she offers any set for *z*. Again, for each of *a*, *b*, *c*, there are at most *n* colours used on their neighbours and so Prover can offer each of *a*, *b*, *c* a set of *n* colours distinct from their neighbours to get a proper colouring for any choice of Adversary.

So we may assume that k = j. In this case, we have Prover offer the set  $\{i, n + 1, n + 4, ..., 2n\}$  for z. Again, for each of a, b, c we have n colours distinct from their neighbours and we can thus complete a proper colouring regardless of Adversary's choices. Thus Prover wins in any situation.

That exhausts all possibilities for when x, y have distinct colours different from n + 1. To finish the proof, it remains to consider the case when x, y have the same colour i from  $\{n + 2, n + 3, ..., 2n\}$  In this case,

Prover offers for the vertices w, q, z the set  $\{n + 1, n + 2, ..., 2n\}$ , while for a, b, c Prover offers the set  $\{1, 2, ..., n\}$ . It is easy to see that any choice of Adversary yields a proper colouring. Thus Prover wins as required.

That concludes the proof.

**Lemma 11.**  $\Phi$  is a yes-instance of  $\{n\}$ -*CSP*( $\mathbb{K}_{2n}$ ) if and only if  $\Psi$  is a yes-instance of QCSP( $\mathbb{K}_n$ ).

**Proof.** We treat the colours n + 2, n + 3, ..., 2n as *don't care* colours, while 1, 2, ..., n will be the actual colours used for colouring *G*. By Lemma 10, the edge gadget makes sure that vertices x, y do not receive the same colours unless at least one of the colours is from  $\{n + 2, n + 3, ..., 2n\}$  (the don't-care colours). This implies that  $\Phi$  correctly simulates  $\Psi$  whereby Prover offers  $\{1, 2, ..., n\}$  for each  $\forall$  variable of  $\Psi$ , and offers  $\{i, n + 2, n + 3, ..., 2n\}$  for each  $\exists$  variable of  $\Psi$  where  $i \in \{1, 2, ..., n\}$ . Note that the construction forces Prover to offer  $\{1, 2, ..., n\}$  for each  $\exists$  variable, while for each  $\exists$  variable Prover must offer n values excluding the value n + 1. In the latter case we may assume that the set of offered values is of the form  $\{i, n + 2, n + 3, ..., 2n\}$ , where  $i \in \{1, 2, ..., n\}$ , since offering more values from  $\{1, 2, ..., n\}$  makes it even easier for Adversary to win (has more choices to force a monochromatic edge).

Thus this shows that  $\Phi$  indeed correctly simulates  $\Psi$  as required.

We finish the proof by remarking that the construction of  $\Phi$  is polynomial in the size of  $\Psi$ . Thus, since QCSP( $\mathbb{K}_n$ ) is Pspace-hard, so is  $\{n\}$ -CSP( $\mathbb{K}_{2n}$ ).

This completes the proof of Theorem 9.

### 4. Algorithm for $\{1, 2\}$ -CSP( $\mathbb{P}_{\infty}$ )

We consider the infinite path  $\mathbb{P}_{\infty}$  to be the graph whose vertex set is  $\mathbb{Z}$  and edges are  $\{ij : |i-j|=1\}$ . In this section, we prove the following theorem.

**Theorem 12.**  $\{1,2\}$ -*CSP*( $\mathbb{P}_{\infty}$ ) *is decidable in polynomial time.* 

An instance to  $\{1,2\}$ -CSP( $\mathbb{P}_{\infty}$ ) is a graph  $G = \mathcal{D}_{\psi}$ , a total order  $\prec$  on V(G), and a function  $\beta : V(G) \rightarrow \{1,2\}$  where

$$\Psi := \exists^{\geq \beta(v_1)} v_1 \exists^{\geq \beta(v_2)} v_2 \cdots \exists^{\geq \beta(v_n)} v_n \bigwedge_{v_i v_j \in E(G)} E(v_i, v_j)$$

### 4.1. Definitions

We write  $X \prec Y$  if  $x \prec y$  for each  $x \in X$  and each  $y \in Y$ . Also, we write  $x \prec Y$  in place of  $\{x\} \prec Y$ . A walk of *G* is a sequence  $x_1, x_2, \ldots, x_r$  of vertices of *G* where  $x_i x_{i+1} \in E(G)$  for all  $i \in \{1, \ldots, r-1\}$ . A walk  $x_1, \ldots, x_r$  is a *closed walk* if  $x_1 = x_r$ . Write |Q| to denote the *length* of the walk *Q* (number of edges on *Q*).

**Definition 3.** If  $Q = x_1, ..., x_r$  is a walk of *G*, we define  $\lambda(Q)$  as follows:

$$\lambda(Q) = |Q| - 2\sum_{i=2}^{r-1} (\beta(x_i) - 1)$$

**Definition 4.** A walk  $x_1, \ldots, x_r$  of *G* is a looping walk if  $x_1 \neq x_r$  and if  $r \geq 3$ 

- (i)  $\{x_1, x_r\} \prec \{x_2, \ldots, x_{r-1}\}$ , and
- (ii) there is  $\ell \notin \{1, r\}$  such that both  $x_1, \ldots, x_\ell$  and  $x_\ell, \ldots, x_r$  are looping walks.

The above is a recursive definition. (The base case is when r = 2 and  $x_1, x_2$  are two distinct adjacent vertices.) Observe that endpoints of a looping walk are always distinct and never appear in the interior of the walk. Other vertices, however, may appear on the walk multiple times as long as the walk obeys (ii). Notably, it is possible that the same vertex is one of  $x_2, \ldots, x_{\ell-1}$  as well as one of  $x_{\ell-1}, \ldots, x_{r-1}$  where  $\ell$  is as defined in (ii). See Figure 4 for examples.

Using looping walks, we define a notion of "distance" in G that will guide Prover in the game.

**Definition 5.** For vertices  $u, v \in V(G)$ , define  $\delta(u, v)$  to be the following:

 $\delta(u, v) = \min \left\{ \lambda(Q) \mid Q = x_1, \dots, x_r \text{ is a looping walk of } G \text{ where } x_1 = u \text{ and } x_r = v \right\}.$ If no looping walk between u and v exists, define  $\delta(u, v) = \infty$ .

Namely,  $\delta(u, v)$  denotes the smallest  $\lambda$ -value of a looping walk between u and v. Note that  $\delta(u, v) = \delta(v, u)$ , since the definition of a looping walk does not prescribe the order of the endpoints of the walk.

The main structural obstruction in our characterization is the following.

**Definition 6.** A bad walk of G is a looping walk  $Q = x_1, \ldots, x_r$  of G such that  $x_1 \prec x_r$  and  $\lambda(Q) \leq \beta(x_r) - 2$ .



Figure 4: Examples of looping walks.

# 4.2. Characterization

**Theorem 13.** Suppose that G is a bipartite graph. Then the following statements are equivalent.

- (1)  $\mathbb{P}_{\infty} \models \Psi$
- (II) Prover has a winning strategy in  $\mathscr{G}(\Psi, \mathbb{P}_{\infty})$ .
- (III) Prover can play  $\mathscr{G}(\Psi, \mathbb{P}_{\infty})$  so that in every instance of the game, the resulting mapping f satisfies the following for all  $u, v \in V(G)$  with  $\delta(u, v) < \infty$ :

$$|f(u) - f(v)| \le \delta(u, v), \tag{(\star)}$$

$$f(u) + f(v) + \delta(u, v)$$
 is an even number.  $(\triangle)$ 

- (IV) There are no  $u, v \in V(G)$  where  $u \prec v$  such that  $\delta(u, v) \leq \beta(v) 2$ .
- (V) There is no bad walk in G.

# 4.3. Proof of Theorem 13

We prove the claim by considering individual implications. The equivalence (I) $\Leftrightarrow$ (II) is proved as Lemma 6. The equivalence (IV) $\Leftrightarrow$ (V) follows immediately from the definitions of  $\delta(\cdot, \cdot)$  and bad walk. The other implications are proved as follows. For (III) $\Rightarrow$ (II), we show that Prover's strategy described in (III) is a winning strategy. For (II) $\Rightarrow$ (III), we show that every winning strategy must satisfy the conditions of (III). For (III) $\Rightarrow$ (IV), we show that having vertices  $u \prec v$  with  $\delta(u, v) \leq \beta(v) - 2$  allows Adversary to win, by playing along the bad walk defined by vertices u, v. Finally, for (IV) $\Rightarrow$ (III), assuming no bad pair u, v, we describe a Prover's strategy satisfying (III).

Before the proof of the remaining implications, we need to show some properties of the function  $\delta(\cdot, \cdot)$ .

**Lemma 14.** Suppose that G is a bipartite graph. If Q is a looping walk of G between u and v, then  $\lambda(Q) + \delta(u, v)$  is an even number.

**Proof.** Let  $Q = x_1, \ldots, x_r$  be a looping walk of G with  $x_1 = u$  and  $x_r = v$ . By definition  $\delta(u, v) \le \lambda(Q) < \infty$ . So there exists a looping walk  $Q' = x'_1, \ldots, x'_s$  with  $x'_1 = u$  and  $x'_s = v$  such that  $\delta(u, v) = \lambda(Q')$ . We calculate:

$$\lambda(Q) + \delta(u, v) = \lambda(Q) + \lambda(Q') = |Q| - 2\sum_{i=2}^{r-1} (\beta(x_i) - 1) + |Q'| - 2\sum_{i=2}^{s-1} (\beta(x'_i) - 1)$$

It follows that  $\lambda(Q) + \delta(u, v)$  is even if and only if |Q| + |Q'| is. Recall that  $x_1 = x'_1 = u$  and  $x_r = x'_s = v$ . Thus  $Q'' = x_1, x_2, \dots, x_{r-1}, x'_s, x'_{s-1}, \dots, x'_1$  is a closed walk of *G* of length |Q| + |Q'|. Since *G* is bipartite, it contains no closed walk of odd length. Thus |Q| + |Q'| is even, and so is  $\lambda(Q) + \delta(u, v)$ .

The following can be viewed as a triangle inequality for  $\delta(\cdot, \cdot)$ .

**Lemma 15.** If  $u, v, w \in V(G)$  satisfy  $u \neq v$  and  $\{u, v\} \prec w$ , then  $\delta(u, v) \leq \delta(u, w) + \delta(v, w) - 2\beta(w) + 2$ . Moreover, if G is a bipartite graph, then  $\delta(u, v) + \delta(u, w) + \delta(v, w)$  is an even number or  $\infty$ .

**Proof.** If  $\delta(u, w) = \infty$  or  $\delta(v, w) = \infty$ , the claim is clearly true. So we may assume that  $\delta(u, w) < \infty$  and  $\delta(v, w) < \infty$ . This means that there exists a looping walk  $Q = x_1, \ldots, x_r$  with  $x_1 = u$  and  $x_r = w$  where  $\lambda(Q) = \delta(u, w)$ , and also a looping walk  $Q' = x'_1, \ldots, x'_s$  with  $x'_1 = v$  and  $x'_s = w$  where  $\delta(v, w) = \lambda(Q')$ .

Note that  $x_r = x'_s = w$ , and define  $Q'' = x_1, x_2, ..., x_{r-1}, x'_s, x'_{s-1}, ..., x'_1$ . We calculate:

$$\begin{split} \lambda(Q) + \lambda(Q') &= |Q| - 2\sum_{i=2}^{r-1} \left(\beta(x_i) - 1\right) + |Q'| - 2\sum_{i=2}^{s-1} \left(\beta(x_i') - 1\right) \\ &= |Q| + |Q'| - 2\sum_{i=2}^{r-1} \left(\beta(x_i) - 1\right) - 2(\beta(w) - 1) - 2\sum_{i=2}^{s-1} \left(\beta(x_i') - 1\right) + 2(\beta(w) - 1) \\ &= \lambda(Q'') + 2\beta(w) - 2 \end{split}$$

Observe that Q'' is a walk of G whose endpoints are u and v. We verify that Q'' is in fact a looping walk of G. We need to check the conditions (i)-(iii) of the definition. For (i), we recall the assumption  $u \neq v$ . For (ii), we note that  $\{x_1, x_r\} \prec \{x_2, \ldots, x_{r-1}\}$  since Q is a looping walk. Similarly,  $\{x'_1, x'_s\} \prec \{x'_2, \ldots, x'_{s-1}\}$  since Q' is a looping walk. We also recall the assumptions  $\{u, v\} \prec w$ . Thus, since  $w = x_r = x'_s$ , we conclude that  $\{u, v\} \prec w \prec \{x_2, \ldots, x_{r-1}, x'_{2'}, \ldots, x'_{s-1}\}$  which shows (ii). Finally, (iii) follows from the fact that both Q and Q' are looping walks. This verifies that Q'' indeed is a looping walk of G between u and v.

So we have  $\delta(u, v) \leq \lambda(Q'')$  by definition, and we can calculate:

$$\delta(u,w) + \delta(v,w) = \lambda(Q) + \lambda(Q') = \lambda(Q'') + 2\beta(w) - 2 \ge \delta(u,v) + 2\beta(w) - 2$$

Thus  $\delta(u, v) \leq \delta(u, w) + \delta(v, w) - 2\beta(w) + 2$  as claimed.

Now, assume that *G* is a bipartite graph. Recall that  $\lambda(Q) + \lambda(Q') = \lambda(Q'') + 2\beta(w) - 2$ . This implies that  $\lambda(Q) + \lambda(Q') + \lambda(Q'')$  is an even number. By Lemma 14 also  $\lambda(Q) + \delta(u, w)$  and  $\lambda(Q') + \delta(v, w)$  are even, since *G* is assumed to be bipartite. For the same reason  $\lambda(Q'') + \delta(u, v)$  is even. So since  $\lambda(Q) + \lambda(Q'')$  is even, it follows that  $\delta(u, v) + \delta(u, w) + \delta(v, w)$  is even, as claimed.

**Proof.** [Theorem 13 (III) $\Rightarrow$ (II)] Assume (III), namely that Prover can play so as to satisfy ( $\star$ ) and ( $\triangle$ ) in every instance of the game. We show that this is a winning strategy for Prover, thus proving (II).

To this end, we need to verify that in every instance of the game the resulting mapping f is a homomorphism of G to  $\mathbb{P}_{\infty}$ . Namely, we verify that for every edge  $uv \in E(G)$ , the mapping f satisfies |f(u) - f(v)| = 1.

Consider an edge  $uv \in E(G)$ . Observe that Q = u, v is a looping walk of G with  $\lambda(Q) = 1$ . Thus  $\delta(u, v) \leq \lambda(Q) = 1$  by the definition of  $\delta(u, v)$ . Using Lemma 14, we deduce that  $\lambda(Q) + \delta(u, v) = 1 + \delta(u, v)$  is even. Thus  $\delta(u, v)$  is odd. By  $(\triangle)$ , we have that  $f(u) + f(v) + \delta(u, v)$  is even. Thus f(u) + f(v) is odd,

since  $\delta(u, v)$  is. Further, by (\*), we observe that  $|f(u) - f(v)| \le \delta(u, v)$ . Thus  $|f(u) - f(v)| \le 1$ , since  $\delta(u, v) \le \lambda(Q) = 1$ . Also  $|f(u) - f(v)| \ge 1$ , since  $|f(u) - f(v)| \ge 0$  and f(u) + f(v) is odd. Thus together we conclude that |f(u) - f(v)| = 1 as required.

**Proof.** [Theorem 13 (II) $\Rightarrow$ (III)] Assume (II), i.e. Prover has a winning strategy. Assume that Prover plays this strategy. Then no matter how Adversary plays, Prover always wins. We show that this strategy satisfies the conditions of (III).

For contradiction, suppose that there is an instance of the game for which the conditions of (III) do not hold. Let g denote the resulting mapping produced by this instance. Let us play the game again to produce a new mapping f by making Adversary play according to the following rules:

- 1. When a vertex v is considered, check if the set  $S_v$  that Prover offers for v contains  $\gamma \in S_v$  for which there exists  $u \prec v$  with  $\delta(u, v) < \infty$  such that  $|f(u) \gamma| > \delta(u, v)$  or such that  $f(u) + \gamma + \delta(u, v)$  is odd.
- 2. If such  $\gamma$  exists, choose  $f(v) = \gamma$ . If such  $\gamma$  does not exist and f(u) = g(u) for all  $u \prec v$ , then choose f(v) = g(v). If neither is possible, then choose any value from  $S_v$  for f(v).

It follows that f does not satisfy the conditions of (III) much like g. Indeed, if in step 2. we find that  $\gamma$  exists, then setting  $f(v) = \gamma$  makes ( $\star$ ) or ( $\triangle$ ) fail for the pair u, v. Thus f fails the conditions of (III). Otherwise, if  $\gamma$  in step 2. never exists, then we conclude that f = g, and hence, f again fails the conditions of (III), since g does. (For this to hold, it is important to note that this is only possible because Prover plays the same deterministic strategy in both instances of the game; thus Prover will offer the same values for f(v) as she did for g(v) as long as Adversary makes the same choices for f as he did for g; i.e., as long as f(u) = g(u) for all vertices  $u \prec v$ .)

Since Prover plays a winning strategy, we must conclude that f is a homomorphism of G to  $\mathbb{P}_{\infty}$ . Namely, we have that each  $uv \in E(G)$  satisfies |f(u) - f(v)| = 1. We show that this leads to a contradiction. From this we will conclude that g does not exist, and hence, Prover's winning strategy satisfies (III) as we advertised earlier.

We say that a vertex v is good if for all  $u \prec v$  with  $\delta(u, v) < \infty$  the conditions ( $\star$ ) and ( $\triangle$ ) hold. Otherwise, we say that v is *bad*. The following is a restatement of Adversary's strategy as described above.

(+) A vertex v is good if and only if

every  $\gamma \in$ 

$$S_v$$
 and every  $u \prec v$  with  $\delta(u, v) < \infty$  are such that  
 $|f(v) - \gamma| \leq \delta(u, v)$  and  $f(v) + \gamma + \delta(u, v)$  is even.

To see this, observe that if v is bad, then, by definition, the statement on the right fails for  $\gamma = f(v) \in S_v$ . Conversely, if for some  $\gamma \in S_v$  there is  $u \prec v$  with  $\delta(u, v) < \infty$  such that  $|f(v) - \gamma| > \delta(u, v)$  or such that  $f(v) + \gamma + \delta(u, v)$  is odd, Adversary chooses  $f(v) = \gamma$  in step 2. Thus v is not a good vertex. This proves (+).

Since *f* fails the conditions of (III), there exists at least one bad vertex. Among all bad vertices, choose *v* to be the bad vertex that is largest with respect to  $\prec$ .

Since v is bad, there exists  $u \prec v$  with  $\delta(u, v) < \infty$  such that  $|f(u) - f(v)| > \delta(u, v)$ , or such that  $f(u) + f(v) + \delta(u, v)$  is odd. In particular, since  $\delta(u, v) < \infty$ , there exists a looping walk  $Q = x_1, \ldots, x_r$  with  $x_1 = u$  and  $x_r = v$  such that  $\lambda(Q) = \delta(u, v)$ . Clearly,  $r \ge 2$  by definition.

Suppose first that r = 2. Then Q = u, v and  $\lambda(Q) = 1$ . In particular,  $uv \in E(G)$  and  $\delta(u, v) = \lambda(Q) = 1$ . Recall that we assume that  $|f(u) - f(v)| > \delta(u, v)$  or that  $f(u) + f(v) + \delta(u, v)$  is odd. If  $|f(u) - f(v)| > \delta(u, v)$ , then |f(u) - f(v)| > 1, since  $\delta(u, v) = 1$ . But then, since  $uv \in E(G)$ , we have that f is not a homomorphism, a contradiction. So  $f(u) + f(v) + \delta(u, v)$  must be odd. Thus f(u) + f(v) is even, because  $\delta(u, v) = 1$ . But then  $|f(u) - f(v)| \neq 1$  again contradicting our assumption that f is a homomorphism.

This excludes the case r = 2. Thus we may assume  $r \ge 3$ . Since Q is a looping walk, this implies that there exists  $\ell \in \{2, ..., r-1\}$  such that both  $Q_1 = x_1, ..., x_\ell$  and  $Q_2 = x_\ell, ..., x_r$  are looping walks of G.

Let us denote  $w = x_{\ell}$ . So  $Q_1$  is a looping walk from u to w, while  $Q_2$  is a looping walk from w to v. This implies that  $\delta(u, w) \leq \lambda(Q_1)$  and  $\delta(v, w) \leq \lambda(Q_2)$ . Note that  $u \prec v \prec w$ , since  $\{u, v\} \prec \{x_2, \ldots, x_{r-1}\}$  because Q is a looping walk. Thus by the maximality of v, we deduce that w is a good vertex. We calculate:

$$\begin{split} \lambda(Q) &= |Q| - 2\sum_{i=2}^{r-1} \left(\beta(x_i) - 1\right) \\ &= |Q_1| + |Q_2| - 2\sum_{i=2}^{\ell-1} \left(\beta(x_i) - 1\right) - 2(\beta(w) - 1) - 2\sum_{i=\ell+1}^{r-1} \left(\beta(x_i) - 1\right) \\ &= \lambda(Q_1) + \lambda(Q_2) - 2\beta(w) + 2 \ge \delta(u, w) + \delta(v, w) - 2\beta(w) + 2 \ge \delta(u, v) = \lambda(Q) \end{split}$$

The last inequality is by Lemma 15. Therefore, we conclude that  $\delta(u, v) = \delta(u, w) + \delta(v, w) - 2\beta(w) + 2$ .

Recall that we assume that  $|f(u) - f(v)| > \delta(u, v)$  or that  $f(u) + f(v) + \delta(u, v)$  is odd. We can exclude the latter as follows. Since w is a good vertex, both  $f(u) + f(w) + \delta(u, w)$  and  $f(v) + f(w) + \delta(v, w)$  are even by  $(\triangle)$ . This implies that  $f(u) + f(v) + \delta(u, w) + \delta(v, w)$  is even. Therefore,  $f(u) + f(v) + \delta(u, v)$  is even, since  $\delta(u, v) = \delta(u, w) + \delta(v, w) - 2\beta(w) + 2$ . In other words,  $f(u) + f(v) + \delta(u, v)$  is not odd, so the remaining possibility, by our assumptions, is that  $|f(u) - f(v)| > \delta(u, v)$ . In particular, since  $f(u) + f(v) + \delta(u, v)$  is even, it follows that  $|f(u) - f(v)| \neq \delta(u, v) + 1$ . Thus  $|f(u) - f(v)| \geq \delta(u, v) + 2$ . Note that  $\beta(w) \in \{1, 2\}$ .

Now, recall that w is a good vertex. Thus by (+) we have that every  $\gamma \in S_w$  satisfies  $|f(u) - \gamma| \le \delta(u, w)$  and  $|f(v) - \gamma| \le \delta(v, w)$ . Using this we can calculate:

$$|f(u) - f(v)| \ge \delta(u, v) + 2 = \delta(u, w) + \delta(v, w) - 2\beta(w) + 2 + 2$$
  
$$\ge |f(u) - \gamma| + |f(v) - \gamma| - 2\beta(w) + 4 \ge |f(u) - f(v)| - 2\beta(w) + 4$$

This implies that  $\beta(w) = 2$  and all the above inequalities are in fact equalities. Namely, the set  $S_w$  contains distinct values  $\gamma_1, \gamma_2$  which satisfy  $|f(u) - \gamma_1| = \delta(u, w) = |f(u) - \gamma_2|$  and  $|f(v) - \gamma_1| = \delta(v, w) = |f(v) - \gamma_2|$ . Since  $\gamma_1 \neq \gamma_2$ , it follows that  $\delta(u, w) \ge 1$  and  $f(u) - \gamma_1 = -f(u) + \gamma_2$ . Similarly, we deduce  $\delta(v, w) \ge 1$  and  $f(v) - \gamma_1 = -f(v) + \gamma_2$ . Thus  $2f(u) = \gamma_1 + \gamma_2 = 2f(v)$  which yields f(u) = f(v). We again calculate:

$$|f(u) - f(v)| \ge \delta(u, v) + 2 = \delta(u, w) + \delta(v, w) - 2\beta(w) + 2 + 2 \ge 1 + 1 - 2 \cdot 2 + 2 + 2 = 2$$

But this is clearly impossible, since f(u) = f(v).

The proof is now complete.

**Proof.** [Theorem 13 (III) $\Rightarrow$ (IV)] We prove the contrapositive. Assume that (IV) fails. Namely, suppose that there are  $u, v \in V(G)$  with  $u \prec v$  such that  $\delta(u, v) \leq \beta(v) - 2$ . We show that Adversary can play to violate (\*) for u, v. This will imply that (III) fails. If  $\delta(u, v) \leq -1$ , then (\*) can never be satisfied, since |f(u) - f(v)| is always non-negative. In that case (III) fails no matter how Adversary plays. So we may assume that  $\delta(u, v) \geq 0$ . This implies that  $\delta(u, v) = 0$  and  $\beta(v) = 2$ , since we assume  $\delta(u, v) \leq \beta(v) - 2$ . So Prover must offer to Adversary two distinct values for v. Since the values are different, at least one of them must be different from f(u). Adversary chooses this value for f(v). This yields  $|f(u) - f(v)| \geq 1$  which violates (\*), since  $\delta(u, v) = 0$ .

This shows that Prover cannot play to always satisfy the conditions of (III), and hence (III) fails as claimed.  $\hfill\square$ 

**Proof.** [Theorem 13 (IV) $\Rightarrow$ (III)] Assume (IV), namely that all  $u, v \in V(G)$  with  $u \prec v$  satisfy  $\delta(u, v) \ge \beta(v) - 1$ . We explain how Prover can play so as to satisfy the conditions of (III). We proceed by induction on the number of processed vertices during the game. As usual let f denote the (partial) assignment constructed by Adversary.

Consider a vertex  $v \in V(G)$  and assume that  $(\star)$  and  $(\triangle)$  hold for all  $u, w \in V(G)$  such that  $\{u, w\} \prec v$ . We show how to construct a set  $S_v$  so that whatever the choice Adversary makes from this set for f(v), it will satisfy  $(\star)$  and  $(\triangle)$  for v and all  $u \in V(G)$  with  $u \prec v$ . This will yield a strategy (for Prover) satisfying (III). Note that  $\delta(u, v) \ge 0$  for all  $u \prec v$ , since  $\delta(u, v) \ge \beta(v) - 1$  by our assumption, and  $\beta(v) \in \{1, 2\}$ .

If v is the first vertex in  $\prec$  or if every  $u \prec v$  is such that  $\delta(u, v) = \infty$ , then there is no condition we need to satisfy; Prover simply offers any set  $S_v \subseteq \mathbb{Z}$ . So we may assume that this is not the case.

For each  $u \prec v$  with  $\delta(u, v) < \infty$ , we write  $\mathcal{I}_u$  to denote the closed interval defined as follows.

$$\mathcal{I}_{u} = \left[f(u) - \delta(u, v), f(u) + \delta(u, v)\right]$$

Let  $\mathcal{L}$  be the intersection of all intervals  $\mathcal{I}_u$  where  $u \prec v$  and  $\delta(u, v) < \infty$ . Observe that in order to satisfy (\*) we must choose  $S_v$  to be a subset of  $\mathcal{L}$ . We show that this is possible while also satisfying ( $\Delta$ ).

Since  $\mathcal{L}$  is the intersection of intervals, there exist x, y such that  $\mathcal{L} = \mathcal{I}_x \cap \mathcal{I}_y$ . Note that  $\{x, y\} \prec v$  and  $\delta(x,v) < \infty$  as well as  $\delta(y,v) < \infty$ . By symmetry, we may assume that  $f(x) + \delta(x,v) \le f(y) + \delta(y,v)$ .

We prove that  $|\mathcal{L}| \geq 2\beta(v) - 1$ . If  $\mathcal{I}_x \subseteq \mathcal{I}_y$ , then we have  $|\mathcal{L}| = |\mathcal{I}_x \cap \mathcal{I}_y| = |\mathcal{I}_x|$  and we recall that we assume  $\delta(x,v) \geq \beta(v) - 1$ , since  $x \prec v$ . Thus  $|\mathcal{I}_x| = 2\delta(x,v) + 1 \geq 2\beta(v) - 1$  and so  $|\mathcal{L}| \geq 2\beta(v) - 1$  as claimed. Similarly, if  $\mathcal{I}_y \subseteq \mathcal{I}_x$ , we conclude that  $|\mathcal{L}| = |\mathcal{I}_y|$  and  $|\mathcal{I}_y| \ge 2\beta(v) - 1$  by our assumption.

Thus we may assume that  $\mathcal{I}_x \not\subseteq \mathcal{I}_y$  and  $\mathcal{I}_y \not\subseteq \mathcal{I}_x$ . Note that this also implies that  $x \neq y$ . Recall that  $f(x) + \delta(x, v) \leq f(y) + \delta(y, v)$ . We claim that also f(x) < f(y). Indeed, if  $f(y) \leq f(x)$ , then  $f(x) + \delta(x, v) \le f(y) + \delta(y, v) \le f(x) + \delta(y, v)$  which yields  $\delta(x, v) \le \delta(y, v)$ . Thus  $f(y) - \delta(y, v) \le f(y) - \delta(y, v) \le f(y) - \delta(y, v) \le \delta(y, v)$ .  $\delta(x,v) \leq f(x) - \delta(x,v)$ . But this leads to  $\mathcal{I}_x \subseteq \mathcal{I}_y$ , since we also assume  $f(x) + \delta(x,v) \leq f(y) + \delta(y,v)$ .

So we conclude that f(x) < f(y) from which we deduce that  $\mathcal{I}_x \cap \mathcal{I}_y = [f(y) - \delta(y, v), f(x) + \delta(x, v)]$ . Thus  $|\mathcal{I}_x \cap \mathcal{I}_y| = f(x) - f(y) + \delta(y, v) + \delta(x, v) + 1$ . Since f(x) < f(y), we can express this as

$$\mathcal{I}_x \cap \mathcal{I}_y| = \delta(x, v) + \delta(y, v) - |f(x) - f(y)| + 1$$

Recall further that  $\{x, y\} \prec v$  and  $x \neq y$ . Thus  $\delta(x, y) \leq \delta(x, v) + \delta(y, v) - 2\beta(v) + 2$  by Lemma 15. Also, note that  $|f(x) - f(y)| \le \delta(x, y)$ , since we assume that (\*) holds for x, y, because  $\{x, y\} \prec v$ . Therefore

$$\begin{aligned} |\mathcal{L}| &= |\mathcal{I}_x \cap \mathcal{I}_y| = \delta(x, v) + \delta(y, v) - |f(x) - f(y)| + 1\\ &\geq \delta(x, y) + 2\beta(v) - 2 - \delta(x, y) + 1 = 2\beta(v) - 1 \end{aligned}$$

This proves that indeed  $|\mathcal{L}| \geq 2\beta(v) - 1$ .

We are finally ready to construct the set  $S_v$ . Recall that  $\mathcal{L} = \mathcal{I}_x \cap \mathcal{I}_y$  and we assume that  $f(x) + \delta(x, v) \leq \delta(x, v)$  $f(y) + \delta(y, v)$ . This implies that the value

$$\gamma = f(x) + \delta(x, v)$$

is the largest element of  $\mathcal{L}$ . We claim that  $f(v) = \gamma$  satisfies ( $\triangle$ ). Suppose that ( $\triangle$ ) fails for  $f(v) = \gamma$ and some  $u \prec v$  with  $\delta(u, v) < \infty$ . Namely, suppose that  $f(u) + \gamma + \delta(u, v)$  is odd. Thus  $f(u) + f(x) + \delta(u, v)$  $\delta(x, v) + \delta(u, v)$  is odd. This implies that  $u \neq x$ . Note that  $\{x, u\} \prec v$ . Thus from Lemma 15, we obtain that  $\delta(x, u) + \delta(x, v) + \delta(u, v)$  is even. Thus  $f(u) + f(x) + \delta(x, u)$  is odd, since  $f(u) + f(x) + \delta(x, v) + \delta(u, v)$  is. Hence,  $(\triangle)$  fails for x, u. However, we assume that  $(\triangle)$  holds for x, u, since  $\{x, u\} \prec v$ , a contradiction. So no such a *u* exists.

Therefore ( $\triangle$ ) holds for  $f(v) = \gamma$  and every  $u \prec v$  with  $\delta(u, v) < \infty$ . Thus ( $\triangle$ ) also clearly holds for  $f(v) = \gamma - 2$ . In particular, if  $\beta(v) = 2$ , then  $\gamma - 2 \in \mathcal{L}$ . This follows from the fact that  $|\mathcal{L}| \geq 2\beta(v) - 1$ and  $\gamma$  is the largest element in  $\mathcal{L}$ . Finally, recall that each element in  $\mathcal{L}$  can be used as f(v) to satisfy (\*). In particular, (\*) holds for  $f(v) = \gamma$  and if  $\beta(v) = 2$ , it also holds for  $f(v) = \gamma - 2$ , since in that case  $\gamma - 2 \in \mathcal{L}.$ 

This together shows that if  $\beta(v) = 1$ , Prover can safely offer  $S_v = \{\gamma\}$  while if  $\beta(v) = 2$ , Prover can safely offer  $S_v = \{\gamma, \gamma - 2\}$ . Any choice Adversary makes for f(v) from such  $S_v$  is guaranteed to make (\*) and  $(\triangle)$  hold for all  $u \prec v$  with  $\delta(u, v) < \infty$ . This is precisely what we set out to prove. 

The proof is now complete.

With this characterization, the proof of Theorem 12 is straightforward.

### 4.4. Proof of Theorem 12

We observe that the values  $\delta(u, v)$  can be easily computed in polynomial time by dynamic programming. This allows us to test conditions of Theorem 13 and thus decide  $\{1,2\}$ -CSP( $\mathbb{P}_{\infty}$ ) in polynomial time as claimed. 

# 5. Algorithm for $\{1, 2\}$ -CSP( $\mathbb{P}_n$ )

The path  $\mathbb{P}_n$  has vertices  $\{1, 2, \dots, n\}$  and edges  $\{ij : |i-j| = 1\}$ . Let  $\Psi$  be an instance of  $\{1, 2\}$ - $CSP(\mathbb{P}_n)$ . As usual, let G be the graph  $\mathcal{D}_{\psi}$  corresponding to  $\Psi$ , and let  $\prec$  be the corresponding total ordering of V(G).

For simplicity, we shall assume that G is connected and bipartite with white and black vertices forming the bipartition. (If it is not bipartite, there is no solution; if disconnected, we solve the problem independently on each component.) No generality is lost this way.

We start by characterizing small cases as follows.

- **Lemma 16.** Assume  $P_{\infty} \models \Psi$ . Let f be the first vertex in the ordering  $\prec$ . Then (i)  $\mathbb{P}_1 \models \Psi \iff G$  is the single  $\exists^{\geq 1}$  vertex f. (ii)  $\mathbb{P}_2 \models \Psi \iff G$  does not contain  $\exists^{\geq 2}$  vertex except possibly for f. (iii)  $\mathbb{P}_3 \models \Psi \iff all \exists^{\geq 2}$  vertices in G have the same colour. (iv)  $\mathbb{P}_4 \models \Psi \iff all \exists^{\geq 2}$  vertices in G are pairwise non-adjacent except possibly for f.
- (v)  $\mathbb{P}_5 \models \Psi \iff$  there is colour C (black or white) such that each edge xy between two  $\exists^{\geq 2}$  vertices where  $x \prec y$  is such that x has colour C

**Proof.** (i) is clear. For (ii), any  $\exists^{\geq 2}$  vertex other than *f* must be offered both 1 and 2, one of which will violate the parity with respect to f (since G is connected). In all other cases,  $\mathbb{P}_2 \models \Psi$  because  $\mathcal{D}_{\psi}$  is bipartite.

Similar argument works for (iii), if there are two  $\exists^{\geq 2}$  vertices u, v of different colour where  $u \prec v$ , then Prover must offer 1 or 3 among the values for *u* and Adversary chooses it. She also must offer 1 or 3 among the values for v and again, Adversary chooses it. This now violates the parity, since u and v are in different sides of the bipartition of *G*. Conversely, if all  $\exists^{\geq 2}$  vertices are, say black, then Prover offers  $\{1,3\}$  to all black  $\exists^{\geq 2}$  vertices, 1 to all black  $\exists^{\geq 1}$  vertices, and 2 to all white vertices. This will allow Prover to win.

For (iv), if G contains adjacent  $\exists^{\geq 2}$  vertices u, v distinct from f where  $u \prec v$ , then Prover must offer  $\{1,3\}$  or  $\{2,4\}$  for u because of the parity with respect to f. Adversary chooses either 1 or 4 and Prover subsequently loses at v. Conversely, if no such vertices u, v exist, Prover first offers  $\{2, 3\}$  for f. By symmetry of the path  $\mathbb{P}_4$ , we may assume that Advesary chooses 2 for f, and that f is black. Prover subsequently offers 2 for each black  $\exists \geq 1$  vertex, and offers 3 for each white  $\exists \geq 1$  vertex. She offers  $\{2,4\}$  for each  $\exists \geq 2$  black vertex and offers  $\{1,3\}$  for each white  $\exists^{\geq 2}$  vertex. Since no two  $\exists^{\geq 2}$  vertices are adjacent, any Adversary's choices lead to a homomorphism, and so Prover always wins.

Finally, for (v), by symmetry, assume that f is black. Suppose that G contains edges xy and wz where x, y, w, z are  $\exists^{\geq 2}$  vertices with  $x \prec y$  and  $w \prec z$ , and where x is black and w is white (possibly f = x). If 2 or 4 is chosen for f, then  $\{1,3\}$  or  $\{1,5\}$  or  $\{3,5\}$  is offered for w because of the parity (f is black and w is white). This allows Adversary to choose 1 or 5 for w and Prover loses at z. If one of 1, 3, 5 is chosen for f, then 1 or 5 is among values offered for x because of parity. So Adversary can choose 1 or 5 for x and Prover loses at y.

Conversely, suppose first that every edge xy where x, y are  $\exists^{\geq 2}$  vertices is such that x is black. Prover offers {2,4} for *f* and for every black  $\exists^{\geq 2}$  vertex. She offers 3 for each white  $\exists^{\geq 1}$  vertex. Each white  $\exists^{\geq 2}$ vertex will be offered  $\{1,3\}$  if 2 was chosen for all its predecessors, or  $\{3,5\}$  if 4 was chosen on all its predecessors. This is always possible as guaranteed by Theorem 13(III) and (IV), since otherwise  $\mathbb{P}_{\infty}$  does not model  $\Psi$ , let alone  $\mathbb{P}_5$ . By the same token, it follows that no black  $\exists^{\geq 1}$  vertex will have two predecessors for which 1 and 5 respectively was chosen. So one of 2,4 can be always successfully offered for each black  $\exists^{\geq 1}$  vertex. This shows that Prover indeed always wins.

Thus we may assume that every edge xy where x, y are  $\exists^{\geq 2}$  vertices is such that x is white. Here Prover offers 3 or  $\{3,5\}$  for f. Then we proceed exactly as in the previous case, just switching colours black and white. 

We now expand this lemma to the general case of  $\{1,2\}$ -CSP( $\mathbb{P}_n$ ) as follows. Recall that we proved that  $\mathbb{P}_{\infty} \models \Psi$  if and only if Prover can play  $\mathscr{G}(\Psi, \mathbb{P}_n)$  so that in every instance of the game, the resulting mapping f satisfies ( $\star$ ) and ( $\triangle$ ). In fact the proof of (III) $\Rightarrow$ (II) from Theorem 13 shows that every winning strategy of Prover has this property. We use this fact in the subsequent text.

We separately investigate the case when n is odd and when n is even.

### 5.1. Even case

In the following definition, we describe for each vertex v the value  $\gamma(v)$  using a recursive definition. We shall use this value to keep track of the distance of f(v) from the center of the path  $\mathbb{P}_n$ .

**Definition 7.** For each vertex v we define  $\gamma(v)$  recursively as follows:

$$\begin{array}{l} \gamma(v) = 0 \quad \text{if } v \text{ is first in the ordering } \prec \\ else \quad \gamma(v) = \beta(v) - 1 + \max\left\{0, \max_{u \prec v}\left(\gamma(u) - \delta(u, v) + \beta(v) - 1\right)\right\}\end{array}$$

**Lemma 17.** Let *M* be real number. Suppose that  $\mathbb{P}_{\infty} \models \Psi$  and that Prover plays a winning strategy in the game  $\mathscr{G}(\Psi, \mathbb{P}_{\infty})$ . Then Adversary can play so that the resulting mapping *f* satisfies  $|f(v) - M| \ge \gamma(v)$  for every vertex  $v \in V(D_{\psi})$ .

**Proof.** As we remarked above the definitions, since Prover plays a winning strategy, she must satisfy  $(\star)$  and  $(\triangle)$ . Adversary when given choice between two values will choose the value that is farther away from *M* (ties broken arbitrarily).

We prove the claim by induction on the number of steps. Consider some step in the game, and let v denote the vertex considered in this step. If v is the first vertex in  $\prec$ , then  $\gamma(v) = 0$  and  $|f(v) - M| \ge 0 = \gamma(v)$  holds. So we may assume that v is not first in  $\prec$ . Thus

$$\gamma(v) = \beta(v) - 1 + \max\left\{0, \max_{u \prec v} \left(\gamma(u) - \delta(u, v) + \beta(v) - 1\right)\right\}$$
(1)

Suppose that no vertex maximizes  $\gamma(v)$ , i.e. we have  $\gamma(v) = \beta(v) - 1$ . If  $\beta(v) = 1$ , then  $\gamma(v) = \beta(v) - 1 = 0$  and so again  $|f(v) - M| \ge 0 = \gamma(v)$ .

So we may assume that  $\beta(v) = 2$  in which case Prover offers two distinct values  $\alpha_1, \alpha_2$ . By symmetry, we may assume  $|M - \alpha_1| \ge |M - \alpha_2|$ . Recall that v is not first in  $\prec$  and  $\mathcal{D}_{\psi}$  is connected. Since the values that Prover offers must satisfy  $(\triangle)$  when used as f(v), we deduce that  $\alpha_1, \alpha_2$  must have the same parity. Indeed, since  $\mathcal{D}_{\psi}$  is connected and v is not first in  $\prec$ , we deduce that there exists a looping walk from u to v where  $u \prec v$ . Then  $(\triangle)$  applied to u and v implies that the values  $\alpha_1, \alpha_2$  have the same parity. Thus  $|\alpha_1 - \alpha_2| \ge 2$ , and since  $|M - \alpha_1| \ge |M - \alpha_2|$ , we deduce that

$$2 \le |\alpha_1 - \alpha_2| \le |M - \alpha_1| + |M - \alpha_2| \le 2|M - \alpha_1|$$

Thus  $|M - \alpha_1| \ge 1$  and Adversary chooses  $f(v) = \alpha_1$ . So, we conclude that  $|f(v) - M| \ge 1 = \beta(v) - 1 = \gamma(v)$  as claimed.

Now, we may assume that there is a vertex  $u \prec v$  that maximizes  $\gamma(v)$ , namely  $\gamma(v) = 2(\beta(v) - 1) + \gamma(u) - \delta(u, v)$ . From the inductive hypothesis, we know that  $|f(u) - M| \ge \gamma(u)$ . Without loss of generality, assume that  $f(u) \ge M$ . From this we calculate

$$f(u) - M = |f(u) - M| \ge \gamma(u) = \gamma(v) + \delta(u, v) - 2(\beta(v) - 1)$$

Now, recall that Prover offers values that satisfy ( $\star$ ) when chosen for f(v). There are two cases.

**Case 1:** assume  $\beta(v) = 1$ . Then Prover offers one value that becomes f(v) where  $|f(u) - f(v)| \le \delta(u, v)$  by (\*). In other words,  $f(u) - \delta(u, v) \le f(v) \le f(u) + \delta(u, v)$  and so we rewrite (since  $\beta(v) - 1 = 0$ )

$$f(v) \ge f(u) - \delta(u, v) \ge f(u) - \delta(u, v) + 2(\beta(v) - 1) \ge M + \gamma(v)$$

from which we conclude  $f(v) - M \ge \gamma(v)$  and therefore  $|f(v) - M| \ge \gamma(v)$ .

**Case 2:** assume  $\beta(v) = 2$ . Then Prover offers two values  $\alpha_1, \alpha_2$  where  $\alpha_1 < \alpha_2$ , and both satisfy (\*) and  $(\triangle)$  in place of f(v). Namely, for i = 1, 2, we have  $f(u) - \delta(u, v) \le \alpha_i \le f(u) + \delta(u, v)$ . From  $(\triangle)$  we know that  $\alpha_1, \alpha_2$  have the same parity. Thus since  $\alpha_1 < \alpha_2$ , we deduce  $\alpha_1 + 2 \le \alpha_2$ . Thus we can can write  $f(u) - \delta(u, v) + 2 \le \alpha_1 + 2 \le \alpha_2$ . Adversary chooses  $f(v) = \alpha_2$  which yields

$$f(v) = \alpha_2 \ge f(u) - \delta(u, v) + 2 = f(u) - \delta(u, v) + 2(\beta(v) - 1) \ge M + \gamma(v)$$

from which we again conclude  $f(v) - M \ge \gamma(v)$  and hence  $|f(v) - M| \ge \gamma(v)$ .

This completes the proof.

**Lemma 18.** Let *M* be a real number. Suppose that  $\mathbb{P}_{\infty} \models \Psi$ . Then there exists a winning strategy for Prover such that in every instance of the game the resulting mapping *f* satisfies  $|f(v) - M| \leq \gamma(v) + 1$  for every  $v \in V(\mathcal{D}_{\psi})$ .

**Proof.** In the desired strategy, Prover offers values closest to *M*. We prove the claim by induction on the number of steps. Since  $\mathbb{P}_{\infty} \models \Psi$ , we conclude that the condition (IV) of Theorem 13 holds. This allows us to follow the induction in the proof of implication (IV) $\Rightarrow$ (III) of said theorem.

In this proof, it is shown that for each vertex v, the set of feasible choices for f(v) forms an interval  $\mathcal{L}$  where either  $\mathcal{L} = \mathbb{R}$  or  $\mathcal{L} = [f(y) - \delta(y, v), f(x) + \delta(x, v)]$  where  $\{x, y\} \prec v$ , possibly x = y. Since (IV) holds, this interval contains at least  $2\beta(v) - 1$  elements and to satisfy ( $\Delta$ ) we may offer for v the value  $f(x) + \delta(x, v)$  or also  $f(x) + \delta(x, v) - 2$  if needed. By the same token we may offer  $f(y) - \delta(y, v)$  or  $f(y) - \delta(y, v) + 2$ , or actually any other value in  $\mathcal{L}$  of the right parity.

**Case 1:** Suppose that  $M \in \mathcal{L}$ . If v is the first vertex in  $\prec$ , then we have Prover offer for v the value  $\lfloor M \rfloor$  if  $\beta(v) = 1$ , and have her offer values  $\lfloor M \rfloor$ ,  $\lfloor M \rfloor + 1$  if  $\beta(v) = 2$ . Adversary chooses f(v) from these values and so  $|f(v) - M| \leq 1 = \gamma(v) + 1$  as required.

So we may assume that v is not first in  $\prec$ . If  $\beta(v) = 1$ , we have Prover offer for v the value  $\lfloor M \rfloor$  or  $\lfloor M \rfloor + 1$ . We choose one of  $\lfloor M \rfloor$ ,  $\lfloor M \rfloor + 1$  that has the right parity so that  $(\triangle)$  holds for all  $u \prec v$  when this value is assigned to f(v). (Since  $\mathcal{D}_{\psi}$  is bipartite, if  $(\triangle)$  holds for one u, it holds for all u; the choice is always possible).

Similarly, if  $\beta(v) = 2$ , then Prover offers either values  $\lfloor M \rfloor - 1$ ,  $\lfloor M \rfloor + 1$ , or values  $\lfloor M \rfloor$ ,  $\lfloor M \rfloor + 2$ . One of the two will satisfy ( $\triangle$ ) for any subsequent choice that Adversary will make. Moreover, since the endpoints of the interval  $\mathcal{L}$  (if  $\mathcal{L} \neq \mathbb{R}$ ) when chosen for f(v) satisfy ( $\triangle$ ), it also follows that the values Prover offers all belong to  $\mathcal{L}$ . This means that any one of these values satisfies ( $\star$ ) when chosen for f(v).

This shows that the value f(v) that Adversary chooses satisfies  $|f(v) - M| \leq \beta(v)$  and  $(\triangle), (\star)$  hold. Recall that

$$\gamma(v) = \beta(v) - 1 + \max\left\{0, \max_{u \prec v} \left(\gamma(u) - \delta(u, v) + \beta(v) - 1\right)\right\}$$
(1)

This implies that  $\gamma(v) \ge \beta(v) - 1$ . Thus  $|f(v) - M| \le \beta(v) \le \gamma(v) + 1$  as required.

**Case 2:** Suppose that  $M \notin \mathcal{L}$ . Thus  $\mathcal{L} \neq \mathbb{R}$  and either  $M < f(y) - \delta(y, v)$  or  $f(x) + \delta(x, v) < M$ . Note that this means that v is not first in  $\prec$ .

By symmetry, we may assume that  $M > f(x) + \delta(x, v)$ . If  $\beta(v) = 1$ , we have Prover offer the value  $f(x) + \delta(x, v)$ . If  $\beta(v) = 2$ , Prover offers values  $f(x) + \delta(x, v)$ ,  $f(x) + \delta(x, v) - 2$ . This implies that the value f(v) that Adversary chooses for v satisfies  $(\star)$ ,  $(\triangle)$  since  $|\mathcal{L}| \ge 2\beta(v) - 1$ , and we have

$$f(x) + \delta(x, v) - 2(\beta(v) - 1) \le f(v) \le f(x) + \delta(x, v)$$

Thus since  $f(x) + \delta(x, v) < M$ , we deduce that f(v) < M. Also f(x) < M since  $\delta(x, v) \ge 0$  by (IV). Applying the inductive hypothesis to x this yields

$$M - f(x) = |f(x) - M| \le \gamma(x) + 1$$

We now recall (1) and thus conclude (when using u := x in (1))

$$\gamma(v) \ge 2(\beta(v) - 1) + \gamma(x) - \delta(x, v)$$

Finally, above we remarked that f(v) < M and so we write

$$\begin{split} |f(v) - M| &= M - f(v) \le M - \left(f(x) + \delta(x, v) - 2(\beta(v) - 1)\right) \\ &\le (\gamma(x) + 1) - \delta(x, v) + 2(\beta(v) - 1) \\ &< \gamma(v) + 1 \end{split}$$

This proves  $|f(v) - M| \le \gamma(v) + 1$  as required. Analogous proof works in the case  $M < f(y) - \delta(y, v)$  where Prover offers  $f(y) - \delta(y, v)$  or  $f(y) - \delta(y, v) + 2$ .

That completes the proof.

With the above lemmas, we are now ready to characterize  $\{1,2\}$ -CSP( $\mathbb{P}_n$ ) for even *n*.

**Theorem 19.** Let  $n \ge 4$  be even. Assume that  $\mathbb{P}_{\infty} \models \Psi$ . Then the following are equivalent.

(1)  $\mathbb{P}_n \models \Psi$ .

(II) Prover has a winning strategy in the game  $\mathscr{G}(\Psi, \mathbb{P}_n)$ .

(III) There is no vertex v with  $\gamma(v) \geq \frac{n}{2}$ .

**Proof.** Note first that since *n* is even, we may assume, without loss of generality, the first vertex in the ordering is quantified  $\exists^{\geq 1}$ . If not, we can freely change its quantifier to  $\exists^{\geq 1}$  without affecting the satisfiability of the instance. Namely, for each value *i*, we can offer a pair of symmetric values *i*, n - i + 1; if Adversary chooses n - i + 1, we simply start offering values n - j + 1 where we would offer *j*.

(I)  $\Leftrightarrow$  (II) is by Lemma 6. For (II) $\Rightarrow$  (III), assume there is v with  $\gamma(v) \geq \frac{n}{2}$  and Prover has a winning strategy in  $\mathscr{G}(\Psi, \mathbb{P}_n)$ . This is also a winning strategy in  $\mathscr{G}(\Psi, \mathbb{P}_\infty)$ . This allows us to apply Lemma 17 for  $M = \frac{n+1}{2}$  to conclude that Adversary can play against Prover so that  $|f(v) - \frac{n+1}{2}| = |f(v) - M| \geq \gamma(v) = \frac{n}{2}$ . Thus either  $f(v) \geq \frac{2n+1}{2} > n$  or  $f(v) \leq \frac{1}{2} < 1$ . But then  $f(v) \notin \{1, \ldots, n\}$  contradicting our assumption that Prover plays a winning strategy.

For (III)  $\Rightarrow$  (II), assume that  $\gamma(v) \leq \frac{n}{2} - 1$  for all vertices v. We apply Lemma 18 for  $M = \frac{n+1}{2}$ . This tells us that Prover has a winning strategy on  $\mathscr{G}(\Psi, \mathbb{P}_{\infty})$  such that in every instance of the game, if f is the resulting mapping, the mapping satisfies  $|f(v) - \frac{n+1}{2}| \leq \gamma(v) + 1$  for every vertex v. From this we conclude that  $f(v) \geq \frac{n+1}{2} - \gamma(v) + 1 \geq \frac{n+1}{2} - \frac{n}{2} = \frac{1}{2}$  and that  $f(v) \leq \frac{2n+1}{2} = n + \frac{1}{2}$ . Therefore  $f(v) \in \{1, 2, \ldots, n\}$  confirming that f is a valid homomorphism to  $\mathbb{P}_n$ .

#### 5.2. Odd case

For odd *n*, we proceed similarly as for even *n* except for a subtle twist. We define  $\gamma'(v)$  using same recursion as  $\gamma(v)$  except that we set  $\gamma'(v) = \beta(v) - 1$  if *v* is first in  $\prec$ . Namely,  $\gamma'(v)$  is defined as follows.

**Definition 8.** For each vertex v we define  $\gamma'(v)$  recursively as follows:

$$\gamma'(v) = \beta(v) - 1 + \max\left\{0, \max_{u \prec v} \left(\gamma'(u) - \delta(u, v) + \beta(v) - 1\right)\right\}$$

**Lemma 20.** Let M be an integer. Suppose that  $\mathbb{P}_{\infty} \models \Psi$  and that Prover plays a winning strategy in the game  $\mathscr{G}(\Psi, \mathbb{P}_{\infty})$ . Then Adversary can play so that the resulting mapping f satisfies  $|f(v) - M| \ge \gamma'(v)$  for every vertex  $v \in V(D_{\psi})$ .

**Proof.** The proof is by induction on the number of steps. For the base case, v is first in the ordering  $\prec$  and we have  $\gamma'(v) = \beta(v) - 1$ . If  $\beta(v) = 1$ , then  $\gamma'(v) = 0$  and  $|f(v) - M| \ge 0 = \gamma'(v)$ . If  $\beta(v) = 2$ , Prover offers two distinct values for v. At least one of them is not M and Adversary chooses this value as f(v). So  $f(v) \ne M$  and since M is an integer, we conclude that  $|f(v) - M| \ge 1 = \gamma'(v)$ .

The rest of the proof (the inductive case) is exactly as in the proof of Lemma 17 (with  $\gamma$  replaced by  $\gamma'$ ). That completes the proof.

**Lemma 21.** Let *M* be an integer and let  $t \in \{0,1\}$ . Suppose that  $\mathbb{P}_{\infty} \models \Psi$ . Then there exists a winning strategy for Prover such that in every instance of the game the resulting mapping *f* satisfies  $|f(v) - M| \le \gamma'(v) + 1$  for every  $v \in V(\mathcal{D}_{\psi})$ , and moreover,  $f(z) \equiv t \pmod{2}$  where *z* is the first vertex in  $\prec$ .

**Proof.** By induction on the number of steps. In the base case, we consider *z*, the first vertex in  $\prec$ . Recall that  $\gamma'(z) = \beta(z) - 1$ . If  $\beta(z) = 1$ , then Prover chooses f(z) to be *M* or *M* + 1 based on the parity of *t*, i.e. so that  $f(z) \equiv t \pmod{2}$ . Thus  $|f(z) - M| \le 1 = \beta(z) = \gamma'(z) + 1$ . If  $\beta(z) = 2$ , then Prover offers values

M, M+2 or M-1, M+1, again chosen so that any value f(z) selected by Adversary from the two values has the right parity, i.e.  $f(z) \equiv t \pmod{2}$ . Thus  $|f(z) - M| \leq 2 = \beta(z) = \gamma'(z) + 1$ , as required. 

The rest is exactly as in the proof of Lemma 18 (with  $\gamma$  replaced by  $\gamma'$ ).

**Theorem 22.** Let  $n \ge 5$  be odd. Assume that  $\mathbb{P}_{\infty} \models \Psi$  and that the vertices of  $\mathcal{D}_{\psi}$  are properly coloured with colours black and white. Then the following are equivalent.

- (1)  $\mathbb{P}_n \models \Psi$ .
- (II) Prover has a winning strategy in the game  $\mathscr{G}(\Psi, \mathbb{P}_n)$ . (III) There are no vertices u, v where  $\gamma'(u) \geq \frac{n-1}{2}$  and  $\gamma'(v) \geq \frac{n-1}{2}$  such that u is black and v is white.

**Proof.** (I) $\Leftrightarrow$ (II) is by Lemma 6. For (II) $\Rightarrow$ (III) assume that Prover plays a winning strategy in  $\mathscr{G}(\Psi, \mathbb{P}_n)$ , but there are vertices u, v where u is black, v is white, and  $\gamma'(u) \ge \frac{n-1}{2}$  and  $\gamma'(v) \ge \frac{n-1}{2}$ . Then Prover's strategy is also winning on  $\mathbb{P}_{\infty}$ . This implies that the resulting mapping f satisfies conditions  $(\star)$  and  $(\triangle)$ . Moreover, by Lemma 20, Adversary can play so that  $|f(u) - \frac{n+1}{2}| \ge \gamma'(u) \ge \frac{n-1}{2}$  and that  $|f(v) - \frac{n+1}{2}| \ge \gamma'(v) \ge \frac{n-1}{2}$ . This means that either  $f(u) \ge n$  or  $f(u) \le 1$ . Same for v. But since Prover wins, f is a homomorphism to  $\mathbb{P}_n$  and so f(u) and f(v) belong to  $\{1, \ldots, n\}$ . Therefore f(u) and f(v) are in  $\{1, n\}$ . Recall that f satisfies  $(\triangle)$  and that the graph  $\mathcal{D}_{\psi}$  is connected. Thus there exists a looping walk from *u* to *v* which by  $(\triangle)$  implies that f(u) and f(v) have different parity (since u is black while v is white). However, both 1 and n are odd numbers and both f(u) and f(v) are from  $\{1, n\}$ , which renders this situation impossible. Therefore Adversary wins showing that no such vertices u, v exist which proves (III).

Conversely, to prove (III) $\Rightarrow$ (II), assume by symmetry that every black vertex u satisfies  $\gamma'(u) \le \frac{n-3}{2}$ . We first show that this implies  $\gamma'(v) \le \frac{n-1}{2}$  for all white vertices v. We proceed by induction on the ordering  $\prec$ . Let v be a white vertex. If  $\gamma'(v) = \beta(v) - 1$ , then  $\gamma'(v) = \beta(v) - 1 \le 1 \le \frac{n-1}{2}$ , since  $n \ge 5$ . So we may assume that there is  $u \prec v$  such that  $\gamma'(v) = 2(\beta(v) - 1) + \gamma'(u) - \delta(u, v)$ . Recall that we assume that  $\mathbb{P}_{\infty} \models \Psi$ . Thus by Theorem 13(IV), we have  $\delta(u, v) \ge \beta(v) - 1$ . If u is black, we have  $\gamma'(u) \le \frac{n-3}{2}$  and so

$$\gamma'(v) = 2(\beta(v) - 1) + \gamma'(u) - \delta(u, v) \le \frac{n-3}{2} + \beta(v) - 1 \le \frac{n-1}{2}$$

If *u* is white, we have by induction (since  $u \prec v$ ) that  $\gamma'(u) \leq \frac{n-1}{2}$ . Moreover, we observe that  $\delta(u, v)$ is even, since both u and v are white. So if  $\beta(v) = 2$ , we never have  $\delta(u, v) = \beta(v) - 1$  since in that case  $\beta(v) - 1$  is odd. Thus we can write  $\delta(u, v) \ge 2(\beta(v) - 1)$  and therefore

$$\gamma'(v) = 2(\beta(v) - 1) + \gamma'(u) - \delta(u, v) \le \frac{n-1}{2}$$

This proves that  $\gamma'(v) \leq \frac{n-1}{2}$  for all white vertices v, as promised. With this we can describe a winning strategy for Prover. Let z be the first vertex in  $\prec$ . Choose t = 0 if z is black and t = 1 if z is white. By Lemma 21 with the above t and  $M = \frac{n+1}{2}$ , Prover can play a strategy in which the resulting mapping

f is a homomorphism to  $\mathbb{P}_{\infty}$  and where  $|f(v) - \frac{n+1}{2}| \leq \gamma'(v) + 1$  for every vertex v and  $f(z) \equiv t \pmod{2}$ . We show that this strategy produces a homomorphism to  $\mathbb{P}_n$ . It suffices to prove that  $f(v) \in \{1, ..., n\}$  for every v. If u is black, then  $|f(u) - \frac{n+1}{2}| \le \gamma'(u) + 1 \le \frac{n-3}{2} + 1 = \frac{n-1}{2}$ . Thus  $1 \le f(u) \le n$  as required. Now consider a white vertex v. Since Prover plays a winning strategy on  $\mathbb{P}_{\infty}$ , the mapping f satisfies

 $(\triangle)$  and  $(\star)$ . We show that this implies that f(v) is odd. Recall that  $f(z) \equiv t \pmod{2}$ . Suppose first that z is black. Then f(z) is even by the choices of t. Thus by  $(\triangle)$ , we deduce that f(z) is odd, since z is black while v is white. We proceed similarly if z is white. In this case, f(z) is odd by the choices of t, and so f(v)is also odd by  $(\triangle)$ , since both z and v are white. This proves that f(v) is indeed odd for every white vertex *v*, and we recall that  $\gamma'(v) \le \frac{n-1}{2}$  as we proved earlier. Therefore, we have  $|f(v) - \frac{n+1}{2}| \le \gamma'(v) + 1 \le \frac{n+1}{2}$ . In other words,  $0 \le f(v) \le n+1$  but since f(v) is odd, we actually have  $1 \le f(v) \le n$ , as required. 

This yields (II) and thus completes the proof (III) $\Rightarrow$ (II) and the theorem.

### 5.3. Proof of Theorem 2

We observe that the values  $\gamma(v)$  and  $\gamma'(v)$  for each vertex v can be calculated using dynamic programming in polynomial time. Thus the conditions of Theorems 19 and 22 can be tested in polynomial time and thus Theorem 2 follows.  $\square$ 

### 6. Proof Corollary 3

We now show how to decide  $\{1,2\}$ -CSP(*H*) when *H* is a forest. Let  $\Psi$  be a given instance to this problem and let  $G = \mathcal{D}_{\psi}$  be the corresponding graph.

First, we may assume that *H* is a tree. This follows easily (with a small caveat mentioned below) as the connected components of *G* have to be mapped to connected components of *H*. Therefore with *H* being a tree, we first claim that if *G* is a yes-instance, then *G* is also a yes instance to  $\{1,2\}$ -CSP( $\mathbb{P}_{\infty}$ ). To conclude this, it can be shown that the condition (III) of Theorem 13 can be generalized to trees by using the distance in *H* in the condition ( $\star$ ), and using proper colouring of *H* for ( $\Delta$ ) in an obvious way. This implies that no two vertices *u*,*v* are mapped in *T* farther away than  $\delta(u, v)$ . So a bad walk cannot exist and *G* is a yes-instance of  $\{1,2\}$ -CSP( $\mathbb{P}_{\infty}$ ).

A similar argument allows us to generalize Lemmas 17 and 18 to trees; namely Adversary will play **away** from some vertex, while Prover **towards** some vertex. The absolute values will be replaced by distances in *H*. From this we conclude by Theorem 19 that Adversary can force each v to be assigned a vertex which is at least  $\gamma'(v)$  resp.  $\gamma(v)$  away from the center vertex of *H*, resp. center edge of *H*. In summary, this proves

**Corollary 23.** Let *H* be a tree. Let *P* be the longest path in *H*. Then  $\Psi$  is a yes-instance of  $\{1,2\}$ -CSP(*H*) if and only if  $\Psi$  is a yes instance of  $\{1,2\}$ -CSP(*P*).

This can be phrased more generally for forests in a straightforward manner. The only caveat is that if two components contain a path of the same length, we can make the first vertex in the instance an  $\exists^{\geq 1}$  vertex without affecting the satisfiability, because if it is  $\exists^{\geq 2}$ , we let Adversary choose which midpoint of the two longest paths to use (which is symmetric).

Finally, we are ready to prove a dichotomy for  $\{1,2\}$ -CSP(*H*) where *H* is a graph.

### 7. Proof of Corollary 4

If *H* is not bipartite, then  $\{1\}$ -CSP(*H*) is NP-hard by [9]; thus  $\{1,2\}$ -CSP(*H*) is also NP-hard. So we may assume that *H* is a bipartite graph. If *H* contains a *C*<sub>4</sub>, then  $\{1,2\}$ -CSP(*H*) is in *L* as shown in [11]. If *H* contains a larger cycle, then the problem is Pspace-complete as we show later in Theorem 5. Thus we may assume that *H* contains no cycles and hence it is a forest. In this case  $\{1,2\}$ -CSP(*H*) is polynomial-time solvable as shown in Corollary 3. That completes the proof.

### 8. Pspace dichotomy for bipartite graphs – Proof of Theorem 5

The following is a slightly simpler version of a subcase of Proposition 5 from [11].

# **Proposition 1.** If $j \ge 3$ , then $\{1, 2\}$ -*CSP*( $\mathbb{C}_{2i}$ ) is Pspace-complete.

**Proof.** We reduce from  $QCSP(\mathbb{K}_j)$ , whose Pspace-completeness follows from [2]. Our reduction borrows a lot from the reduction from  $CSP(\mathbb{K}_j)$  to the retraction problem  $Ret(\mathbb{C}_{2j})$  used to prove the NP-hardness of the latter in [7]. For an input  $\Psi := Q_1 x_1 Q_2 x_2 \dots Q_m x_m \ \psi(x_1, x_2, \dots, x_m)$  for  $QCSP(\mathbb{K}_j)$  we build an input  $\Theta$  for  $\{1, 2\}$ - $CSP(\mathbb{C}_{2j})$  as follows. We begin by considering the graph  $\mathcal{D}_{\psi}$ , from which we first build a graph  $G' := \mathcal{D}_{\psi} \uplus \mathbb{C}_{2j}$ , with the latter cycle on new vertices  $\{w_1, \dots, w_{2j}\}$ . Now we build G'' from G' by replacing every edge  $(x, y) \in \mathcal{D}_{\psi}$  with a gadget which involves 3j new copies of  $\mathbb{C}_{2j}$  connected in a prismic path (cartesian product with  $\mathbb{P}_{3j}$ ) to the fixed copy of  $\mathbb{C}_{2j}$  in G' – induced by  $\{w_1, \dots, w_{2j}\}$ . The vertex x sits at the end of a pendant path of length j - 1 which joins the final, leftmost copy of  $\mathbb{C}_{2j}$  diametrically opposite y; and the shortest path from x to y is of length j - 1 (if j odd) or j (if j even). Finally, for universal variables v of  $\Psi$  we add a new path  $v_1, \dots, v_j$  culminating in v of length j. These gadgets are drawn in [11].

The quantifier-free part  $\theta$  of  $\Theta$  will be  $\phi_{G''}$  we now explain how to add the quantification. The variables  $\{w_1, \ldots, w_{2i}\}$  will be quantified outermost and last. Aside from this, and moving inwards through the

quantifier order of  $\Psi$ , when we encounter an existential variable v, we apply existential quantification to it in  $\Theta$ . On the other hand, when we encounter a universal variable v, we apply existential (or  $\exists^{\geq 2}$ ) quantification to the first element  $v_1$  of its associated path, followed by  $\exists^{\geq 2}$  quantification to the remaining  $v_2, \ldots, v_j$  of this path and finally existential quantification to the v that this path culminates in. All the remaining parts of the edge gadgets can be existentially quantified innermost.

Suppose that the variables  $w_1, \ldots, w_{2j}$  are evaluated truly, i.e. up to some automorphism of  $\mathbb{C}_{2j}$ . It is simple to see that the gadgets enforce that variables x and y of  $\Psi$  are evaluated at distinct points of  $\mathbb{C}_{2j}$  with the same parity. Furthermore, the universally quantified variables are forced, under suitable evaluation of the  $\exists^{\geq 2}$  variables within their gadgets, to all positions of  $\mathbb{C}_{2j}$  of the respective parity.

Finally we need to enforce on the cycle  $\mathbb{C}_{2i}$  with the variables  $w_1, \ldots, w_{2i}$ , the outermost quantification

$$\exists^{\geq 2} w_1, w_2, w_3, \ldots, w_{j+1} \exists w_{j+2}, \ldots, w_{2j-1}.$$

The point is that there is some evaluation of  $w_1, \ldots, w_{j+1}$  that enforces that  $w_1, \ldots, w_{2j}$  is evaluated automorphically to  $\mathbb{C}_{2j}$ . Clearly, the outermost quantification on  $w_1$  and  $w_2$  could be pure existential. The other possibilities (the *degenerate* cases) involve a mapping to a homomorphic image of  $\mathbb{C}_{2j}$  which we can extend to homomorphism through careful assignment of the parts of the chains of the edge gadgets in which no  $\exists^{\geq 2}$  quantification subsists. The degenerate cases are the reason our chain is of length 3j rather than the much shorter j - 1 that appears in [7]; for when  $w_1, \ldots, w_{2j}$  are mapped degenerately, the universal variables can still appear anywhere on the cycle.

**Corollary 24.** If *H* is a bipartite graph whose smallest cycle in  $\mathbb{C}_{2j}$  for  $j \ge 3$ , then  $\{1,2\}$ -*CSP*(*H*) is Pspace-complete.

**Proof.** There are two difficulties arising from simply using the proof of the previous proposition as it is. Firstly, let us imagine that the variables  $w_1, \ldots, w_{2j}$  are indeed mapped to a fixed copy of  $\mathbb{C}_{2j}$  in H. We need to ensure that variables x, y derived from the original instance  $\Psi$  are mapped to the cycle also. For y variables in our gadget one can check this must be true – the successive cycles in the edge gadget may never deviate from the fixed  $\mathbb{C}_{2j}$ , since H contains no smaller cycle than  $\mathbb{C}_{2j}$  – but for x variables off on the pendant this might not be true. There are various fixes for this. If the instance of  $\Psi$  were symmetric, i.e. contained an atom E(x, y) iff it also contained E(y, x) then this property would automatically hold (and it is easy to see from [2] that QCSP( $\mathbb{K}_j$ ) remains Pspace-complete on symmetric instances). An alternative is to add a system of 3j cycles to tether variables x also to the fixed copy of  $\mathbb{C}_{2j}$ .

The second difficulty is in isolating a copy of the cycle  $\mathbb{C}_{2j}$  with the variables  $w_1, \ldots, w_{2j}$ , but since H does not contain a cycle smaller than  $\mathbb{C}_{2j}$  a simple argument shows that one such cycle must be identified.

### 9. Cases in NP: Dominating vertices

We consider here the class of undirected graphs with a single dominating vertex w which is also a self-loop.

**Proposition 2.** If *H* has a reflexive dominating vertex *w* and  $H \setminus \{w\}$  contains a loop or is irreflexive bipartite, then  $\{1,2\}$ -*CSP*(*H*) is in *P*.

**Proof.** If  $H \setminus \{w\}$  contains a loop then H contains adjacent looped vertices and  $\{1,2\}$ -CSP(H) is trivial (all instances are yes-instances). Assume  $H \setminus \{w\}$  is irreflexive bipartite and consider an input  $\Psi$  for  $\{1,2\}$ -CSP(H). All variables quantified by  $\exists$  can be evaluated as w and can be safely removed while preserving satisfaction. So, let  $\Psi'$  be the subinstance of  $\Psi$  induced by the variables quantified by  $\exists^{\geq 2}$  and let  $\psi'$  be the associated quantifier-free part. If  $\mathcal{D}_{\psi'}$  is bipartite, the instance is a yes-instance, otherwise it is a no-instance.

**Proposition 3.** If *H* has a reflexive dominating vertex *w* and  $H \setminus \{w\}$  is irreflexive non-bipartite, then  $\{1,2\}$ -*CSP*(*H*) is *NP*-complete.

**Proof.** For membership of NP we note the following. Let *U* be a unary predicate defining the set  $H \setminus \{w\}$ . From an input  $\Psi$  for  $\{1,2\}$ -CSP(*H*) we will build an instance  $\Psi'$  for CSP(*H*; *U*) so that  $H \models \Psi$  iff  $(H; U) \models \Psi'$ . The latter is clearly in NP, so the result follows. To build  $\Psi'$  we take  $\Psi$  and add U(v) to the quantifier-free part for all  $\exists^{\geq 2}$  quantified variables v, before converting in the quantification  $\exists^{\geq 2}v$  to  $\exists v$ .

For NP-hardness we reduce from  $CSP(H \setminus \{w\})$  which is NP-hard by [9]. For an input  $\Psi$  to this, we build an input  $\Psi'$  for  $\{1,2\}$ -CSP(H) by converting each  $\exists$  to  $\exists^{\geq 2}$ . It is easy to see that  $(H \setminus \{w\}) \models \Psi$  iff  $H \models \Psi'$ and the result follows.

**Corollary 25.** If H has a reflexive dominating vertex, then  $\{1,2\}$ -CSP(H) is either in P or is NP-complete.

### 10. Small graphs

It follows from Proposition 3 that there is a partially reflexive graph on four vertices,  $\mathbb{K}_4$  with a single reflexive vertex, so that the corresponding  $\{1,2\}$ -CSP is NP-complete. We can argue this phenomenon is not visible on smaller graphs.

**Proposition 4.** Let *H* be a (partially reflexive) graph on at most three vertices, then either  $\{1,2\}$ -CSP(*H*) is in *P* or it is Pspace-complete.

**Proof.** The Pspace-complete cases are  $\mathbb{K}_3$  (see [11]) and  $\mathcal{P}_{101}$ , which is the path of length two whose internal vertex is loopless while the end vertices are looped. It is known QCSP( $\mathcal{P}_{101}$ ) is Pspace-complete [12]. One can reduce this problem to  $\{1,2\}$ -CSP( $\mathcal{P}_{101}$ ) by substituting  $\forall x$  in the former by  $\exists^{\geq 2}x, x' E(x, x')$  in the latter (where x' is a newly introduced variable).

It is clear that  $\{1,2\}$ -CSP(H) is trivial if H contains a reflexive clique of size 2,  $\mathbb{K}_2^*$ . If H is irreflexive bipartite, i.e. is a forest, then  $\{1,2\}$ -CSP(H) is in P according to [11]. When H contains just isolated loops and non-loops then it is easy to give a tailored algorithm. If H contains at least two loops then: any input with a subinstance  $Qx \exists^{\geq 2}x' E(x, x')$  (Q any quantifier,  $x \neq x'$ ) is false; and all other inputs are true. If H contains only one loop: any input with a subinstance  $Qx \exists^{\geq 2}x' E(x, x')$  (Q any quantifier, possibly x = x') is false; and all other inputs are true. If H contains just isolated loops then it is bipartite. We henceforth assume these cases solved.

We are left with one remaining two-vertex graph,  $\mathcal{P}_{10}$ . For this problem, any input with a subinstance  $\exists^{\geq 2}x, x' E(x, x')$  (*Q* any quantifier, possibly x = x') is false; and all other inputs are true.

We continue with graphs of exactly three vertices. Among the remaining possibilities where *H* has exactly two loops is only  $\mathcal{P}_{10} \uplus \mathbb{K}_1^*$ . For this problem, any input with a subinstance  $\exists^{\geq 2}x, x' E(x, x')$  (*Q* any quantifier,  $x \neq x'$ ) is false; and all other inputs are true.

We now address the case in which there is precisely one loop. If it dominates, then we have tractability by Proposition 2. If it is isolated, then the remaining case is  $\mathbb{K}_1^* \uplus \mathbb{K}_2$ . For this, any input  $\Psi$  with subinstance  $\exists^{\geq 2}x \ E(x, x)$  or an  $\exists^{\geq 2}v$  attached to a sequence (connected component of v in  $\mathcal{D}_{\psi}$ ) that is non-bipartite is false; and all other inputs are true. The remaining possibilities are  $\mathcal{P}_{100}$  and  $\mathcal{P}_{10} \uplus \mathbb{K}_1$ . For the latter, we have the same {1,2}-CSP as for  $\mathcal{P}_{10}$ , which has already been resolved. {1,2}-CSP( $\mathcal{P}_{100}$ ) requires some subtlety and appears as its own result in Proposition 5.

Finally, we come to the irreflexive cases, and realise these are either bipartite or  $\mathbb{K}_3$  and are hence resolved.  $\hfill\square$ 

We denote  $\mathcal{P}_{100}$  the path on three vertices 0, 1, 2 with loop at 0.

**Proposition 5.**  $\{1, 2\}$ -*CSP*( $\mathcal{P}_{100}$ ) is in *P*.

**Proof.** The following four types of subinstance in  $\Psi$  result in it being false (always consider the symmetric closure).

- (*i*.)  $\exists^{\geq 2}x_1, x_2, x_3 E(x_1, x_2) \land E(x_2, x_3).$
- (*ii.*)  $Qx_1 \exists \geq x_2, x_3 E(x_1, x_3) \land E(x_2, x_3)$  (*Q* any quantifier).
- (*iii.*)  $\exists \geq x_1, x_2, x_3 E(x_2, x_3) \land E(x_1, y) \land E(x_3, y)$ ; where y is quantified anywhere existentially.
- (*iv.*)  $\exists^{\geq 2}x_1, x_2, x_3, x_4 \ E(x_1, x_2) \land E(x_3, x_4) \land E(x_2, y_1) \land E(x_4, y_2) \land E(y_1, y_2)$ ; where  $y_1, y_2$  are quantified anywhere existentially.

We claim that all other inputs are yes-instances. We give the following strategy for Prover. Consider  $\mathcal{P}_{100}$  to be  $\{0, 1, 2\}$  and take the canonical sequence 0, 1, 2 to be the path of  $\mathcal{P}_{100}$  from the loop 0. For  $\exists^{\geq 2}$  variables Prover offers  $\{0, 1\}$ , unless constrained by adjacency of a variable already played to 1, in which case she offers  $\{0, 2\}$ . For  $\exists$  variables Prover offers  $\{0\}$ , unless constrained by adjacency of a variable already played to 2, in which case she offers  $\{1\}$ . We argue this strategy must be winning. This is tantamount to saying that Prover is never offered (A.) an  $\exists^{\geq 2}$  variable that is simultaneously adjacent to 0 and 1; (B.) an  $\exists^{\geq 2}$  variable that is adjacent to 2; and (C.) an  $\exists$  variable that is simultaneously adjacent to both 2 and 1. (B) follows from Rule (*i*) and (A) follows from Rule (*ii*). (C) follows from (*iii*) and (*iv*).

# 11. Final remarks

In this paper we have settled the major questions left open in [11] and it might reasonably be said we have now concluded our preliminary investigations into constraint satisfaction with counting quantifiers. Of course there is still a wide vista of work remaining, not the least of which is to improve our P/ NP-hard dichotomy for  $\{1,2\}$ -CSP on undirected graphs to a P/ NP-complete/ Pspace-complete trichotomy (if indeed the latter exists). The absence of a similar trichotomy for QCSP, together with our reliance on [9], suggests this could be a challenging task. Some more approachable questions include lower bounds for  $\{2\}$ -CSP( $\mathbb{K}_4$ ) and  $\{1,2\}$ -CSP( $\mathbb{P}_{\infty}$ ). For example, intuition suggests these might be NL-hard (even P-hard for the former). Another question would be to study X-CSP( $\mathbb{P}_{\infty}$ ), for  $\{1,2\} \not\subseteq X \subset \mathbb{N}$ .

Since we initiated our work on constraint satisfaction with counting quantifiers, a possible algebraic approach has been published in [5, 6]. It is clear reading our expositions that the combinatorics associated with our counting quantifiers is complex, and unfortunately the same seems to be the case on the algebraic side (where the relevant "expanding" polymorphisms have not previously been studied in their own right). At present, no simple algebraic method, generalizing results from [2], is known for counting quantifiers with majority operations. This would be significant as it might help simplify our tractability result of Theorem 2. So far, only the Mal'tsev case shows promise in this direction.

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