Tree-based Coarsening and Partitioning of Complex Networks

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Abstract. Many applications produce massive complex networks whose analysis would benefit from parallel processing. Parallel algorithms, in turn, often require a suitable network partition. For solving optimization tasks such as graph partitioning on large networks, multilevel methods are preferred in practice. Yet, complex networks pose challenges to established multilevel algorithms, in particular to their coarsening phase. One way to specify a (recursive) coarsening of a graph is to rate its edges and then contract the edges as prioritized by the rating. In this paper we (i) define weights for the edges of a network that express the edges importance for connectivity, (ii) compute a minimum weight spanning tree T^m w.r.t. these weights, and (iii) rate the network edges based on the conductance values of T^m 's fundamental cuts. To this end, we also (iv) develop the first optimal linear-time algorithm to compute the conductance values of all fundamental cuts of a given spanning tree. We integrate the new edge rating into a leading multilevel graph partitioner and equip the latter with a new greedy postprocessing for optimizing the maximum communication volume (MCV). Experiments on bipartitioning frequently used benchmark networks show that the postprocessing already reduces MCV by 11.3%. Our new edge rating further reduces MCV by 10.3% compared to the previously best rating with the postprocessing in place for both ratings. In total, with a modest increase in running time, our new approach reduces the MCV of complex network partitions by 20.4%.

Keywords: Graph coarsening, multilevel graph partitioning, complex networks, fundamental cuts, spanning trees

1 Introduction

Complex networks such as social networks or web graphs have become a focus of investigation recently [7]. Such networks are often scale-free, i. e. they have a power-law degree distribution with many low-degree vertices and few high-degree vertices. They also have a small diameter (small-world property), so that the whole network is discovered within a few hops from any vertex. Complex networks arise in a variety of applications; several of them generate massive data sets. As an example, the social network Facebook currently contains a billion active users (http://newsroom.fb.com/Key-Facts). On this scale many algorithmic tasks benefit from parallel processing. The efficiency of parallel algorithms on huge networks, in turn, is usually improved by graph partitioning (GP).

Given a graph G=(V,E) and a number of blocks k>0, the GP problem asks for a division of V into k pairwise disjoint subsets V_1,\ldots,V_k (blocks) such that no block is larger than $(1+\varepsilon)\cdot \left\lceil \frac{|V|}{k}\right\rceil$, where $\varepsilon\geq 0$ is the allowed imbalance. When GP is used for parallel processing, each processing element (PE) usually receives one block, and edges running between two blocks model communication between PEs. The most widely used objective function (whose minimization is \mathcal{NP} -hard) is the edge cut, the total weight of the edges between different blocks. However, it has been pointed out more than a decade ago [12] that the determining factor for modeling the communication cost of parallel iterative graph algorithms is the maximum communication volume (MCV), which has received growing attention recently, e.g. in a GP challenge [1]. MCV considers the worst communication volume taken over all blocks V_p ($1 \leq p \leq k$) and thus penalizes imbalanced communication: $MCV(V_1,\ldots,V_k) := \max_p \sum_{v \in V_p} |\{V_i \mid \exists \{u,v\} \in E \text{ with } u \in V_i \neq V_p\}|$. Note that parallel processing is only one of many applications for graph partitioning; more can be found in recent surveys [3, 4].

All state-of-the-art tools for partitioning very large graphs in practice rely on the multilevel approach [4]. In the first phase a hierarchy of graphs G_0, \ldots, G_l is built by recursive coarsening. G_l is supposed to be very small in size, but similar in structure to the input G_0 . In the second phase a very good initial solution for G_l is computed. In the final phase, the solution is prolongated to the next-finer graph, where it is improved using a local improvement algorithm. This process of prolongation and local improvement is repeated up to G_0 .

Partitioning static meshes and similar non-complex networks this way is fairly mature. Yet, the structure of complex networks (skewed degree distribution, small-world property) distinguishes complex networks from traditional inputs and makes finding small cuts challenging with current tools.

One reason for the difficulties of established multilevel graph partitioners is the coarsening phase. Most tools rely on edge contractions for coarsening. Traditionally, only edge weights have guided the selection of the edges to be contracted [16]. Holtgrewe et al. [13] recently presented a two-phase approach that makes contraction more systematic by separating two issues: An edge rating and a matching algorithm. The rating of an edge indicates how much sense it makes to contract the edge. The rating then forms the input to an approximate maximum weight matching algorithm, and the edges of the resulting matching are contracted. As one contribution of this paper, we define a new edge rating geared towards complex network partitions with low MCV.

Outline and Contribution. After the introduction we sketch the state of the art (Section 2) and settle necessary notation (Section 3). Our first technical contribution, described briefly in Section 4, results from our goal to minimize MCV rather than the edge cut: We equip a leading multilevel graph partitioner with greedy postprocessing that trades in small edge cuts for small MCVs.

Our main contributions follow in Sections 5 and 6. The first one is a new edge rating, designed for complex networks by combining local and non-local information. Its rationale is to find moderately balanced cuts of high quality quickly

(by means of the clustering measure conductance [15] and its loose connection to MCV via isoperimetric graph partitioning [11]) and to use this information to indicate whether an edge is part of a small cut or not. Finding such cuts is done by evaluating conductance for all fundamental cuts of a minimum spanning tree of the input graph with carefully chosen edge weights. (a fundamental cut is induced by the removal of exactly one spanning tree edge, cf. Section 3). The second main contribution facilitates an efficient computation of our new edge rating. We present the first optimal linear-time algorithm to compute the conductance values of all fundamental cuts of a spanning tree.

We have integrated both MCV postprocessing and our new edge rating $ex_cond(\cdot)$ into KaHIP [21, 22], a state-of-the-art graph partitioner with a reference implementation of the edge rating $ex_alg(\cdot)$, that yielded the best quality for complex networks so far (see Section 2).

Experiments in Section 7 show that greedy MCV postprocessing improves the partitions of our complex network benchmark set in terms of MCV by 11.3% on average with a comparable running time.

Additional extensive bipartitioning experiments (MCV postprocessing included) show that, compared to $ex_alg(\cdot)$, the fastest variant of our new edge rating further improves the MCVs by 10.3%, at the expense of an increase in running time by a factor of 1.79. Altogether, compared to previous work on partitioning complex networks with state-of-the-art methods [20], the total reduction of MCV by our new techniques amounts to 20.4%.

2 State of the Art

Multilevel graph partitioners such as METIS [16] and KAHIP [21, 22] (more are described in recent surveys [3, 4]) typically employ recursive coarsening by contracting edges, which are often computed as those of a matching. Edge ratings are important in guiding the matching algorithm; a successful edge rating is

$$expansion^{*2}(\{u,v\}) = \omega(\{u,v\})^2/(c(u)c(v)),$$
 (2.1)

where the weights of the vertices $u, v \in V$ and of the edges $\{u, v\} \in E$ are given by $c(\cdot)$ and $\omega(\cdot)$, respectively [13].

To broaden the view of the myopic rating above (it does not look beyond its incident vertices), Safro *et al.* [20] precompute the algebraic distance $\rho_{\{u,v\}}$ [5] for the end vertices of each edge $\{u,v\}$ and use the edge rating

$$ex_{alg}(\{u, v\}) = (1/\rho_{\{u, v\}}) \cdot expansion^{*2}(u, v)$$
(2.2)

For graphs with power-law degree distributions, ex_alg(·) yields considerably higher partition quality than $expansion^{*2}(\cdot)$ [20]. This is due to the fact that algebraic distance expresses a semi-local connection strength of an edge $\{u, v\}$ [5]. Specifically, $\rho_{\{u,v\}}$ is computed from R randomly initialized vectors that are

smoothed by a Jacobi-style over-relaxation for a few iterations. The idea is that the vector entries associated with well-connected vertices even out more quickly than those of poorly connected vertices. Thus, a high value of $\rho_{\{u,v\}}$ indicates that the edge $\{u,v\}$ constitutes a bottleneck and should not be contracted.

Another strategy for matching-based multilevel schemes in complex networks (e. g. for agglomerative clustering [8]) is to match unconnected vertices at 2-hop distance in order to eliminate star-like structures. Also, alternatives to matching-based coarsening exist, e. g. weighted aggregation schemes [6, 18].

Pritchard and Thurimella [19] use a spanning tree to sample the *cycle space* of a graph in a uniform way and thus find small cuts (consisting of a single edge, two edges or a cut vertex) with high probability [19]. Our method uses a minimum weight spanning tree on a graph with carefully chosen edge weights. Moreover, we sample the *cut-space*. The aim of the sampling is to create a collection \mathcal{C} of moderately balanced cuts which form the basis of our new edge rating.

We integrate our new algorithms into KAHIP [21,22]. KAHIP focuses on solution quality and has been shown recently to be a leading graph partitioner for a wide variety of graphs such as road networks, meshes, and complex networks [23]. It implements several advanced multilevel graph partitioning algorithms, metaheuristics, and sophisticated local improvement schemes.

3 Preliminaries

Let $G = (V, E, \omega)$ be a finite, undirected, connected, and simple graph. Its edge weights are given by $\omega : E \mapsto \mathbb{R}^+$. We write $\omega_{u,v}$ for $\omega(\{u,v\})$ and extend ω to subsets of E through $\omega(E') = \sum_{e \in E'} \omega(e)$.

For subsets V_1 , V_2 of V with $V_1 \cap V_2 = \emptyset$, the set $S(V_1, V_2)$ consists of those edges in E that have one end vertex in V_1 and the other end vertex in V_2 . If, in addition to $V_1 \cap V_2 = \emptyset$, it holds that (i) $V = V_1 \cup V_2$ and (ii) $V_1, V_2 \neq \emptyset$, then the pair (V_1, V_2) is called a cut of G, and $S(V_1, V_2)$ is called the cut-set of (V_1, V_2) . The weight of a cut (V_1, V_2) is given by $\omega(S(V_1, V_2))$. The volume of any subset V' of V is the total weight of the edges incident on V' (which equals the sum over the weighted degrees of the vertices in V'):

$$vol(V') = \omega(\{e = \{v', v\} \in E \mid v' \in V', v \in V\}), \tag{3.1}$$

Definition 3.1 (Fundamental cut, cut-set $S_T(e_T)$, cond (e_T, T))

Let T be a spanning tree of G, and let $e_T \in E(T)$. If T_1 and T_2 are the connected components (trees) of the graph $(V, E(T) \setminus \{e_T\})$, then $(V(T_1), V(T_2))$ is the fundamental cut of G with respect to T and e_T , and

$$S_T(e_T) = S(V(T_1), V(T_2)).$$
 (3.2)

is the fundamental cut-set of G with respect to T and e_T . Conductance is a common quality measure in graph clustering [15]. Its value for $(V(T_1), V(T_2))$ is

$$cond(e_T, T) = cond(V_1, V_2) = \frac{\omega(S_T(e_T))}{\min\{vol(V(T_1)), vol(V(T_2))\}}$$
(3.3)

4 Greedy MCV Optimization

The ultimate applications we target with our graph partitioning algorithm are iterative parallel algorithms executed on complex networks. As argued in Section 1, the maximum communication volume (MCV) is a more accurate optimization criterion than the edge cut. The graph partitioner KAHIP has so far solely focused on the edge cut, though. That is why, as a new feature, we equip KAHIP with a postprocessing that greedily optimizes MCV. This postprocessing is executed after local improvement on the finest level of the multilevel hierarchy and works in rounds. In each round, we iterate over all boundary vertices of the input partition in a random order and check whether moving the vertex from its own block to the opposite block reduces or keeps the MCV value. If this is the case, the current vertex will be moved to the opposite block. One round of the algorithm can be implemented in $\mathcal{O}(|E|)$ time (see Section C in the appendix for more details). The total number of rounds of the algorithm is a tuning parameter. After preliminary experiments we have set it to 20.

5 A New Conductance-based Edge Rating for Partitioning

An edge rating in a multilevel graph partitioner should yield a low rating for an edge e if e is likely to be contained in the cut-set of a "good" cut, e.g. if the cut-set consists of a bridge.

In our approach a good cut is one that (i) has a low conductance and (ii) is at least moderately balanced. In complex networks (i) does not always imply (ii) (see below). A loose connection between conductance and MCV in bipartitions can be established via isoperimetric graph partitioning [11]. Our approach to define an edge rating and use it for partitioning is as follows.

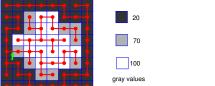
- 1. Generate a collection \mathcal{C} of moderately balanced bipartitions (cuts of G) that contain cuts with a low conductance value.
- 2. Define a measure $\operatorname{Cond}(\cdot)$ such that $\operatorname{Cond}(e)$ is low [high] if e is [not] contained in the cut-set of a cut in $\mathcal C$ with low conductance.
- 3. Instead of multiplying the edge rating $expansion^{*2}(\{u,v\})$ with the factor $(1/\rho_{\{u,v\}})$ as in [20], we replace one of the two (identical) myopic factors $\omega(\{u,v\})$ in $expansion^{*2}(\{u,v\})$ by the more far-sighted factor Cond(·). This yields the new edge rating

$$\operatorname{ex_cond}(\{u, v\}) = \omega(\{u, v\}) \operatorname{Cond}(\{u, v\}) / (c(u)c(v))$$
(5.1)

The higher Cond(e), the higher $ex_cond(e)$, and thus the higher the chances for e to be contracted during coarsening.

4. Run a multilevel graph partitioner capable of handling edge ratings such as KAHIP with $\exp(\cdot)$.

To specify $\exp(\cdot)$, we need to define \mathcal{C} and $\operatorname{Cond}(\cdot)$.



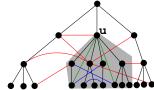


Fig. 1. (a) Example of MST (red) in GBIS. For the green arrow see the text. (b) Vertex attributes intraWeight and interWeight. Tree T is formed by the black edges, and the subtree with root u is contained in the shaded area. The weights of the blue and green edges contribute to intraWeight[u], and the weights of the red edges contribute to interWeight[u].

Specifics of C. For the definition of C, we resort to a basic concept of graph-based clustering, i. e. the use of minimum weight spanning trees (MSTs). We describe this concept in the context of graph-based image segmentation (GBIS) [9, 25] for illustration purposes (see Figure 1a).

In GBIS one represents an image by a graph G whose vertices [edges] represent pixels [neighborhood relations of pixels]. The edges are equipped with weights that reflect the contrast between the gray values at the edges' end vertices. An MST T^m of G with respect to contrast has the following property (see [14, Thm. 4.3.3]). The contrast value associated with any $e \in E(T^m)$ is minimal compared to the contrast values of the edges in the fundamental cut-set $S_{T^m}(e)$ (see Eq. 3.2). Thus, for any $e \in E(T^m)$ with a high contrast value (see the green arrow in Figure 1a), the fundamental cut $S_{T^m}(e)$ yields a segmentation into two regions with a high contrast anywhere on the common border.

Here, we arrive at a collection \mathcal{C} of |V|-1 moderately balanced bipartitions (cuts) of G by (i) computing connectivity-based contrast values for the edges of G, (ii) computing an MST T^m of G w.r.t. these values, and (iii) letting \mathcal{C} consist of G's fundamental cuts w.r.t. T^m . The contrast value of an edge $e=\{u,v\}$ should be low [high] if the edge is indispensable for "many" connections via shortest paths. Thus, the higher the contrast, the stronger the negative effect on G's connectivity if e is cut, and thus the more reasonable it is to cut e. To define the contrast values, we generate a random collection \mathcal{T} of breadth-first-traversal (BFT) trees. A tree in \mathcal{T} is formed by first choosing a root randomly. As usual, we let the trees grow out of the roots using a queue, but we process the edges incident on a vertex in a randomized order. Alternatively, SSSP trees may be used if edge weights are to be included.

Let $n_{\mathcal{T}}(u,v)$ denote the number of trees in \mathcal{T} that contain e and in which u is closer to the tree's root than v (u and v cannot have the same distance to the root). We set the *contrast* value of an edge $\{u,v\}$ to

$$\gamma(\{u, v\}) = \min\{n_{\mathcal{T}}(u, v), n_{\mathcal{T}}(v, u)\}. \tag{5.2}$$

Just considering the number of trees in \mathcal{T} which contain e, turned out to yield poorer partitions than using Eq. 5.2. We believe that this is due to small subgraphs which are connected to the graphs' "main bodies" via very few edges.

Just considering the number of trees in \mathcal{T} which contain such an edge would result in a high contrast of the edge although the cut is far from moderately balanced. Even worse, the conductance of the cut may be small (e.g. if the cut-set contains only one edge). This would protect edges in cut-sets of very unbalanced cuts from being contracted — an undesired feature.

Specifics of Cond(·). Our plan is to define a measure Cond(·) such that Cond(e) is low [high] if e is [not] contained in the cut-set of a cut in \mathcal{C} with low conductance. Hence, we set

$$Cond(e) = \min_{C \in \mathcal{C}, e \in S(C)} (cond(C)), \tag{5.3}$$

where S(C) denotes the cut-set of the cut C. Let FC_e denote the set of edges in the (fundamental) cycle that arises if e is inserted into T^m . Then, the cuts $C \in \mathcal{C}$ with $e \in S(C)$ (see Eq. 5.3) are precisely the fundamental cuts $S_T(e_T)$ (see Eq. 3.2) with $e_T \in FC_e$ and $e_T \neq e$. Note that e is the only edge in FC_e that is not in $E(T^m)$. This suggests to first compute the Cond-values for all edges $e_T \in E(T^m)$ as specified in Section 6. For $e \notin E(T^m)$ the value of Cond(e) is then obtained by forming the minimum of the Cond-values of $FC_e \setminus \{e\}$. If $e = \{u, v\}$, then $FC_e \setminus \{e\}$ is the set of edges on the unique path in T^m that connects u to v.

6 An $\mathcal{O}(|E|)$ -Algorithm for Computing All cond (e_T, T)

In this section we demonstrate how, for a rooted given spanning tree T of a graph G(V, E), one can compute all conductance values $\operatorname{cond}(e_T, T), e_T \in E(T)$, in time $\mathcal{O}(|E|)$ (the root can be chosen randomly). This algorithm facilitates an efficient computation of the edge rating introduced in the previous section. The key to achieving optimal running time is to aggregate information on fundamental cuts during a postorder traversal of T. The aggregated information is kept in the three vertex attributes subtree Vol, intraWeight and interWeight defined in Definition 6.1 below. Technically, the three vertex attributes take the form of arrays, where indices represent vertices.

Definition 6.1 Let $C_T(u)$ be the children of vertex u in T. Moreover, let T(u) denote the subtree rooted at u (including u), and let D(u) (descendants of u) denote the set that contains the vertices of T(u), i. e. D(u) = V(T(u)). We use the following three vertex attributes to aggregate information that we need to compute the conductance values:

- subtree Vol[u] = vol(D(u)).
- intraWeight[u] equals twice the total weight of all edges $e = \{v, w\}$ with (i) $v, w \in D(u)$, (ii) $v, w \neq u$ and (iii) the lowest common ancestor of v and w in T is u (blue edges in Figure 1b) plus the total weight of all edges not in T with one end vertex being u and the other end vertex being contained in D(u) (green edges in Figure 1b).

Algorithm 1 Procedure NonLeaf(T, u) called during postorder traversal of T

```
1: parentEdge \leftarrow undefined_edge
 2: for all f = \{u, t\} \in E do
         if f \in E(T) then
 3:
 4:
              if label[u] < label[t] then
 5:
                    subtree Vol[u] \leftarrow subtree Vol[u] + subtree Vol[t]
 6:
                    interWeight[u] \leftarrow interWeight[u] + interWeight[t]
 7:
              else
 8:
                   parentEdge \leftarrow f
 9:
              end if
10:
11:
              if ((label[t] < label[u]) \lor (label[t] > maxLabelDescendants[u])) then
12:
                                                                            \triangleright equivalent to test if t \notin D(u)
                    lca \leftarrow LCA(T, u, t)
13:
                   intraWeight[lca] \leftarrow intraWeight[lca] + \omega(f)
14:
15:
                    interWeight[u] \leftarrow interWeight[u] + \omega(f)
16:
               end if
17:
          end if
18: end for
19: subtree Vol[u] \leftarrow subtree Vol[u] + vol(\{u\})
20: interWeight[u] \leftarrow interWeight[u] - intraWeight[u];
21: if parentEdge \neq undefined\_edge then
          \operatorname{cond}(\operatorname{parentEdge}, T) \leftarrow \frac{\operatorname{interWeight}[u] + \omega(\operatorname{parentEdge})}{\min\{\operatorname{subtreeVol}[u], \operatorname{vol}(V) - \operatorname{subtreeVol}[u]\}}
22:
23: end if
```

- interWeight[u] equals the total weight of all edges not in T with exactly one end vertex in D(u) (red edges in Figure 1b).

If u has a parent edge e_T , Eq. 3.3 takes the form

$$cond(e_T, T) = \frac{interWeight[u] + \omega(e_T)}{\min\{subtreeVol[u], vol(V) - subtreeVol[u]\}}$$
(6.1)

When computing subtree Vol, intraWeight and interWeight, we employ two vertex labellings (stored in arrays indexed by the vertices): label[u] indicates the preorder label of u in T, and maxLabelDescendants[u] indicates the maximum of label[t] over all $t \in T(u)$. We also need lowest common ancestors (LCAs). Queries LCA(T, u, v), i. e. the LCA of u and v on T, require constant time after an $\mathcal{O}(n)$ -time preprocessing [2].

We start by initializing labels and vertex attributes to arrays of length |V| with all entries set to 0 (for details see Algorithm 2 in Section A in the appendix). Then we compute the entries of label and maxLabelDescendants in a single depth-first traversal of T and perform the preprocessing for $LCA(\cdot,\cdot,\cdot)$. Finally, we call a standard postorder traversal in T starting at the root of T. When visiting a vertex, either one of the subroutines $LEAF(\cdot)$ (see Algorithm 3 in Section A of the appendix) or $NonLEAF(\cdot)$ (see Algorithm 1) is called depending on the vertex type.

If u is a leaf, Algorithm 3 sets subtree Vol[u] to $vol(\{u\})$ and interWeight[u] to the total weight of all edges in $E \setminus E(T)$ that are incident on u. Likewise, the entry intraWeight[LCA(T, u, t)] is updated for any t with $\{u, t\} \notin E(T)$.

If u is not a leaf (Algorithm 1), and if u has a parent edge in T, this edge is found in line 8 and the corresponding conductance value is computed in line 22 using subtreeVol[u] and interWeight[u]. The entry intraWeight[LCA(T,u,t)] is updated multiple times until the postorder traversal ascends from u towards the root of T (line 14). The update of interWeight is justified in the proof of Theorem 6.2 (see Section B (appendix), it also contains the proof of Proposition 6.3). Eq. 6.5 in Theorem 6.2 guarantees that the conductance values computed in line 22 are correct.

Theorem 6.2 After having finished processing $u \in V$ in a traversal of T, the equalities given below hold (where in the last one we assume that u is not the root of T and that e_T is the parent edge of u in T).

$$subtree Vol[u] = vol(D(u)), (6.2)$$

$$intraWeight[u] = \sum_{c_i \neq c_j \in C(u)} \omega(S(D(c_i), D(c_j)))$$
(6.3)

$$+\omega(S(D(u)\setminus\{u\},\{u\})\setminus E(T))$$
 and (6.4)

$$interWeight[u] = \omega(S_T(e_T)) - \omega(e_T).$$
 (6.5)

Proposition 6.3 Given a rooted spanning tree T of G = (V, E), the computation of all $cond(e_T, T)$, $e_T \in E(T)$, takes $\mathcal{O}(|E|)$ time.

7 Experimental Results

Approach and settings. The multilevel partitioner within the KaHIP package has three different algorithm configurations: strong, eco and fast. We use the eco configuration since this configuration was chosen in [20], too. The variable ε in the balance constraint is set to the common value 0.03.

We evaluate the postprocessing and compare ex_cond with ex_alg on the basis of the 15 complex networks listed in Table 7.1 and further described in Table D.1 (appendix). The networks are from two popular archives [1,17]. The same networks have been used previously in [20] to evaluate ex_alg.

Name	#vertices	#edges
p2p-Gnutella	6 405	29 215
PGPgiantcompo	10 680	24 316
email-EuAll	16 805	60 260
as-22july06	22 963	48 436
soc-Slashdot0902	28550	379 445
loc-brightkite_edges	56 739	212 945
loc-gowalla_edges	196591	950327
coAuthorsCiteseer	227320	814 134
wiki-Talk	232 314	1 458 806
citationCiteseer	268495	1156647
coAuthorsDBLP	299 067	977 676
web-Google	356 648	2093324
coPapersCiteseer	434 102	16036720
coPapersDBLP	540486	15 245 729
as-skitter	554930	5 797 663

Table 7.1. Complex networks used as benchmark set.

Table 7.2. Geometric means of the performance quotients minMCV, avgMCV and avgTime over all networks in Table 7.1. Number of trees: 20, 100 and 200. Reference is the edge rating ex_alg. A quotient < 1.0 means that ex_cond yields better results than ex_alg.

	minMCV	avgMCV	avgTime
$\overline{\text{Ratios ex_cond}_{20} / \text{ex_alg}}$	0.892	0.897	1.793
Ratios ex_cond_{100} / ex_alg	0.874	0.893	5.278
Ratios ex_cond_{200} / ex_alg	0.865	0.890	9.411

All computations are done on a workstation with two 8-core Intel(R) Xeon(R) E5-2680 processors at 2.70GHz. Our code is implemented in C/C++ and compiled with GCC 4.7.1. Note that we do not exploit parallelism here and run sequential experiments only. First of all, we focus in this paper on solution quality, not on speed. Secondly, the standard of reference, ex_alg, is also implemented sequentially.

Since the results produced by KAHIP depend on many factors including random seeds, we perform 50 runs with different seeds for each network and compute the following three *performance indicators*:

- minMCV and avgMCV: minimal and average MCV found by KAHIP.
- minCut and avgCut: minimal and average cut found by KAHIP.
- avgTime: average time KAHIP needs for the complete partitioning process.

Postprocessing results. For ex_alg, the average reduction of avgMCV due to postprocessing amounts to 11.3% (see Table E.1 in Section E of the appendix). Since the postprocessing trades in small edge cuts for small MCVs, values for minMCV and avgMCV [minCut and avgCut] are with [without] postprocessing. The increase in running time due to postprocessing is negligible.

Edge rating results. Intriguingly, using an asymptotically optimal Range Minimum Query (RMQ) code (by Fischer and Heun [10]) within ex_cond for the algorithms in Section 6 does not decrease the running time. The straightforward asymptotically slower algorithm is slightly faster (1.1% in total) in our experiments. To investigate this effect further, we compare the results on a set of non-complex networks, Walshaw's graph partitioning archive [24]. Again, the implementation of the (in theory faster) RMQ algorithm does not play out, running time and quality remain comparable. Therefore, the running times in all tables refer to the implementation not using the Fischer/Heun RMQ code.

The edge rating ex_cond depends on the number of random spanning trees, i. e. $|\mathcal{T}|$. To make this clear we write ex_cond_{$|\mathcal{T}|$} instead of ex_cond.

For a given network we measure the quality of the edge rating $ex_cond_{|\mathcal{T}|}$ through (three) quotients of the form (performance indicator using $ex_cond_{|\mathcal{T}|}$ divided by the same performance indicator using ex_alg). Tables E.2, E.3, and E.4 in Section E of the appendix show the performance quotients of ex_cond_{20} , ex_cond_{100} and ex_cond_{200} . The geometric means of the performance quotients over all networks are shown in Table 7.2.

As the main result we state that buying quality through increasing $|\mathcal{T}|$ is expensive in terms of running time. The rating ex_cond₂₀ already yields avgMCV that is 10.3% lower than avgMCV from ex_alg — at the expense of a relative increase in running time by only 1.79. The total reduction of average MCV from postprocessing and replacing ex_alg by ex_cond₂₀ amounts to 20.4% (see Tables E.1 and E.3 in the appendix).

It is further interesting to note that, when we omit the postprocessing step and compare the average edge cut instead of MCV, ex_alg and ex_cond perform comparably well. While ex_cond yields a slightly better minimum cut, ex_alg yields a slightly better average cut (see Table E.5 in Section E of the appendix).

8 Conclusions and Future Work

Motivated by the deficits of coarsening complex networks during multilevel graph partitioning, we have devised a new edge rating for guiding edge contractions. The new rating of an edge indicates whether it is part of a good moderately balanced conductance-based cut or not. To compute the necessary conductance values efficiently, we have developed the first linear-time algorithm to compute the conductance values of all fundamental cuts of a spanning tree. Our evaluation shows a significant improvement over a previously leading code for partitioning complex networks. The new edge rating and additional greedy postprocessing combined result in a 20.4% better maximum communication volume.

We would like to stress that good coarsening is not only of interest for graph partitioning, but can be employed in many other methods and applications that exploit hierarchical structure in networks. Future work should investigate the concurrence of the contrast γ and the conductance values Cond — possibly replacing γ by an even better contrast yet to be found. Our overall coarsening scheme is agnostic to such a replacement and would require no further changes. Moreover, we would like to extend our methods to an arbitrary number of blocks. While the proposed edge rating should work out of the box, the greedy MCV minimization has to be adapted to work effectively for a larger number of blocks.

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A Pseudocode of Algorithms in Section 6

Algorithm 2 Given a spanning tree T of G = (V, E) with root r, compute $\operatorname{cond}(e_T, T)$ for all $e_T \in E(T)$

- 1: Set label, maxLabelDescendants, subtreeVol, intraWeight, interWeight to 0
- 2: Compute label[u] and maxLabelDescendants[u] for all $u \in V$.
- 3: Perform LCA preprocessing
- 4: Postorder(T, r)

Algorithm 3 Procedure Leaf(T, u) called during postorder traversal of T

```
1: subtreeVol[u] \leftarrow vol(\{u\})
 2: parentEdge \leftarrow undefined_edge
 3: for all f = \{u, t\} \in E do
 4:
           if f \in E(T) then
                parentEdge \leftarrow f
 5:
 6:
           else
 7:
                lca \leftarrow LCA(T, u, t)
                intraWeight[lca] \leftarrow intraWeight[lca] + \omega(f)
 8:
 9:
                interWeight[u] \leftarrow interWeight[u] + \omega(f)
10:
           end if
11: end for
12: if parentEdge \neq undefined\_edge then
           \operatorname{cond}(\operatorname{parentEdge},T) \leftarrow \frac{\operatorname{interWeight}[u] + \omega(\operatorname{parentEdge})}{\min\{\operatorname{subtreeVol}[u],\operatorname{vol}(V) - \operatorname{subtreeVol}[u]\}}
13:
14: end if
```

B Proofs of Section 6

B.1 Proof of Theorem 6.2

Proof. We prove Eq.s 6.2, 6.4 and 6.5 one after the other. Colors correspond to the edge types introduced in Definition 6.1.

- 1. Due to line 1 of Algorithm 3, $subtree Vol[u] = vol(\{u\}) = vol(D(u))$ for any leaf u of T. If u is not a leaf of T, we proceed by induction. Specifically, we assume inductively $subtree Vol[c_i] = vol(D(c_i))$ for all children c_i of u. Due to lines 5 and 19 of Algorithm 1, $subtree Vol[u] = \sum_{c_i \in C(u)} vol(D(c_i)) + vol(\{u\}) = vol(D(u))$.
- 2. Due to line 8 of Algorithm 3 and line 14 of Algorithm 1,

$$intraWeight[u] = \sum (\omega_{v,t} \mid v \in D(c_i) \text{ for some } c_i \in C(u),$$

$$\{v,t\} \in E \setminus E(T), \ LCA(v,t) = u\}$$

$$= \sum (\omega_{v,t} \mid v \in D(c_i) \text{ for some } c_i \in C(u), \ t \in D(c_j))$$
for some $c_j \in C(u), c_j \neq c_i +$

$$\sum (\omega_{v,u} \mid v \in D(c_i) \text{ for some } c_i \in C(u))$$

$$\{v,u\} \in E \setminus E(T)\}$$

$$= \sum_{c_i \neq c_j \in C(u)} \omega(S(D(c_i), D(c_j))) +$$

$$\omega((E \setminus E(T)) \cap S(D(u) \setminus \{u\}, \{u\})).$$

Note that a blue edge contributes twice to intraWeight[u] since it is encountered from both endpoints. A green edge, on the other hand, contributes only once.

3. Due to line 9 of Algorithm 3, $interWeight[u] = \sum_{e=\{u,t\},e\in E\setminus E(T)} \omega(e) = \omega(S_T(e_T)) - \omega(e_T)$ for any leaf u of T. If u is not a leaf of T, let $e_i = \{u, c_i\}$ be the edges between u and its children c_i . In particular, $e_i \in E(T)$. By induction we may assume $interWeight[c_i] = \omega(S_T(e_i)) - \omega(e_i)$. Thus, line 6 of Algorithm 1, where we take the sum over all interWeight values of u's children c_i , yields

$$\begin{split} interWeight[u] &= \sum (\omega_{v,t} \mid \{v,t\} \in E \setminus E(T) \wedge \exists c_i \in C(u): \\ &\quad v \in D(c_i), t \notin D(c_i)) \\ &= \sum (\omega_{v,t} \mid v \in D(c_i) \text{ for some } c_i \in C(u) \wedge t \in D(c_j)) \\ &\quad \text{for some } c_j \in C(u), c_j \neq c_i) \\ &\quad + \sum (\omega_{v,u} \mid \{v,u\} \in E \setminus E(T) \wedge v \in D(u) \setminus \{u\}) \\ &\quad + \sum (\omega_{v,t} \mid \{v,t\} \in E \setminus E(T) \wedge v \in D(u) \setminus \{u\}, \ t \notin D(u)) \end{split}$$

Note that this is an intermediate result of interWeight[u]. The blue and green terms make up intraWeight[u], which we still have to subtract. Moreover, the red term does not yet contain the (weights of) edges in T with one end vertex being u and the other one not being contained in D(u). In the following, we replace the blue and the green term by intraWeight[u] and rewrite the red term.

$$\begin{split} interWeight[u] &= intraWeight[u] + \sum (\omega_{v,t} \mid \{v,t\} \in E \setminus E(T)), \\ &\quad v \in D(u) \setminus \{u\}, \ t \notin D(u) \\ &= intraWeight[u] + \sum (\omega_{v,t} \mid \{v,t\} \in E \setminus E(T), \\ &\quad v \in D(u), \ t \notin D(u)) - \\ &\quad \sum (\omega_{u,t} \mid \{u,t\} \in E \setminus E(T), \ t \notin D(u)) \\ &= intraWeight[u] + \omega(S_T(e_T)) - \omega(e_T) - \\ &\quad \sum (\omega_{u,t} \mid \{u,t\} \in E \setminus E(T), \ t \notin D(u)) \end{split}$$

Finally, line 15 of Algorithm 1 results in $interWeight[u] = intraWeight[u] + \omega(S_T(e_T)) - \omega(e_T)$, and line 20 of Algorithm 1 yields Eq. 6.5.

B.2 Proof of Proposition 6.3

Proof. All initialization and preprocessing steps can be done in $\mathcal{O}(n)$ time. During the postorder traversal of T each $v \in V$ explores its direct neighborhood, either in Leaf or in Nonleaf. Two observations are crucial now. First, all elementary operations within Leaf and Nonleaf take constant time, including the LCA queries. Second, for each edge of G the respective operations are executed at most twice.

C MCV Postprocessing

We now show how one round of the postprocessing algorithm that optimizes MCV can be implemented in $\mathcal{O}(|E|)$ time. The crucial step is to decide if moving a vertex v to the opposite block reduces MCV in $O(\deg(v))$ time. To do so, we need a few notations. An *internal vertex* is a vertex of the graph which is not a boundary vertex, and the external degree of a vertex is defined as the number of neighbors in the opposite block. Let (V_1, V_2) be a bipartition of G and, without loss of generality, let v be a random boundary vertex from block V_1 . During the course of the algorithm, we keep track of the communication volumes of the blocks. Let C_1 and C_2 be the initial communication volume of V_1 and V_2 , respectively. We do the following to decide if moving v to the opposite block reduces MCV. First, we move v to the opposite block V_2 . Afterwards, the communication volume C_2 is reduced by the number of boundary vertices in V_2 that are also neighbors of v and become internal vertices after the movement. Moreover, the communication volume C_1 is increased by the amount of internal vertices in V_1 that become boundary vertices after v is moved to V_2 . Additionally, since we move v, the communication volume C_1 is reduced by one, and if the number of neighbors of v in V_1 is not zero, then C_2 is increased by one. Note that we can check in constant time if a vertex is a boundary vertex or an internal vertex by storing the external degree of all vertices in an array and updating the external degree of a vertex and its neighbors when the vertex is moved. We move v back to its origin if the movement did not yield an improvement in MCV.

D Test Set of Complex Networks

Table D.1. Complex networks used for comparing ex_cond and ex_alg.

Name	#vertices	#edges	Network Type
p2p-Gnutella	6 405	29 215	filesharing network
PGPgiantcompo	10680	24 316	largest connected component in network of PGP users
email-EuAll	16805	60 260	network of connections via email
as-22july06	22 963	48 436	network of autonomous systems in the internet
soc-Slashdot0902	28 550	379 445	news network
loc-brightkite_edges	56 739	212 945	location-based friendship network
loc-gowalla_edges	196 591	950 327	location-based friendship network
coAuthorsCiteseer	227 320	814 134	citation network
wiki-Talk	232 314	1458806	network of user interactions through edits
citationCiteseer	268 495	1156647	citation network
coAuthorsDBLP	299 067	977 676	citation network
web-Google	356 648	2093324	hyperlink network of web pages
coPapersCiteseer	434 102	16036720	citation network
coPapersDBLP	540 486	15245729	citation network
as-skitter	554 930	5 797 663	network of internet service providers

E Details of Experimental Results

Table E.1. Effect of postprocessing on performance of ex_alg. Numbers are ratios of the performance indicators minMCV and avgMCV with and without postprocessing. A ratio r < 1.0 means that postprocessing reduces MCV by 100(1-r)%.

	minMCV	avgMCV
PGPgiantcompo	0.951	0.924
as-22july06	0.913	0.824
email-EuAll	0.858	0.859
loc -brightkite_edges	0.765	0.753
p2p-Gnutella04	0.866	0.862
soc-Slashdot0902	0.921	0.912
citationCiteseer	0.939	0.919
coAuthorsCiteseer	0.975	0.955
coAuthorsDBLP	0.905	0.898
loc-gowalla_edges	0.887	0.870
web-Google	0.965	0.926
wiki-Talk	0.994	0.974
as-skitter	0.939	0.937
coPapersCiteseer	0.892	0.877
coPapersDBLP	0.856	0.841
Geometric mean	0.907	0.887

Table E.2. Performance quotients of ex_cond_{20} with postprocessing for minMCV, avgMCV and avgTime. Reference is ex_alg with postprocessing. A quotient < 1.0 means that ex_cond_{20} yields better results than ex_alg .

	minMCV	avgMCV	avgTime
PGPgiantcompo	0.962	1.004	3.068
as-22july06	0.898	0.766	1.387
email-EuAll	0.904	0.918	1.537
loc -brightkite_edges	0.714	0.714	1.190
p2p-Gnutella04	1.003	1.000	2.255
soc-Slashdot0902	0.991	0.999	3.045
citationCiteseer	1.001	0.974	1.938
coAuthorsCiteseer	1.090	1.053	2.842
coAuthorsDBLP	0.743	0.774	1.461
loc-gowalla_edges	0.608	0.607	0.642
web-Google	0.820	0.729	3.479
wiki-Talk	0.992	1.023	1.142
as-skitter	0.769	0.760	0.924
coPapersCiteseer	1.011	1.260	2.575
coPapersDBLP	1.035	1.135	2.448
Geometric mean	0.892	0.897	1.793

Table E.3. Performance quotients of ex_cond_{100} with postprocessing for minMCV, avgMCV and avgTime. Reference is ex_alg with postprocessing. A quotient < 1.0 means that ex_cond_{100} yields better results than ex_alg .

	minMCV	avgMCV	avgTime
PGPgiantcompo	0.986	0.998	12.379
as-22july06	0.750	0.760	2.373
email-EuAll	0.874	0.917	3.183
loc-brightkite_edges	0.715	0.715	3.836
p2p-Gnutella04	0.995	0.995	6.670
soc-Slashdot0902	0.920	0.987	10.801
citationCiteseer	1.017	0.972	7.485
coAuthorsCiteseer	1.068	1.035	10.567
coAuthorsDBLP	0.737	0.776	5.282
loc-gowalla_edges	0.624	0.609	1.929
web-Google	0.837	0.729	13.507
wiki-Talk	0.987	1.015	1.194
as-skitter	0.793	0.769	2.986
coPapersCiteseer	0.991	1.245	8.567
coPapersDBLP	0.970	1.118	7.951
Geometric mean	0.874	0.893	5.278

Table E.4. Performance quotients of ex_cond_{200} with postprocessing for minMCV, avgMCV and avgTime. Reference is ex_alg with postprocessing. A quotient < 1.0 means that ex_cond_{200} yields better results than ex_alg .

	minMCV	avgMCV	avgTime
PGPgiantcompo	0.976	1.000	24.258
as-22july06	0.694	0.735	3.645
email-EuAll	0.874	0.927	5.262
$\overline{\text{loc-brightkite_edges}}$	0.692	0.711	7.215
p2p-Gnutella04	0.997	0.997	12.313
soc-Slashdot0902	0.913	0.959	20.883
citationCiteseer	1.013	0.974	14.589
coAuthorsCiteseer	1.026	1.035	20.412
coAuthorsDBLP	0.740	0.773	10.157
loc-gowalla_edges	0.615	0.611	3.574
web-Google	0.843	0.729	26.421
wiki-Talk	0.986	1.017	1.217
as-skitter	0.775	0.760	5.588
coPapersCiteseer	1.035	1.255	16.131
coPapersDBLP	0.958	1.122	14.881
Geometric mean	0.865	0.890	9.411

Table E.5. Performance quotients of ex_cond_{100} without postprocessing for minCut, avgCut and avgTime. Reference is ex_alg without postprocessing. A quotient < 1.0 means that ex_cond_{100} yields better results than ex_alg .

	minMCV	avgMCV	avgTime
PGPgiantcompo	1.036	1.021	13.405
as-22july06	0.858	0.904	2.387
email-EuAll	0.976	0.894	3.248
loc-brightkite_edges	1.026	1.032	3.964
p2p-Gnutella04	0.951	0.959	7.554
soc-Slashdot0902	0.943	1.258	13.083
citationCiteseer	0.986	0.941	8.000
coAuthorsCiteseer	1.118	1.110	11.420
coAuthorsDBLP	0.795	0.885	5.554
loc-gowalla_edges	1.100	1.093	1.948
web-Google	0.826	0.699	15.660
wiki-Talk	1.032	1.033	1.194
as-skitter	1.007	1.007	3.085
coPapersCiteseer	1.150	1.391	13.578
coPapersDBLP	1.034	1.252	12.045
Geometric mean	0.984	1.018	5.915