The $PH/PH/1$ Multi-threshold Queue

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Abstract. We consider a $PH/PH/1$ queue in which a threshold policy determines the stage of the system. The arrival and service processes follow a Phase-Type (PH) distribution depending on the stage of the system. Each stage has both a lower and an upper threshold at which the stage of the system changes, and a new stage is chosen according to a prescribed distribution. The $PH/PH/1$ multi-threshold queue is a Quasi-Birth-and-Death process with a tri-diagonal block structured boundary state which we model as a Level Dependent Quasi-Birth-and-Death process. An efficient algorithm is presented to obtain the stationary queue length vectors using Matrix Analytic methods.

Keywords: $PH/PH/1$ queue, multiple thresholds, Matrix Analytic methods, Quasi-Birth-and-Death process, tri-diagonal block structured boundary state.

1 Introduction

We consider a $PH/PH/1$ queue in which a threshold policy determines the stage of the system. The arrival and service processes follow a Phase-Type (PH) distribution depending on the stage of the system. Each stage has both a lower and an upper threshold at which the stage of the system changes. At these thresholds a new stage is chosen according to a prescribed distribution.

In literature, threshold policies are often used to activate or deactivate servers when the queue length reaches certain thresholds. The $M/M/2$ queue in which the second server is activated when the queue length reaches an upper threshold and deactivated when it reaches a lower threshold is studied in [11], where a closed form expression is obtained for the steady-state probabilities. In [13], see also Section 4.2, closed form expressions are obtained for the steady-state distributions for the $M/M/c$ with c heterogeneous servers. Using Green's function, Ibe and Keilson [9] studied the $M/M/c$ queue with homogeneous servers and the $M/M/2$ queue with heterogeneous servers. The $M/M/c$ with heterogeneous servers is also studied in [14] where the steady-state probabilities are obtained using a stochastic complement analysis for uncoupling Markov Chains. A $MAP/M/c$ with homogeneous servers is analysed in [4] and the $PH/M/2$ queue with heterogeneous servers is studied by Neuts [16]. In [5], see also Section 4.3, a very general setting is studied in which the generator of the queueing

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system forms a nested Quasi-Birth-and-Death process. In this model a threshold policy controls the stage of the system which, in turn, determines the arrival process and the service process. An upper threshold increases the stage by one whereas the the lower threshold decreases the stage by one, creating a *staircase* threshold policy. In [12] an $M/M/2$ queue is studied with two heterogeneous servers in which the second server is exponentially delayed before activation.

Threshold policies are also used to send servers to a certain queue, as is shown in [7]. In this paper, a system is studied containing two queues and two servers where both interarrival times and service times are exponentially distributed. After each service completion, the server chooses a queue to serve according to a threshold policy. A generalisation of this model is analysed in [6] where customers from multiple classes arrive according to a Poisson process and require an exponential amount of service. The queueing system contains a fixed number of servers which are allocated to a customer class according to a threshold policy. Each server experiences an exponential delay once it is assigned to a different customer class. In [17], the joint queue length distribution is obtained for an $M/G/1$ queue with multiple customer classes in which customers from higher class are blocked when thresholds are reached.

Motivating Example. The queueing system in this paper is motivated by the hysteretic relation between density and speed of traffic flows observed on a highway, see Helbing [8]. In [8] it is stated that this hysteretic behaviour is controlled by two critical densities, denoted by ρ_1 and ρ_2. When the density of cars on the highway increases vehicles are more and more affected by each other and the driving speeds decrease. Once the density reaches ρ_2 the highway becomes congested and driving speeds decrease drastically. The density must reduce to ρ_1 for the highway to become non-congested. In Baer, Boucherie and van Ommeren [2], an $M/M/1$ threshold queue was used to model a particular highway section. In [2], the arrival rates were kept constant, whereas the service rates where altered according to a 2-stage threshold policy. When the queue length surpasses an upper threshold the service rates decreased. The service rates were increased again when the queue length dropped below a lower threshold. In [2], the mean sojourn time is determined. Since a single queue represents a highway section, this directly gives the average time to cross the highway section and the mean speed of a vehicle. The motivating example in Figure 1 is an extension to the model in [2], where not only the service rates are controlled by a threshold policy, but also the arrival rates. This models the hysteretic relation within a highway section, but also between two consecutive highway sections. We will, get back to this example in Section 4.1.

Contribution. This paper generalises the model of [5] to an arbitrary threshold policy and introduces a novel dedicated solution method based on the Level Dependent Quasi-Birth-and-Death process of [3]. In particular, a class of $PH/PH/1$ multi-threshold queueing systems is described for which the solution method in [3] can be decomposed to find the stationary queue length vector for each stage separately. The stationary distribution of the $PH/PH/1$ multi-threshold queue

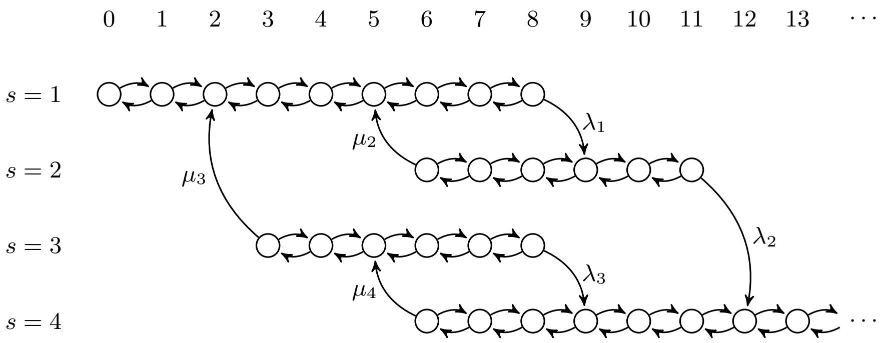

Fig. 1. State Diagram

can be obtained using the results in [3] but for a large number of stages, this may result in computational demanding calculations. In this paper we use the structure of the $PH/PH/1$ multi-threshold queue to form, based on the results in [3], smaller and easier equations to obtain the stationary distribution.

Overview. Section 2 introduces the $PH/PH/1$ multi-threshold queue and presents the queueing system as a Level Dependent Quasi-Birth-and-Death process. In Section 3 we analyse the multi-threshold queue using Matrix Analytic methods and obtain the stationary queue length probabilities. Furthermore, we present a decomposition theorem for a class of multi-threshold queues providing an explicit description of the stationary queue length probability vectors. In Section 4 we illustrate our results via three multi-threshold queues obtained from literature. Section 5 gives concluding remarks.

2 Model Description

Consider a $PH/PH/1$ queue, controlled by a threshold policy. The system can be in different stages $s = 1 \ldots, S$, where every stage s is associated with a set of feasible queue lengths $\{L_s, \ldots, U_s\}$. The quantities L_s and U_s are the lower, respectively upper thresholds for stage s. In case $U_s = \infty$, we say that stage s has no upper threshold. For each queue length $n = 0, 1, \ldots$, a stage s is a *potential* stage when $L_s \leq i \leq U_s$. If the system is in stage s and a departure or arrival causes the queue length to drop below L_s or to exceed U_s, the stage of the system changes (the threshold policy). If the queue length increases to $U_s + 1$ the stage changes from s to t with probability $p_{s,t}$. Note that $p_{s,t} > 0$ implies that t is a potential stage for queue length $U_s + 1$. If the queue length decreases to $L_s - 1$ the stage changes from s to t with probability $q_{s,t}$. See Figure 1 for an illustration with exponential service times and Poisson arrivals.

The arrival process in stage s follows a $PH(\boldsymbol{\Lambda}_s, \boldsymbol{\lambda}_s)$ distribution of $v_s + 1$ phases (v_s transient phases and 1 absorbing phase). We define $\boldsymbol{\Lambda}_s^0 = -\boldsymbol{\Lambda}_s \boldsymbol{e}_{v_s}$, with $\boldsymbol{e}_{v_s}$ a $v_s \times 1$ vector of ones. Furthermore we assume that the absorbing state is never chosen as initial state, i.e. $\boldsymbol{\lambda}_s \boldsymbol{e}_{v_s} = \mathbf{1}$. Similarly, the service process in stage s is $PH(\boldsymbol{M}_s, \boldsymbol{\mu}_s)$ distributed with w_s+1 phases. We define $\boldsymbol{M}_s^0 = -\boldsymbol{M}_s \boldsymbol{e}_{w_s}$ and assume $\boldsymbol{\mu}_s \boldsymbol{e}_{w_s} = \mathbf{1}$. The mean interarrival times and mean service time if given by $-\boldsymbol{\lambda}_j \boldsymbol{\Lambda}_j^{-1} \boldsymbol{e}_{v_j}$ and $-\boldsymbol{\mu}_j \boldsymbol{M}_j^{-1} \boldsymbol{e}_{w_j}$, see Neuts [15].

When an arrival or departure changes the stage of the system both the arrival process and service process are reset by choosing a new initial phase for both processes according to the distributions of the new stage.

This $PH/PH/1$ multi-threshold queue can be modelled as a four-dimensional Markov Chain (i, s, x, y) where i and s represent the queue length and stage of the system, $x = 1, \ldots, v_s$ the phase of the arrival process and $y = 1, \ldots, w_s$ the phase of the service process. This queueing system is a Quasi-Birth-and-Death process (QBD) [10] in which the levels are represented by the queue length i, with $i > \max_s\{U_s\}$. Modelling the system as a QBD-process results in a boundary level (level 0) containing the entire threshold policy. By ordering the states lexicographically a tri-diagonal block structure emerges in the boundary level. This structure is utilised by modelling the queueing system as a Level Dependent Quasi-Birth-and-Death process (LDQBD) [3] in which the levels of the LDQBD are the queue length i. We stress that, from here on, we refer to the queue lenght as the level of the LDQBD. The other three variables represent the phase within a level. The states are ordered lexicographically in (i, s, x, y).

The generator $\boldsymbol{Q}$ for this LDQBD is:

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{L}^{(0)} & \boldsymbol{F}^{(0)} & \mathbf{0} & \cdots & & & \\ \boldsymbol{B}^{(1)} & \boldsymbol{L}^{(1)} & \boldsymbol{F}^{(1)} & \ddots & & & \\ \mathbf{0} & \boldsymbol{B}^{(2)} & \boldsymbol{L}^{(2)} & \ddots & & & \\ \vdots & \ddots & \ddots & \ddots & \boldsymbol{F}^{(i-1)} & & \\ & & & \boldsymbol{B}^{(i)} & \boldsymbol{L}^{(i)} & \ddots & \\ & & & & \ddots & \ddots & \end{bmatrix}, \tag{1}$$

where $\boldsymbol{B}^{(i)}$ denotes the *backward* transitions (departures) from level i to level $i-1$, $\boldsymbol{L}^{(i)}$ the *local* transitions within level i and $\boldsymbol{F}^{(i)}$ the *forward* transitions (arrivals) from level i to level $i+1$.

If the number of potential stages for level $i-1$, i and $i+1$, are ℓ, m and n respectively, $\boldsymbol{B}^{(i)}$ is a $m \times \ell$ matrix of submatrices $\boldsymbol{B}_{(j,k)}^{(i)}$, $\boldsymbol{L}^{(i)}$ is a $m \times m$ matrix of submatrices $\boldsymbol{L}_{(j,k)}^{(i)}$ and $\boldsymbol{F}^{(i)}$ is a $m \times n$ matrix of submatrices $\boldsymbol{F}_{(j,k)}^{(i)}$, describing the backward, local and forward transition rates from stage j to stage k. Let $\boldsymbol{I}_t$ denote the $t \times t$ identity matrix and let $\otimes$ denote the Kronecker product. For $s = 1, \ldots, S$, the forward, local and backward submatrices are given by:

$$\boldsymbol{F}^{(i)}_{(s,j)} = \begin{cases} \boldsymbol{\Lambda}^0_s \otimes \boldsymbol{\lambda}_s \otimes \boldsymbol{I}_{w_s}, & \text{if } j = s \text{ and } L_s \le i < U_s, \\ p_{s,j} \cdot \boldsymbol{\Lambda}^0_s \otimes \boldsymbol{e}_{w_s} \otimes \boldsymbol{\lambda}_j \otimes \boldsymbol{\mu}_j, & \text{if } i = U_s, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \tag{2}$$

$$\boldsymbol{L}^{(i)}_{(s,j)} = \begin{cases} \boldsymbol{\Lambda}_s \otimes \boldsymbol{I}_{w_s} + \boldsymbol{I}_{v_s} \otimes \boldsymbol{M}_s, & \text{if } j = s, i > 0 \text{ and } L_s \le i \le U_s, \\ \boldsymbol{\Lambda}_s \otimes \boldsymbol{I}_{w_s}, & \text{if } j = s, i = 0 \text{ and } L_s = 0, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \tag{3}$$

$$\boldsymbol{B}^{(i)}_{(s,j)} = \begin{cases} \boldsymbol{I}_{v_s} \otimes \boldsymbol{M}^0_s \otimes \boldsymbol{\mu}_s, & \text{if } j = s \text{ and } L_s < i \le U_s, \\ q_{s,j} \cdot \boldsymbol{e}_{v_s} \otimes \boldsymbol{M}^0_s \otimes \boldsymbol{\lambda}_j \otimes \boldsymbol{\mu}_j, & \text{if } i = L_s, \\ \boldsymbol{0}, & \text{otherwise.} \end{cases} \tag{4}$$

These formulas can be obtained by closely observing the queueing system. Consider, for instance, the forward transition matrices $\boldsymbol{F}^{(i)}_{(s,j)}$. When $L_s \le i < U_s$ the stage cannot change upon an arrival, so $j = s$. Now, with rate $\boldsymbol{\Lambda}^0_s$ an arrival occurs at which an initial state is chosen with probability $\boldsymbol{\lambda}_s$, independent of the phase of the service process. The stage will change when an arrival occurs when $i = U_s$. Now, with rate $\boldsymbol{\Lambda}^0_s$, independent of the phase of the service process, an arrival occurs and the stage changes from s to j with probability $p_{s,j}$. During this event an initial phase is chosen for both the arrival process *and* the service process respectively probability $\boldsymbol{\lambda}_j$ and $\boldsymbol{\mu}_j$. Similar reasoning gives the relations for $\boldsymbol{L}^{(i)}_{(s,j)}$ and $\boldsymbol{B}^{(i)}_{(s,j)}$.

Remark 1. Note that modelling the queueing system as a QBD-process results in the following generator

$$\widetilde{\boldsymbol{Q}} = \begin{bmatrix} \widetilde{\boldsymbol{Q}}_{00} & \widetilde{\boldsymbol{Q}}_{01} & \boldsymbol{0} & \cdots & \cdots \\ \widetilde{\boldsymbol{Q}}_{10} & \boldsymbol{L} & \boldsymbol{F} & \ddots & \\ \boldsymbol{0} & \boldsymbol{B} & \boldsymbol{L} & \boldsymbol{F} & \ddots \\ \vdots & \ddots & \boldsymbol{B} & \boldsymbol{L} & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{5}$$

The threshold policy, in the LDQBD-process described by the levels $0, \ldots, U_{max}$, is now described in the submatrix $\widetilde{\boldsymbol{Q}}_{00}$ with

$$U_{max} = 1 + \max\{U_s \ : \ s = 1, \ldots, S, \ U_s < \infty\}.$$

Finding the stationary distribution for the QBD-process, i.e. solving $\pi\widetilde{\boldsymbol{Q}} = 0$, would also include solving

$$\pi_0\widetilde{\boldsymbol{Q}}_{00} + \pi_1\widetilde{\boldsymbol{Q}}_{10} = 0,$$

with π_0 and π_1 denoting the stationary distribution of the entire threshold policy and of the first level in the QBD-process respectively. By modelling the queueing system as the LDQBD-process in (1) we split up level 0 in the QBD-process (5) into smaller blocks such that the stationary distribution π is easier obtained.

3 Steady-State Analysis

In the previous section we modelled the $PH/PH/1$ multi-threshold queue as a LDQBD. In this section, following the analysis in [3] we obtain the steady-state probabilities of the Markov Chain using Matrix Analytic methods. The special structure of our generator allows us to obtain an efficient algorithm for the $\boldsymbol{R}$-matrices.

We assume the queueing system is stable, i.e., the mean service time is less than the mean interarrival time, see [15], in stages without upper threshold:

$$-\boldsymbol{\mu}_j \boldsymbol{M}_j^{-1} \boldsymbol{e}_{w_j} < -\boldsymbol{\lambda}_j \boldsymbol{\Lambda}_j^{-1} \boldsymbol{e}_{v_j}, \qquad \text{for } j \text{ such that } U_j = \infty.$$

The equilibrium distribution $\boldsymbol{\pi} = [\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots]$ is then given, see Bright and Taylor [3], by

$$\boldsymbol{\pi}_n = \boldsymbol{\pi}_0 \prod_{i=0}^{n-1} \boldsymbol{R}^{(i)},$$

where $\boldsymbol{R}^{(i)}$ is the minimal non-negative solution to

$$\boldsymbol{F}^{(i)} + \boldsymbol{R}^{(i)} \boldsymbol{L}^{(i+1)} + \boldsymbol{R}^{(i)} \boldsymbol{R}^{(i+1)} \boldsymbol{B}^{(i+2)} = \boldsymbol{0}, \tag{6}$$

with $\boldsymbol{0}$ the zero matrix, see [3]. The element $[\boldsymbol{R}^{(i)}]_{(r,t)}$ describes the mean sojourn time in state $(i+1,t)$ per unit sojourn time in the state (i,r) before returning to level i, given that the process started in state (i,r) see p. 499 in [3]. The $\boldsymbol{R}^{(i)}$-matrices can be obtained using the algorithm for LDQBD's by Bright and Taylor [3]. For later convenience, by analogy of $\boldsymbol{F}^{(i)}_{(j,k)}$, $\boldsymbol{L}^{(i)}_{(j,k)}$ and $\boldsymbol{B}^{(i)}_{(j,k)}$, we define the submatrix $\boldsymbol{R}^{(i)}_{(j,k)}$ of $\boldsymbol{R}^{(i)}$ in which the element $[\boldsymbol{R}^{(i)}_{(j,k)}]_{(r,t)}$ describes the mean sojourn time in state $(i+1,t)$ and stage k per unit sojourn time in state (i,r) and stage j before returning returning to level i, given that the process started in state (i,r) and stage j.

We obtain $\boldsymbol{\pi_0}$ by solving the boundary condition:

$$\boldsymbol{\pi}_0 \boldsymbol{L}^{(0)} + \boldsymbol{\pi}_1 \boldsymbol{B}^{(1)} = \boldsymbol{\pi}_0 \left(\boldsymbol{L}^{(0)} + \boldsymbol{R}^{(0)} \boldsymbol{B}^{(1)} \right) = \boldsymbol{0},$$

and the normalising equation:

$$1 = \sum_{n=0}^{\infty} \boldsymbol{\pi}_n \boldsymbol{e} = \boldsymbol{\pi}_0 \left(\boldsymbol{I} + \sum_{n=1}^{\infty} \prod_{i=0}^{n-1} \boldsymbol{R}^{(i)} \right) \boldsymbol{e}.$$

Above level $U_{\max}$ only stages without upper threshold are active and we may define $\boldsymbol{F} = \boldsymbol{F}^{(i)}$, $\boldsymbol{L} = \boldsymbol{L}^{(i)}$ and $\boldsymbol{B} = \boldsymbol{B}^{(i)}$, $i \geq U_{max}$, i.e., the LDQBD is level independent from level U_{max} upwards. We have $\boldsymbol{R}^{(i)} = \boldsymbol{R}$, $i \geq U_{max}$, where $\boldsymbol{R}$ is the minimal nonnegative solution of

$$\boldsymbol{F} + \boldsymbol{R}\boldsymbol{L} + \boldsymbol{R}^2 \boldsymbol{B} = \boldsymbol{0}. \tag{7}$$

The LDQBD is level independent from level U_{max}. Therefore, the matrices $\boldsymbol{F}$, $\boldsymbol{L}$, $\boldsymbol{B}$ and $\boldsymbol{R}$ are diagonal block matrices. As a consequence, (7) reduces to the matrix equation for the submatrices $\boldsymbol{R}_{(s,s)}$ of $\boldsymbol{R}$

$$\boldsymbol{F}_{(s,s)} + \boldsymbol{R}_{(s,s)} \boldsymbol{L}_{(s,s)} + \boldsymbol{R}^2_{(s,s)} \boldsymbol{B}_{(s,s)} = \boldsymbol{0}, \qquad \text{for } s \text{ such that } U_s = \infty. \tag{8}$$

For $i < U_{max}$, the matrices $\boldsymbol{R}^{(i)}$ are obtained from (6) by iteration

$$\boldsymbol{R}^{(i)} = -\boldsymbol{F}^{(i)} \left[\boldsymbol{L}^{(i+1)} + \boldsymbol{R}^{(i+1)}\boldsymbol{B}^{(i+2)}\right]^{-1}, \qquad i = 0, 1, \ldots, U_{max} - 1. \tag{9}$$

Following the appendix in [3] the inverse exists and has only non-positive elements so that $\boldsymbol{R}^{(i)}$, given by (9), is the unique non-negative solution to (6).

Notice that, unlike [3], we do not need to truncate the iteration for large i, as the structure of our multi-threshold queue guarantees the existence of $U_{max} < \infty$, or for $U_{max} = \infty$ reduces to a single stage.

For a special class of multi-threshold queue the submatrices $\boldsymbol{R}^{(i)}_{(j,k)}$ of $\boldsymbol{R}^{(i)}$ can be obtained efficiently by considering the block elements of the l.h.s. of (6). This result is presented in Theorem 1.

Theorem 1. *For a multi-threshold queue consisting of S stages such that*

(i) $\boldsymbol{F}^{(i)}_{(j,k)} = \boldsymbol{0}$, *for $k < j$ and $i = 0, 1, \ldots,$ and*
(ii) *if $\boldsymbol{B}^{(i)}_{(j,k)} \neq \boldsymbol{0}$, for $k < j$, then $\boldsymbol{L}^{(i-1)}_{(x,x)} = \boldsymbol{0}$, for $k < x \leq j$,*

the submatrices $\boldsymbol{R}^{(i)}_{(j,k)}$ of $\boldsymbol{R}^{(i)}$ are given by

$$\boldsymbol{R}^{(i)}_{(j,j)} = -\boldsymbol{F}^{(i)}_{(j,j)} \left[\boldsymbol{L}^{(i+1)}_{(j,j)} + \sum_{b=j}^{S} \boldsymbol{R}^{(i+1)}_{(j,b)} \boldsymbol{B}^{(i+2)}_{(b,j)}\right]^{-1}, \tag{10}$$

$$\boldsymbol{R}^{(i)}_{(j,k)} = \begin{cases} \boldsymbol{0}, & \text{if } k < j, \\ -\left[\boldsymbol{F}^{(i)}_{(j,k)} + \sum_{a=j}^{k-1}\sum_{b=a}^{S} \boldsymbol{R}^{(i)}_{(j,a)} \boldsymbol{R}^{(i+1)}_{(a,b)} \boldsymbol{B}^{(i+2)}_{(b,k)}\right] & \\ \quad \cdot \left[\boldsymbol{L}^{(i+1)}_{(k,k)} + \sum_{b=k}^{S} \boldsymbol{R}^{(i+1)}_{(k,b)} \boldsymbol{B}^{(i+2)}_{(b,k)}\right]^{-1}, & \text{if } k > j. \end{cases} \tag{11}$$

and

$$\boldsymbol{R}^{(i)}_{(x,y)} = \boldsymbol{0} \text{ if } \boldsymbol{B}^{(i+1)}_{(j,k)} \neq \boldsymbol{0} \text{ for } k < x \leq y \leq j. \tag{12}$$

Proof. Assuming $\boldsymbol{R}^{(i+1)}$ is an upper triangular block matrix one can verify that the unique solution to the block elements of the l.h.s. of (6), i.e.

$$\begin{aligned} \boldsymbol{0} &= \boldsymbol{F}^{(i)}_{(j,k)} + \sum_{a=1}^{S} \boldsymbol{R}^{(i)}_{(j,a)} \boldsymbol{L}^{(i+1)}_{(a,k)} + \sum_{a=1}^{S}\sum_{b=1}^{S} \boldsymbol{R}^{(i)}_{(j,a)} \boldsymbol{R}^{(i+1)}_{(a,b)} \boldsymbol{B}^{(i+2)}_{(b,k)} \\ &= \boldsymbol{F}^{(i)}_{(j,k)} + \boldsymbol{R}^{(i)}_{(j,k)} \boldsymbol{L}^{(i+1)}_{(k,k)} + \sum_{a=1}^{S}\sum_{b=a}^{S} \boldsymbol{R}^{(i)}_{(j,a)} \boldsymbol{R}^{(i+1)}_{(a,b)} \boldsymbol{B}^{(i+2)}_{(b,k)}. \end{aligned}$$

is given by (10), (11) and (12). Since $\boldsymbol{R}$ is a diagonal block matrix this proves by induction that $\boldsymbol{R}^{(i)}$, $i = 0, 1, \ldots$, is an upper triangular block matrix and that its submatrices are uniquely determined by (10), (11) and (12).

The conditions of Theorem 1 can be interpreted as (i) at upper thresholds the stage of the system can only change to higher stages, and (ii) at lower thresholds the stage of the system can change to higher stages and to at most one lower stage. If at level i the stage of the system changes from s to t, with $t < s$, then all stages, $r = t+1, \ldots, s-1$ must not be potential stage for level $i-1$.

Remark 2 (Upper triangularity of $\boldsymbol{R}^{(i)}$). Note that under the conditions of Theorem 1, $\boldsymbol{R}^{(i)}$ must be an upper triangular block matrix for all i. This implies that only stage 1 has no lower threshold.

To prove this, we extend the interpretation of $\boldsymbol{R}^{(i)}$ to the product $\boldsymbol{R}^{(i)}\boldsymbol{R}^{(i+1)}$. Observe that the element $\left[\boldsymbol{R}^{(i)}\boldsymbol{R}^{(i+1)}\right]_{(r,t)}$ describes the mean sojourn time in state $(i+2,t)$ per unit sojourn time in state (i,r) before returning to level i, given that the process started in state (i,r). If the element $\left[\boldsymbol{R}^{(i)}\boldsymbol{R}^{(i+1)}\right]_{(r,t)} = 0$ then state $(i+2,t)$ cannot be reached from state (i,r) without visiting level i. The same interpretation holds for the submatrices of the product

$$\boldsymbol{R}(n) = \prod_{i=0}^{n-1} \boldsymbol{R}^{(i)}.$$

If the submatrix $\boldsymbol{R}(n)_{(j,k)}$ of $\boldsymbol{R}(n)$ is $\boldsymbol{0}$, then stage k at level n can never be reached from stage j at level 0. Under the conditions of Theorem 1, $\boldsymbol{R}^{(i)}$ is an upper triangular block matrix for $i \geq 0$, therefore, $\boldsymbol{R}(n)$ is also an upper triangular block matrix for $n \geq 0$. Suppose now that stage $j \neq 1$ has no lower threshold, then stages $k < j$ can never be reached from stage j since $\boldsymbol{R}(n)_{(j,k)} = \boldsymbol{0}$ for $k < j$ and $n \geq 0$. This implies that stages $k < j$ can be removed from the threshold policy. Since the Markov Chain is irreducible, $j = 1$. ■

In Corollary 1, we provide an efficient algorithm to compute the stationary queue length vectors $\boldsymbol{\pi}_i$, $i = 0, 1, \ldots$, using the submatrices of $\boldsymbol{R}^{(i)}$ defined in Theorem 1 and equation (8).

Corollary 1. *Define the vector $\boldsymbol{p}_i = \left[\boldsymbol{p}_i^1 \; \boldsymbol{p}_i^2 \; \cdots \; \boldsymbol{p}_i^S\right]$ for $i = 0, 1, \ldots$ such that*

$$\boldsymbol{p}_i^j = \begin{cases} \sum_{a=1}^{j} \boldsymbol{p}_{i-1}^a \boldsymbol{R}_{(a,j)}^{(i-1)}, & i = 1, \ldots, U_{max}, \\ \boldsymbol{p}_{U_{max}}^j \left[\boldsymbol{R}_{(j,j)}\right]^{i-U_{max}}, & i = U_{max}+1, U_{max}+2, \ldots, \end{cases} \tag{13}$$

with $\boldsymbol{p}_0^1$ the solution to

$$\boldsymbol{p}_0^1 \left[\boldsymbol{L}_{(1,1)}^{(0)} + \sum_{a=1}^{S} \boldsymbol{R}_{(i,a)}^{(0)} \boldsymbol{B}_{(a,i)}^{(1)}\right] = \boldsymbol{0}, \tag{14}$$

such that

$$\boldsymbol{p}_0^1 \boldsymbol{e} = 1, \tag{15}$$

and $\boldsymbol{p}_0^j = \mathbf{0}$ *for* $j = 2, \ldots, S$. *Under the conditions of Theoren 1, the stationary probability vector,* $\boldsymbol{\pi}_i = \left[\boldsymbol{\pi}_i^1 \; \boldsymbol{\pi}_i^2 \; \cdots \; \boldsymbol{\pi}_i^S\right]$, *is given by*

$$\boldsymbol{\pi}_i^j = \frac{\boldsymbol{p}_i^j}{\sum_{k=1}^S \beta_k}, \tag{16}$$

with

$$\beta_k = \begin{cases} \displaystyle\sum_{i=L_k}^{U_k} \boldsymbol{p}_i^k \boldsymbol{e}, & \text{if } U_k < \infty, \\ \displaystyle\sum_{i=L_k}^{U_{max}-1} \boldsymbol{p}_i^k \boldsymbol{e} + \boldsymbol{p}_{U_{max}}^k \left[\boldsymbol{I} - \boldsymbol{R}_{(k,k)}\right]^{-1} \boldsymbol{e}, & \text{if } U_k = \infty, \end{cases}$$

where $\boldsymbol{e}$ *is a vector of ones and* $\boldsymbol{I}$ *the identity matrix of appropriate size.*

Proof. From (13) is follows directly that

$$\boldsymbol{p}_i = \boldsymbol{p}_{i-1} \boldsymbol{R}^{(i-1)},$$

and from (16)

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_{i-1} \boldsymbol{R}^{(i-1)}.$$

At level 0, only stage 1 is active (see Remark 1), it then follows from (14) that

$$\boldsymbol{p}_0 \left[\boldsymbol{L}^{(0)} + \boldsymbol{R}^{(0)} \boldsymbol{B}^{(1)}\right] = \mathbf{0},$$

and that

$$\boldsymbol{\pi}_0 \left[\boldsymbol{L}^{(0)} + \boldsymbol{R}^{(0)} \boldsymbol{B}^{(1)}\right] = \mathbf{0}.$$

Stability of the multi-threshold queue guarantees that

$$\begin{aligned} \sum_{j=1}^S \sum_{i=0}^\infty \boldsymbol{p}_i^j \boldsymbol{e} &= \sum_{\{j\,:\,U_j<\infty\}} \sum_{i=L_j}^{U_j} \boldsymbol{p}_i^j \boldsymbol{e} + \sum_{\{j\,:\,U_j=\infty\}} \left\{ \sum_{i=L_j}^{U_{max}-1} \boldsymbol{p}_i^j \boldsymbol{e} + \sum_{i=U_{max}}^{\infty} \boldsymbol{p}_i^j \boldsymbol{e} \right\} \\ &= \sum_{\{j\,:\,U_j<\infty\}} \beta_j + \sum_{\{j\,:\,U_j=\infty\}} \left\{ \sum_{i=L_j}^{U_{max}-1} \boldsymbol{p}_i^j \boldsymbol{e} + \boldsymbol{p}_{U_{max}}^j \sum_{i=0}^{\infty} \left[\boldsymbol{R}_{(j,j)}\right]^i \boldsymbol{e} \right\} \\ &= \sum_{\{j\,:\,U_j<\infty\}} \beta_j + \sum_{\{j\,:\,U_j=\infty\}} \left\{ \sum_{i=L_j}^{U_{max}-1} \boldsymbol{p}_i^j \boldsymbol{e} + \boldsymbol{p}_{U_{max}}^j \left[\boldsymbol{I} - \boldsymbol{R}_{(j,j)}\right]^{-1} \boldsymbol{e} \right\} \\ &= \sum_{j=1}^S \beta_j < \infty, \end{aligned}$$

and that $\boldsymbol{\pi}$ is the stationary queue length distribution.

Remark 3 (Permutations of stages). Consider a multi-threshold queue with S stages. If there exists a permutation of the S stages such that the conditions of Theorem 1 hold, its stationary queue length vector can efficiently be obtained using this permutation and the results from Theorem 1 and Corollary 1.

4 Examples

In this section expressions for $\boldsymbol{R}^{(i)}_{(j,k)}$ and the stationary queue length distribution $\boldsymbol{\pi}^j_i$ are obtained for three multi-threshold queueing systems. These expressions follow using Theorem 1 and Corollary 1 and are obtained by straightforward but tedious derivations. The three multi-threshold queueing systems we will consider are the multi-threshold queue from Figure 1, the staircase multi-threshold with exponential service and arrival rates from [13] and the staircase multi-threshold queue in a general setting from [5].

4.1 Extended Traffic Model

Consider the multi-threshold queue in Figure 1. Observe that the threshold policy in Figure 1 satisfies both conditions of Theorem 1. In this multi-threshold queueing system, inspired by the traffic model in [2], we assume that

$$0 = L_1 < L_3 < L_2 = L_4 < U_1 = U_3 < U_2 < U_4 = \infty$$

and we define $\rho_i = \frac{\lambda_i}{\mu_i}$. Note that by assuming exponential arrival and service rates, each submatrix $\boldsymbol{R}^{(i)}_{(j,k)}$ reduces to a single element. Therefore, the solution to equation (8) is ρ_4 and each submatrix $\boldsymbol{R}^{(i)}_{(j,k)}$ is given by:

$$\boldsymbol{R}^{(i)}_{(1,1)} = \begin{cases} \rho_1, & i = 0, \ldots, L_3 - 2, \\ \rho_1 \frac{\left(1-\rho_1^{U_1-i}\right)\left(\rho_2^{U_2-U_1}-\rho_2^{U_2-L_2+2}\right)+\left(1-\rho_1^{U_1-L_2+2}\right)\left(1-\rho_2^{U_2-U_1}\right)}{\left(1-\rho_1^{U_1+1-i}\right)\left(\rho_2^{U_2-U_1}-\rho_2^{U_2-L_2+2}\right)+\left(1-\rho_1^{U_1-L_2+2}\right)\left(1-\rho_2^{U_2-U_1}\right)}, & \\ & i = L_3 - 1, \ldots, L_2 - 2, \\ \frac{\rho_1-\rho_1^{U_1+1-i}}{1-\rho_1^{U_1+1-i}}, & i = L_2 - 1, \ldots, U_1 - 1, \end{cases}$$

$$\boldsymbol{R}^{(i)}_{(1,2)} = \frac{\lambda_1}{\mu_2} \frac{\left(\rho_1^{U_1-i}-\rho_1^{U_1-i+1}\right)\left(1-\rho_2^{U_2-U_1}\right)}{\left(1-\rho_1^{U_1+1-i}\right)\left(1-\rho_2^{U_2+1-i}\right)}, \qquad i = L_2 - 1, \ldots, U_1,$$

$$\boldsymbol{R}^{(i)}_{(1,3)} = \frac{\lambda_1}{\mu_3} \frac{\left(\rho_1^{U_1-i}-\rho_1^{U_1+1-i}\right)\left(\rho_2^{U_2-U_1}-\rho_2^{U_2-L_2+2}\right)}{\left(1-\rho_1^{U_1+1-i}\right)\left(\rho_2^{U_2-U_1}-\rho_2^{U_2-L_2+2}\right)+\left(1-\rho_1^{U_1-L_2+2}\right)\left(1-\rho_2^{U_2-U_1}\right)},$$
$$i = L_3 - 1, \ldots, L_4 - 2,$$

$$\boldsymbol{R}^{(i)}_{(1,4)} = \frac{\lambda_1}{\mu_4} \frac{\left(\rho_1^{U_1-i}-\rho_1^{U_1+1-i}\right)\left(\rho_2^{U_2-U_1}-\rho_2^{U_2+1-i}\right)}{\left(1-\rho_1^{U_1+1-i}\right)\left(1-\rho_2^{U_2+1-i}\right)}, \qquad i = L_4 - 1, \ldots, U_1,$$

$$\boldsymbol{R}^{(i)}_{(2,2)} = \frac{\rho_2-\rho_2^{U_2+1-i}}{1-\rho_2^{U_2+1-i}}, \qquad i = L_2, \ldots, U_2 - 1,$$

$$\boldsymbol{R}^{(i)}_{(2,3)} = 0, \qquad \forall i,$$

$$\boldsymbol{R}^{(i)}_{(2,4)} = \frac{\lambda_2}{\mu_4} \frac{\rho_2^{U_2-i}-\rho_2^{U_2+1-i}}{1-\rho_2^{U_2+1-i}}, \qquad i = L_2, \ldots, U_2,$$

$$\boldsymbol{R}^{(i)}_{(3,3)} = \begin{cases} \rho_3, & i = L_3, \ldots, L_4 - 2, \\ \frac{\rho_3-\rho_3^{U_3+1-i}}{1-\rho_3^{U_3+1-i}}, & i = L_4 - 1, \ldots, U_3 - 1, \end{cases}$$

$$\boldsymbol{R}^{(i)}_{(3,4)} = \frac{\lambda_3}{\mu_4}\frac{\rho_3^{U_3-i}-\rho_3^{U_3+1-i}}{1-\rho_3^{U_3+1-i}}, \qquad i = L_4-1,\dots,U_3,$$
$$\boldsymbol{R}^{(i)}_{(4,4)} = \rho_4, \qquad i = L_4, L_4+1,\dots.$$

The stationary queue length probability of i customers in stage j, $\boldsymbol{\pi}^j_i$, follows from Corollary 1 by normalising $\boldsymbol{p}^j_i$. For $i=0$:

$$\boldsymbol{p}^j_0 = \begin{cases} 1, & j=1, \\ 0, & j \neq 1, \end{cases}$$

and for $i>0$:

$$\boldsymbol{p}^1_i = \boldsymbol{p}^1_{i-1}\boldsymbol{R}^{(i-1)}_{(1,1)}, \qquad 0 < i \leq U_1,$$
$$\boldsymbol{p}^2_i = \begin{cases} \boldsymbol{p}^1_{i-1}\boldsymbol{R}^{(i-1)}_{(1,2)}, & i = L_2, \\ \boldsymbol{p}^1_{i-1}\boldsymbol{R}^{(i-1)}_{(1,2)} + \boldsymbol{p}^2_{i-1}\boldsymbol{R}^{(i-1)}_{(2,2)}, & L_2 < i \leq U_1+1, \\ \boldsymbol{p}^2_{i-1}\boldsymbol{R}^{(i-1)}_{(2,2)}, & U_1+1 < i \leq U_2, \end{cases}$$
$$\boldsymbol{p}^3_i = \begin{cases} \boldsymbol{p}^1_{i-1}\boldsymbol{R}^{(i-1)}_{(1,3)}, & i = L_3, \\ \boldsymbol{p}^1_{i-1}\boldsymbol{R}^{(i-1)}_{(1,3)} + \boldsymbol{p}^3_{i-1}\boldsymbol{R}^{(i-1)}_{(3,3)}, & L_3 < i \leq L_4-1, \\ \boldsymbol{p}^3_{i-1}\boldsymbol{R}^{(i-1)}_{(3,3)}, & L_4-1 < i \leq U_3, \end{cases}$$
$$\boldsymbol{p}^4_i = \begin{cases} \boldsymbol{p}^1_{i-1}\boldsymbol{R}^{(i-1)}_{(1,4)} + \boldsymbol{p}^3_{i-1}\boldsymbol{R}^{(i-1)}_{(3,4)}, & i = L_4, \\ \boldsymbol{p}^1_{i-1}\boldsymbol{R}^{(i-1)}_{(1,4)} + \boldsymbol{p}^2_{i-1}\boldsymbol{R}^{(i-1)}_{(2,4)} + \boldsymbol{p}^3_{i-1}\boldsymbol{R}^{(i-1)}_{(3,4)} + \boldsymbol{p}^4_{i-1}\boldsymbol{R}^{(i-1)}_{(4,4)}, & \\ & L_4 < i \leq U_1+1, \\ \boldsymbol{p}^2_{i-1}\boldsymbol{R}^{(i-1)}_{(2,4)} + \boldsymbol{p}^4_{i-1}\boldsymbol{R}^{(i-1)}_{(4,4)}, & U_1+1 < i \leq U_2+1, \\ \boldsymbol{p}^4_{U_2+1}\left[\boldsymbol{R}^{(U_2+1)}_{(4,4)}\right]^{i-U_2-1}, & U_2+1 < i. \end{cases}$$

4.2 Le Ny and Tuffin [13]

Consider a multi-threshold queue of S stages as analysed by Le Ny and Tuffin in [13]. In each stage i arrivals are Poisson distributed with rate λ_i, service times are exponentially distributed with rate μ_i and we define $\rho_i = \frac{\lambda_i}{\mu_i}$. An arrival changes the stage from j to $j+1$ at U_j and a departure changes the stage from j to $j-1$ at L_j. We assume

$$0 = L_1 < L_2 < \cdots < L_S \leq U_1 < \cdots < U_{S-1} < U_S = \infty.$$

The state diagram created by this threshold policy forms a *staircase* as schematically shown in Figure 2.

As in Section 4.1 each submatrix $\boldsymbol{R}^{(i)}_{(j,k)}$ consists of a single element and equation (7), and in particular (8), gives

$$\boldsymbol{R}^{(U_{max})}_{(S,S)} = \rho_S.$$

Both conditions of Theorem 1 are satisfied by the threshold policy and $\boldsymbol{R}^{(i)}_{(j,k)}$ is given by:

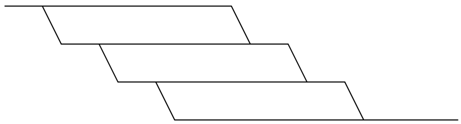

Fig. 2. Schematic representation of the state diagram of a staircase threshold policy with 4 stages.

$$
\begin{aligned}
\boldsymbol{R}^{(i)}_{(j,j)} &= \begin{cases} \rho_j, & L_j \le i \le L_{j+1}-2, \\ \frac{\rho_j - \rho_j^{U_j+1-i}}{1-\rho_j^{U_j+1-i}}, & L_{j+1}-1 \le i \le U_j, \end{cases} \\
\boldsymbol{R}^{(i)}_{(S,S)} &= \rho_S, \qquad\qquad L_S \le i, \\
\boldsymbol{R}^{(i)}_{(j,k)} &= \begin{cases} \frac{\lambda_j}{\mu_k} \frac{\rho_j^{U_j-i} - \rho_j^{U_j+1-i}}{1-\rho_j^{U_j+1-i}} \\ \quad \cdot \prod_{a=j+1}^{k-1} \frac{\rho_a^{U_a-U_{a-1}} - \rho_a^{U_a+1-i}}{1-\rho_a^{U_a+1-i}}, & L_k - 1 \le i \le L_{k+1}-2, \\ \frac{\lambda_j}{\mu_k} \frac{\left(\rho_j^{U_j-i} - \rho_j^{U_j+1-i}\right)\left(1-\rho_k^{U_k-U_{k-1}}\right)}{\left(1-\rho_j^{U_j+1-i}\right)\left(1-\rho_k^{U_k+1-i}\right)} \\ \quad \cdot \prod_{a=j+1}^{k-1} \frac{\rho_a^{U_a-U_{a-1}} - \rho^{U_a+1-i}}{1-\rho_a^{U_a+1-i}}, & L_{k+1}-1 \le i \le U_j, \end{cases} \\
\boldsymbol{R}^{(i)}_{(j,S)} &= \frac{\lambda_j}{\mu_S} \frac{\rho_j^{U_j-i} - \rho_j^{U_j+1-i}}{1-\rho_j^{U_j+1-i}} \prod_{a=j+1}^{S-1} \frac{\rho_a^{U_a-U_{a-1}} - \rho_a^{U_a+1-i}}{1-\rho_a^{U_a+1-i}}, \quad L_S - 1 \le i.
\end{aligned}
$$

The stationary queue length distribution $\boldsymbol{\pi}_i^j$ follows from Corollary 1 by normalising $\boldsymbol{p}_i^j$. For $i = 0$:

$$
\boldsymbol{p}_0^j = \begin{cases} 1, & j = 1, \\ 0, & j \ne 1, \end{cases}
$$

for $i > 0$ and $j = 1$ or $j = 2$:

$$
\boldsymbol{p}_i^1 = \boldsymbol{p}_{i-1}^1 \boldsymbol{R}^{(i-1)}_{(1,1)}, \qquad 0 < i \le U_1, \tag{17}
$$

$$
\boldsymbol{p}_i^2 = \begin{cases} \boldsymbol{p}_{i-1}^1 \boldsymbol{R}^{(i-1)}_{(1,2)}, & i = L_2, \\ \boldsymbol{p}_{i-1}^1 \boldsymbol{R}^{(i-1)}_{(1,2)} + \boldsymbol{p}_{i-1}^2 \boldsymbol{R}^{(i-1)}_{(2,2)}, & L_2 < i \le U+1, \\ \boldsymbol{p}_{i-1}^2 \boldsymbol{R}^{(i-1)}_{(2,2)}, & U+1 < i \le U_2, \end{cases} \tag{18}
$$

for $i > 0$ and $j = 3, \ldots, S-1$:

$$
\boldsymbol{p}_i^j = \begin{cases} \sum_{a=1}^{j-1} \boldsymbol{p}_{i-1}^a \boldsymbol{R}^{(i-1)}_{(a,j)}, & i = L_j, \\ \sum_{a=1}^{j} \boldsymbol{p}_{i-1}^a \boldsymbol{R}^{(i-1)}_{(a,j)}, & L_j < i \le U_1 + 1, \\ \sum_{a=k}^{j} \boldsymbol{p}_{i-1}^a \boldsymbol{R}^{(i-1)}_{(a,j)}, & U_{k-1}+1 < i \le U_k + 1,\ k = 2, \ldots, j-1, \\ \boldsymbol{p}_{i-1}^j \boldsymbol{R}^{(i-1)}_{(j,j)}, & U_{j-1}+1 < i \le U_j, \end{cases} \tag{19}
$$

and for $i > 0$ and $j = S$

$$\boldsymbol{p}_i^S = \begin{cases} \sum_{a=1}^{S-1} \boldsymbol{p}_{i-1}^a \boldsymbol{R}_{(a,S)}^{(i-1)}, & i = L_S, \\ \sum_{a=1}^{S} \boldsymbol{p}_{i-1}^a \boldsymbol{R}_{(a,S)}^{(i-1)}, & L_S < i \leq U_1 + 1, \\ \sum_{a=k}^{S} \boldsymbol{p}_{i-1}^a \boldsymbol{R}_{(a,S)}^{(i-1)}, & U_{k-1} + 1 < i \leq U_k + 1,\ k = 2, \dots, S-1, \\ \boldsymbol{p}_{i-1}^S \left[\boldsymbol{R}_{(S,S)}^{(i-1)}\right]^{i-U_{max}}, & U_{max} < i. \end{cases} \tag{20}$$

4.3 Choi et al [5]

Consider the multi-threshold queue of S stages as analysed by Choi et al [5]. This model generalises the staircase model of [13] to $PH(\Lambda_s, \lambda_s)$ arrivals and $PH(M_s, \mu_s)$ services in stage s. The forward, local and backward transition matrices are given by (2), (3) and (4) respectively. In this case, the submatrices $\boldsymbol{R}_{(j,k)}^{(i)}$ are not single elements and the matrix equation (8) must be solved numerically. The submatrices $\boldsymbol{R}_{(j,k)}^{(i)}$, $i = 0, \dots, U_{max} - 1$, are iteratively given, following Theorem 1, by

$$\boldsymbol{R}_{(j,j)}^{(i)} = \begin{cases} -\boldsymbol{F}_{(j,j)}^{(i)} \left[\boldsymbol{L}_{(j,j)}^{(i+1)} + \boldsymbol{R}_{(j,j)}^{(i+1)} \boldsymbol{B}_{(j,j)}^{(i+2)}\right]^{-1}, & L_j \leq i < U_j - 1,\ i \neq L_{j+1} - 2, \\ -\boldsymbol{F}_{(j,j)}^{(i)} \left[\boldsymbol{L}_{(j,j)}^{(i+1)} + \sum_{b=j}^{j+1} \boldsymbol{R}_{(j,b)}^{(i+1)} \boldsymbol{B}_{(b,j)}^{(i+2)}\right]^{-1}, & \\ & i = L_{j+1} - 2, \\ -\boldsymbol{F}_{(j,j)}^{(i)} \left[\boldsymbol{L}_{(j,j)}^{(i+1)}\right]^{-1}, & i = U_j - 1, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$\boldsymbol{R}_{(j,k)}^{(i)} = \begin{cases} -\left[\sum_{a=j}^{k-1} \boldsymbol{R}_{(j,a)}^{(i)} \boldsymbol{R}_{(a,k)}^{(i+1)} \boldsymbol{B}_{(k,k)}^{(i+2)}\right] & \\ \quad \cdot \left[\boldsymbol{L}_{(k,k)}^{(i+1)} + \boldsymbol{R}_{(k,k)}^{(i+1)} \boldsymbol{B}_{(k,k)}^{(i+2)}\right]^{-1}, & L_k - 1 \leq i < U_j,\ i \neq L_{k+1} - 2, \\ -\left[\sum_{b=k}^{k+1} \sum_{a=j}^{k-1} \boldsymbol{R}_{(j,a)}^{(i)} \boldsymbol{R}_{(a,b)}^{(i+1)} \boldsymbol{B}_{(b,k)}^{(i+2)}\right] & \\ \quad \cdot \left[\boldsymbol{L}_{(k,k)}^{(i+1)} + \sum_{b=k}^{k+1} \boldsymbol{R}_{(k,b)}^{(i+1)} \boldsymbol{B}_{(b,k)}^{(i+2)}\right]^{-1}, & i = L_{k+1} - 2, \\ -\left[\boldsymbol{F}_{(j,k)}^{(i)} \mathbb{1}_{\{k=j+1\}} + \sum_{a=j+1}^{k-1} \boldsymbol{R}_{(j,a)}^{(i)} \boldsymbol{R}_{(a,k)}^{(i+1)} \boldsymbol{B}_{(k,k)}^{(i+2)}\right] & \\ \quad \cdot \left[\boldsymbol{L}_{(k,k)}^{(i+1)} + \boldsymbol{R}_{(k,k)}^{(i+1)} \boldsymbol{B}_{(k,k)}^{(i+2)}\right]^{-1}, & i = U_j, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

for $j = 1, \dots, S-1$, and

$$\boldsymbol{R}_{(S,S)}^{(i)} = \begin{cases} \boldsymbol{R}_{(S,S)}, & L_S \leq i, \\ \mathbf{0}, & \text{otherwise}. \end{cases}$$

The stationary queue length distribution $\boldsymbol{\pi}_i^j$ follows from Corollary 1 by normalising $\boldsymbol{p}_i^j$. The vectors $\boldsymbol{p}_i^j$, $i > 0$, are given by equations (17), (18), (19) and (20). Finally, $\boldsymbol{p}_0^1$ is obtained from (14) and (15) and $\boldsymbol{p}_0^j = \mathbf{0}$, $j > 1$.

5 Summary and Conclusion

We introduced the $PH/PH/1$ multi-threshold queue where the arrival process and service process are controlled by a threshold policy. The threshold policy determines, based on the queue length, the stage of system, and the stage determines the arrival and service processes. We modelled this queue as a Level Dependent Quasi-Birth-and-Death process and obtained the stationary queue length probabilities using Matrix Analytic methods.

A special class of multi-threshold queues is presented and explicit description of the $\boldsymbol{R}$-matrices has been obtained in terms of its submatrices. This decomposition theorem allows an efficient computation of each $\boldsymbol{R}$-submatrix as well as the stationary queue length probability vectors.

Future work consists of a network of PH/PH/1 threshold queues in which the threshold policy can control the service rates of previous queue, see Baer, Al Hanbali, Boucherie and van Ommeren [1].

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