# Incremental QBF Solving\*

Florian Lonsing and Uwe Egly

Vienna University of Technology Institute of Information Systems Knowledge-Based Systems Group http://www.kr.tuwien.ac.at/

**Abstract.** We consider the problem of incrementally solving a sequence of quantified Boolean formulae (QBF). Incremental solving aims at using information learned from one formula in the process of solving the next formulae in the sequence. Based on a general overview of the problem and related challenges, we present an approach to incremental QBF solving which is application-independent and hence applicable to QBF encodings of arbitrary problems. We implemented this approach in our incremental search-based QBF solver **DepQBF** and report on implementation details. Experimental results illustrate the potential benefits of incremental solving in QBF-based workflows.

## 1 Introduction

The success of SAT technology in practical applications is largely driven by *incremental solving*. SAT solvers based on conflict-driven clause learning (CDCL) [32] gather information about a formula in terms of learned clauses. When solving a sequence of closely related formulae, it is beneficial to keep clauses learned from one formula in the course of solving the next formulae in the sequence.

The logic of quantified Boolean formulae (QBF) extends propositional logic by universal and existential quantification of variables. QBF potentially allows for more succinct encodings of PSPACE-complete problems than SAT. Motivated by the success of incremental SAT solving, we consider the problem of incrementally solving a sequence of syntactically related QBFs in prenex conjunctive normal form (PCNF). Building on search-based QBF solving with clause and cube learning (QCDCL) [8,13,21,24,36], we present an approach to incremental QBF solving, which we implemented in our solver DepQBF.<sup>1</sup>

Different from many incremental SAT and QBF [27] solvers, DepQBF allows to add clauses to and delete clauses from the input PCNF in a stack-based way by *push* and *pop* operations. A related stack-based framework was implemented in

<sup>\*</sup> Supported by the Austrian Science Fund (FWF) under grant S11409-N23. We would like to thank Armin Biere and Paolo Marin for helpful discussions. This article will appear in the proceedings of the 20th International Conference on Principles and Practice of Constraint Programming, LNCS, Springer, 2014.

<sup>&</sup>lt;sup>1</sup> DepQBF is free software: http://lonsing.github.io/depqbf/

the SAT solver PicoSAT [5]. A solver API with *push* and *pop* increases the usability from the perspective of a user. Moreover, we present an optimization based on this stack-based framework which reduces the size of the learned clauses.

Incremental QBF solving was introduced for QBF-based bounded model checking (BMC) of partial designs [26,27]. This approach, like ours, relies on selector variables and assumptions to support the deletion of clauses from the current input PCNF [1,11,20,28]. The quantifier prefixes of the incrementally solved PCNFs resulting from the BMC encodings are modified only at the left or right end. In contrast to that, we consider incremental solving of *arbitrary* sequences of PCNFs. For the soundness it is crucial to determine which of the learned clauses and cubes can be kept across different runs of an incremental QBF solver. We aim at a general presentation of incremental QBF solving and illustrate problems related to clause and cube learning. Our approach is *application-independent* and applicable to QBF encodings of *arbitrary* problems.

We report on experiments with constructed benchmarks. In addition to experiments with QBF-based conformant planning using DepQBF [12], our results illustrate the potential benefits of incremental QBF solving in application domains like synthesis [6,33], formal verification [4], testing [17,25,34], planning [9], and model enumeration [3], for example.

# 2 Preliminaries

We introduce terminology related to QBF and search-based QBF solving necessary to present a general view on incremental solving.

For a propositional variable x, l := x or  $l := \neg x$  is a *literal*, where v(l) = x denotes the variable of l. A *clause* (*cube*) is a disjunction (conjunction) of literals. A *constraint* is a clause or a cube. The empty constraint  $\emptyset$  does not contain any literals. A clause (cube) C is *tautological* (*contradictory*) if  $x \in C$  and  $\neg x \in C$ .

A propositional formula is in *conjunctive (disjunctive) normal form* if it consists of a conjunction (disjunction) of clauses (cubes), called CNF (DNF). For simplicity, we regard CNFs and DNFs as sets of clauses and cubes, respectively.

A quantified Boolean formula (QBF)  $\psi := \hat{Q} \cdot \phi$  is in prenex CNF (PCNF) if it consists of a quantifier-free CNF  $\phi$  and a quantifier prefix  $\hat{Q}$  with  $\hat{Q} := Q_1 B_1 \dots Q_n B_n$  where  $Q_i \in \{\forall, \exists\}$  are quantifiers and  $B_i$  are blocks (i.e. sets) of variables such that  $B_i \neq \emptyset$  and  $B_i \cap B_j = \emptyset$  for  $i \neq j$ , and  $Q_i \neq Q_{i+1}$ .

The blocks in the quantifier prefix are *linearly ordered* such that  $B_i < B_j$ if i < j. The linear ordering is extended to variables and literals:  $x_i < x_j$  if  $x_i \in B_i, x_j \in B_j$  and  $B_i < B_j$ , and l < l' if v(l) < v(l') for literals l and l'.

We consider only *closed* PCNFs, where every variable which occurs in the CNF is quantified in the prefix, and vice versa.

A variable  $x \in B_i$  is universal, written as  $q(x) = \forall$ , if  $Q_i = \forall$  and existential, written as  $q(x) = \exists$ , if  $Q_i = \exists$ . A literal l is universal if  $q(v(l)) = \forall$  and existential if  $q(v(l)) = \exists$ , written as  $q(l) := \forall$  and  $q(l) := \exists$ , respectively.

An assignment is a mapping from variables to the truth values *true* and *false*. An assignment A is represented as a set of literals  $A := \{l_1, \ldots, l_k\}$  such that, for  $l_i \in A$ , if  $v(l_i)$  is assigned to false (true) then  $l_i = \neg v(l_i)$  ( $l_i = v(l_i)$ ).

A PCNF  $\psi$  under an assignment A is denoted by  $\psi[A]$  and is obtained from  $\psi$  as follows: for  $l_i \in A$ , if  $l_i = v(l_i)$   $(l_i = \neg v(l_i))$  then all occurrences of  $v(l_i)$  in  $\psi$  are replaced by the syntactic truth constant  $\top (\bot)$ , respectively. All constants are eliminated from  $\psi[A]$  by the usual simplifications of Boolean algebra and superfluous quantifiers and blocks are deleted from the quantifier prefix of  $\psi[A]$ . Given a cube C and a PCNF  $\psi$ ,  $\psi[C] := \psi[A]$  is the formula obtained from  $\psi$  under the assignment  $A := \{l \mid l \in C\}$  defined by the literals in C.

The semantics of closed PCNFs is defined recursively. The QBF  $\top$  is satisfiable and the QBF  $\perp$  is unsatisfiable. The QBF  $\psi = \forall (B_1 \cup \{x\}) \dots Q_n B_n. \phi$  is satisfiable if  $\psi[\neg x]$  and  $\psi[x]$  are satisfiable. The QBF  $\psi = \exists (B_1 \cup \{x\}) \dots Q_n B_n. \phi$  is satisfiable if  $\psi[\neg x]$  or  $\psi[x]$  are satisfiable.

A PCNF  $\psi$  is satisfied under an assignment A if  $\psi[A] = \top$  and falsified under A if  $\psi[A] = \bot$ . Satisfied and falsified clauses are defined analogously.

Given a constraint C,  $L_Q(C) := \{l \in C \mid q(l) = Q\}$  for  $Q \in \{\forall, \exists\}$  denotes the set of universal and existential literals in C. For a clause C, universal reduction produces the clause  $UR(C) := C \setminus \{l \mid l \in L_{\forall}(C) \text{ and } \forall l' \in L_{\exists}(C) : l' < l\}$ .

*Q-resolution* of clauses is a combination of resolution for propositional logic and universal reduction [7]. Given two non-tautological clauses  $C_1$  and  $C_2$  and a pivot variable p such that  $q(p) = \exists$  and  $p \in C_1$  and  $\neg p \in C_2$ . Let C' := $(UR(C_1) \setminus \{p\}) \cup (UR(C_2) \setminus \{\neg p\})$  be the *tentative Q-resolvent* of  $C_1$  and  $C_2$ . If C' is non-tautological then it is the *Q-resolvent* of  $C_1$  and  $C_2$  and we write  $C' = C_1 \otimes C_2$ . Otherwise,  $C_1$  and  $C_2$  do not have a Q-resolvent.

Given a PCNF  $\psi := \hat{Q}. \phi$ , a *Q*-resolution derivation of a clause *C* from  $\psi$  is the successive application of Q-resolution and universal reduction to clauses in  $\psi$  and previously derived clauses resulting in *C*. We represent a derivation as a directed acyclic graph (DAG) with edges (1)  $C'' \to C'$  if C' = UR(C'') and (2)  $C_1 \to C'$  and  $C_2 \to C'$  if  $C' = C_1 \otimes C_2$ . We write  $\hat{Q}.\phi \vdash C$  if there is a derivation of a clause *C* from  $\psi$ . Otherwise, we write  $\hat{Q}.\phi \nvDash C$ . Q-resolution is a sound and refutationally-complete proof system for QBFs [7]. A *Q*-resolution proof of an unsatisfiable PCNF  $\psi$  is a Q-resolution derivation of the empty clause.

# 3 Search-Based QBF Solving

We briefly describe search-based QBF solving with conflict-driven clause learning and solution-driven cube learning (QCDCL) [8,13,21,24,36] and related properties. In the context of *incremental* QBF solving, clause and cube learning requires a special treatment, which we address in Section 4.

Given a PCNF  $\psi$ , a QCDCL-based QBF solver successively assigns the variables to generate an assignment A. If  $\psi$  is falsified under A, i.e.  $\psi[A] = \bot$ , then a new learned clause C is derived by Q-resolution and added to  $\psi$ . If  $\psi$  is unsatisfiable, then finally the empty clause will be derived by clause learning. If  $\psi$  is

$$C_{3} = (\neg x_{1} \lor x_{4}) \qquad C_{4} = (\neg y_{8} \lor \neg x_{4})$$

$$C_{7} = (\neg y_{8} \lor \neg x_{1})$$

$$C_{8} = (\neg x_{1})$$

$$C_{10} = (\neg y_{8}) \qquad C_{12} = (y_{8} \land \neg x_{1})$$

$$C_{13} = (\neg x_{1})$$

$$C_{14} = \emptyset$$

Cube derivation:

Clause derivation:

Fig. 1. Derivation DAGs of the clauses and cubes from Example 1. The literals in the initial cubes  $C_9$  and  $C_{11}$  have been omitted in the figure to save space.

satisfied under A, i.e.  $\psi[A] = \top$ , then a new learned *cube* is constructed based on the following *model generation rule*, *existential reduction* and *cube resolution*.

**Definition 1 (model generation rule [13]).** Given a PCNF  $\psi := \hat{Q}.\phi$ , an assignment A such that  $\psi[A] = \top$  is a model<sup>2</sup> of  $\psi$ . An initial cube  $C = (\bigwedge_{l_i \in A} l_i)$  is a conjunction over the literals of a model A.

**Definition 2 ([13]).** Given a cube C, existential reduction produces the reduced cube  $ER(C) := C \setminus \{l \mid l \in L_{\exists}(C) \text{ and } \forall l' \in L_{\forall}(C) : l' < l\}.$ 

**Definition 3 (cube resolution [13,36]).** Given two non-contradictory cubes  $C_1$  and  $C_2$ , cube resolution is defined analogously to Q-resolution for clauses, except that existential reduction is applied and the pivot variable must be universal. The cube resolvent of  $C_1$  and  $C_2$  (if it exists) is denoted by  $C := C_1 \otimes C_2$ .

If  $\psi$  is satisfiable, then finally the empty cube will be derived by cube learning (Theorem 5 in [13]). Whereas in clause learning initially clauses of the input PCNF  $\psi$  can be resolved, in cube learning first initial cubes have to be generated by the model generation rule, which can then be used to produce cube resolvents. Similar to Q-resolution derivations (DAGs) of clauses and Q-resolution proofs, we define *cube resolution derivations* of cubes and *proofs of satisfiability*.

*Example 1.* Given the satisfiable PCNF  $\psi := \exists x_1 \forall y_8 \exists x_5, x_2, x_6, x_4, \phi$ , where  $\phi := \bigwedge_{i:=1,...,6} C_i$  with  $C_1 := (y_8 \lor \neg x_5), C_2 := (x_2 \lor \neg x_6), C_3 := (\neg x_1 \lor x_4), C_4 := (\neg y_8 \lor \neg x_4), C_5 := (x_1 \lor x_6), \text{ and } C_6 := (x_4 \lor x_5).$ 

Figure 1 shows the derivation of the clauses  $C_7 := C_3 \otimes C_4 = (\neg y_8 \vee \neg x_1)$ and  $C_8 := UR(C_7) = (\neg x_1)$  by Q-resolution and universal reduction.

The assignment  $A_1 := \{x_6, x_2, \neg y_8, \neg x_5, x_4\}$  is a model of  $\psi$  by Definition 1. Hence  $C_9 := (x_6 \land x_2 \land \neg y_8 \land \neg x_5 \land x_4)$  is an initial cube. Existential reduction of  $C_9$  produces the cube  $C_{10} := ER(C_9) = (\neg y_8)$ . Similarly,  $A_2 := \{y_8, \neg x_4, \neg x_1, x_5, x_6, x_2\}$  is a model of  $\psi$  and  $C_{11} := (y_8 \land \neg x_4 \land \neg x_1 \land x_5 \land x_6 \land x_2)$  is an initial cube. Existential reduction of  $C_{11}$  produces the cube  $C_{12} := ER(C_{11}) = C_{12}$ 

 $<sup>^{2}</sup>$  We adopted this definition of models from [21].

 $(y_8 \wedge \neg x_1)$ . The cube  $C_{13} := (\neg x_1)$  is obtained by resolving  $C_{10} = (\neg y_8)$  and  $C_{12} = (y_8 \wedge \neg x_1)$ . Finally, existential reduction of  $C_{13}$  produces the empty cube  $C_{14} := ER(C_{13}) = \emptyset$ , which proves that the PCNF  $\psi$  is satisfiable.

A QCDCL-based solver implicitly constructs derivation DAGs in constraint learning. However, typically only selected constraints of these derivations are kept as learned constraints in an *augmented CNF* [36].

**Definition 4.** Let  $\psi := \hat{Q}.\phi$  be a PCNF. The augmented CNF (ACNF) of  $\psi$  has the form  $\psi' := \hat{Q}.(\phi \land \theta \lor \gamma)$ , where  $\hat{Q}$  is the quantifier prefix,  $\phi$  is the set of original clauses,  $\theta$  is a CNF containing the learned clauses, and  $\gamma$  is a DNF containing the learned cubes obtained by clause and cube learning in QCDCL.

Given an ACNF  $\psi'$  and an assignment A, the notation  $\psi'[A]$  is defined similarly to PCNFs. Analogously to clause derivations, we write  $\hat{Q}. \phi \vdash C$  if there is a derivation of a cube C from the PCNF  $\hat{Q}.\phi$ . During a run of a QCDCL-based solver the learned constraints can be derived from the current PCNF.

**Proposition 1.** Let  $\psi' := \hat{Q}. (\phi \land \theta \lor \gamma)$  be the ACNF obtained by QCDCL from a PCNF  $\psi := \hat{Q}. \phi$ . It holds that (1)  $\forall C \in \theta : \hat{Q}. \phi \vdash C$  and (2)  $\forall C \in \gamma : \hat{Q}. \phi \vdash C$ .

Proposition 1 follows from the correctness of constraint learning in *non-incremental* QCDCL. That is, we assume that the PCNF  $\psi$  is not modified over time. However, as we point out below, in *incremental* QCDCL the constraints learned previously might no longer be derivable after the PCNF has been modified.

**Definition 5.** Given the ACNF  $\psi' := \hat{Q}$ .  $(\phi \land \theta \lor \gamma)$  of the PCNF  $\psi := \hat{Q}$ .  $\phi$ , a clause  $C \in \theta$  (cube  $C \in \gamma$ ) is derivable with respect to  $\psi$  if  $\psi \vdash C$ . Otherwise, if  $\psi \nvDash C$ , then C is non-derivable.

Due to the correctness of model generation, existential/universal reduction, and resolution, constraints which are derivable from the PCNF  $\psi$  can be added to the ACNF  $\psi'$  of  $\psi$ , which results in a satisfiability-equivalent ( $\equiv_{sat}$ ) formula.

**Proposition 2 ([13]).** Let  $\psi' := \hat{Q}.(\phi \land \theta \lor \gamma)$  be the ACNF of the PCNF  $\psi := \hat{Q}.\phi$ . Then (1)  $\hat{Q}.\phi \equiv_{sat} \hat{Q}.(\phi \land \theta)$  and (2)  $\hat{Q}.\phi \equiv_{sat} \hat{Q}.(\phi \lor \gamma)$ .

# 4 Incremental Search-Based QBF Solving

We define *incremental QBF solving* as the problem of solving a sequence of PCNFs  $\psi_0, \psi_1, \ldots, \psi_n$  using a QCDCL-based solver. Thereby, the goal is to not discard all the learned constraints after the PCNF  $\psi_i$  has been solved. Instead, to the largest extent possible we want to re-use the constraints that were learned from  $\psi_i$  in the process of solving the next PCNF  $\psi_{i+1}$ . To this end, the ACNF  $\psi'_{i+1} = \hat{Q}_{i+1}. (\phi_{i+1} \wedge \theta_{i+1} \vee \gamma_{i+1})$  of  $\psi_{i+1}$  for i > 0, which is maintained by the solver, must be initialized with a set  $\theta_{i+1}$  of learned clauses and a set  $\gamma_{i+1}$  of learned cubes such that  $\theta_{i+1} \subseteq \theta_i, \gamma_{i+1} \subseteq \gamma_i$  and Proposition 2 holds with respect

to  $\psi_{i+1}$ . The sets  $\theta_i$  and  $\gamma_i$  contain the clauses and cubes that were learned from the previous PCNF  $\psi_i$  and potentially can be used to derive further constraints from  $\psi_{i+1}$ . If  $\theta_{i+1} \neq \emptyset$  and  $\gamma_{i+1} \neq \emptyset$  at the beginning, then the solver solves the PCNF  $\psi_{i+1}$  incrementally. For the first PCNF  $\psi_0$  in the sequence, the solver starts with empty sets of learned constraints in the ACNF  $\psi'_0 = \hat{Q}_0. (\phi_0 \wedge \theta_0 \vee \gamma_0)$ .

Each PCNF  $\psi_{i+1}$  for  $0 \leq i < n$  in the sequence  $\psi_0, \psi_1, \ldots, \psi_n$  has the form  $\psi_{i+1} = \hat{Q}_{i+1}. \phi_{i+1}$ . The CNF part  $\phi_{i+1}$  of  $\psi_{i+1}$  results from  $\phi_i$  of the previous PCNF  $\psi_i = \hat{Q}_i. \phi_i$  in the sequence by addition and deletion of clauses. We write  $\phi_{i+1} = (\phi_i \setminus \phi_{i+1}^{add}) \cup \phi_{i+1}^{add}$ , where  $\phi_{i+1}^{del}$  and  $\phi_{i+1}^{add}$  are the sets of deleted and added clauses. The quantifier prefix  $\hat{Q}_{i+1}$  of  $\psi_{i+1}$  is obtained from  $\hat{Q}_i$  of  $\psi_i$  by deletion and addition of variables and quantifiers, depending on the clauses in  $\phi_{i+1}^{add}$  and  $\phi_{i+1}^{del}$ . That is, we assume that the PCNF  $\psi_{i+1}$  is closed and that its prefix  $\hat{Q}_{i+1}$  does not contain superfluous quantifiers and variables.

When solving the PCNF  $\psi_i$  using a QCDCL-based QBF solver, learned clauses and cubes accumulate in the corresponding ACNF  $\psi'_i = \hat{Q}_i. (\phi_i \wedge \theta_i \vee \gamma_i)$ . Assume that the learned constraints are derivable with respect to  $\psi_i$ . The PCNF  $\psi_i$  is modified to obtain the next PCNF  $\psi_{i+1}$  to be solved. The learned constraints in  $\theta_i$  and  $\gamma_i$  might become non-derivable with respect to  $\psi_{i+1}$  in the sense of Definition 5. Consequently, Proposition 2 might no longer hold for the ACNF  $\psi'_{i+1} = \hat{Q}_{i+1}. (\phi_{i+1} \wedge \theta_{i+1} \vee \gamma_{i+1})$  of the new PCNF  $\psi_{i+1}$  if previously learned constraints from  $\theta_i$  and  $\gamma_i$  appear in  $\theta_{i+1}$  and  $\gamma_{i+1}$ . In this case, the solver might produce a wrong result when solving  $\psi_{i+1}$ .

#### 4.1 Clause Learning

Assume that the PCNF  $\psi_i = \hat{Q}_i \cdot \phi_i$  has been solved and learned constraints have been collected in the ACNF  $\psi'_i = \hat{Q}_i \cdot (\phi_i \wedge \theta_i \vee \gamma_i)$ . The clauses in  $\phi_{i+1}^{del}$  are deleted from  $\phi_i$  to obtain the CNF part  $\phi_{i+1} = (\phi_i \setminus \phi_{i+1}^{del}) \cup \phi_{i+1}^{add}$  of the next PCNF  $\psi_{i+1} = \hat{Q}_{i+1} \cdot \phi_{i+1}$ . If the derivation of a learned clause  $C \in \theta_i$  depends on deleted clauses in  $\phi_{i+1}^{del}$ , then we might have that  $\psi_i \vdash C$  but  $\psi_{i+1} \nvDash C$ . In this case, C is non-derivable with respect to the next PCNF  $\psi_{i+1}$ . Hence C must be discarded before solving  $\psi_{i+1}$  starts so that  $C \notin \theta_{i+1}$  in the initial ACNF  $\psi'_{i+1} = \hat{Q}_{i+1} \cdot (\phi_{i+1} \wedge \theta_{i+1} \vee \gamma_{i+1})$ . Otherwise, if  $C \in \theta_{i+1}$  then the solver might construct a bogus Q-resolution proof for the PCNF  $\psi_{i+1}$  and, if  $\psi_{i+1}$  is satisfiable, erroneously conclude that  $\psi_{i+1}$  is unsatisfiable.

Example 2. Consider the PCNF  $\psi$  from Example 1. The derivation of the clause  $C_8 = (\neg x_1)$  shown in Fig. 1 depends on the clause  $C_4 = (\neg y_8 \lor \neg x_4)$ . We have that  $\psi \vdash C_8$ . Let  $\psi_1$  be the PCNF obtained from  $\psi$  by deleting  $C_4$ . Then  $\psi_1 \nvDash C_8$  because  $C_3 = (\neg x_1 \lor x_4)$  is the only clause which contains the literal  $\neg x_1$ . Hence a possible derivation of the clause  $C_8 = (\neg x_1)$  must use  $C_3$ . However, no such derivation exists in  $\psi_1$ . There is no clause C' containing a literal  $\neg x_4$  which can be resolved with  $C_3$  to produce  $C_8 = (\neg x_1)$  after a sequence of resolution steps.

Consider the PCNF  $\psi_{i+1} = \hat{Q}_{i+1}$ .  $\phi_{i+1}$  with  $\phi_{i+1} = \phi_i \cup \phi_{i+1}^{add}$  which is obtained from  $\hat{Q}_i \cdot \phi_i$  by only adding the clauses  $\phi_{i+1}^{add}$ , but not deleting any clauses.

Assuming that  $\hat{Q}_i.\phi_i \vdash C$  for all  $C \in \theta_i$  in the ACNF  $\psi'_i = \hat{Q}_i.(\phi_i \land \theta_i \lor \gamma_i)$ , also  $\hat{Q}_{i+1}.(\phi_i \cup \phi_{i+1}^{add}) \vdash C$ . Hence all the learned clauses in  $\theta_i$  are derivable with respect to the next PCNF  $\psi_{i+1}$  and can be added to the ACNF  $\psi'_{i+1}$ .

### 4.2 Cube Learning

Like above, let  $\psi'_i = \hat{Q}_i . (\phi_i \land \theta_i \lor \gamma_i)$  be the ACNF of the previously solved PCNF  $\psi_i = \hat{Q}_i . \phi_i$ . Dual to clause deletions, the addition of clauses to  $\phi_i$  can make learned cubes in  $\gamma_i$  non-derivable with respect to the next PCNF  $\psi_{i+1} = \hat{Q}_{i+1} . \phi_{i+1}$  to be solved. The clauses in  $\phi_{i+1}^{add}$  are added to  $\phi_i$  to obtain the CNF part  $\phi_{i+1} = (\phi_i \setminus \phi_{i+1}^{del}) \cup \phi_{i+1}^{add}$  of  $\psi_{i+1}$ . An initial cube  $C \in \gamma_i$  has been obtained from a model A of the previous PCNF  $\psi_i$ , i.e.  $\psi_i[A] = \top$ . We might have that  $\psi_{i+1}[A] \neq \top$  with respect to the next PCNF  $\psi_{i+1}$  because of an added clause  $C' \in \phi_{i+1}^{add}$  (and hence also  $C' \in \phi_{i+1}$ ) such that  $C'[A] \neq \top$ . Therefore, A is not a model of  $\psi_{i+1}$  and the initial cube C is non-derivable with respect to  $\psi_{i+1}$ , i.e.  $\hat{Q}_i . \phi_i \vdash C$  but  $\hat{Q}_{i+1} . \phi_{i+1} \nvDash C$ . Hence C and every cube whose derivation depends on C must be discarded to prevent the solver from generating a bogus cube resolution proof for  $\psi_{i+1}$ . If  $\psi_{i+1}$  is unsatisfiable, then the solver might erroneously conclude that  $\psi_{i+1}$  is satisfiable. That is, Proposition 2 might not hold with respect to non-derivable cubes and the ACNF  $\psi'_{i+1}$  of  $\psi_{i+1}$ .

Example 3. Consider the PCNF  $\psi$  from Example 1. The derivation of the cube  $C_{10} = (\neg y_8)$  shown in Fig. 1 depends on the initial cube  $C_9 = (x_6 \land x_2 \land \neg y_8 \land \neg x_5 \land x_4)$ , which has been generated from the model  $A_1 = \{x_6, x_2, \neg y_8, \neg x_5, x_4\}$ . The cube  $C_9$  is derivable with respect to  $\psi$  since  $\psi[A_1] = \top$ , and hence  $\psi \vdash C_9$ . The cube  $C_{10}$  is also derivable since  $C_{10} = ER(C_9)$ . Assume that the clause  $C_0 := (\neg x_2 \lor \neg x_4)$  is added to  $\psi$  resulting in the unsatisfiable PCNF  $\psi_2$ . Now  $C_9$  is non-derivable with respect to  $\psi_2$  since  $C_0[A_1] = \bot$ . Further,  $\psi_2 \nvDash C_{10}$ .

Consider the PCNF  $\psi_{i+1} = \hat{Q}_{i+1} \cdot \phi_{i+1}$  with  $\phi_{i+1} = \phi_i \setminus \phi_{i+1}^{del}$  which is obtained from  $\hat{Q}_i \cdot \phi_i$  by only deleting the clauses  $\phi_{i+1}^{del}$ , but not adding any clauses. If after the clause deletions some variable x does not occur anymore in the resulting PCNF  $\psi_{i+1}$ , then x is removed from the quantifier prefix of  $\psi_{i+1}$  and from every cube  $C \in \gamma_i$  which was learned when solving the previous PCNF  $\psi_i$ . Proposition 2 holds for the cleaned up cubes  $C' = C \setminus \{l \mid v(l) = x\}$  for all  $C \in \gamma_i$  with respect to  $\psi_{i+1}$  and hence C' can be added to the ACNF  $\psi'_{i+1}$ .

**Proposition 3.** Let  $\psi'_i := \hat{Q}_i$ .  $(\phi_i \land \theta_i \lor \gamma_i)$  be the ACNF of the PCNF  $\psi_i := \hat{Q}_i . \phi_i$ . Let  $\psi_{i+1} := \hat{Q}_{i+1} . \phi_{i+1}$  be the PCNF resulting from  $\psi_i$  with  $\phi_{i+1} = (\phi_i \land \phi_{i+1}^{del})$ , where the variables  $V_{i+1}^{del}$  no longer occur in  $\phi_{i+1}$  and are removed from  $\hat{Q}_i$  to obtain  $\hat{Q}_{i+1}$ . Given a cube  $C \in \gamma_i$ , let  $C' := C \setminus \{l \mid v(l) \in V_{i+1}^{del}\}$ . Proposition 2 holds for C' with respect to  $\hat{Q}_{i+1} . \phi_{i+1} : \hat{Q}_{i+1} . \phi_{i+1} \equiv_{sat} \hat{Q}_{i+1} . (\phi_{i+1} \lor C')$ .

*Proof (Sketch).* By induction on the structure of the derivations of cubes in  $\gamma_i$ . Let  $C \in \gamma_i$  be an initial cube due to the assignment A with  $\psi_i[A] = \top$ . For

 $A' := A \setminus \{l \mid v(l) \in V_{i+1}^{del}\}$ , we have  $\psi_{i+1}[A'] = \top$  since all the clauses containing

the variables in  $V_{i+1}^{del}$  were deleted from  $\psi_i$  to obtain  $\psi_{i+1}$ . Then the claim holds for the initial cube  $C' = C \setminus \{l \mid v(l) \in V_{i+1}^{del}\} = (\bigwedge_{l_i \in A'} l_i)$  since  $\psi_{i+1} \vdash C'$ .

Let  $C \in \gamma_i$  be obtained from  $C_1 \in \gamma_i$  by existential reduction such that  $C = ER(C_1)$ . Assuming that the claim holds for  $C'_1 = C_1 \setminus \{l \mid v(l) \in V_{i+1}^{del}\}$ , it also holds for  $C' = C \setminus \{l \mid v(l) \in V_{i+1}^{del}\} = ER(C'_1)$  since existential reduction removes existential literals which are maximal with respect to the prefix ordering.

Let  $C \in \gamma_i$  be obtained from  $C_1, C_2 \in \gamma_i$  by resolution on variable x with  $x \in C_1, \ \neg x \in C_2$ . Assume that the claim holds for  $C'_1 = C_1 \setminus \{l \mid v(l) \in V^{del}_{i+1}\}$ and  $C'_2 = C_2 \setminus \{l \mid v(l) \in V^{del}_{i+1}\}$ , i.e.  $\hat{Q}_{i+1}.\phi_{i+1} \equiv_{sat} \hat{Q}_{i+1}.(\phi_{i+1} \vee C'_1)$  and  $\hat{Q}_{i+1}.\phi_{i+1} \equiv_{sat} \hat{Q}_{i+1}.(\phi_{i+1} \vee C'_2)$ . If  $x \notin V^{del}_{i+1}$  then the claim also holds for  $C' = C \setminus \{l \mid v(l) \in V^{del}_{i+1}\} = C'_1 \otimes C'_2$  with  $x \in C'_1, \ \neg x \in C'_2$  due to the correctness of resolution (Proposition 2). If  $x \notin V^{del}_{i+1}$  then the claim also holds for  $C' = C \setminus \{l \mid v(l) \in V^{del}_{i+1}\} = (C'_1 \wedge C'_2)$  since  $\{y, \neg y\} \not\subseteq (C'_1 \cup C'_2)$  for all variables y, which can be proved by reasoning with tree-like models of QBFs [30].  $\Box$ 

If a variable x no longer occurs in the formula, then cubes where x has been removed might become non-derivable. However, due to Propositions 2 and 3 it is sound to keep all the cleaned up cubes (resolution is not inferentially-complete). Moreover, due to the correctness of resolution and existential reduction, Proposition 2 also holds for new cubes derived from the cleaned up cubes.

In practice, the goal is to keep as many learned constraints as possible because they prune the search space and can be used to derive further constraints. Therefore, subsets  $\theta_{i+1} \subseteq \theta_i$  and  $\gamma_{i+1} \subseteq \gamma_i$  of the learned clauses  $\theta_i$  and cubes  $\gamma_i$ must be selected so that Proposition 2 holds with respect to the initial ACNF  $\psi'_{i+1} = \hat{Q}_{i+1}.(\phi_{i+1} \land \theta_{i+1} \lor \gamma_{i+1})$  of the PCNF  $\psi_{i+1}$  to be solved next.

# 5 Implementing an Incremental QBF Solver

We describe the implementation of our incremental QCDCL-based solver DepQBF. Our approach is general and fits any QCDCL-based solver. For incremental solving we do not apply a sophisticated analysis of variable dependencies by dependency schemes in DepQBF [22]. Instead, as many other QBF solvers, we use the linear ordering given by the quantifier prefix. We implemented a stackbased representation of the CNF part of PCNFs based on selector variables and assumptions. Assumptions were also used for incremental QBF-based BMC of partial designs [27] and are common in incremental SAT solving [1,11,20,28].

We address the problem of checking which learned constraints can be kept across different solver runs after the current PCNF has been modified. To this end, we present approaches to check if a constraint learned from the previous PCNF is still derivable from the next one, which makes sure that Proposition 2 holds. Similar to incremental SAT solving, selector variables are used to handle the learned clauses. Regarding learned cubes, selector variables can also be used (although in a way asymmetric to clauses), in addition to an alternative approach relying on full derivation DAGs, which have to be kept in memory. Learned cubes might become non-derivable by the deletion of clauses and superfluous variables, but still can be kept due to Proposition 3. We implemented a simple approach which, after clauses have been added to the formula, allows to keep only initial cubes but not cubes obtained by resolution or existential reduction.

#### 5.1 QBF Solving under Assumptions

Let  $\psi := Q_1 B_1 Q_2 B_2 \dots Q_n B_n$ ,  $\phi$  be a PCNF. We define a set  $A := \{l_1, \dots, l_k\}$ of assumptions as an assignment such that  $v(l_i) \in B_1$  for all literals  $l_i \in A$ . The variables assigned by A are from the first block  $B_1$  of  $\psi$ . Solving the PCNF  $\psi$ under the set A of assumptions amounts to solving the PCNF  $\psi[A]$ . The definition of assumptions can be applied recursively to the PCNF  $\psi[A]$ . If A assigns all the variables in  $B_1$ , then variables from  $B_2$  can be assigned as assumptions with respect to  $\psi[A]$ , since  $B_2$  is the first block in the quantifier prefix of  $\psi[A]$ .

We implemented the handling of assumptions according to the *literal-based* single instance (LS) approach (in the terminology of [28]). Thereby, the assumptions in A are treated in a special way so that the variables in A are never selected as pivots in the resolution derivation of a learned constraint according to QCDCL-based learning. Similar to SAT-solving under assumptions, LS allows to keep all the constraints that were learned from the PCNF  $\psi[A]$  under a set A of assumptions when later solving  $\psi[A']$  under a different set A' of assumptions.

### 5.2 Stack-Based CNF Representation

In DepQBF, the CNF part  $\phi$  of an ACNF  $\psi'_i = \hat{Q}_i . (\phi_i \land \theta_i \lor \gamma_i)$  to be solved is represented as a stack of clauses. The clauses on the stack are grouped into *frames*. The solver API provides functions to push new frames onto the stack, pop present frames from the stack, and to add new clauses to the current topmost frame. Each *push* operation opens a new topmost frame  $f_j$ . New clauses are always added to the topmost frame  $f_j$ . Each new frame  $f_j$  opened by a *push* operation is associated with a fresh *frame selector variable*  $s_j$ . Frame selector variables are existentially quantified and put into a separate, leftmost quantifier block  $B_0$  i.e. the current ACNF  $\psi'_i$  has the form  $\psi'_i = \exists B_0 \hat{Q}_i . (\phi_i \land \theta_i \lor \gamma_i)$ . Before a new clause C is added to frame  $f_j$ , the frame selector variable  $s_j$  of  $f_j$  is inserted into C so that in fact the clause  $C' = C \cup \{s_j\}$  is added to  $f_j$ . If all the selector variables are assigned to *false* then under that assignment every clause  $C' = C \cup \{s_i\}$  is syntactically equivalent to C.

The purpose of the frame selector variables is to *enable* or *disable* the clauses in the CNF part  $\phi_i$  with respect to the *push* and *pop* operations applied to the clause stack. If the selector variable  $s_j$  of a frame  $f_j$  is assigned to *true* then all the clauses of  $f_j$  are satisfied under that assignment. In this case, these satisfied clauses are considered disabled because they can not be used to derive new learned clauses in QCDCL. Otherwise, the assignment of *false* to  $s_j$  does not satisfy any clauses in  $f_j$ . Therefore these clauses are considered enabled.

Before the solving process starts, the clauses of frames popped from the stack are disabled and the clauses of frames still on the stack are enabled by assigning the selector variables to *true* and *false*, respectively. The selector variables are assigned as assumptions. This is possible because these variables are in the leftmost quantifier block  $B_0$  of the ACNF  $\psi'_i = \exists B_0 \hat{Q}_i . (\phi_i \wedge \theta_i \vee \gamma_i)$  to be solved.

The idea of enabling and disabling clauses by selector variables and assumptions originates from incremental SAT solving [11]. This approach was also applied to bounded model checking of partial designs by incremental QBF solving [27]. In DepQBF, we implemented the *push* and *pop* operations related to the clause stack by selector variables similarly to the SAT solver PicoSAT [5].

In the implementation of DepQBF, frame selector variables are maintained entirely by the solver. Depending on the *push* and *pop* operations, selector variables are automatically inserted into added clauses and assigned as assumptions. This approach saves the user the burden of inserting selector variables manually into the QBF encoding of a problem and assigning them as assumptions via the solver API. Manual insertion is typically applied in incremental SAT solving based on assumptions as pioneered by MiniSAT [10,11]. We argue that the usability of an incremental QBF solver is improved considerably if the selector variables are maintained by the solver. For example, from the perspective of the user, the QBF encoding contains only variables relevant to the encoded problem.

In the following, we consider the problem of maintaining the sets of learned constraints across different solver runs. As pointed out in Section 4, Proposition 2 still holds for learned clauses (cubes) after the addition (deletion) of clauses to (from) the PCNF. Therefore, we present the maintenance of learned constraints separately for clause additions and deletions.

### 5.3 Handling Clause Deletions

A clause  $C \in \theta_i$  in the current ACNF  $\psi'_i = \hat{Q}_i \cdot (\phi_i \wedge \theta_i \vee \gamma_i)$  might become non-derivable if its derivation depends on clauses in  $\phi^{del}_{i+1}$  which are deleted to obtain the CNF part  $\phi_{i+1} = (\phi_i \setminus \phi^{del}_{i+1}) \cup \phi^{add}_{i+1}$  of the next PCNF  $\psi_{i+1}$ .

In DepQBF, learned clauses in  $\theta_i$  are deleted as follows. As pointed out in the previous section, clauses of popped off frames are disabled by assigning the respective frame selector variables to *true*. Since the formula contains only positive literals of selector variables, these variables cannot be chosen as pivots in derivations. Therefore, learned clauses whose derivations depend on disabled clauses of a popped off frame  $f_j$  contain the selector variable  $s_j$  of  $f_j$ . Hence these learned clauses are also disabled by the assignment of  $s_j$ . This approach to handling learned clauses is also applied in incremental SAT solving [11].

The disabled clauses are physically deleted in a garbage collection phase if their number exceeds a certain threshold. Variables which no longer occur in the CNF part of the current PCNF are removed from the quantifier prefix and, by Proposition 3, from learned cubes in  $\gamma_i$  to produce cleaned up cubes. We initialize the set  $\gamma_{i+1}$  of learned cubes in the ACNF  $\psi'_{i+1} = \hat{Q}_{i+1}$ .  $(\phi_{i+1} \wedge \theta_{i+1} \vee \gamma_{i+1})$  of the next PCNF  $\psi_{i+1}$  to be solved to contain the cleaned up cubes.

The deletion of learned clauses based on selector variables is not optimal in the sense of Definition 5. There might be another derivation of a disabled learned clause C which does not depend on the deleted clauses  $\phi_{i+1}^{del}$ . This observation also applies to the use of selector variables in incremental SAT solving.

As illustrated in the context of incremental SAT solving, the size of learned clauses might increase considerably due to the additional selector variables [1,20]. In the stack-based CNF representation of DepQBF, the clauses associated to a frame  $f_j$  all contain the selector variable  $s_j$  of  $f_j$ . Therefore, the maximum number of selector variables in a new clause learned from the current PCNF  $\psi_i$ is bounded by the number of currently enabled frames. The sequence of *push* operations introduces a linear ordering  $f_0 < f_1 < \ldots < f_k$  on the enabled frames  $f_i$  and their clauses in the CNF with respect to the point of time where that frames and clauses have been added. In DepQBF, we implemented the following optimization based on this temporal ordering. Let C and C' be clauses which are resolved in the course of clause learning. Assume that  $s_i \in C$  and  $s_j \in C'$ are the only selector variables of currently enabled frames  $f_i$  and  $f_j$  in C and C'. Instead of computing the usual Q-resolvent  $C'' := C \otimes C'$ , we compute  $C'' := (C \otimes C') \setminus \{l \mid l = s_i \text{ if } f_i < f_j \text{ and } l = s_j \text{ otherwise}\}.$  That is, the selector variable of the frame which is smaller in the temporal ordering is discarded from the resolvent. If  $f_i < f_j$  then the clauses in  $f_i$  were pushed onto the clause stack before the clauses in  $f_j$ . The frame  $f_j$  will be popped off the stack before  $f_i$ . Therefore, in order to properly disable the learned clause C'' after *pop* operations, it is sufficient to keep the selector variable  $s_j$  of the frame  $f_j$  in C''. With this optimization, every learned clause contains exactly one selector variable. In the SAT solver PicoSAT, an optimization which has similar effects is implemented.

### 5.4 Handling Clause Additions

Assume that the PCNF  $\psi_i := \hat{Q}_i \cdot \phi_i$  has been solved and that all learned constraints in the ACNF  $\psi'_i = \hat{Q}_i \cdot (\phi_i \wedge \theta_i \vee \gamma_i)$  of  $\psi_i$  are derivable with respect to  $\psi_i$ . The set  $\phi_{i+1}^{add}$  of clauses is added to  $\phi_i$  to obtain the CNF part  $\phi_{i+1} = (\phi_i \setminus \phi_{i+1}^{del}) \cup \phi_{i+1}^{add}$  of the next PCNF  $\psi_{i+1} = \hat{Q}_{i+1} \cdot \phi_{i+1}$ . For learned clauses, we can set  $\theta_{i+1} := \theta_i$  in the ACNF  $\psi'_{i+1} = \hat{Q}_{i+1} \cdot (\phi_{i+1} \wedge \theta_{i+1} \vee \gamma_{i+1})$  of  $\psi_{i+1}$ . The following example illustrates the effects of adding  $\phi_{i+1}^{add}$  on the cubes.

Example 4. Consider the cube derivation shown in Fig. 1. As illustrated in Example 3, the cubes  $C_9 = (x_6 \land x_2 \land \neg y_8 \land \neg x_5 \land x_4)$  and  $C_{10} = (\neg y_8)$  are non-derivable with respect to the PCNF  $\psi_2$  obtained from  $\psi$  by adding the clause  $C_0 := (\neg x_2 \lor \neg x_4)$ . The initial cube  $C_{11} := (y_8 \land \neg x_4 \land \neg x_1 \land x_5 \land x_6 \land x_2)$  still is derivable because the underlying model  $A_2 := \{y_8, \neg x_4, \neg x_1, x_5, x_6, x_2\}$  of  $\psi$  is also a model of  $\psi_2$ . Therefore, when solving  $\psi_2$  we can keep the derivable cubes  $C_{11}$  and  $C_{12} = ER(C_{11})$ . The non-derivable cubes  $C_9$  and  $C_{10}$  must be discarded. Otherwise, QCDCL might produce the cube resolution proof shown in Fig. 1 when solving the unsatisfiable PCNF  $\psi_2$ , which is incorrect.

We sketch an approach to identify the cubes in a cube derivation DAG G which are non-derivable with respect to the next PCNF  $\psi_{i+1} = \hat{Q}_{i+1} \cdot \phi_{i+1}$ . Starting at the initial cubes, G is traversed in a topological order. An initial cube C is marked as derivable if  $\psi_{i+1}[C] = \top$ , otherwise if  $\psi_{i+1}[C] \neq \top$  then C is marked as non-derivable. This test can be carried out syntactically by checking whether every clause of  $\psi_{i+1}$  is satisfied under the assignment given by C. A cube C obtained by existential reduction or cube resolution is marked as derivable if all its predecessors in G are marked as derivable. Otherwise, C is marked as non-derivable. Finally, all cubes in G marked as non-derivable are deleted.

The above procedure allows to find a subset  $\gamma_{i+1} \subseteq \gamma_i$  of the set  $\gamma_i$  of cubes in the solved ACNF  $\psi'_i = \hat{Q}_i$ .  $(\phi_i \land \theta_i \lor \gamma_i)$  so that all cubes in  $\gamma_{i+1}$  are derivable and Proposition 2 holds for the next ACNF  $\psi'_{i+1} = \hat{Q}_{i+1}$ .  $(\phi_{i+1} \land \theta_{i+1} \lor \gamma_{i+1})$ . However, this procedure is not optimal because it might mark a cube  $C \in G$  as non-derivable with respect to the next PCNF  $\psi_{i+1}$  although  $\psi_{i+1} \vdash C$ .

Example 5. Given the satisfiable PCNF  $\psi := \exists x_1 \forall y_8 \exists x_5, x_2, x_6, x_4, \phi$ , where  $\phi := \bigwedge_{i:=1,\ldots,5} C_i$  with the clauses  $C_i$  from Example 1 where  $C_1 := (y_8 \vee \neg x_5)$ ,  $C_2 := (x_2 \vee \neg x_6), C_3 := (\neg x_1 \vee x_4), C_4 := (\neg y_8 \vee \neg x_4), C_5 := (x_1 \vee x_6)$ . Consider the model  $A_3 := \{\neg x_1, y_8, \neg x_5, x_2, x_6, \neg x_4\}$  of  $\psi$  and the initial cube  $C_{15} := (\neg x_1 \wedge y_8 \wedge \neg x_5 \wedge x_2 \wedge x_6 \wedge \neg x_4)$  generated from  $A_3$ . Existential reduction of  $C_{15}$  produces the cube  $C_{16} := ER(C_{15}) = (\neg x_1 \wedge y_8)$ . Assume that the clause  $C_0 := (x_4 \vee x_5)$  is added to  $\psi$  to obtain the PCNF  $\psi_3$ . The initial cube  $C_{16}$  derived from  $C_{15}$  it holds that  $\psi_3 \vdash C_{16}$ . The assignment  $A_4 := \{\neg x_1, y_8, x_5, x_2, x_6, \neg x_4\}$  is a model of  $\psi_3$ . Let  $C_{17} := (\neg x_1 \wedge y_8 \wedge x_5 \wedge x_2 \wedge x_6 \wedge \neg x_4)$  be the initial cube generated from  $A_4$ . Then  $C_{16} = ER(C_{17})$  is derivable with respect to  $\psi_3$ .

In practice, QCDCL-based solvers typically store only the learned cubes, which might be a small part of the derivation DAG G, and no edges. Therefore, checking the cubes in a traversal of G is not feasible. Even if the full DAG Gis available, the checking procedure is not optimal as pointed out in Example 5. Furthermore, it cannot be used to check cubes which have become non-derivable after cleaning up by Proposition 3. Hence, it is desirable to have an approach to checking the derivability of *individual* learned cubes which is independent from the derivation DAG G. To this end, we need a condition which is sufficient to conclude that some *arbitrary* cube C is derivable with respect to a PCNF  $\psi$ , i.e. to check whether  $\psi \vdash C$ . However, we are not aware of such a condition.

As an alternative to keeping the full derivation DAG in memory, a *fresh* selector variable can be added to *each* newly learned initial cube. Similar to selector variables in clauses, these variables are transferred to all derived cubes. Potentially non-derivable cubes are then disabled by assigning the selector variables accordingly. However, different from clauses, it must be checked *explicitly* which initial cubes are non-derivable by checking the condition in Definition 1 for all initial cubes in the set  $\gamma_i$  of learned cubes. This amounts to an asymmetric treatment of selector variables in clauses and cubes. Clauses are added to and removed from the CNF part by *push* and *pop* operations provided by the solver API. This way, it is known precisely which clauses are removed. In contrast to that, cubes are added to the set of learned cubes  $\gamma_i$  on the fly during cube learning. Moreover, the optimization based on the temporal ordering of selector variables from the previous section is not applicable to generate shorter cubes since cubes are not associated to stack frames.

Due to the complications illustrated above, we implemented the following simple approach in DepQBF to keep only initial cubes. Every initial cube computed by the solver is stored in a linked list L of bounded capacity, which is increased dynamically. The list L is separate from the set of learned clauses. Assume that a set  $\phi_{i+1}^{add}$  of clauses is added to the CNF part  $\phi_i$  of the current PCNF to obtain the CNF part  $\phi_{i+1} = (\phi_i \setminus \phi_{i+1}^{add}) \cup \phi_{i+1}^{add}$  of the next PCNF  $\psi_{i+1} = \hat{Q}_{i+1} \cdot \phi_{i+1}$ . All the cubes in the current set  $\gamma_i$  of learned cubes are discarded. For every added clause  $C \in \phi_{i+1}^{add}$  and for every initial cube  $C' \in L$ , it is checked whether the assignment A given by C' is a model of the next PCNF  $\psi_{i+1}$ . Initial cubes C' for which this check succeeds are added to the set  $\gamma_{i+1}$  of learned cubes in the ACNF  $\psi'_{i+1}$  of the next PCNF  $\psi_{i+1}$  after existential reduction has been applied to them. If the check fails, then C' is removed from L. It suffices to check the initial cubes in L only with respect to the clauses  $C \in \phi_{i+1}^{add}$ , and not the full CNF part  $\phi_{i+1}$ , since the assignments given by the cubes in L are models of the current PCNF  $\psi_i$ . In the end, the set  $\gamma_{i+1}$ . If clauses are removed from the formula, then by Proposition 3 variables which do not occur anymore in the formula are removed from the initial cubes in L.

In the incremental QBF-based approach to BMC for partial designs [26,27], all cubes are kept across different solver calls under the restriction that the quantifier prefix is modified only at the left end. This restriction does not apply to incremental solving of PCNF where the formula can be modified arbitrarily.

### 5.5 Incremental QBF Solver API

The API of DepQBF [23] provides functions to manipulate the prefix and the CNF part of the current PCNF. Clauses are added and removed by the *push* and *pop* operations described in Section 5.2. New quantifier blocks can be added at any position in the quantifier prefix. New variables can be added to any quantifier block. Variables which no longer occur in the formula and empty quantifier blocks can be explicitly deleted. The quantifier block  $B_0$  containing the frame selector variables is invisible to the user. The solver maintains the learned constraints as described in Sections 5.3 and 5.4 without any user interaction.

The *push* and *pop* operations are a feature of DepQBF. Additionally, the API supports the manual insertion of selector variables into the clauses by the user. Similar to incremental SAT solving [11], clauses can then be enabled and disabled manually by assigning the selector variables as assumptions via the API. In this case, these variables are part of the QBF encoding and the optimization based on the frame ordering presented in Section 5.3 is not applicable. After a PCNF has been found unsatisfiable (satisfiable) under assumptions where the leftmost quantifier block is existential (universal), the set of relevant assumptions which were used by the solver to determine the result can be extracted.<sup>3</sup>

<sup>&</sup>lt;sup>3</sup> This is similar to the function "analyzeFinal" in MiniSAT, for example.

	QBFEVAL'12-SR				QBFEVAL'12-SR-Bloqqer					
	$discard \ LC$	$keep \ LC$	diff.(%)			$discard \ LC$	$keep \ LC$	diff.(%)		
$\overline{a}$ :	$29.37 \times 10^6$	$26.18 \times 10^6$	-10.88	$\overline{a}$	:	$39.75 \times 10^{6}$	$34.03 \times 10^6$	-14.40		
$\tilde{a}$ :	$3,\!833,\!077$	$2,\!819,\!492$	-26.44	$ \tilde{a} $	:	$1.71 \times 10^6$	$1.65 \times 10^6$	-3.62		
$\overline{b}$ :	139,036	116,792	-16.00	$\overline{b}$	:	117,019	91,737	-21.61		
$\tilde{b}$ :	8,243	6,360	-22.84	$ \tilde{b} $	:	10,322	8,959	-13.19		
$\overline{t}$ :	99.03	90.90	-8.19	$\overline{t}$		100.15	95.36	-4.64		
$\tilde{t}$ :	28.56	15.74	-44.88	$ \tilde{t} $		4.18	2.83	-32.29		

**Table 1.** Average and median number of assignments ( $\overline{a}$  and  $\tilde{a}$ , respectively), backtracks ( $\overline{b}$ ,  $\widetilde{b}$ ), and wall clock time ( $\overline{t}$ ,  $\widetilde{t}$ ) in seconds on sequences  $S = \psi_0, \ldots, \psi_{10}$  of PCNFs which were fully solved by DepQBF both if all learned constraints are discarded (*discard LC*) and if constraints which are correct in the sense of Propositions 2 and 3 are kept (keep LC). Clauses are added to  $\psi_i$  to obtain  $\psi_{i+1}$  in S.

	ODEE	)	QBFEVAL'12-SR-Blogger					
	QBFEVAL'12-SR				•			
	$discard \ LC$	$keep \ LC$	diff.(%)		discar	$rd \ LC$	$keep \ LC$	diff.(%)
$\overline{a}$ :	$5.48 \times 10^6$	$0.73 \times 10^6$	-86.62	$\overline{a}$	5.88	$\times 10^{6}$	$1.29 \times 10$	$^{6}$ -77.94
$\tilde{a}$ :	186,237	15,031	-91.92	ã	103	,330	$^{8,199}$	-92.06
$\overline{b}$ :	36,826	1,228	-96.67	$\overline{b}$ :	31,	489	$3,\!350$	-89.37
$ \tilde{b}$ :	424	0	-100.00	$ $ $\tilde{b}$ :	8	27	5	-99.39
$\overline{t}$ :	21.94	4.32	-79.43	$\overline{t}$ :	30	.29	9.78	-67.40
$ \tilde{t}$ :	0.75	0.43	-42.66	$ $ $\tilde{t}$ :	0.	50	0.12	-76.00
		1 1 1		· ,	0	1 /	1 0	DONE

**Table 2.** Like Table 1 but for the reversed sequences  $S' = \psi_9, \ldots, \psi_0$  of PCNFs after the original sequence  $S = \psi_0, \ldots, \psi_9, \psi_{10}$  has been solved. Clauses are *deleted* from  $\psi_i$  to obtain  $\psi_{i-1}$  in S'.

# 6 Experimental Results

To demonstrate the basic feasibility of general incremental QBF solving, we evaluated our incremental QBF solver DepQBF based on the instances from QBFE-VAL'12 Second Round (SR) with and without preprocessing by Bloqqer.<sup>4</sup> We disabled the sophisticated dependency analysis in terms of dependency schemes in DepQBF and instead applied the linear ordering of the quantifier prefix in the given PCNFs. For experiments, we constructed a sequence of related PCNFs for each PCNF in the benchmark sets as follows. Given a PCNF  $\psi$ , we divided the number of clauses in  $\psi$  by 10 to obtain the size of a slice of clauses. The first PCNF  $\psi_0$  in the sequence contains the clauses of one slice. The clauses of that slice are removed from  $\psi$ . The next PCNF  $\psi_1$  is obtained from  $\psi_0$  by adding another slice of clauses, which is removed from  $\psi$ . The other PCNFs in the sequence  $S = \psi_0, \psi_1, \ldots, \psi_{10}$  are constructed similarly so that finally the last PCNF  $\psi_{10}$  contains all the clauses from the original PCNF  $\psi$ . In our tests, we constructed each PCNF  $\psi_i$  from the previous one  $\psi_{i-1}$  in the sequence by adding a slice of clauses to a new frame after a *push* operation. We ran DepQBF

<sup>&</sup>lt;sup>4</sup> http://www.kr.tuwien.ac.at/events/qbfgallery2013/benchmarks/.

on the sequences of PCNFs constructed this way with a wall clock time limit of 1800 seconds and a memory limit of 7 GB.

Tables 1 and 2 show experimental results<sup>5</sup> on sequences  $S = \psi_0, \ldots, \psi_{10}$  of PCNFs and on the reversed ones  $S' = \psi_9, \ldots, \psi_0$ , respectively. To generate S', we first solved the sequence S and then started to discard clauses by popping the frames from the clause stack of DepQBF via its API. In one run (*discard* LC), we always discarded all the constraints that were learned from the previous PCNF  $\psi_i$  so that the solver solves the next PCNF  $\psi_{i+1}$  ( $\psi_{i-1}$  with respect to Table 2) starting with empty sets of learned clauses and cubes. In another run (*keep* LC), we kept learned constraints as described in Sections 5.3 and 5.4. This way, 70 out of 345 total PCNF sequences were fully solved from the set QBFEVAL'12-SR by both runs, and 112 out of 276 total sequences were fully solved from the set QBFEVAL'12-SR-Blogger.

The numbers of assignments, backtracks, and wall clock time indicate that keeping the learned constraints is beneficial in incremental QBF solving despite the additional effort of checking the collected initial cubes. In the experiment reported in Table 1 clauses are always added but never deleted to obtain the next PCNF in the sequence. Thereby, across all incremental calls of the solver in the set QBFEVAL'12-SR on average 224 out of 364 (61%) collected initial cubes were identified as derivable and added as learned cubes. For the set QBFEVAL'12-SR-Bloqqer, 232 out of 1325 (17%) were added.

Related to Table 2, clauses are always removed but never added to obtain the next PCNF to be solved, which allows to keep learned cubes based on Proposition 3. Across all incremental calls of the solver in the set QBFEVAL'12-SR on average 820 out of 1485 (55%) learned clauses were disabled and hence effectively discarded because their Q-resolution derivation depended on removed clauses. For the set QBFEVAL'12-SR-Blogger, 704 out of 1399 (50%) were disabled.

# 7 Conclusion

We presented a general approach to incremental QBF solving which integrates ideas from incremental SAT solving and which can be implemented in any QCDCL-based QBF solver. The API of our incremental QBF solver DepQBF provides *push* and *pop* operations to add and remove clauses in a PCNF. This increases the usability of our implementation. Our approach is applicationindependent and applicable to arbitrary QBF encodings.

We illustrated the problem of keeping the learned constraints across different calls of the solver. To improve cube learning in incremental QBF solving, it might be beneficial to maintain (parts of) the cube derivation in memory. This would allow to check the cubes more precisely than with the simple approach we implemented. Moreover, the generation of proofs and certificates [2,14,29] is supported if the derivations are kept in memory rather than in a trace file.

Dual reasoning [15,16,19,35] and the combination of preprocessing and certificate extraction [18,26,31] are crucial for the performance and applicability

<sup>&</sup>lt;sup>5</sup> Experiments were run on AMD Opteron 6238, 2.6 GHz, 64-bit Linux.

of CNF-based QBF solving. The combination of incremental solving with these techniques has the potential to further advance the state of QBF solving.

Our experimental analysis demonstrates the feasibility of incremental QBF solving in a general setting and motivates further applications, along with the study of BMC of partial designs using incremental QBF solving [27]. Related experiments with conformant planning based on incremental solving by DepQBF showed promising results [12]. Further experiments with problems which are inherently incremental can provide more insights and open new research directions.

# References

- Audemard, G., Lagniez, J.M., Simon, L.: Improving Glucose for Incremental SAT Solving with Assumptions: Application to MUS Extraction. In: Järvisalo, M., Van Gelder, A. (eds.) SAT. LNCS, vol. 7962, pp. 309–317. Springer (2013)
- Balabanov, V., Jiang, J.R.: Unified QBF certification and its applications. Formal Methods in System Design 41(1), 45–65 (2012)
- Becker, B., Ehlers, R., Lewis, M.D.T., Marin, P.: ALLQBF Solving by Computational Learning. In: Chakraborty, S., Mukund, M. (eds.) ATVA. LNCS, vol. 7561, pp. 370–384. Springer (2012)
- Benedetti, M., Mangassarian, H.: QBF-Based Formal Verification: Experience and Perspectives. JSAT 5, 133–191 (2008)
- 5. Biere, A.: PicoSAT Essentials. JSAT 4(2-4), 75–97 (2008)
- Bloem, R., Könighofer, R., Seidl, M.: SAT-Based Synthesis Methods for Safety Specs. In: McMillan, K.L., Rival, X. (eds.) VMCAI. LNCS, vol. 8318, pp. 1–20. Springer (2014)
- Büning, H.K., Karpinski, M., Flögel, A.: Resolution for Quantified Boolean Formulas. Inf. Comput. 117(1), 12–18 (1995)
- Cadoli, M., Schaerf, M., Giovanardi, A., Giovanardi, M.: An Algorithm to Evaluate Quantified Boolean Formulae and Its Experimental Evaluation. J. Autom. Reasoning 28(2), 101–142 (2002)
- Cashmore, M., Fox, M., Giunchiglia, E.: Planning as Quantified Boolean Formula. In: Raedt, L.D., Bessière, C., Dubois, D., Doherty, P., Frasconi, P., Heintz, F., Lucas, P.J.F. (eds.) ECAI. Frontiers in Artificial Intelligence and Applications, vol. 242, pp. 217–222. IOS Press (2012)
- Eén, N., Sörensson, N.: An Extensible SAT-Solver. In: Giunchiglia, E., Tacchella, A. (eds.) SAT. LNCS, vol. 2919, pp. 502–518. Springer (2003)
- Eén, N., Sörensson, N.: Temporal Induction by Incremental SAT Solving. Electr. Notes Theor. Comput. Sci. 89(4), 543–560 (2003)
- Egly, U., Kronegger, M., Lonsing, F., Pfandler, A.: Conformant Planning as a Case Study of Incremental QBF Solving. CoRR abs/1405.7253 (2014)
- Giunchiglia, E., Narizzano, M., Tacchella, A.: Clause/Term Resolution and Learning in the Evaluation of Quantified Boolean Formulas. J. Artif. Intell. Res. (JAIR) 26, 371–416 (2006)
- Goultiaeva, A., Van Gelder, A., Bacchus, F.: A Uniform Approach for Generating Proofs and Strategies for Both True and False QBF Formulas. In: Walsh, T. (ed.) IJCAI. pp. 546–553. IJCAI/AAAI (2011)
- Goultiaeva, A., Bacchus, F.: Recovering and Utilizing Partial Duality in QBF. In: Järvisalo, M., Van Gelder, A. (eds.) SAT. LNCS, vol. 7962, pp. 83–99. Springer (2013)

- Goultiaeva, A., Seidl, M., Biere, A.: Bridging the Gap between Dual Propagation and CNF-based QBF Solving. In: Macii, E. (ed.) DATE. pp. 811–814. EDA Consortium San Jose, CA, USA / ACM DL (2013)
- Hillebrecht, S., Kochte, M.A., Erb, D., Wunderlich, H.J., Becker, B.: Accurate QBF-Based Test Pattern Generation in Presence of Unknown Values. In: Macii, E. (ed.) DATE. pp. 436–441. EDA Consortium San Jose, CA, USA / ACM DL (2013)
- Janota, M., Grigore, R., Marques-Silva, J.: On QBF Proofs and Preprocessing. In: McMillan, K.L., Middeldorp, A., Voronkov, A. (eds.) LPAR. LNCS, vol. 8312, pp. 473–489. Springer (2013)
- Klieber, W., Sapra, S., Gao, S., Clarke, E.M.: A Non-prenex, Non-clausal QBF Solver with Game-State Learning. In: Strichman, O., Szeider, S. (eds.) SAT. pp. 128–142. LNCS, Springer (2010)
- Lagniez, J.M., Biere, A.: Factoring Out Assumptions to Speed Up MUS Extraction. In: Järvisalo, M., Van Gelder, A. (eds.) SAT. LNCS, vol. 7962, pp. 276–292. Springer (2013)
- Letz, R.: Lemma and Model Caching in Decision Procedures for Quantified Boolean Formulas. In: Egly, U., Fermüller, C.G. (eds.) TABLEAUX. LNCS, vol. 2381, pp. 160–175. Springer (2002)
- Lonsing, F., Biere, A.: Integrating Dependency Schemes in Search-Based QBF Solvers. In: Strichman, O., Szeider, S. (eds.) SAT. pp. 158–171. LNCS, Springer (2010)
- Lonsing, F., Egly, U.: Incremental QBF Solving by DepQBF (Extended Abstract). In: The 4th International Congress on Mathematical Software, ICMS 2014, Seoul, Korea, August 2014. Proceedings. LNCS, vol. 8592. Springer (2014), to appear.
- Lonsing, F., Egly, U., Van Gelder, A.: Efficient Clause Learning for Quantified Boolean Formulas via QBF Pseudo Unit Propagation. In: Järvisalo, M., Van Gelder, A. (eds.) SAT. LNCS, vol. 7962, pp. 100–115. Springer (2013)
- Mangassarian, H., Veneris, A.G., Benedetti, M.: Robust QBF Encodings for Sequential Circuits with Applications to Verification, Debug, and Test. IEEE Trans. Computers 59(7), 981–994 (2010)
- Marin, P., Miller, C., Becker, B.: Incremental QBF Preprocessing for Partial Design Verification - (Poster Presentation). In: Cimatti, A., Sebastiani, R. (eds.) SAT. LNCS, vol. 7317, pp. 473–474. Springer (2012)
- Marin, P., Miller, C., Lewis, M.D.T., Becker, B.: Verification of Partial Designs using Incremental QBF Solving. In: Rosenstiel, W., Thiele, L. (eds.) DATE. pp. 623–628. IEEE (2012)
- Nadel, A., Ryvchin, V.: Efficient SAT Solving under Assumptions. In: Cimatti, A., Sebastiani, R. (eds.) SAT. LNCS, vol. 7317, pp. 242–255. Springer (2012)
- Niemetz, A., Preiner, M., Lonsing, F., Seidl, M., Biere, A.: Resolution-Based Certificate Extraction for QBF - (Tool Presentation). In: Cimatti, A., Sebastiani, R. (eds.) SAT. LNCS, vol. 7317, pp. 430–435. Springer (2012)
- Samulowitz, H., Davies, J., Bacchus, F.: Preprocessing QBF. In: Benhamou, F. (ed.) CP. LNCS, vol. 4204, pp. 514–529. Springer (2006)
- Seidl, M., Könighofer, R.: Partial witnesses from preprocessed quantified Boolean formulas. In: DATE. pp. 1–6. IEEE (2014)
- Silva, J.P.M., Lynce, I., Malik, S.: Conflict-Driven Clause Learning SAT Solvers. In: Biere, A., Heule, M., van Maaren, H., Walsh, T. (eds.) Handbook of Satisfiability, FAIA, vol. 185, pp. 131–153. IOS Press (2009)
- Staber, S., Bloem, R.: Fault Localization and Correction with QBF. In: Marques-Silva, J., Sakallah, K.A. (eds.) SAT. LNCS, vol. 4501, pp. 355–368. Springer (2007)

- Sülflow, A., Fey, G., Drechsler, R.: Using QBF to Increase Accuracy of SAT-Based Debugging. In: ISCAS. pp. 641–644. IEEE (2010)
- Van Gelder, A.: Primal and Dual Encoding from Applications into Quantified Boolean Formulas. In: Schulte, C. (ed.) CP. LNCS, vol. 8124, pp. 694–707. Springer (2013)
- Zhang, L., Malik, S.: Towards a Symmetric Treatment of Satisfaction and Conflicts in Quantified Boolean Formula Evaluation. In: Hentenryck, P.V. (ed.) CP. LNCS, vol. 2470, pp. 200–215. Springer (2002)