

# Deterministically Computing Reduction Numbers of Polynomial Ideals

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## Abstract

We discuss the problem of determining reduction numbers of a polynomial ideal  $\mathcal{I}$  in  $n$  variables. We present two algorithms based on parametric computations. The first one determines the absolute reduction number of  $\mathcal{I}$  and requires computations in a polynomial ring with  $(n - \dim \mathcal{I}) \dim \mathcal{I}$  parameters and  $n - \dim \mathcal{I}$  variables. The second one computes via a Gröbner system the set of all reduction numbers of the ideal  $\mathcal{I}$  and thus in particular also its big reduction number. However, it requires computations in a ring with  $n \dim \mathcal{I}$  parameters and  $n$  variables.

## 1 Introduction

One of the fundamental ideas behind Gröbner bases is the reduction of questions about general polynomial ideals to monomial ideals. In the context of determining invariants of an ideal like projective dimension or Castelnuovo-Mumford regularity, it is therefore interesting to know when these invariants possess the same values for an ideal and its leading ideal. It is well-known that in many instances the invariants of the leading ideal provide an upper bound for those of the polynomial ideal and that in generic position, i. e. when the leading ideal is the generic initial ideal, the values even coincide.

From an algorithmic point of view, it is not easy to work with the generic initial ideal. While it is comparatively easy to determine it with probabilistic method, there exists no simple test to verify that one has really obtained the generic initial ideal. However, relaxing the conditions on the leading ideal somewhat one can introduce generic positions which share many properties with the generic initial ideal and which are effectively checkable with deterministic algorithms. In [9], the authors showed that for many purposes it suffices to ensure that the leading ideal is quasi-stable (i. e. that the given ideal possesses

a Pommaret basis [17, 18]) in order to achieve that many invariants can be immediately read off the Pommaret basis.

Our article [9] was mainly concerned with invariants and concepts related to the minimal free resolution of the given ideal. In this work, we study the *reduction number*, an invariant which was introduced by Northcott and Rees [15] and which intuitively measures the complexity of computations in the associated factor ring. It is also related to some other invariants like the degree, the arithmetic degree and the Castelnuovo-Mumford regularity (see [3, 20, 22] for more details). Independently, Conca [4] and Trung [21] proved that the reduction number of an ideal is bounded by the one of its leading ideal (for an arbitrary term order) and Trung [20] showed that for the generic initial ideal (for the degree reverse lexicographic order) equality holds.

Trung [21] also presented an approach to the effective determination of various reduction numbers. However, his method is very expensive. We will show that it is indeed impossible to design a “simple” algorithm for reduction numbers where we mean by “simple” an approach based solely on the analysis of leading terms. Nevertheless, we will provide two alternative methods which we believe to be more efficient than the one presented by Trung. Our first method is based on directly adding the right number of sufficiently generic linear forms and yields the absolute reduction number. Our second method determines the whole set of possible reduction numbers (and thus in particular both the absolute and the big reduction number) using a Gröbner system.

Throughout this article, we will use the following notations.  $\mathcal{P} = \mathbb{k}[x_1, \dots, x_n]$  is an  $n$ -dimensional polynomial ring over some infinite field  $\mathbb{k}$  with homogeneous maximal ideal  $\mathfrak{m}$ . If not stated otherwise, the term order will always be the degree reverse lexicographic order induced by  $x_n \prec \dots \prec x_1$ . We assume that we are given a fixed homogeneous ideal  $\mathcal{I} \subseteq \mathcal{P}$  of dimension  $D$  and write for the corresponding factor ring  $\mathcal{R} = \mathcal{P}/\mathcal{I}$ . A non-singular matrix  $A = (a_{ij}) \in \mathrm{GL}(n, \mathbb{k})$  induces on  $\mathcal{P}$  the linear change of coordinates  $x \mapsto A \cdot x$  transforming the given ideal  $\mathcal{I}$  into a new ideal  $A \cdot \mathcal{I} \subseteq \mathcal{P}$ . Finally, given a term  $t \in \mathcal{P}$ , we denote by  $w(t)$  the largest integer  $\ell$  such that  $x_\ell \mid t$ .

The article is organised as follows. The next section collects some known facts about reduction numbers and generic initial ideals. Section 3 introduces some novel generalised notions of stability for monomial ideals. The following section extends the for us crucial notion of weak  $D$ -stability to polynomial ideals and presents a deterministic algorithm to transform any ideal into weakly  $D$ -stable position. After these preparations, we present in Section 5 an algorithm for computing the absolute reduction number. In the final section, we exploit Gröbner systems to compute the set of all possible reduction numbers.

## 2 Reduction Numbers and the Generic Initial Ideal

We recall some basic facts about reduction numbers. There exist several equivalent approaches to defining them; for our purposes the following one is particularly convenient. Let  $y_1, \dots, y_D \in \mathcal{P}_1$  be  $D$  linear forms defining a Noether normalisation of  $\mathcal{R}$ . Then the ideal  $\mathcal{J} = \mathcal{I} + \langle y_1, \dots, y_D \rangle$  is called a *minimal reduction* of  $\mathcal{I}$  and the *reduction number*  $r_{\mathcal{J}}(\mathcal{R})$  with respect to  $\mathcal{J}$  is the largest non-vanishing degree in the factor ring  $\mathcal{P}/\mathcal{J}$ . We write for the set of all possible reduction numbers  $\text{rSet}(\mathcal{R}) = \{r_{\mathcal{J}}(\mathcal{R}) \mid \mathcal{J} \text{ minimal reduction of } \mathcal{I}\}$ . The (*absolute*) *reduction number*  $r(\mathcal{R})$  is the minimal element of  $\text{rSet}(\mathcal{R})$ , the *big reduction number*  $br(\mathcal{R})$  the maximal one. As already mentioned above, the former one appeared first in the work of Northcott and Rees [15]; the latter one was much later introduced by Vasconcelos [23].

While it is easy to construct some minimal reduction  $\mathcal{J}$ , the obvious key problem in computing  $r(\mathcal{R})$  consists of identifying a  $\mathcal{J}$  with  $r_{\mathcal{J}}(\mathcal{R}) = r(\mathcal{R})$ . In the sequel, we will use the following three results. The first one characterises all minimal reductions of a *monomial* ideal in Noether position. Any such ideal has a minimal generator of the form  $x_{n-D}^\alpha$ . The second result relates for a strongly stable ideal (which is always in Noether position)  $r(\mathcal{R})$  with the exponent  $\alpha$ . The final result bounds for arbitrary ideals  $r(\mathcal{R})$  by  $r(\mathcal{P}/\text{lt } \mathcal{I})$ .

**Lemma 2.1** ([3, Lemma 5]) *Let  $\mathcal{I} \subseteq \mathcal{P}$  be a monomial ideal such that the variables  $x_{n-D+1}, \dots, x_n$  induce a minimal reduction. Then every minimal reduction is induced by linear forms*

$$y_i = x_{n-D+i} + \sum_{j=1}^{n-D} a_{ij} x_j, \quad a_{ij} \in \mathbb{k}. \quad (1)$$

**Theorem 2.2** ([3, Thm. 11]) *Let  $\mathcal{I} \subseteq \mathcal{P}$  be a strongly stable monomial ideal. Then  $\mathcal{I}$  has a minimal generator  $x_{n-D}^\alpha$  and we have  $r(\mathcal{R}) = r_{\mathcal{J}}(\mathcal{R}) = \alpha - 1$  for any minimal reduction  $\mathcal{J}$  of  $\mathcal{I}$ .*

**Theorem 2.3** ([4, Thm. 1.1], [21, Cor. 3.4]) *For any ideal  $\mathcal{I} \subseteq \mathcal{P}$  and any term order  $\prec$ , the inequality  $r(\mathcal{R}) \leq r(\mathcal{P}/\text{lt } \mathcal{I})$  holds.*

Galligo [5] proved for a base field  $\mathbb{k}$  of characteristic 0 that almost any linear coordinate transformation leads to the same leading ideal, the *generic initial ideal*  $\text{gin } \mathcal{I}$  (for more information see [7]). Bayer and Stillman [2] extended this result to positive characteristic. A for us important result of Trung asserts that for the generic initial ideal the inequality in Theorem 2.3 becomes an equality.

**Theorem 2.4** (Galligo, [5], [2]) *There exists a nonempty Zariski open subset  $\mathcal{U} \subseteq \text{GL}(n, \mathbb{k})$  such that  $\text{lt}(A \cdot \mathcal{I}) = \text{lt}(A' \cdot \mathcal{I})$  for all matrices  $A, A' \in \mathcal{U}$ .*

**Theorem 2.5** ([20, Thm. 4.3]) *For the degree reverse lexicographic order, we always find  $r(\mathcal{R}) = r(\mathcal{P}/\text{gin } \mathcal{I})$ .*

### 3 Some Generalised Notions of Stability

Stable and strongly stable ideals form two important classes of monomial ideals. We introduce now generalisations of these concepts depending on an integer  $\ell$ . In the context of determining reduction numbers, it will turn out that the case  $\ell = D$  is of particular interest. Like for the classical stability notions, it is easy to see that it always suffices, if the defining property is satisfied by the minimal generators of the ideal.

**Definition 3.1** *Let  $0 \leq \ell < n$  be an integer. The monomial ideal  $\mathcal{I}$  is  $\ell$ -stable, if for every term  $t \in \mathcal{I}$  with  $w(t) \geq n - \ell$  and every  $i < w(t)$  the term  $x_i t / x_{w(t)}$  also lies in  $\mathcal{I}$ . For a weakly  $\ell$ -stable ideal  $\mathcal{I}$ , the above condition must be satisfied only for all  $i \leq n - \ell$ . Finally,  $\mathcal{I}$  is strongly  $\ell$ -stable, if for every term  $t \in \mathcal{I}$  with  $w(t) \geq n - \ell$ , every  $j \geq n - \ell$  with  $x_j \mid t$  and every  $i < j$  the term  $x_i t / x_j$  also lies in  $\mathcal{I}$ .*

**Example 3.2** *We consider first for  $n = 6$  the ideal*

$$\mathcal{I} = \langle x_1, x_4^2, x_3x_4, x_2x_4, x_2x_3, x_2^2, x_3^3, x_4x_5^2, x_3x_5^2, x_2x_5^2, x_3^2x_5, x_3^3, x_5^2x_6^2, x_4x_5x_6^2, x_3x_5x_6^2, x_2x_5x_6^2, x_3^2x_6^2, x_5x_6^4, x_4x_6^4, x_3x_6^4, x_2x_6^4, x_6^6 \rangle,$$

*the leading ideal of the fifth Katsura ideal. As one can easily see that here  $D = 0$ , it suffices to check the defining property for the generators containing  $x_6$  and it turns out that  $\mathcal{I}$  is 0-stable. However,  $\mathcal{I}$  is not stable, as for example  $x_3x_4 \in \mathcal{I}$  but  $x_3^2 \notin \mathcal{I}$ .*

*Consider now for  $n = 5$  the monomial ideal*

$$\mathcal{I} = \langle x_1^2, x_2^3, x_1x_2^2, x_3^2x_2^2, x_2x_3^2x_1, x_3^5, x_2x_3^4, x_1x_3^4, x_3^4x_4^2, x_2x_3^3x_4^2, x_1x_3^3x_4^2, x_3^3x_4^4, x_3^2x_2x_4^4, x_1x_3^2x_4^4, x_3x_2^2x_4^4, x_3x_2x_1x_4^4, x_1x_2x_3x_4^3x_5^2, x_1x_3x_4^6, x_2^2x_4^6, x_1x_2x_4^6, x_2^2x_4^5x_5^2, x_1x_2x_4^5x_5^2, x_3x_2^2x_4^3x_5^4, x_1x_3x_4^5x_5^3, x_2x_3^2x_4^3x_5^5, x_1x_3^2x_4^3x_5^5, x_1x_2x_4^4x_5^6, x_1x_4^4x_5^5, x_2^2x_4^4x_5^7, x_2x_3x_4^5x_5^7 \rangle.$$

*Since here  $D = 2$ , we must check the defining property of a weakly  $D$ -stable ideal only for the terms containing  $x_3, x_4, x_5$  and one readily verifies that  $\mathcal{I}$  is weakly  $D$ -stable. However, it is not  $D$ -stable because  $t = x_1x_4^6x_5^5 \in \mathcal{I}$  but  $tx_4/x_5 \notin \mathcal{I}$ .*

The generic initial ideal is always Borel-fixed, i. e. invariant under the natural action of the Borel group [2, 6]. In general, it depends on the characteristic of the base field whether a given ideal is Borel-fixed. In characteristic zero, the Borel-fixed ideals are precisely the strongly stable ones. We provide now the analogous result for strong  $\ell$ -stability.

**Definition 3.3** *The Borel group is the subgroup  $\mathcal{B} < \text{GL}(n, \mathbb{k})$  consisting of all lower triangular invertible  $n \times n$  matrices. For any integer  $0 \leq \ell < n$ , we define the  $\ell$ -Borel group as the subgroup  $\mathcal{B}_\ell \leq \mathcal{B}$  consisting of all matrices  $A \in \mathcal{B}$  such that for  $i < n - \ell$  we have  $a_{ii} = 1$  and  $a_{ij} = 0$  for  $i \neq j$ .*

**Proposition 3.4** *Assume that  $\text{char } \mathbb{k} = 0$ . The monomial ideal  $\mathcal{I} \leq \mathcal{P}$  is strongly  $\ell$ -stable, if and only if it is invariant under the  $\ell$ -Borel group  $\mathcal{B}_\ell$ .*

**Proof** Assume first that  $\mathcal{I}$  is  $\ell$ -stable and consider a generating set  $\mathcal{H}$  of it. The transformation induced by an element  $A = (a_{ij}) \in \mathcal{B}_\ell$  is of the form

$$\begin{aligned} x_i &\rightarrow x_i && \text{if } i < n - \ell, \\ x_i &\rightarrow a_{ii}x_i + \sum_{j=n-\ell}^{i-1} a_{ij}x_j && \text{if } i \geq n - \ell. \end{aligned} \quad (2)$$

One immediately sees that any generator  $t \in \mathcal{H}$  with  $w(t) < n - \ell$  remains unchanged under the action of  $A$ . If  $w(t) \geq n - \ell$ , then  $t$  is transformed into a polynomial  $f_t = A \cdot t$ . It follows again from (2) that any term in the support of  $f_t$  is obtained from  $t$  by applying a sequence of “elementary moves” of the form  $s \rightarrow x_j s / x_k$  with  $j < k$  where  $x_k \mid s$ . In this sequence we always have  $k \geq n - \ell$  and thus the strong  $\ell$ -stability of  $\mathcal{I}$  implies that all appearing terms  $s$  lie in  $\mathcal{I}$ . Furthermore,  $t$  itself always lies in the support of  $f_t$ .

Consider now the elements  $t$  of  $\mathcal{H}$  with  $w(t) \geq n - \ell$  sorted reverse lexicographically. If  $t$  is the largest term among these, then  $w(s) < w(t)$  for all  $s \neq t$  appearing in the support of  $f_t$ . Thus they are multiples of elements of  $\mathcal{H}$  which remain unchanged under the operation of  $A$  and can be eliminated. If  $t$  is the second largest term, then the support may in addition contain multiples of the largest term; otherwise we can apply the same argument. By iteration, we obtain that the whole ideal remains invariant.

For the converse, we need the assumption on the characteristic. If  $\text{char } \mathbb{k} = 0$  (and thus no coefficient drops out when we transform a term), then we may revert the above arguments: if  $\mathcal{I}$  is invariant under  $\mathcal{B}_\ell$ , then all terms appearing in the support of  $f_t$  must lie in  $\mathcal{I}$  and hence  $\mathcal{I}$  is strongly  $\ell$ -stable.  $\square$

In relation to our previous work [9], it is of interest to show that a  $D$ -stable ideal is automatically quasi-stable. The proof depends on the following characterisation of  $\ell$ -stability which is of independent interest.

**Proposition 3.5** *The monomial ideal  $\mathcal{I} \trianglelefteq \mathcal{P}$  is  $\ell$ -stable, if and only if it satisfies for all  $0 \leq i \leq \ell$*

$$\langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : x_{n-i} = \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}. \quad (3)$$

**Proof** Assume first that  $\mathcal{I}$  is  $\ell$ -stable and let  $t$  be a term such that  $x_{n-i}t \in \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle$  for some  $i \leq \ell$ . If  $w(t) > n - i$ , then  $t \in \langle x_n, \dots, x_{n-i+1} \rangle$  and nothing is to be proven. Otherwise we have  $x_{n-i}t \in \mathcal{I}$  and  $w(x_{n-i}t) = n - i \geq n - \ell$ . Because of the  $\ell$ -stability, this entails that  $x_j t = x_j(x_{n-i}t)/x_{n-i} \in \mathcal{I}$  for all  $j \leq n - \ell$ . Hence  $t\langle x_1, \dots, x_{n-i} \rangle \subseteq \mathcal{I}$  implying  $t\mathfrak{m} \subseteq \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle$ .

For the converse consider a term  $t \in \mathcal{I}$  with  $w(t) = n - i \geq n - \ell$ . Because of (3), we have  $t/x_{n-i} \in \mathcal{I} : x_{n-i} \subseteq \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}$ . Hence  $x_j t / x_{n-i} \in \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle$  for all  $j \leq n$ . If  $j \leq n - i$ , then  $w(x_j t / x_{n-i}) \leq n - i$  and thus we must have  $x_j t / x_{n-i} \in \mathcal{I}$  so that  $\mathcal{I}$  is  $\ell$ -stable.  $\square$

**Corollary 3.6** *A  $D$ -stable monomial ideal  $\mathcal{I}$  is quasi-stable.*

**Proof** According to the previous proposition, (3) holds for all  $0 \leq i \leq D$ . As a preparatory step, we claim that this fact implies that for these values of  $i$  also

$$\langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : x_{n-i}^\infty = \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}^\infty. \quad (4)$$

Indeed, if the term  $t$  lies in the ideal on the left hand side, then an integer  $s$  exists such that  $x_{n-i}^s t \in \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle$  and therefore

$$x_{n-i}^{s-1} t \in \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : x_{n-i} = \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}.$$

Applying this argument a second time yields

$$\begin{aligned} x_{n-i}^{s-2} t &\in (\langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}) : x_{n-i} \\ &= (\langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : x_{n-i}) : \mathfrak{m} \\ &= \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}^2. \end{aligned}$$

Thus we find by iteration that  $t \in \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}^s$  proving the claim.

It follows that  $x_{n-i}$  is not a zero divisor in  $\mathcal{P}/(\langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}^\infty)$  for all  $0 \leq i < D$ . Indeed, if  $f \in \mathcal{P}$  satisfies  $x_{n-i} f \in \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}^\infty$ , then an exponent  $s$  exists such that  $x_{n-i} f \mathfrak{m}^s \subseteq \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle$  and hence  $x_{n-i}^{s+1} f \in \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle$ . But this implies  $f \in \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : x_n^\infty = \langle \mathcal{I}, x_n, \dots, x_{n-i+1} \rangle : \mathfrak{m}^\infty$ . Now the assertion follows from [18, Prop. 4.4].  $\square$

**Example 3.7** *Weak  $D$ -stability is not sufficient for quasi-stability, as one can see from the ideal  $\langle x_1^2, x_1 x_3 \rangle$  where  $n = 3$  and  $D = 2$ . One easily verifies that it is weakly  $D$ -stable but not quasi-stable. And for the same values of  $n$  and  $D$  the ideal  $\langle x_1^3, x_1 x_2 \rangle$  shows that the converse of Corollary 3.6 does not hold, as it is quasi-stable but not (weakly)  $D$ -stable.*

**Remark 3.8** *Assume that the monomial ideal  $\mathcal{I}$  is weakly  $\ell$ -stable for some  $\ell$  and that  $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{I}$ . It follows immediately from Definition 3.1 that any term of the form  $x_1^{\alpha_1 + \beta_1} \cdots x_{n-\ell}^{\alpha_{n-\ell} + \beta_{n-\ell}}$  with  $\beta_1 + \cdots + \beta_{n-\ell} = \alpha_{n-\ell+1} + \cdots + \alpha_n$  is then also contained in  $\mathcal{I}$ . If we introduce for  $1 \leq j \leq \ell$  the homogeneous polynomials*

$$g_j = \sum_{\beta_1^{(j)} + \cdots + \beta_{n-\ell}^{(j)} = \alpha_{n-\ell+1} + \cdots + \alpha_n} a_{\beta_1^{(j)}, \dots, \beta_{n-\ell}^{(j)}}^{(j)} x_1^{\beta_1^{(j)}} \cdots x_{n-\ell}^{\beta_{n-\ell}^{(j)}}$$

*with arbitrary coefficients  $a_{\beta_1^{(j)}, \dots, \beta_{n-\ell}^{(j)}}^{(j)} \in \mathbb{k}$ , then it follows from the observation above that the polynomial*

$$f_t = x_1^{\alpha_1} \cdots x_{n-\ell}^{\alpha_{n-\ell}} g_1 \cdots g_\ell$$

*also lies in  $\mathcal{I}$ . Each term in its support is of the form  $x_1^{\alpha_1 + \beta_1} \cdots x_{n-\ell}^{\alpha_{n-\ell} + \beta_{n-\ell}}$  with  $\beta_i = \beta_i^{(1)} + \cdots + \beta_i^{(\ell)}$  and by construction  $\beta_1 + \cdots + \beta_{n-\ell} = \alpha_{n-\ell+1} + \cdots + \alpha_n$ .*

**Proposition 3.9** *A weakly  $D$ -stable ideal  $\mathcal{I}$  is always in Noether position.*

**Proof** A  $D$ -dimensional monomial ideal is in Noether position, if and only if for all  $1 \leq j \leq n - D$  a pure power  $x_j^{e_j}$  is contained in  $\mathcal{I}$ . Assume first that there exists a term  $t \in \mathcal{I} \cap \mathbb{k}[x_{n-D+1}, \dots, x_n]$ . Then Remark 3.8 immediately implies for  $e = \deg t$  that  $x_j^e \in \mathcal{I}$  for all  $1 \leq j \leq n - D$  and we are done. If  $\mathcal{I} \cap \mathbb{k}[x_{n-D+1}, \dots, x_n] = \emptyset$ , then the  $D$ -dimensional cone  $1 \cdot \mathbb{k}[x_{n-D+1}, \dots, x_n]$  lies completely in the complement of  $\mathcal{I}$ . Assume that for some  $1 \leq j \leq n - D$  no power of  $x_j$  was contained in  $\mathcal{I}$ . Since  $D = \dim \mathcal{I}$ , it is not possible that the complement of  $\mathcal{I}$  contains a  $(D + 1)$ -dimensional cone. Thus we must have  $\mathcal{I} \cap \mathbb{k}[x_j, x_{n-D+1}, \dots, x_n] \neq \emptyset$ . But if a term  $t$  of degree  $e$  lies in this intersection, then again by Remark 3.8  $x_j^e \in \mathcal{I}$  in contradiction to our assumption.  $\square$

The simple Algorithm 1 verifies whether a given monomial ideal is weakly  $D$ -stable without a priori knowledge of the dimension  $D$  of  $\mathcal{I}$ . For showing its correctness, we note that if  $\mathcal{I}$  is weakly  $D$ -stable, then the number  $d$  computed in Line 2 equals  $D$  by Proposition 3.9 and by Definition 3.1 of weak  $D$ -stability we never get to Line 6. If  $\mathcal{I}$  is not weakly  $D$ -stable, then  $d \geq D$  (this estimate holds for any monomial ideal) and soon or later we will reach Line 6. The bit complexity of the algorithm is polynomial in  $kn$ , as one can easily see that the number of operations in the two **for**-loops is at most  $k^2 n^3$ .

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**Algorithm 1** WDS-TEST: Test for weak  $D$ -stability

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**Input:** minimal basis  $G = \{m_1, \dots, m_k\}$  of monomial ideal  $\mathcal{I} \triangleleft \mathcal{P}$

**Output:** The answer to: is  $\mathcal{I}$  weakly  $D$ -stable?

```

1:  $e := \max \{\deg(m_1), \dots, \deg(m_k)\}$ 
2:  $d :=$  smallest  $\ell$  such that  $x_i^e \in \mathcal{I}$  for  $i = 1, \dots, n - \ell$ 
3: for all  $x_1^{e_1} \dots x_h^{e_h} \in G$  with  $h \geq n - d$  and  $e_h > 0$  do
4:   for  $j = 1, \dots, n - d$  do
5:     if  $x_1^{e_1} \dots x_{h-1}^{e_{h-1}} x_h^{e_h-1} x_j \notin \langle G \rangle$  then
6:       return false
7:     end if
8:   end for
9: end for
10: return true
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## 4 Weak $D$ -Stability for Polynomial Ideals

In the previous section, we considered exclusively monomial ideals. All the notions introduced in Definition 3.1 can be straightforwardly extended to polynomial ideals by saying that an ideal  $\mathcal{I}$  satisfies some form of stability, if its leading ideal  $\text{lt } \mathcal{I}$  satisfies this form of stability. Galligo's Theorem 2.4 immediately implies that after a generic change of coordinate  $A \in \text{GL}(n, \mathbb{k})$  the transformed ideal  $A \cdot \mathcal{I}$  possesses any stability property here considered. Thus in principle

a random coordinate transformation (almost) always provides a “nice” leading ideal.

However, from a computational point of view, random transformations are rather unpleasant, as they destroy all sparsity typically present in ideal bases. It is therefore of great interest to see whether for some notion of stability it is possible to design a *deterministic* algorithm which yields a fairly sparse transformation  $A$  such that  $A \cdot \mathcal{I}$  has the desired stability property. In a forthcoming work [1], we will study this question in depth and provide such an algorithm for many important stability notions. Here, we only present a variation of this algorithm for the case of weak  $D$ -stability. For lack of space, we omit the (non-trivial) termination proof which will be given in [1].

Algorithm 2 works by performing incrementally very sparse transformations where all variables except one remain unchanged and this one undergoes a transformation of the form  $x_i \rightarrow x_i + ax_j$  where  $j < i$  and  $a \in \mathbb{k} \setminus \{0\}$  is a generic parameter. The pair  $(i, j)$  is chosen in such a way that each transformation leads to true progress towards a weakly  $D$ -stable position, if  $a$  does not take one of finitely many “bad” values. In practice, we always use the value  $a = 1$ . If this accidentally represents a “bad” value, then we will automatically perform the same transformation a second time which corresponds to  $a = 2$ . Obviously, after a finite number of iterations (which can be bounded via the degrees of the generators), we will reach a “good” value, since  $\mathbb{k}$  is an infinite field.

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**Algorithm 2** WDS-TRAFO: Transformation to weakly  $D$ -stable position

---

**Input:** Gröbner basis  $G$  of homogeneous ideal  $\mathcal{I} \trianglelefteq \mathcal{P}$

**Output:** a linear change of coordinates  $\Psi$  such that  $\Psi(\mathcal{I})$  is weakly  $D$ -stable

- 1:  $D := \dim \mathcal{I}$ ;  $\Psi := \text{id}$
  - 2: **while**  $\exists g \in G, 1 \leq j \leq n - D : i = w(\text{lt } g) \geq n - D \wedge x_j \text{lt } (g)/x_i \notin \langle \text{lt } G \rangle$   
    **do**
  - 3:    $\psi := (x_i \mapsto x_i + x_j)$ ;  $\Psi = \psi \circ \Psi$
  - 4:    $G := \text{GRÖBNERBASIS}(\psi(G))$
  - 5: **end while**
  - 6: **return**  $\Psi$
- 

Algorithm 2 is not in an optimised form. In practice, if one finds more than one suitable pair  $(i, j)$ , it appears natural to perform several transformations simultaneously, as each iteration of the **while** loop requires a Gröbner basis computation. Furthermore, one should take into account that the input for these computations is typically already fairly close to a Gröbner basis. Hence it is probably useful to apply some specialised algorithm exploiting this fact. A prototype implementation of Algorithm 2 in MAPLE can be found at <http://amirhashemi.iut.ac.ir/software.s>.

**Example 4.1** We consider for  $n = 3$  the ideal  $\mathcal{I} = \langle x_1^3, x_2^2 x_3, x_2^3 \rangle$  with  $D = 1$ . This ideal is not weakly  $D$ -stable, since  $x_1(x_2^2 x_3)/x_3 \notin \mathcal{I}$  and, according to Algorithm 2, we perform the change of coordinates  $\psi_1 : x_3 \mapsto x_1 + x_3$ . The transformed ideal  $\mathcal{I}_1 = \psi_1(\mathcal{I})$  has the leading ideal  $\langle x_1^3, x_1 x_2^2, x_2^3, x_2^2 x_3 \rangle$  and is



also not  $D$ -stable, since  $x_1(x_1x_2^2)/x_2 \notin \text{lt } \mathcal{I}_1$ . Thus in the second iteration the **while** loop performs the change of coordinate  $\psi_2 : x_2 \mapsto x_1 + x_2$ . The leading ideal of the transformed ideal  $\mathcal{I}_2 = \psi_2(\mathcal{I}_1)$  is by chance even the generic initial ideal  $\text{gin } \mathcal{I} = \langle x_1^3, x_1^2x_2, x_1x_2^2, x_2^4, x_1^2x_3^3 \rangle$  and thus of course weakly  $D$ -stable.

## 5 Computing the Absolute Reduction Number

We consider first the case of a monomial ideal and extend Theorem 2.2 from strongly stable ideals to weakly  $D$ -stable ones. Our proof follows closely the arguments of the original proof by Bresinsky and Hoa [3].

**Theorem 5.1** *Let  $\mathcal{I} \leq \mathcal{P}$  be a weakly  $D$ -stable monomial ideal. Then  $\mathcal{I}$  has a minimal generator  $x_{n-D}^\alpha$  and  $r(\mathcal{R}) = r_{\mathcal{J}}(\mathcal{R}) = \alpha - 1$  for any minimal reduction  $\mathcal{J}$  of  $\mathcal{I}$ .*

**Proof** Since  $\mathcal{I}$  is assumed to be weakly  $D$ -stable,  $x_{n-D+1}, \dots, x_n$  induce a minimal reduction by Proposition 3.9 and we can apply Lemma 2.1. Consider the  $D$  linear forms  $y_i = x_{n-D+i} + a_{i,1}x_1 + \dots + a_{i,n-D}x_{n-D}$  with  $1 \leq i \leq D$  and arbitrary coefficients  $a_{i,j} \in \mathbb{k}$  and set  $\mathcal{J}_1 = \mathcal{I} + \langle y_1, \dots, y_D \rangle$ .

We claim that  $r_{\mathcal{J}_1}(\mathcal{R}) = r_{\mathcal{J}_2}(\mathcal{R})$  where  $\mathcal{J}_2 = \mathcal{I} + \langle x_{n-D+1}, \dots, x_n \rangle$ . It is enough to show the identity  $\mathcal{I}_1 = \mathcal{I}_2$  where  $\mathcal{P}/\mathcal{J}_1 \simeq \mathbb{k}[x_1, \dots, x_{n-D}]/\mathcal{I}_1$  and  $\mathcal{P}/\mathcal{J}_2 \simeq \mathbb{k}[x_1, \dots, x_{n-D}]/\mathcal{I}_2$ . One easily sees that  $\mathcal{I}_2 = \mathcal{I} \cap \mathbb{k}[x_1, \dots, x_{n-D}]$  and thus trivially  $\mathcal{I}_2 \subseteq \mathcal{I}_1$ . The converse inclusion  $\mathcal{I}_1 \subseteq \mathcal{I}_2$  follows by Remark 3.8 which entails that for any term  $t = x_1^{\alpha_1} \dots x_n^{\alpha_n} \in \mathcal{I}$  the corresponding term

$$\tilde{t} = x_1^{\alpha_1} \dots x_{n-D}^{\alpha_{n-D}} \prod_{j=1}^D (-a_{j,1}x_1 - \dots - a_{j,n-D}x_{n-D})^{\alpha_{n-D+j}} \in \mathcal{I}_1$$

also lies in  $\mathcal{I}$  and hence in  $\mathcal{I}_2$ .

Proposition 3.9 also implies that  $\mathcal{I}$  has a minimal generator of the form  $x_{n-D}^\alpha$  for some  $\alpha \in \mathbb{N}$ . Hence,  $r_{\mathcal{J}_2}(\mathcal{R}) \geq \alpha - 1$ . On the other hand,  $x_{n-D}^\alpha \in \mathcal{I}$  implies by Remark 3.8 that any term  $x_1^{\alpha_1} \dots x_{n-D}^{\alpha_{n-D}}$  of degree  $\alpha$  also belongs to  $\mathcal{I}$  and thus  $r_{\mathcal{J}_2}(\mathcal{R}) \leq \alpha - 1$ . Therefore  $r_{\mathcal{J}_2}(\mathcal{R}) = \alpha - 1$  proving the second assertion.  $\square$

We have thus identified a class of monomial ideals, the weakly  $D$ -stable ideals, for which it is particularly simple to determine their reduction number. Given a polynomial ideal  $\mathcal{I}$ , we may use Algorithm 2 to render it weakly  $D$ -stable and obtain then immediately the reduction number of its leading ideal  $\text{lt } \mathcal{I}$ . According to Theorem 2.3, this number gives us an upper bound for  $r(\mathcal{R})$ . We introduce now a more specialised class of ideals for which we can guarantee that  $\mathcal{I}$  and  $\text{lt } \mathcal{I}$  have the same reduction number. We denote here for a monomial ideal  $\mathcal{L}$  by  $\deg_{x_k} \mathcal{L}$  the maximal  $x_k$ -degree of a minimal generator of  $\mathcal{L}$ .

**Definition 5.2** *Let  $0 \leq \ell < n$  be an integer. The homogeneous ideal  $\mathcal{I} \leq \mathcal{P}$  is weakly  $\ell$ -minimal stable, if its leading ideal  $\text{lt } \mathcal{I}$  is weakly  $\ell$ -stable and if for*

any linear change of coordinates  $A \in \text{GL}(n, \mathbb{k})$  such that  $\text{lt}(A \cdot \mathcal{I})$  is still weakly  $\ell$ -stable, we have  $\deg_{x_{n-\ell}} \text{lt } \mathcal{I} \leq \deg_{x_{n-\ell}} \text{lt}(A \cdot \mathcal{I})$ .

Again it is easy to see that this is a generic notion, as any coordinate transformation  $A$  with  $\text{lt}(A \cdot \mathcal{I}) = \text{gin } \mathcal{I}$  leads to a weakly  $\ell$ -minimal stable position.

**Example 5.3** Consider for  $n = 3$  the ideal  $\mathcal{I} = \langle x_1x_3, x_1x_2 + x_2^2, x_1^2 \rangle$  introduced by Green [7]. One finds that the leading ideal  $\text{lt } \mathcal{I} = \langle x_1^2, x_1x_2, x_1x_3, x_2^3, x_2^2x_3 \rangle$  is even strongly stable and thus of course weakly  $D$ -stable (with  $D = 1$  here). However,  $\mathcal{I}$  is not weakly  $D$ -minimal stable, as  $\text{gin}(\mathcal{I}) = \langle x_1^2, x_1x_2, x_2^2, x_1x_3^2 \rangle$  and thus has a lower degree in  $x_2$ .

**Example 5.4** We consider for  $n = 4$  the ideal

$$\mathcal{I} = \langle x_1x_4 - x_2x_3, x_2^3 - x_1x_3^2, x_2^2x_4 - x_1^3 \rangle;$$

it represents the special case  $a = 2, b = 3$  of [3, Example 15]. Here  $D = 2$  and the ideal  $\mathcal{I}$  is not weakly  $D$ -stable. The following linear change of coordinates  $\Psi : x_2 \mapsto x_1 + x_2, x_3 \mapsto x_1 + x_3$  transforms  $\mathcal{I}$  into a weakly  $D$ -stable (in fact, even strongly stable) ideal  $\mathcal{I}_1$  with leading ideal

$$\text{lt } \mathcal{I}_1 = \langle x_1^2, x_1x_2^2, x_2^3, x_1x_2x_3^2, x_1x_3^3, x_2^2x_3^3, x_2x_3^4 \rangle.$$

Note that although this leading ideal is different from

$$\text{gin } \mathcal{I} = \langle x_1^2, x_1x_2^2, x_2^3, x_1x_2x_3^2, x_2^2x_3^2, x_1x_3^4, x_2x_3^4 \rangle,$$

both ideals have the same minimal generator  $x_2^3$ . Thus  $\mathcal{I}_1$  is weakly  $D$ -minimal stable and we see that in this example the set of transformations leading to weakly  $D$ -minimal position is strictly larger than the one leading to the generic initial ideal.

**Theorem 5.5** Let  $\mathcal{I} \trianglelefteq \mathcal{P}$  be a weakly  $D$ -minimal stable homogeneous ideal. Then  $\text{lt } \mathcal{I}$  has a minimal generator  $x_{n-D}^\alpha$  and  $r(\mathcal{R}) = r(\mathcal{P}/\text{lt } \mathcal{I}) = \alpha - 1$ .

**Proof** Since  $\text{lt } \mathcal{I}$  is weakly  $D$ -stable, it possesses by Proposition 3.9 a minimal generator  $x_{n-D}^\alpha$  and thus  $r(\mathcal{P}/\text{lt } \mathcal{I}) = \alpha - 1$  by Proposition 5.1. As  $\mathcal{I}$  is assumed to be weakly  $D$ -minimal stable,  $x_{n-D}^\alpha$  must also be a minimal generator of  $\text{gin } \mathcal{I}$  and hence  $r(\mathcal{R}) = r(\mathcal{P}/\text{gin } \mathcal{I}) = \alpha - 1$  by Theorem 2.5.  $\square$

Unfortunately, Theorem 5.5 is mainly of theoretical interest, as we are not able to provide a simple deterministic algorithm for the construction of a change of coordinates leading to be weakly  $D$ -minimal stable position. We present now Algorithm 3 for the computation of  $r(\mathcal{R})$ . Instead of a coordinate transformation, it is based on a parametric computation. The main point will be to keep the number of parameters as small as possible.

The algorithm simply adds  $D$  linear forms  $y_i$  of the special form (1). The occurring coefficients  $a_{ij}$  are then considered as undetermined parameters. Replacing in the ideal  $\mathcal{I}$  every variable  $x_{n-D+i}$  with  $i > 0$  by  $-\sum_{j=1}^{n-D} a_{ij}x_j$ , we obtain

---

**Algorithm 3** REDNUM: (Absolute) Reduction Number

---

**Input:** Gröbner basis  $G$  of a homogeneous ideal  $\mathcal{I} \triangleleft \mathcal{P}$

**Output:** the absolute reduction number  $r(\mathcal{R})$

- 1:  $D := \dim \mathcal{I}$
  - 2:  $\tilde{G} := G$  with  $x_{n-D+i}$  replaced by  $-\sum_{j=1}^{n-D} a_{ij}x_j$  for all  $i > 0$
  - 3:  $\tilde{\mathcal{I}} := \langle \tilde{G} \rangle_{\tilde{\mathcal{P}}}$
  - 4:  $\mathcal{H} := \text{POMMARETBASIS}(\tilde{\mathcal{I}})$
  - 5: **return**  $\deg \mathcal{H} - 1$
- 

a new homogeneous ideal  $\tilde{\mathcal{I}}$  in the polynomial ring  $\tilde{\mathcal{P}} = \mathbb{k}(a_{ij})[x_1, \dots, x_{n-D}]$  over the field of rational functions in the  $D(n-D)$  parameters  $a_{ij}$  and compute its Pommaret basis (see [17, 18] and references therein).

**Theorem 5.6** *Algorithm 3 correctly determines  $r(\mathcal{R})$ .*

**Proof** We consider first the addition of  $D$  generic linear forms  $z_i = \sum_{j=1}^n b_{ij}x_j$  to the ideal  $\mathcal{I}$ . This leads to an ideal  $\hat{\mathcal{I}}$  in the polynomial ring  $\hat{\mathcal{P}} = \mathbb{k}(b_{ij})[x_1, \dots, x_n]$  depending on  $Dn$  parameters and  $n$  variables. It follows from the classical proof of the existence of a Noether normalisation (see e.g. [8, Thm. 3.4.1]) over an infinite field that  $\hat{\mathcal{I}}$  is a zero-dimensional ideal (which thus possesses a finite Pommaret basis).

We now claim that the absolute reduction number  $r(\mathcal{R})$  is one less than the Castelnuovo-Mumford regularity  $\text{reg } \hat{\mathcal{I}}$ . According to [18, Cor. 9.5],  $\text{reg } \hat{\mathcal{I}}$  is given by the degree of the Pommaret basis of  $\hat{\mathcal{I}}$ , so that this claim implies that  $r(\mathcal{R})$  can be read off the Pommaret basis of  $\hat{\mathcal{I}}$ . The correctness of the claim follows from a simple genericity argument.

We build recursively  $\mathbb{k}(b_{ij})$ -linear generating systems of the vector spaces  $\hat{\mathcal{I}}_q$  for all degrees  $q = 1, 2, \dots$  by taking all elements of  $\mathcal{H}$  of degree  $q$  and adding all products of elements of the previous generating system multiplied with a variable  $x_j$ . We collect the coefficients of the obtained generators in a matrix. Entering generic values for the parameters  $b_{ij}$  leads to the maximal possible rank of this matrix and thus to the lowest possible dimension of the complement of the degree  $q$  component of the corresponding specialisation of  $\hat{\mathcal{I}}$ . The absolute reduction number is the largest value of  $q$  for which we cannot achieve a zero-dimensional complement. Hence a generic choice of the parameters leads to the correct value of the absolute reduction number  $r(\mathcal{R})$ . Since computing over  $\mathbb{k}(b_{ij})$  corresponds to the generic branch of the parametric computation and since for a zero-dimensional ideal  $\text{reg } \hat{\mathcal{I}}$  is the lowest degree  $q$  where  $\hat{\mathcal{I}}_q = \hat{\mathcal{P}}_q$ , we conclude that our claim is correct.

Now consider the  $(D \times n)$ -matrix  $(b_{ij})$ : if the determinant of the submatrix composed of the last  $D$  column does not vanish, then by a Gaussian elimination we obtain a set of linear forms  $y_i$  in the “reduced” triangular form (1) leading to the same ideal  $\hat{\mathcal{I}}$ . As the intersection of two Zariski open sets is again Zariski open, this observation proves that generically also the reduced ansatz (1) used

in our algorithm yields the correct absolute reduction number. Because of the special form of this ansatz, we may solve the linear forms for the variables  $x_{n-D+i}$  and then perform the computations in the polynomial ring  $\tilde{\mathcal{P}}$  depending only on  $D(n-D)$  parameters and  $n-D$  variables.  $\square$

**Remark 5.7** *Since the Algorithms 2 and 3 are based on Gröbner or Pommaret bases and the worst case complexity of computing Gröbner bases is doubly exponential in the number of variables (as shown by Mayr and Meyer [12]), we conclude that the complexity of these algorithms is also doubly exponential in the number of variables.*

**Example 5.8** *For  $n = 4$ , the homogenised Weispfenning94 ideal  $\mathcal{I} \triangleleft \mathbb{k}[x_1, \dots, x_4]$  is generated by the polynomials*

$$\begin{aligned} f_1 &= x_2^4 + x_1x_2^2x_3 + x_1^2x_4^2 - 2x_1x_2x_4^2 + x_2^2x_4^2 + x_3^2x_4^2, \\ f_2 &= x_1x_2^4 + x_2x_3^4 - 2x_1^2x_2x_4^2 - 3x_4^5, \\ f_3 &= -x_1^3x_2^2 + x_1x_2x_3^3 + x_2^4x_4 + x_1x_2^2x_3x_4 - 2x_1x_2x_4^3. \end{aligned}$$

*Here  $D = 2$  and we replace  $x_4$  by  $-(a_{4,1}x_1 + a_{4,2}x_2)$  and  $x_3$  by  $-(a_{3,1}x_1 + a_{3,2}x_2)$  in  $\mathcal{I}$  to obtain the new ideal  $\tilde{\mathcal{I}} \triangleleft \mathbb{k}(a_{3,1}, a_{3,2}, a_{4,1}, a_{4,2})[x_1, x_2]$ . We compute a Pommaret basis  $\mathcal{H}$  of  $\tilde{\mathcal{I}}$  and get as leading terms*

$$\text{lt } \mathcal{H} = \{x_1^4, x_1^3x_2^2, x_1^2x_2^3, x_1x_2^5, x_2^6\}.$$

*Therefore  $r(\mathcal{R}) = 6 - 1 = 5$ .*

Our second example proves that there cannot exist a “simple” algorithm for computing the (absolute) reduction number. By “simple” we mean that the algorithm uses exclusively information obtained from the leading terms (like for instance Algorithm 2 to transform into weakly  $D$ -stable position).

**Example 5.9** *We consider again Example 5.3 of Green. It follows immediately from the above presented bases that here  $r(\mathcal{R}) = 1 < 2 = r(\mathcal{P}/\text{lt } \mathcal{I})$ . Following Algorithm 3, we replace  $x_3$  by  $-(a_1x_1 + a_2x_2)$  in order to obtain the ideal  $\tilde{\mathcal{I}}$ . Then we compute a Pommaret basis  $\mathcal{H}$  of  $\tilde{\mathcal{I}}$  and get for the leading terms*

$$\text{lt } \mathcal{H} = \{x_1^2, x_1x_2, x_2^2\}.$$

*Hence our algorithm yields the correct result  $r(\mathcal{R}) = 1$ . Since  $\mathcal{L} = \text{lt } \mathcal{I}$  is in fact even strongly stable, we conclude that  $\text{gin } \mathcal{L} = \mathcal{L}$ . Hence the leading terms of the generators of  $\mathcal{I}$  cannot contain any information on how to transform  $\mathcal{I}$  into a position such that the transformed ideal and its leading ideal share the same reduction number.*

## 6 Big Reduction Numbers and Gröbner Systems

We present now an approach that is able to determine the whole reduction number set  $\text{rSet}(\mathcal{R})$  and thus in particular both the absolute and the big reduction number. Our method is based on the theory of Gröbner systems, a

notion introduced by Weispfenning [24] who also provided a first algorithm for computing such systems. Subsequently, improvements and alternatives were presented by many authors [10, 11, 13, 14, 16]. Our calculations were done using a MAPLE implementation of the DISPGB algorithm of Montes which is available at <http://amirhashemi.iut.ac.ir/software>.

In the sequel, we denote by  $\tilde{\mathcal{P}} = \mathcal{P}[\mathbf{a}] = \mathbb{k}[\mathbf{a}, \mathbf{x}]$  a *parametric* polynomial ring where  $\mathbf{a} = a_1, \dots, a_m$  represents the parameters and  $\mathbf{x} = x_1, \dots, x_n$  the variables. Let  $\prec_{\mathbf{x}}$  (resp.  $\prec_{\mathbf{a}}$ ) be a term order for the power products of the variables  $x_i$  (resp. the parameters  $a_i$ ). Then we introduce the block elimination term order  $\prec_{\mathbf{x}, \mathbf{a}}$  in the usual manner: for all  $\alpha, \gamma \in \mathbb{N}_0^n$  and all  $\beta, \delta \in \mathbb{N}_0^m$ , we define  $\mathbf{a}^\delta \mathbf{x}^\gamma \prec_{\mathbf{x}, \mathbf{a}} \mathbf{a}^\beta \mathbf{x}^\alpha$ , if either  $\mathbf{x}^\gamma \prec_{\mathbf{x}} \mathbf{x}^\alpha$  or  $\mathbf{x}^\gamma = \mathbf{x}^\alpha$  and  $\mathbf{a}^\delta \prec_{\mathbf{a}} \mathbf{a}^\beta$ .

**Definition 6.1** *A finite set of triples  $\{(\tilde{G}_i, N_i, W_i)\}_{i=1}^\ell$  with finite sets  $\tilde{G}_i \subset \tilde{\mathcal{P}}$  and  $N_i, W_i \subset \mathcal{Q} = \mathbb{k}[\mathbf{a}]$  is a Gröbner system for a parametric ideal  $\tilde{\mathcal{I}} \trianglelefteq \tilde{\mathcal{P}}$  with respect to the block order  $\prec_{\mathbf{x}, \mathbf{a}}$ , if for every index  $1 \leq i \leq \ell$  and every specialisation homomorphism  $\sigma : \mathcal{Q} \rightarrow \mathbb{k}$  such that*

$$(i) \ \forall g \in N_i : \sigma(g) = 0, \quad (ii) \ \forall h \in W_i : \sigma(h) \neq 0 \quad (5)$$

*$\sigma(\tilde{G}_i)$  is a Gröbner basis of  $\sigma(\tilde{\mathcal{I}}) \trianglelefteq \mathcal{P}$  with respect to the order  $\prec_{\mathbf{x}}$  and if for any point  $\mathbf{a} \in \mathbb{k}^m$  an index  $1 \leq i \leq \ell$  exists such that  $\mathbf{a} \in \mathcal{V}(N_i) \setminus \mathcal{V}(W_i)$ .*

Thus a Gröbner systems yields a Gröbner basis for all possible values of the parameters  $\mathbf{a}$ . Weispfenning [24, Theorem 2.7] proved that every parametric ideal  $\mathcal{I} \trianglelefteq \mathcal{S}$  possesses a Gröbner system, but in general the system is not unique. Basically every algorithm (in particular the DISPGB algorithm used by us) produces Gröbner systems such that given one specific triple  $(\tilde{G}_i, N_i, W_i)$  all specialisations  $\sigma$  satisfying (5) yield the same leading terms  $\text{lt } \sigma(G_i)$  so that we can speak of a monomial ideal  $\mathcal{L}_i \trianglelefteq \mathcal{P}$  determined by the conditions  $(N_i, W_i)$ . In the sequel, we will always assume that a Gröbner system with this property is used. As a simple corollary, we find then that the reduction number set of an ideal  $\mathcal{I} \trianglelefteq \mathcal{P}$  is always finite. Our proof also yields an explicit method for computing it.

**Theorem 6.2** *Let  $\mathcal{I} \trianglelefteq \mathcal{P}$  be a homogeneous ideal. Then its reduction number set  $\text{rSet}(\mathcal{R})$  is finite.*

**Proof** By definition, any minimal reduction of  $\mathcal{I}$  is induced by  $D$  linear forms

$$y_i = \sum_{j=1}^n a_{i,j} x_j, \quad i = 1, \dots, D \quad (6)$$

with  $a_{i,j} \in \mathbb{k}$  and minimality is equivalent to  $\mathcal{J} = \mathcal{I} + \langle y_1, \dots, y_D \rangle$  being a zero-dimensional ideal. Considering the coefficients  $a_{i,j}$  as parameters, we may identify  $\mathcal{J}$  with a parametric ideal  $\tilde{\mathcal{I}} \trianglelefteq \tilde{\mathcal{P}}$ . Let  $\{(\tilde{G}_i, N_i, W_i)\}_{i=1}^\ell$  be a Gröbner system for  $\tilde{\mathcal{I}}$ . Without loss of generality, we may assume that for the first

$s$  triples the associated monomial ideals  $\mathcal{L}_i$  are zero-dimensional, whereas all other triples lead to monomial ideals of positive dimension. Hence precisely the parameter values satisfying one of the conditions  $(N_i, W_i)$  with  $1 \leq i \leq s$  define minimal reductions. If  $d_i$  is the highest degree such that  $(\mathcal{L}_i)_{d_i} \neq \mathcal{P}_{d_i}$ , then it follows that  $\text{rSet}(\mathcal{R}) = \{d_1, \dots, d_s\}$ .  $\square$

**Remark 6.3** Any Gröbner system for a parametric ideal  $\tilde{\mathcal{I}}$  contains one generic branch where the set  $N_i$  of equations is empty. Obviously, the corresponding leading ideal  $\mathcal{L}_i$  must be the generic initial ideal  $\text{gin } \mathcal{I}$  and we have  $d_i = r(\mathcal{R})$ . This observation immediately yields an alternative proof of [21, Cor. 2.2]: for almost all minimal reductions  $\mathcal{J}$  we find  $r_{\mathcal{J}}(\mathcal{R}) = r(\mathcal{R})$ .

**Example 6.4** Let us consider again Green's Example 5.3 where  $D = 1$ . Hence we set  $\tilde{\mathcal{I}} = \langle x_1^2, x_1x_3, x_2^2 + x_1x_2, a_1x_1 + a_2x_2 + a_3x_3 \rangle$ . The Gröbner system for  $\tilde{\mathcal{I}}$  consists of 4 triples. For simplicity, we present in the following list for each branch as first entry only the corresponding leading ideal  $\mathcal{L}_i$ ; the other two entries are the equations  $N_i$  and the inequations  $W_i$ , respectively.

$$\begin{array}{lll} \{x_1, x_2^2, x_3^2, x_2x_3\} & \{\} & \{a_1, a_2, a_1 - a_2\} \\ \{x_1, x_2^2, x_3^2, x_2x_3\} & \{a_1 - a_2\} & \{a_2\} \\ \{x_1, x_2^2, x_3^2\} & \{a_2\} & \{a_1\} \\ \{x_2, x_1^2, x_3^2, x_1x_3\} & \{a_1\} & \{\} \end{array}$$

We observe that all four branches lead to zero-dimensional leading ideals and their reduction numbers are 1, 1, 2, 1, respectively. Therefore,  $\text{rSet}(\mathcal{R}) = \{1, 2\}$  and  $\text{br}(\mathcal{R}) = 2$ .

**Remark 6.5** For comparison, we briefly outline Trung's constructive characterisation [21] of the big reduction number of an ideal. He also takes  $D$  linear forms (6) with undetermined coefficients  $a_{i,j}$  and proceeds with the ideal  $\mathcal{J} = \mathcal{I} + \langle y_1, \dots, y_D \rangle \trianglelefteq \mathcal{P}$  (note that he does not work in the parametric polynomial ring  $\tilde{\mathcal{P}}$ ). Then he introduces the matrix  $M_d$  of the coefficients of the generators in a  $\mathbb{k}$ -linear basis of  $\mathcal{J}_d$  (which are elements in  $\mathcal{Q}$ ). Let  $\mathcal{V}_d$  be the variety of the ideal generated in  $\mathcal{Q}$  by all the minors of  $M_d$  of the size of the number of terms of degree  $d$ . Then,  $\text{br}(\mathcal{R})$  is the largest  $d$  such that  $\mathcal{V}_d \neq \mathcal{V}_{d+1}$  [21, Cor. 2.3].

Note, however, that a priori it is unclear how to detect that one has obtained the largest  $d$  with this property. Thus his approach becomes truly algorithmic only by combining it with another result of his, namely that  $\text{br}(\mathcal{R}) + 1$  is bounded by the Castelnuovo-Mumford regularity  $\text{reg}(\mathcal{I})$  [19, Prop. 3.2]. Now one can check all degrees  $d$  until  $\text{reg}(\mathcal{I})$ —which has to be computed first—and then finally decide on the value of  $\text{br}(\mathcal{R})$ . While the computation of a Gröbner system is surely a rather expensive operation, we strongly believe that it is much more efficient than the determination and subsequent analysis of large determinantal ideals. Furthermore, our approach yields directly all possible values for the reduction number, whereas Trung must consider one determinantal ideal after the other (of increasing size).

Finally, we note that Trung [21] proved that  $br(\mathcal{R}) \leq br(\mathcal{P}/\text{lt } \mathcal{I})$  if  $\mathcal{R}$  is Cohen-Macaulay. He also claimed that generally one cannot compare  $br(\mathcal{R})$  and  $br(\mathcal{P}/\text{lt } \mathcal{I})$ . However, he did not provide a concrete example where the above inequality is violated—which we will do now.

**Example 6.6** *Consider for  $n = 3$  the ideal*

$$\mathcal{I} = \langle x_1^2x_2 + x_1x_2^2, x_2^3 + x_2^2x_3, x_1x_3^5, x_2^2x_3^5, x_1^2x_3 + x_1x_2x_3, x_1^3 - x_1x_2^2 \rangle.$$

*The given generators form already a Gröbner basis and thus  $D = 1$ .  $\text{lt } \mathcal{I}$  is quasi-stable and, using Pommaret bases, one easily shows that the depth of  $\mathcal{R}$  is 0 and  $\mathcal{R}$  is not Cohen-Macaulay. With  $\mathcal{J} = \mathcal{I} + \langle x_1 + x_2 + x_3 \rangle$ , a simple computation yields that  $\text{lt } \mathcal{J} = \langle x_1, x_2x_3^2, x_2^2x_3, x_3^3, x_3^7 \rangle$  and thus  $br(\mathcal{R}) \geq r_{\mathcal{J}}(\mathcal{R}) = 6$ . For showing that  $br(\mathcal{R}) = 6$ , we set  $\tilde{\mathcal{I}} = \mathcal{I} + \langle a_1x_1 + a_2x_2 + a_3x_3 \rangle \subseteq \mathcal{P}$ . The Gröbner system of this ideal shows that the reduction numbers of the zero-dimensional branches are 3, 5, 6, respectively, and therefore  $br(\mathcal{R}) = 6$ . On the other hand,  $\text{lt } \mathcal{I} = \langle x_1^2x_3, x_2^3, x_1^2x_2, x_1^3, x_1x_3^5, x_2^2x_3^5 \rangle$ . We set  $\tilde{\mathcal{I}}_1 = \text{lt } \mathcal{I} + \langle a_1x_1 + a_2x_s + a_3x_3 \rangle$ , and compute its Gröbner system. Only three branches are zero-dimensional and they all have as reduction number 3. This shows that  $br(\mathcal{P}/\text{lt } \mathcal{I}) = 3 < br(\mathcal{R})$ .*

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