# Parameterized Directed $k$-Chinese Postman Problem and $k$ Arc-Disjoint Cycles Problem on Euler Digraphs 

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#### Abstract

In the Directed $k$-Chinese Postman Problem ( $k$-DCPP), we are given a connected weighted digraph $G$ and asked to find $k$ non-empty closed directed walks covering all arcs of $G$ such that the total weight of the walks is minimum. Gutin, Muciaccia and Yeo (Theor. Comput. Sci. 513 (2013) 124-128) asked for the parameterized complexity of $k$-DCPP when $k$ is the parameter. We prove that the $k$-DCPP is fixed-parameter tractable. We also consider a related problem of finding $k$ arc-disjoint directed cycles in an Euler digraph, parameterized by $k$. Slivkins (ESA 2003) showed that this problem is W[1]-hard for general digraphs. Generalizing another result by Slivkins, we prove that the problem is fixed-parameter tractable for Euler digraphs. The corresponding problem on vertex-disjoint cycles in Euler digraphs remains W[1]-hard even for Euler digraphs.


## 1 Introduction

A digraph $H$ is connected if the underlying undirected graph of $H$ is connected. Let $G=(V, A)$ be a connected digraph, where each arc $a \in A$ is assigned a non-negative integer weight $\omega(a)$ ( $G$ is a weighted digraph). The Directed Chinese Postman Problem is a well-studied polynomial-time solvable problem in combinatorial optimization [19]13].

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Directed Chinese Postman Problem (DCPP)
Input: A connected weighted digraph G}=(V,A)
Task: Find a minumum total weight closed directed walk }
    on G such that every arc of G is contained in T.
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In this paper, we will investigate the following generalisation of DCPP.

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Directed k-Chinese Postman Problem ( }k\mathrm{ -DCPP)
Input: A connected weighted digraph G}=(V,A)\mathrm{ and an integer k.
Task: Find a minimum total weight set of }k\mathrm{ non-empty
    closed directed walks such that every arc of G is
    contained in at least one of them.
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Note that the $k$-DCPP can be extended to directed multigraphs (that may include parallel arcs but no loops), but the extended version can be reduced to the one on digraphs by subdividing parallel arcs and adjusting weights appropriately. Since it is more convenient, we consider the $k$-DCPP for digraphs only.

In the literature, the undirected version of $k$-DCPP, abbreviated $k$-UCPP, has also been studied. If a vertex $v$ of $G$ is part of the input and we require that each of the $k$ walks contains $v$ then the $k$-DCPP and $k$-UCPP are polynomial-time solvable [24|16]. However, in general the $k$-DCCP is NP-complete [12], as is the $k$-UCPP [12[23].

Lately research in parameterized algorithms and complexity 1 for the CPP and its generalizations was summarized in [2] and reported in [20]. Several recent results described there are of Niedermeier's group who identified a number of practically useful parameters for the CPP and its generalizations, obtained several interesting results and posed some open problems, see, e.g. [8|21|22]. van Bevern et al. [2] and Sorge [20] suggested to study the $k$-UCPP as a parameterized problem with parameter $k$ and asked whether the $k$-UCPP is fixed-parameter tractable, i.e. can be solved by an algorithm of running time $O\left(f(k) n^{O(1)}\right)$, where $f$ is a function of $k$ only and $n=|V|$.

Gutin, Muciaccia and Yeo [12] proved that the $k$-UCPP is fixed-parameter tractable. Observing that their approach for the $k$-UCPP is not applicable to the $k$-DCPP, the authors of [12] asked for the parameterized complexity of $k$-DCPP parameterized by $k$. In this paper, we show that the $k$-DCPP is also fixed-parameter tractable.

## Theorem 1. The $k$-DCPP is fixed-parameter tractable.

Our proof is very different from that in [12] for the $k$-UCPP. While the latter proof was based on a simple reduction to a polynomial-size kernel, we give a fixed-parameter algorithm directly using significantly more powerful tools. In particular, we use an approximation algorithm of Grohe and Grüber [11] for the problem of finding the maximum number $\nu_{0}(D)$ of vertex-disjoint directed cycles in a digraph $D$ (this algorithm is based on the celebrated paper by Reed et al. [17] on bounding $\nu_{0}(D)$ by a function of $\tau_{0}(D)$, the minimum size of a feedback vertex set of $D$ ). We also use the well-known fixed-parameter algorithm of Chen et al. [4] for the feedback vertex set problem on digraphs.

We also consider the following well-known problem related to the $k$-DCPP.

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k-Arc-Disjoint Cycles Problem ( }k\mathrm{ -ADCP)
Input: A digraph D and an integer }k\mathrm{ .
Task: Decide whether D has k arc-disjoint directed cycles.
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Crucially, we are interested in the $k$-ADCP because given a set of $k$ arc-disjoint cycles, we can solve the $k$-DCPP in polynomial time (see Lemma 5). However, this problem is important in its own right.

The problem is NP-hard in general but polynomial-time solvable for planar digraphs [14]. In fact, for planar digraphs the maximum number of arc-disjoint directed cycles equals the minimum size of a feedback arc set, see, e.g, [1]. It is natural to consider $k$ as the parameter for the $k$-ADCP. It follows easily from the results of Slivkins [19] that the $k$ - ADCP is $\mathrm{W}[1]$-hard. It remains $\mathrm{W}[1]$-hard for quite restricted classes of directed multigraphs, e.g., for directed multigraphs which become acyclic after deleting two sets of parallel arcs [19]. Here we show that the $k$-ADCP-EULER, the $k$-ADCP on Euler digraphs, is fixed-parameter tractable, generalizing a result in [19] (Theorem 4).

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## Theorem 2. The $k$-ADCP-EULER is fixed-parameter tractable.

Interestingly, the problem of deciding whether a digraph has $k$ vertex-disjoint directed cycles, which is W[1]-hard (also easily follows from the results of Slivkins [19]), remains W[1]-hard on Euler digraphs. Indeed, consider a digraph $D$ and let $\nu_{0}(D)$ denote the maximum number of vertex-disjoint directed cycles in $D$. Construct a new digraph $H$ from $D$ by adding two new vertices $x$ and $y$, arcs $x y$ and $y x$ and the following extra arcs between $x$ and the vertices of $D$ : for each $v \in V(D)$ add $\max \left\{d^{-}(v)-d^{+}(v), 0\right\}$ parallel arcs $v x$ and $\max \left\{d^{+}(v)-d^{-}(v), 0\right\}$ parallel arcs $x v$, where $d^{-}(v)$ and $d^{+}(v)$ are the in-degree and out-degree of $v$, respectively. To eliminate parallel arcs, it remains to subdivide all arcs between $x$ and $V(D)$. Now it is sufficient to observe that $H$ is Euler and $\nu_{0}(H)=\nu_{0}(D)+1$.

To prove Theorems 1 and 2 we study the following problem that generalizes the $k$ DCPP (in the case when an optimal solution exists in which the number of times each arc is visited by every closed walk is restricted) and $k$-ADCP. Let $b \leq c$ be non-negative integers.

| Directed $k$-WALK $[b, c]$-Covering Problem $(k[b, c]$-DWCP) |  |
| :--- | :--- |
| Input: | A connected weighted digraph $G=(V, A)$ and <br> an integer $k$. |
| Task: | Find a minimum total weight set of $k$ non-empty <br> closed directed walks in which every arc of $G$ appears <br> between $b$ and $c$ times. |
|  |  |

Let $D$ be a digraph. For a vertex ordering $\nu=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ of $V(D)$, the cutwidth of $\nu$ is the maximum number of arcs between $\{1, \ldots, i\}$ and $\{i+1, \ldots n\}$ over all $i \in[n]$. The cutwidth of $D$ is the minimum cutwidth of all vertex orderings of $V(D)$.

In Section 3 we will prove the following theorem.
Theorem 3. Let $(G, k)$ be an instance of $k[b, c]-D W C P$ and suppose we are given a vertex ordering $\nu=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ of $G$ with cutwidth at most $p$. Then $(G, k)$ can be solved in time $O^{*}\left(\left(c 2^{k}\right)^{p} 4^{k}\right)$.

Note that when $c$ and $p$ are upper-bounded by functions of $k$, the algorithm of this theorem is fixed-parameter.

In order to apply Theorem 3 to the $k$-DCPP and $k$-ADCP-EULER, we first need to find a vertex ordering of bounded cutwidth. This is done using Lemma 3, which given an Euler directed graph, either finds a vertex ordering with cutwidth bounded by a function of $k$, or finds $k$ arc-disjoint cycles. (For the $k$-DCPP, we apply Lemma 3 to an Euler directed multigraph derived from a solution to the DCPP on $G$.) If $k$ arc-disjoint cycles are found, then the $k$-ADCP-EULER is solved. In the case of the $k$-DCPP, it remains to use Lemma 5, which shows that given $k$ arc-disjoint cycles (in the derived directed multigraph), we can solve the $k$-DCPP on $G$ in polynomial time.

If we find a vertex ordering of cutwidth $p(k)$, we can solve the $k$-ADCP-EULER by applying Theorem 3 with $b=0, c=1$. In the case of the $k$-DCPP, $b=1$ and it remains to find an upper bound on $c$. This is done using Lemma 6 proved in Section 2 , which
shows that if an optimal solution of DCPP traverses each arc less than $k$ times then there is an optimal solution for the $k$-DCPP such that no arc is visited more than $k$ times in total by the $k$ walks of the solution. If an optimal solution of DCPP visits an arc at least $k$ times, then the derived graph for this solution contains at least $k$ arc-disjoint cycles and again we may use Lemma 5 Thus, starting from an arbitrary optimal solution of DCPP, we may either apply Theorem 3 with $c=k$, or Lemma 5 .

The paper is organised as follows. In Section 2, we prove six lemmas providing structural results for the $k$-DCPP and $k$-ADCP-EULER. In Sections 3 and 4 we prove Theorem 3 and the main two results of the paper, Theorems 1 and 2 We conclude the paper with brief discussions of open problems in Section 5

In what follows, all walks and cycles in directed multigraphs are directed. For a positive integer $p,[p]$ will denote the set $\{1,2, \ldots, p\}$. For integers $a \leq b,[a, b]$ will denote the set $\{a, a+1, \ldots, b\}$. Given a directed graph $D$, a feedback vertex set for $D$ is a set $S$ of vertices such that $D-S$ contains no directed cycles. A feedback arc set for $D$ is a set $F$ of arcs such that $D-F$ contains no directed cycles. A vertex $v$ of a digraph is balanced if the in-degree of $v$ equals its out-degree.

## 2 Structural Results and Fixed-Parameter Algorithms

Recall that a directed multigraph $H$ is Euler (i.e., has an Euler trail) if and only if $H$ is connected and every vertex of $H$ is balanced [1].

The next lemma is a simple sufficient condition for an Euler digraph to contain $k$ arc-disjoint cycles.

Lemma 1. Every Euler digraph $D$ having a vertex of out-degree at least $k \geq 1$, contains $k$ arc-disjoint cycles that can be found in polynomial time.

Proof. For $k=1$, it is true as $D$ has a cycle that can be found in polynomial time. Let $k \geq 2$ and let $C$ be a cycle in $D$. Observe that after deleting the arcs of $C, D$ has a vertex of out-degree at least $k-1$ and we are done by induction hypothesis.

Reed et al. [17] proved that there is a function $f: \mathbb{N} \rightarrow \mathbb{N}$ such that for every $k$, if a digraph $D$ does not have $k$ arc-disjoint cycles, then it has a feedback arc set with at most $f(k)$ arcs. The celebrated result of Reed et al. [17] can be easily extended to directed multigraphs by subdividing parallel arcs. Using this result, Grohe and Grüber [11] showed that there is a non-decreasing and unbounded function $h: \mathbb{N} \rightarrow \mathbb{N}$ and a fixed-parameter algorithm that for a digraph $D$ returns at least $h(k)$ arc-disjoint cycles if $D$ has at least $k$ arc-disjoint cycles.

Let $h^{-1}: \mathbb{N} \rightarrow \mathbb{N}$ be defined by $h^{-1}(q)=\min \{p: h(p) \geq q\}$. Since $h$ is a nondecreasing and unbounded function, $h^{-1}$ is a non-decreasing and unbounded function. Combining the above results, we find that for every digraph $D$, either the algorithm of Grohe and Grüber returns at least $k$ arc-disjoint cycles, or $D$ has a feedback arc set of size at most $f\left(h^{-1}(k)\right)$.

Chen et al. [4] designed a fixed-parameter algorithm that decides whether a digraph $D$ contains a feedback vertex set of size $k$ ( $k$ is the parameter). As this is an iterative compression algorithm, it can be easily modified to an algorithm for finding a minimum
feedback vertex set in $D$ (the running time of the latter algorithm is $q\left(\tau_{0}(D)\right) n^{O(1)}$, where $\tau_{0}(D)$ is the minimum size of a feedback vertex set in $D, n=|V(D)|$ and $q(k)=4^{k} k!$ ). The modified algorithm can be used for finding a minimum feedback arc set in $D$ as $D$ can be transformed, in polynomial time, into another digraph $H$ such that $D$ has a feedback arc set of size $k$ if and only if $H$ has a feedback vertex set of size $k$, see, e.g., [1] (Proposition 15.3.1).

Lemma 2. There is a function $g: \mathbb{N} \rightarrow \mathbb{N}$ and a fixed-parameter algorithm such that for a digraph $D$, the algorithm returns either $k$ arc-disjoint cycles or a feedback arc set of size at most $g(k)$.

Proof. Run the Grohe-Grüber algorithm on $D$. Either the algorithm returns at least $k$ arc-disjoint cycles, or we know that $D$ has no $h^{-1}(k)$ arc-disjoint cycles and so by the result of Reed et al. $D$ has a feedback arc set of size at most $f\left(h^{-1}(k)\right)$. We can use the algorithm of Chen et al. to find in $D$ a minimum feedback arc set. We may set $g(k)=f\left(h^{-1}(k)\right)$.

Lemma 3. Let $g: \mathbb{N} \rightarrow \mathbb{N}$ be the function in Lemma 2 Let $D$ be an Euler directed multigraph. We can obtain either $k$ arc-disjoint cycles of $D$ or a vertex ordering of cutwidth at most $2 g(k)$.

Proof. Let us run the procedure of Lemma 2 for $D$ and $k$. If we get $k$ arc-disjoint cycles, we are done. Otherwise, we get a feedback arc set $F$ of $D$ such that $|F| \leq g(k)$. Then $D^{\prime}=D-F$ is an acyclic digraph. We let $\nu=\left(v_{1}, \ldots, v_{n}\right)$ be an acyclic ordering of $D^{\prime}$, i.e., $D^{\prime}$ has no arc of the form $v_{i} v_{j}, i>j$, (it is well-known that such an ordering exists [1]). Now $\nu$ is a vertex ordering for $D$ with at most $|F| \operatorname{arcs}$ from $\left\{v_{i+1}, \ldots, v_{n}\right\}$ to $\left\{v_{1}, \ldots, v_{i}\right\}$ for each $i \in[n-1]$, and because $D$ is Euler there are the same number of arcs from $\left\{v_{1}, \ldots, v_{i}\right\}$ to $\left\{v_{i+1}, \ldots, v_{n}\right\}$ [1] Corollary 1.7.3]. So $\nu$ is a vertex ordering with cutwidth at most $2 g(k)$.

In the rest of this section, $G=(V, A)$ is a connected weighted directed graph. For a solution $T=\left\{T_{1}, \ldots, T_{k}\right\}$ to the $k$-DCPP on $G(k \geq 1)$, let $G_{T}=\left(V, A_{T}\right)$, where $A_{T}$ is a multiset containing all arcs of $A$, each as many times as it is traversed in total by $T_{1} \cup \cdots \cup T_{k}$.

Lemmas 4 and 5 are similar to two simple results obtained for the $k$-UCPP in [12]. Note that given $k$ closed walks which cover all the arcs of a digraph, their union is a closed walk covering all the arcs and, therefore, it is a solution for the DCPP. Hence, the following proposition holds.

Lemma 4. The weight of an optimal solution for the $k$-DCPP on $G$ is not smaller than the weight of an optimal solution for the DCPP on $G$.

Lemma 5. Let $T$ be an optimal solution for the DCPP on $G$. If $G_{T}$ contains at least $k$ arc-disjoint cycles, then the weight of an optimal solution for the $k$-DCPP on $G$ is equal to the weight of an optimal solution of the DCPP on $G$. Furthermore if $k$ arcdisjoint cycles in $G_{T}$ are given, then an optimal solution for the $k-D C P P$ can be found in polynomial time.

Proof. Note that $G_{T}$ is an Euler directed multigraph and so every vertex of $G_{T}$ is balanced. Let $\mathcal{C}$ be any collection of $k$ arc-disjoint cycles in $G_{T}$. Delete all arcs of $\mathcal{C}$ from $G_{T}$ and observe that every vertex in the remaining directed multigraph $G^{\prime}$ is balanced. Find an optimal DCPP solution for every connected component of $G^{\prime}$ and append each such solution $F$ to a cycle in $\mathcal{C}$ which has a common vertex with $F$. As a result, in polynomial time, we obtain a collection $Q$ of $k$ closed walks for the $k$-DCPP on $G$ of the same weight as $T$. So $Q$ is optimal by Lemma4

For a directed multigraph $D$, let $\mu_{D}(x y)$ denote the multiplicity of an arc $x y$ of $D$. The multiplicity $\mu(D)$ of $D$ is the maximum of the multiplicities of its arcs. Thus, Lemma 5 implies that if $\mu\left(G_{T}\right) \geq k$ for any optimal solution $T$ of the DCPP on $G$, then there is an optimal solution of the $k$-DCPP on $G$ with weight equal to the weight of $G_{T}$. The next lemma helps us in the case that $\mu\left(G_{T}\right) \leq k-1$.

Lemma 6. Let $T$ be an optimal solution of the $D C P P$ on $G$ such that $\mu\left(G_{T}\right) \leq k-1$. Then there is an optimal solution $W$ for the $k-D C P P$ on $G$ such that $\mu\left(G_{W}\right) \leq k$.

Proof. Let $T$ be an optimal solution of DCPP on $G$ and let $\mu\left(G_{T}\right) \leq k-1$. Suppose that there is an optimal solution $W$ of the $k$-DCPP on $G$ such that $\mu\left(G_{W}\right)>k$.

Let $\rho(x y)=\mu_{G_{W}}(x y)-\mu_{G_{T}}(x y)$ for each arc $x y$ of $G$. Consider a directed multigraph $H^{\prime}$ with the same vertex set as $G$ and in which $x y$ is an arc of multiplicity $|\rho(x y)|$ if it is an arc in $G$ and $\rho(x y) \neq 0$. We say that an arc $x y$ of $H^{\prime}$ is positive (negative) if $\rho(x y)>0(\rho(x y)<0)$. Now reverse every negative arc of $H^{\prime}$ (i.e., replace every negative arc $u v$ by the negative arc $v u$ ) keeping the weight of the arcs the same. We denote the resulting directed multigraph by $H$.

For a digraph $D$ and its vertex $x$, let $N_{D}^{+}(x)$ and $N_{D}^{-}(x)$ denote the sets of outneighbors and in-neighbors of $x$, respectively. Since $G_{T}$ and $G_{W}$ are both Euler directed multigraphs, we have that

$$
\sum_{y \in N_{H^{\prime}}^{+}(x)} \rho(x y)=\sum_{z \in N_{H^{\prime}}^{-}(x)} \rho(z x) \text { implying } \sum_{u \in N_{H}^{+}(x)} \mu(x u)=\sum_{v \in N_{H}^{-}(x)} \mu(v x)
$$

for each vertex $x$ in $G$. So, every vertex in $H$ has the same in-degree as out-degree. Thus, the arcs of $H$ can be decomposed into a collection $\mathcal{C}=\left\{C_{1}, \ldots, C_{t}\right\}$ of cycles. We define the weight $\omega\left(C_{i}\right)$ of a cycle $C_{i}$ of $\mathcal{C}$ as the sum of the weights of its positive arcs minus the sum of the weights of its negative arcs, and assume that $\omega\left(C_{1}\right) \leq \cdots \leq$ $\omega\left(C_{t}\right)$.

Set $F_{0}=G_{T}$ and for $i \in[t]$, construct $F_{i}$ from $F_{i-1}$ as follows: for each arc $x y$ of $C_{i}$, if $x y$ is a positive arc in $H$ add a copy of $x y$ to $F_{i-1}$ and if $x y$ is a negative arc in $H$ remove a copy of $y x$ from $F_{i-1}$. Since for each arc $u v$ of $G, \mu_{G_{T}}(u v) \geq 1$ and $\mu_{G_{W}}(u v) \geq 1$, we have $\mu_{F_{i}}(u v) \geq 1$. Each vertex of $F_{i}$ is balanced, so $F_{i}$ is a solution of DCPP on $G$. Since $T$ is optimal, $\omega\left(F_{0}\right) \leq \omega\left(F_{1}\right)=\omega\left(F_{0}\right)+\omega\left(C_{1}\right)$ and so $\omega\left(C_{1}\right) \geq 0$. Due to the ordering of cycles of $\mathcal{C}$ according to their weights, $\omega\left(C_{i}\right) \geq 0$ for $i \in[t]$. Thus, $\omega\left(F_{i}\right) \geq \omega\left(F_{i-1}\right)$ for $i \in[t]$.

Since $\mu\left(F_{0}\right) \leq k-1$ and $\mu\left(F_{t}\right)>k$, there is an index $j$ such that $\mu\left(F_{j}\right)=k$. Then the out-degree of some vertex of $F_{j}$ is at least $k$ and so by Lemma $1 F_{j}$ has $k$ arcdisjoint cycles. Similarly to Lemma5, it is not hard to show that there is a solution $U$
of $k$-DCPP on $G$ of weight $\omega\left(F_{j}\right)$. Since $W$ is optimal and $\omega\left(F_{j}\right) \leq \omega\left(F_{t}\right)=\omega\left(G_{W}\right)$, $U$ is also optimal and we are done.

## 3 Proof of Theorem 3

Theorem 3 is proved by providing a dynamic programming (DP) algorithm of required complexity. We first make an observation to simplify the DP algorithm.

Lemma 7. Let $G=(V, A)$ and $k$ define an instance of $k[b, c]-D W C P$. The instance is positive and and the weight of an optimal solution is $\rho$ if and only if there are (not necessarily connected) non-empty directed multigraphs $G_{1}, \ldots, G_{k}$ with the following properties:

- All multigraphs $G_{1}, \ldots, G_{k}$ use only arcs of $G$ (each, possibly, multiple number of times);
- $G_{1}$ is a balanced multigraph;
- For $2 \leq i \leq k, G_{i}$ is a balanced digraph (with no parallel arcs);
- Each arc $a \in A$ occurs between $b$ and $c$ times in the multigraph $G_{1} \cup \cdots \cup G_{k}$, and the total weight of this multigraph is $\rho$.

Proof. On the one hand, let $W_{1}, \ldots, W_{k}$ be a solution to the $k[b, c]$-DWCP instance, where each $W_{i}$ is a closed directed walk. For each $i \in[k]$, let $Q_{i}$ be the directed multigraph whose vertices are the vertices visited by $W_{i}$ and which contains an arc $u v$ of multiplicity $\mu$ if $u v$ is traversed exactly $\mu$ times by $W_{i}$. For each $i \geq 2$, if $Q_{i}$ has parallel arcs, let $G_{i}$ be a cycle in $Q_{i}$ and let $Q_{i}^{\prime}=Q_{i} \backslash A\left(G_{i}\right)$ and, otherwise (i.e., $Q_{i}$ has no parallel arcs), let $G_{i}=Q_{i}$ and let $Q_{i}^{\prime}$ be empty. Now let $G_{1}=Q_{1} \cup Q_{2}^{\prime} \cup \cdots \cup Q_{k}^{\prime}$. Observe that all properties of the lemma are satisfied.

On the other hand, consider directed multigraphs $G_{1}, \ldots, G_{k}$ satisfying the properties of the lemma. If all multigraphs $G_{i}$ are connected, then we are done. If $b=0$, then we may replace each graph $G_{i}$ with a cycle $C_{i}$ contained in $G_{i}$, and produce a solution to $k[b, c]$-DWCP that consists of $k$ (not necessarily pairwise arc-disjoint) cycles.

Finally, if not all multigraphs are connected and $b>0$, we proceed as follows. First, select for each multigraph $G_{i}, i>1$ an arbitrary connected component $H_{i}$, and move all other components of $G_{i}$ to $G_{1}$, increasing arc multiplicity as appropriate. Next, as long as $G_{1}$ remains unconnected, let $H$ be an arbitrary connected component of $G_{1}$. As $b>0$ and $G$ is connected, some component $H_{i}, i>1$ must intersect a vertex of $H$; we may move $H$ to the multigraph $G_{i}$ and maintain that $G_{i}$ is connected. Repeat this until $G_{1}$ (and hence each multigraph $G_{i}$ ) is connected. Note that this does not change the arc multiplicity or the weight of the solution. Now every multigraph $G_{i}$ for $i \in[k]$ is balanced and connected, i.e., Euler, and we can find an Euler tour $W_{i}$ for each graph $G_{i}$, which forms the solution to the $k[b, c]$-DWCP instance.

Let $\nu=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ be a vertex ordering of a digraph $G$ of cutwidth at most $p$. For each $i \in\{0,1, \ldots, n\}$, let $E_{i}$ be the set of arcs of the form $v_{j} v_{h}$ or $v_{h} v_{j}$, where

[^1]$j \leq i$ and $h>i$. Note that in particular $E_{0}=\emptyset$ and $E_{n}=\emptyset$. As $\nu$ has cutwidth at $\operatorname{most} p,\left|E_{i}\right| \leq p$ for each $i$. We refer to $E_{0}, E_{1}, \ldots, E_{n}$ as the arc bags of $\nu$. For each $i \in\{0,1, \ldots, n\}$, let $\gamma(i)=\bigcup_{0 \leq j \leq i} E_{j}$. For a vertex $v \in V$, let $A^{+}(v)=\{v u \in A$ : $u \in V\}$ and $A^{-}(v)=\{u v \in A: u \in V\}$.

We now give an intuitive description of the DP algorithm before giving technical details. Our DP algorithm will process each arc bag of $\nu$ in turn, from $E_{0}$ to $E_{n}$. For each arc bag $E_{i}$, we store the weights of a range of partial solutions. A partial solution consists of a multiset $A_{1}$ and sets $A_{2}, \ldots, A_{k}$ of arcs in $\gamma(i)$. Each $A_{j}$ is to be thought of the (multi)set of arcs in $G_{j}$ (defined in Lemma 7) taken from $\gamma(i)$. A function $\phi$ is used to represent how many times each arc in the bag $E_{i}$ is used by each (multi)set $A_{j}$ in the solution. Finally, a set $S$ tracks which (multi)sets are non-empty. This is to ensure we don't produce a solution which uses less than $k$ non-empty walks. For each arc bag $E_{i}$, and every choice of $\phi, S$ respecting the conditions of Lemma 7 we will calculate the minimum weight of a partial solution corresponding to these choices.

Let us make these notions more precise. Let $E_{i}$ be an arc bag in $\nu$, and let $\phi$ be a function $E_{i} \times[k] \rightarrow[0, c]$ such that for each $a \in E_{i}$ we have $\sum_{j} \phi(a, j) \in[b, c]$ and $\phi(a, j) \leq 1$ for $2 \leq j \leq k$. Let $S$ be a subset of $[k]$. For a vertex $v$ and multiset $M$ of arcs, let $A^{+}(v, M)$ be the multiset of arcs from $M$ leaving $v$, and similarly let $A^{-}(v, M)$ be the multiset of arcs from $M$ entering $v$. Then we define $\chi\left(E_{i}, \phi, S\right)$ to be the minimum integer $\rho$ for which there exist arc multisets $A_{1}, \ldots, A_{k}$ satisfying the following conditions:

1. For every arc $a \in E_{i}$ and every $j \in[k], A_{j}$ contains exactly $\phi(a, j)$ copies of $a$;
2. For every arc $a \in \gamma(i)$, the multiset $A_{1} \cup \cdots \cup A_{k}$ contains between $b$ and $c$ copies of $a$;
3. For every $h \leq i$ and every $j \in S,\left|A^{+}\left(v_{h}, A_{j}\right)\right|=\left|A^{-}\left(v_{h}, A_{j}\right)\right|$;
4. For every $j \in[k], A_{j} \neq \emptyset$ if and only if $j \in S$; and
5. $\sum_{j \in[k]} \sum_{a \in A_{j}} \omega(a)=\rho$.

Note that $\left|A^{+}\left(v_{h}, A_{j}\right)\right|$ and $\left|A^{-}\left(v_{h}, A_{j}\right)\right|$ are the numbers of arcs in $A_{j}$ leaving and entering $v_{h}$, respectively, and that the second sum in Condition 5 is taken over all arcs in multiset $A_{j}$, i.e., over every copy of an arc in $A_{j}$.

If no such integer $\rho$ exists, then we let $\chi\left(E_{i}, \phi, S\right)=\infty$.
Observe that if $E_{i}, \phi, S$ and $\rho$ together with arc multisets $\left(A_{1}, \ldots, A_{k}\right)$ satisfy the above conditions, then $\chi\left(E_{i}, \phi, S\right) \leq \rho$. In such a case we will call $\left(A_{1}, \ldots, A_{k}\right)$ a witness for $\chi\left(E_{i}, \phi, S\right) \leq \rho$. Thus, $\chi\left(E_{i}, \phi, S\right)$ is the minimum $\rho$ such that there exists a witness for $\chi\left(E_{i}, \phi, S\right) \leq \rho$.

The next lemma shows that we can solve the $k[b, c]$-DCPP by finding the values $\chi\left(E_{i}, \phi, S\right)$. Since $E_{n}=\emptyset$, the only function $\phi: E_{n} \times[k] \rightarrow[b, c]$ is the empty function.

Lemma 8. Let $\phi: E_{n} \times[k] \rightarrow[b, c]$ be the empty function. Then $\chi\left(E_{n}, \phi,[k]\right)=\infty$ if there is no solution for the $k[b, c]-D C P P$ on $G$, and otherwise $\chi\left(E_{n}, \phi,[k]\right)$ is the minimum total weight of a solution for $k[b, c]-D C P P$.

Proof. We will show that (a) if $\chi\left(E_{n}, \phi,[k]\right)=\rho \neq \infty$, then there exists a solution for the $k[b, c]$-DCPP on $G$ with weight $\rho$; and that (b) if there exists a solution for the $k[b, c]$-DCPP on $G$ with weight $\rho$, then there exists a witness for $\chi\left(E_{n}, \phi,[k]\right) \leq \rho$.

In what follows it will be useful to observe that $\gamma(n)=A(G)$.
Suppose that $\chi\left(E_{n}, \phi,[k]\right)=\rho \neq \infty$ and $\left(A_{1}, \ldots, A_{k}\right)$ is a witness for $\chi\left(E_{n}, \phi,[k]\right) \leq \rho$. By Condition 3 of $\chi\left(E_{n}, \phi,[k]\right)$, every vertex is balanced with respect to each arc (multi)set $A_{j}$, and by Condition 4 each $A_{j}$ is non-empty. Thus, $A_{1}$ forms the arc (multi)set of a balanced directed multigraph and $A_{j}, j>1$ the arc set of a balanced digraph, and by Condition 2, every arc in $G$ appears between $b$ and $c$ times in these (multi)sets. By Lemma77, the arcs of the multiset $A_{1} \cup \ldots \cup A_{k}$ can be partitioned into a solution for the $k[b, c]$-DCPP, which by Condition 5 and minimality of $\rho$ has total weight exactly $\rho$. Thus there exists a solution for the $k[b, c]$-DCPP on $G$ with weight $\rho$.

Now suppose that there exists a solution for the $k[b, c]$-DCPP on $G$ with weight $\rho$; by Lemma 7 there then exist non-empty balanced directed multigraphs $G_{1}, \ldots, G_{k}$ of total weight $\rho$, where every arc appears between $b$ and $c$ times in total, and where $G_{j}$ for $j>1$ has no parallel arcs. Letting $A_{j}$ be the arc (multi)set of $G_{j}$ for each $j \in[k]$, we find that $\left(A_{1}, \ldots, A_{k}\right)$ is a witness for $\chi\left(E_{n}, \phi,[k]\right) \leq \rho$. As $E_{n}=\emptyset$, Condition 1 of $\chi\left(E_{n}, \phi,[k]\right)$ is trivially satisfied. Condition 2 is satisfied by the conditions in Lemma 7 Since every vertex in a balanced directed multigraph is balanced, Condition 3 is satisfied. As each of the $k$ multigraphs is non-empty, Condition 4 is satisfied. Finally, as the multigraphs have total weight $\rho$, Condition5is satisfied. Thus $\left(A_{1}, \ldots, A_{k}\right)$ is a witness for $\chi\left(E_{n}, \phi,[k]\right) \leq \rho$, as required.

Due to the space limit, we place the proof of the next lemma in the Appendix.
Lemma 9. Consider an arc bag $E_{i}$, for $i \geq 1$. Let $E_{i}^{*}=E_{i} \backslash E_{i-1}$. For any $\phi$ : $E_{i} \times[k] \rightarrow[0, c]$ and $S \subseteq[k]$, let $Y=\sum_{j \in S} \sum_{a \in E_{i}^{*}} \phi(a, j) \cdot \omega(a)$.

If there exists $a \in E_{i}$ such that $\sum_{j \in[k]} \phi(a, j)<b$ or $\sum_{j \in[k]} \phi(a, j)>c$, then $\chi\left(E_{i}, \phi, S\right)=\infty$.

Otherwise, the following recursion holds:

$$
\chi\left(E_{i}, \phi, S\right)=Y+\min _{\phi^{\prime}, S^{\prime}} \chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)
$$

where the minimum is taken over all $\phi^{\prime}: E_{i-1} \times[k] \rightarrow[0, c]$, and $S^{\prime} \subseteq[k]$ satisfying the following conditions:

- For all $a \in E_{i} \cap E_{i-1}$ and all $j \in[k], \phi^{\prime}(a, j)=\phi(a, j)$;
- For all $j \in[k]$,

$$
\begin{aligned}
& \sum_{a \in A^{+}\left(v_{i}\right) \cap E_{i-1}} \phi^{\prime}(a, j)+\sum_{a \in A^{+}\left(v_{i}\right) \cap E_{i}} \phi(a, j) \\
= & \sum_{a \in A^{-}\left(v_{i}\right) \cap E_{i-1}} \phi^{\prime}(a, j)+\sum_{a \in A^{-}\left(v_{i}\right) \cap E_{i}} \phi(a, j) .
\end{aligned}
$$

$$
-S=S^{\prime} \cup\left\{j \in[k]: \sum_{a \in E_{i}^{*}} \phi(a, j)>0\right\} .
$$

If there are no $\phi^{\prime}, S^{\prime}$ satisfying these conditions, then $\chi\left(E_{i}, \phi, S\right)=\infty$.
Furthermore, if there exist $\phi^{\prime}, S^{\prime}$ satisfying the above conditions and we are given a witness $\left(A_{1}^{\prime}, \ldots, A_{k}^{\prime}\right)$ for $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right) \leq \rho^{\prime}$, then we can construct a witness for $\chi\left(E_{i}, \phi, S\right) \leq Y+\rho^{\prime}$ in polynomial time.

We are now ready to prove Theorem 3 .
Theorem 3 Let $(G, k)$ be an instance of $k[b, c]-D W C P$ and suppose we are given a vertex ordering $\nu=\left(v_{1}, v_{2}, \ldots, v_{n}\right)$ of $G$ with cutwidth at most $p$. Then $(G, k)$ can be solved in time $O^{*}\left(\left(c 2^{k}\right)^{p} 4^{k}\right)$.

Proof. Our DP algorithm calculates all values $\chi\left(E_{i}, \phi, S\right)$ with $\phi(\cdot, j) \leq 1$ for $j>$ 1 in a bottom-up manner, that is, we only calculate values $\chi\left(E_{i}, \cdot, \cdot\right)$ after all values $\chi\left(E_{j}, \cdot, \cdot\right)$ have been calculated for $0 \leq j<i$ (we use the recursion of Lemma9).

Each arc bag $E_{i}$ of $\nu$ contains at most $p$ arcs. For each arc $a$, there are $c+1$ options for $\phi(a, 1)$ and 2 options for $\phi(a, j)$ for each $j>1$, i.e., $(c+1) 2^{k-1} \leq c 2^{k}$ options per arc. Thus there are at most $\left(c 2^{k}\right)^{p}$ valid choices for $\phi: E_{i} \times[k] \rightarrow[0, c]$. As there are $2^{k}$ choices for a set $S \subseteq[k]$, the total size of each DP table is $O\left(\left(c 2^{k}\right)^{p} 2^{k}\right)$.

Since $E_{0}=\emptyset$, the only function $\phi: E_{0} \times[k] \rightarrow[0, c]$ is the empty function. It is easy to see that $\chi\left(E_{0}, \phi, S\right)=0$ if $S=\emptyset$, and $\infty$ otherwise. To speed up the application of Lemma 9 for $E_{i}, 1 \leq i \leq n$, we form an intermediate table $T$ from the data for bag $E_{i-1}$. Call two entries $\chi\left(E_{i}, \phi, S\right)$ and $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$ compatible when the conditions in Lemma 9 are met (i.e., $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$ is one of the entries included in the minimisation for $\chi\left(E_{i}, \phi, S\right)$ ). Let the signature of entry $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$ be $\left(\phi^{\prime \prime}, d_{1}, \ldots, d_{k}, S^{\prime}\right)$, where $\phi^{\prime \prime}$ is $\phi^{\prime}$ restricted to arcs $E_{i-1} \cap E_{i}$, and where $d_{j}=\sum_{a \in A^{+}\left(v_{i}\right) \cap E_{i-1}} \phi^{\prime}(a, j)-\sum_{a \in A^{-}\left(v_{i}\right) \cap E_{i-1}} \phi^{\prime}(a, j)$ is the imbalance at $v_{i}$ in walk number $j$. Observe that whether an entry $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$ is compatible with the entry $\chi\left(E_{i}, \phi, S\right)$ can be determined from the signature alone. Thus, for every signature $\left(\phi^{\prime \prime}, d_{1}, \ldots, d_{k}, S^{\prime}\right)$ we let $T\left(\phi^{\prime \prime}, d_{1}, \ldots, d_{k}, S^{\prime}\right)$ contain the minimum value over all entries $\chi\left(E_{i-1}, \ldots\right)$ with matching signature; this can be computed in a single loop over the entries $\chi\left(E_{i-1}, \ldots\right)$. Then, for every entry $\chi\left(E_{i}, \phi, S\right)$ of the new table, we look in $T$ through all signatures that would be compatible with $(\phi, S)$ and keep the minimum value (and add $Y$ to it, by Lemma 9). The reason we may have to look at several signatures is the set $S$; for simplicity, we may simply loop over all sets $S^{\prime} \subseteq S$ such that $S^{\prime} \cup\left\{j \in[k]: \phi(a, j)>0\right.$, some $\left.a \in E_{i}\right\}=S$. Note that the size of the intermediate table $T$ is immaterial; the time taken consists of first one loop through $\chi\left(E_{i-1}, \ldots\right)$, then $2^{k}$ queries to $T$ for each entry in $\chi\left(E_{i}, \ldots\right)$. Thus, the entries $\chi\left(E_{i}, \ldots\right)$ can all be computed in total time $O^{*}\left(\left(c 2^{k}\right)^{p} 4^{k}\right)$. As $E_{n}=\emptyset$ there is only one function $\phi: E_{n} \times[k] \rightarrow[b, c]$. By Lemma $8, \chi\left(E_{n}, \phi,[k]\right)$ is the minimum total weight of a solution for $k[b, c]$-DCPP, and $\infty$ if there is no such solution. Thus to solve $k[b, c]$-DCPP it suffices to check the value of $\chi\left(E_{n}, \phi,[k]\right)$.

Thus the algorithm finds the value $\rho$ in time $O^{*}\left(\left(c 2^{k}\right)^{p} 4^{k}\right)$.
The algorithm can easily be made constructive using the method of Lemma 9 For each arc bag $E_{i}, \phi: E_{i} \times[k] \rightarrow[0, c], S \subseteq[k]$, in addition to calculating the value $\chi\left(E_{i}, \phi, S\right)=\rho$, we also calculate a witness for $\chi\left(E_{i}, \phi, S\right) \leq \rho$, in the cases where $\rho \neq \infty$. Just as we can calculate the values of all $\chi\left(E_{i}, \cdot, \cdot\right)$ given the values of all $\chi\left(E_{i-1}, \cdot, \cdot\right)$, we may construct witnesses for all $\chi\left(E_{i}, \cdot, \cdot\right)$ given witnesses for all $\chi\left(E_{i-1}, \cdot, \cdot\right)$, using an intermediate table $T$ as before. (Note that $\left(A_{1}, \ldots, A_{k}\right)$, where each $A_{i}=\emptyset$, is a witness for $\chi\left(E_{0}, \phi, \emptyset\right)=0$, where $\phi$ is the empty function. This gives us the base case in our construction of witnesses.) Given a witness for $\chi\left(E_{n}, \phi,[k]\right)$, Lemma 8 shows how to construct a solution to $k[b, c]$-DCPP on $G$ from this witness.

## 4 Proofs of Theorems 1 and 2

## Theorem 2 The $k$-ADCP-EULER is fixed-parameter tractable.

Proof. Let $D$ be an Euler digraph. We may assume that $D$ has no vertex of out-degree at least $k$ as otherwise we are done by Lemma 1 By Lemma 3 for $D$ we can either obtain $k$ arc-disjoint cycles or a vertex ordering $\nu$ of cutwidth at most $2 g(k)$ for some function $g: \mathbb{N} \rightarrow \mathbb{N}$. Note that $D$ is a positive instance of the $k$-ADCP-EULER if and only if ( $D, k$ ) has a finite solution for $k[0,1]$-DWCP (as every closed walk contains a cycle). It remains to observe that the algorithm of Theorem 3 for the $k[0,1]$-DWCP is fixed-parameter when the out-degree of every vertex of $D$ is upper-bounded by $k$ and the cutwidth of $\nu$ is bounded by a function of $k$.

## Theorem 1 The $k$-DCPP admits a fixed-parameter algorithm.

Proof. Let $G=(V, A)$ be a digraph and let $T$ be an optimal solution of DCPP on $G$. If we get a collection $\mathcal{C}$ of $k$ arc-disjoint cycles in $G_{T}$, then using $\mathcal{C}$, by Lemma [5, we can solve the $k$-DCPP on $G$ in (additional) polynomial time. Otherwise, by lemma 3 we have a vertex ordering of $G_{T}$ of cutwidth bounded by a function of $k$. We may assume that every vertex of $G_{T}$ is of out-degree at most $k-1$ (otherwise by Lemma $1 G_{T}$ has a collection of $k$ arc-disjoint cycles). Since every vertex of $G_{T}$ is of out-degree at most $k-1$, the multiplicity of $G_{T}$ is at most $k-1$. Now Lemma 6 implies that there is an optimal solution $W$ for the $k$-DCPP on $G$ such that the multiplicity of $G_{W}$ is at most $k$. Thus, we may treat the $k$-DCPP on $G$ as an instance $(G, k)$ of $k[1, k]$-DWCP. It remains to observe that the algorithm of Theorem 3 to solve the $k[1, k]$-DWCP on $G$ will be fixed-parameter.

## 5 Discussions

Our algorithms for solving both $k$-DCPP and $k$-ADCP on Euler digraphs have very large running time bounds, mainly because the bound $f\left(h^{-1}(k)\right)$ on the size of feedback arc set is very large. Function $f(k)$ obtained in [17] is a multiply iterated exponential, where the number of iterations is also a multiply iterated exponential and, as a result, $h^{-1}(k)$ grows very quickly. So obtaining a significantly smaller upper bound for $f(k)$ on Euler digraphs would significantly reduce $h^{-1}(k)$ as well and is of certain interest in itself. In particular, is it true that $f(k)=O\left(k^{O(1)}\right)$ for Euler digraphs? Note that for planar digraphs, $f(k)=k$ [1] Corollary 15.3.10] and Seymour [18] proved the same result for a wide family of Euler digraphs. It would also be interesting to check whether the $k$-DCPP or $k$-ADCP admits a polynomial-size kernel.

Cechlárová and Schlotter [3] introduced the following somewhat related problem in the context of housing markets: can we delete at most $k$ arcs in a given digraph such that each strongly connected component of the resulting digraph is Euler? They asked for the parameterized complexity of this problem, where $k$ is the parameter. Crowston et al. [5] showed that the problem restricted to tournaments is fixed-parameter tractable, but in general the complexity still remains an open question. See also the recent paper [6] for other related problems.

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## References

1. J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, 2nd Ed., Springer, 2009.
2. R. van Bevern, R. Niedermeier, M. Sorge, and M. Weller, Complexity of Arc Rooting Problems. Chapter 2 in A. Corberán and G. Laporte (eds.), Arc Routing: Problems, Methods and Applications, SIAM, Phil., in press.
3. K. Cechlárová and I. Schlotter, Computing the deficiency of housing markets with duplicate houses, Proc. IPEC 2010, Lect. Notes Comput. Sci. 6478 (2010) 72-84.
4. J. Chen, Y. Liu, S. Lu, B. O'Sullivan and I. Razgon, A fixed-parameter algorithm for the directed feedback vertex set problem, J. ACM 55(5) (2008) 1-19.
5. R. Crowston, G. Gutin, M. Jones, and A. Yeo, Parameterized eulerian strong component arc deletion problem on tournaments. Inform. Proc. Lett. 112 (2012) 249-251.
6. M. Cygan, D. Marx, M. Pilipczuk, M. Pilipczuk, and I. Schlotter, Parameterized Complexity of Eulerian Deletion Problems, Algorithmica 68 (2014) 41-61.
7. R. G. Downey and M. R. Fellows, Parameterized Complexity, Springer, 1999.
8. F. Dorn, H. Moser, R. Niedermeier, and M. Weller, Efficient algorithms for Eulerian extension. SIAM J. Discrete Math. 27(1):75-94, 2013.
9. J. Edmonds and E. L. Johnson, Matching, Euler tours and the Chinese postman. Mathematical Programming 5 (1973) 88-124.
10. J. Flum and M. Grohe, Parameterized Complexity Theory, Springer, 2006.
11. M. Grohe and M. Grüber, Parameterized Approximability of the Disjoint Cycle Problem, in ICALP 2007, Lect. Notes Comput. Sci. 4596 (2007) 363-374.
12. G. Gutin, G. Muciaccia and A. Yeo, Parameterized Complexity of $k$-Chinese Postman Problem. Theor. Comput. Sci. 513 (2013) 124-128.
13. Y. Lin and Y. Zhao, A new algorithm for the directed Chinese postman problem. Comput. \& Oper. Res. 15(6) (1988) 577-584.
14. C.L. Lucchesi, A minimax equality for directed graphs. PhD thesis, Univ. Waterloo, Ontario, Canada, 1976.
15. R. Niedermeier. Invitation to Fixed-Parameter Algorithms. Oxford University Press, 2006.
16. W.L. Pearn, Solvable cases of the $k$-person Chinese postman problem. Oper. Res. Lett. 16(4) (1994) 241-244.
17. B. Reed, N. Robertson, P.D. Seymour and R. Thomas, Packing directed circuits. Combinatorica 16(4) 1996 535-554.
18. P.D. Seymour, Packing circuits in Eulerian digraphs. Combinatorica 16(2) (1996) 223-231.
19. A. Slivkins, Parameterized tractability of edge-disjoint paths on directed acyclic graphs. In ESA 2003, Lect. Notes Comput. Sci. 2832 (2003) 482-493.
20. M. Sorge, Some Algorithmic Challenges in Arc Routing, talk at NII Shonan Seminar no. 18, May 2013.
21. M. Sorge, R. van Bevern, R. Niedermeier and M. Weller, From Few Components to an Eulerian Graph by Adding Arcs, in WG'2011, Lect. Notes Comput. Sci. 6986 (2011) 307-319.
22. M. Sorge, R. van Bevern, R. Niedermeier and M. Weller, A new view on Rural Postman based on Eulerian Extension and Matching, J. Discrete Alg., 16 (2012) 12-33.
23. C. Thomassen, On the complexity of finding a minimum cycle cover of a graph, SIAM J. Comput. 26 (3) (1997) 675-677.
24. L. Zhang, Polynomial Algorithms for the $k$-Chinese Postman Problem, in Information Processing '92, vol. 1 (1992) 430-435.

## Appendix: Proof of Lemma 9

Lemma 9 Consider an arc bag $E_{i}$, for $i \geq 1$. Let $E_{i}^{*}=E_{i} \backslash E_{i-1}$. For any $\phi$ : $E_{i} \times[k] \rightarrow[0, c]$ and $S \subseteq[k]$, let $Y=\sum_{j \in S} \sum_{a \in E_{i}^{*}} \phi(a, j) \cdot \omega(a)$.

If there exists $a \in E_{i}$ such that $\sum_{j \in[k]} \phi(a, j)<b$ or $\sum_{j \in[k]} \phi(a, j)>c$, then $\chi\left(E_{i}, \phi, S\right)=\infty$.

Otherwise, the following recursion holds:

$$
\chi\left(E_{i}, \phi, S\right)=Y+\min _{\phi^{\prime}, S^{\prime}} \chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)
$$

where the minimum is taken over all $\phi^{\prime}: E_{i-1} \times[k] \rightarrow[0, c]$, and $S^{\prime} \subseteq[k]$ satisfying the following conditions:

- For all $a \in E_{i} \cap E_{i-1}$ and all $j \in[k], \phi^{\prime}(a, j)=\phi(a, j)$;
- For all $j \in[k]$,

$$
\begin{aligned}
& \sum_{a \in A^{+}\left(v_{i}\right) \cap E_{i-1}} \phi^{\prime}(a, j)+\sum_{a \in A^{+}\left(v_{i}\right) \cap E_{i}} \phi(a, j) \\
= & \sum_{a \in A^{-}\left(v_{i}\right) \cap E_{i-1}} \phi^{\prime}(a, j)+\sum_{a \in A^{-}\left(v_{i}\right) \cap E_{i}} \phi(a, j) .
\end{aligned}
$$

$-S=S^{\prime} \cup\left\{j \in[k]: \sum_{a \in E_{i}^{*}} \phi(a, j)>0\right\}$.
If there are no $\phi^{\prime}, S^{\prime}$ satisfying these conditions, then $\chi\left(E_{i}, \phi, S\right)=\infty$.
Furthermore, if there exist $\phi^{\prime}, S^{\prime}$ satisfying the above conditions and we are given a witness $\left(A_{1}^{\prime}, \ldots, A_{k}^{\prime}\right)$ for $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right) \leq \rho^{\prime}$, then we can construct a witness for $\chi\left(E_{i}, \phi, S\right) \leq Y+\rho^{\prime}$ in polynomial time.
Proof We will prove the last claim of the lemma first. Suppose we are given a witness $\left(A_{1}^{\prime}, \ldots, A_{k}^{\prime}\right)$ for $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right) \leq \rho^{\prime}$. For each $j \in[k]$, let $A_{j}$ be the multiset $A_{j}^{\prime}$ together with $\phi(a, j)$ copies of each arc in $E_{i}^{*}$. We now show that $\left(A_{1}, \ldots, A_{k}\right)$ is a witness for $\chi\left(E_{i}, \phi, S\right)=Y+\rho^{\prime}$.

By construction of $A_{j}$, definition of $A_{j}^{\prime}$ and the fact that $\phi^{\prime}(a, j)=\phi(a, j)$ for all $a \in E_{i} \cap E_{i-1}, j \in[k]$, we have that for all $a \in E_{i}$ and $j \in[k], A_{j}$ contains exactly $\phi(a, j)$ copies of $a$, satisfying Condition 1 of $\chi\left(E_{i}, \phi, S\right) \leq Y+\rho^{\prime}$.

By definition of $A_{j}^{\prime}$ and the fact that $b \leq \sum_{j \in[k]} \phi(a, j) \leq c$ for each $a \in E_{i}^{*}$, we have that every arc appears at least $b$ times and at most $c$ times in $A_{1} \cup \cdots \cup A_{k}$, satisfying Condition 2

Observe that $E_{i}^{*}$ consists of all arcs of the form $v_{i} v_{h}$ or $v_{h} v_{i}$ for $h>i$. It follows by construction that for any $h<i$ and $j \in[k], A^{+}\left(v_{h}, A_{j}\right)=A^{+}\left(v_{h}, A_{j}^{\prime}\right)$ and $A^{-}\left(v_{h}, A_{j}\right)=A^{-}\left(v_{h}, A_{j}^{\prime}\right)$. Then as $\left|A^{+}\left(v_{h}, A_{j}^{\prime}\right)\right|=\left|A^{-}\left(v_{h}, A_{j}^{\prime}\right)\right|$ for all $h<i$, we have that $\left|A^{+}\left(v_{h}, A_{j}\right)\right|=\left|A^{-}\left(v_{h}, A_{j}\right)\right|$ for all $h<i$. As every arc incident with $v_{i}$ is in exactly one of $E_{i-1}$ or $E_{i}$, we have that for all $j \in$ $[k],\left|A^{+}\left(v_{i}, A_{j}\right)\right|=\sum_{a \in A^{+}\left(v_{i}\right) \cap E_{i-1}} \phi^{\prime}(a, j)+\sum_{a \in A^{+}\left(v_{i}\right) \cap E_{i}} \phi(a, j)$, and similarly $\left|A^{-}\left(v_{i}, A_{j}\right)\right|=\sum_{a \in A^{-}\left(v_{i}\right) \cap E_{i-1}} \phi^{\prime}(a, j)+\sum_{a \in A^{-}\left(v_{i}\right) \cap E_{i}} \phi(a, j)$. It follows by the second condition of the lemma that $\left|A^{+}\left(v_{i}, A_{j}\right)\right|=\left|A^{-}\left(v_{i}, A_{j}\right)\right|$. Therefore $\left|A^{+}\left(v_{h}, A_{j}\right)\right|=\left|A^{-}\left(v_{h}, A_{j}\right)\right|$ for all $h \leq i$, satisfying Condition3,

By the fact that $S=S^{\prime} \cup\left\{j \in[k]: \sum_{a \in E_{i} \backslash E_{i-1}} \phi(a, j)>0\right\}$, definition of $\left(A_{1}^{\prime}, \ldots, A_{k}^{\prime}\right)$ and construction of $\left(A_{1}, \ldots, A_{k}\right)$, we have that $S=S^{\prime} \cup\{j \in[k]$ : $\left.A_{j} \backslash A_{j}^{\prime} \neq \emptyset\right\}=\left\{j \in[k]: A_{j} \neq \emptyset\right\}$. This satisfies Condition4

Finally, by construction of $\left\{A_{1}, \ldots, A_{k}\right\}$ and $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$, we have that $\chi\left(E_{i}, \phi, S\right)=\sum_{j \in[k]} \sum_{a \in A_{j}} \omega(a)=Y+\sum_{j \in[k]} \sum_{a \in A_{j}^{\prime}} \omega(a)=Y+$ $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$, satisfying Condition 5 ,

Thus, we have that $\left(A_{1}, \ldots, A_{k}\right)$ is a witness for $\chi\left(E_{i}, \phi, S\right) \leq Y+\rho^{\prime}$.
We now prove the other claims of the lemma. If there exists $a \in E_{i}$ such that $\sum_{j \in[k]} \phi(a, j)<b$ or $\sum_{j \in[k]} \phi(a, j)>c$, then any arc multisets $A_{1}, \ldots, A_{k}$ that satisfy Condition 1 of $\chi\left(E_{i}, \phi, S\right)$ will falsify Condition2, and so $\chi\left(E_{i}, \phi, S\right)=\infty$. So now assume that $b \leq \sum_{j \in[k]} \phi(a, j) \leq c$ for every $a \in E_{i}$.

Let $\phi^{\prime}: E_{i-1} \times[k] \rightarrow[b, c], S^{\prime} \subseteq[k]$ be such that the conditions of the lemma are satisfied and $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$ is minimised. If $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)=\infty$ then trivially $\chi\left(E_{i}, \phi, S\right) \leq Y+\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$. Otherwise, $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)=\rho^{\prime} \neq \infty$ and so there exists a witness for $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right) \leq \rho^{\prime}$ Then by the argument above, there exists a witness for $\chi\left(E_{i}, \phi, S\right) \leq Y+\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$. In either case $\chi\left(E_{i}, \phi, S\right) \leq$ $Y+\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$.

It remains to show that if $\chi\left(E_{i}, \phi, S\right) \neq \infty$, then there exist $\phi^{\prime}, S^{\prime}$ such that $\chi\left(E_{i}, \phi, S\right)=Y+\chi\left(E_{i-1} \phi^{\prime}, S^{\prime}\right)$.

Suppose that $\chi\left(E_{i}, \phi, S\right)=\rho \neq \infty$. Let $\left(A_{1}, \ldots, A_{k}\right)$ be a witness for $\chi\left(E_{i}, \phi, S\right)=\rho$. Then for each $j \in[k]$, let $A_{j}^{\prime}$ be the multiset of arcs from $A_{j}$ not incident to $v_{i}$ and let $A_{j}^{*}$ be the multiset of arcs from $A_{j}$ incident to $v_{i}$. For a multiset $M$ of arcs from $G$, let $\omega(M)=\sum_{a \in M} \omega(a)$, where each arc $a$ is taken in the sum as many times as it has copies in $M$. Observe that $Y=\sum_{j \in[k]} \omega\left(A_{j}^{*}\right)$. Let $Z=\sum_{j \in[k]} \omega\left(A_{j}^{\prime}\right)$; then $\chi\left(E_{i}, \phi, S\right)=Y+Z$.

Let $\phi^{\prime}: E_{i-1} \times[k] \rightarrow[b, c]$ be the function such that $\phi^{\prime}(a, j)$ is the number of copies of $a$ in $A_{j}^{\prime}$, for each $a \in E_{i-1}, j \in[k]$. Finally let $S^{\prime}=\left\{j \in[k]: A_{j}^{\prime} \neq \emptyset\right\}$.

As $\gamma(i) \backslash \gamma(i-1)$ contains no arcs incident to $v_{h}$ for any $h<i$, we have that for any $h<i,\left|A(v) \cap A_{j}^{\prime}\right|=\left|A(v) \cap A_{j}\right|$ for each $j \in[k]$. Therefore $\left(A_{1}^{\prime}, \ldots, A_{k}^{\prime}\right)$ satisfies Conditon 3 of a witness for $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)=Z$. It is easy to see that $\left(A_{1}^{\prime}, \ldots, A_{k}^{\prime}\right)$ satisfies the other conditions for a witness for $\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)=Z$, from which it follows that $\chi\left(E_{i}, \phi, S\right)=Y+\chi\left(E_{i-1}, \phi^{\prime}, S^{\prime}\right)$.


[^0]:    ${ }^{1}$ For terminology and results on parameterized algorithms and complexity we refer the reader to the monographs [7|10|15].

[^1]:    ${ }^{2}$ Here, as in the proof, the union of multigraphs means that the multiplicity of an arc in the union equals the sum of multiplicities of this arc in the multigraphs of the union.

