

The generalized 3-edge-connectivity of lexicographic product graphs*

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Abstract

The generalized k -edge-connectivity $\lambda_k(G)$ of a graph G is a generalization of the concept of edge-connectivity. The lexicographic product of two graphs G and H , denoted by $G \circ H$, is an important graph product. In this paper, we mainly study the generalized 3-edge-connectivity of $G \circ H$, and get upper and lower bounds of $\lambda_3(G \circ H)$. Moreover, all bounds are sharp.

Keywords: edge-disjoint paths, edge-connectivity, Steiner tree, edge-disjoint Steiner trees, packing, generalized edge-connectivity,

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1 Introduction

All graphs considered in this paper are simple, finite and undirected. We follow the terminology and notation of Bondy and Murty [3]. For a graph G , the local edge-connectivity between distinct vertices u and v , denoted by $\lambda(u, v)$, is the maximum number of pairwise edge disjoint uv -paths. A nontrivial graph G is k -edge-connected if $\lambda(u, v) \geq k$ for any two distinct vertices u and v of G . The edge-connectivity $\lambda(G)$ of a graph G is the maximum value of k for which G is k -edge-connected, see [19].

Naturally, the concept of edge-connectivity can be extended to a new concept, the generalized k -edge-connectivity, which was introduced by Li et al. [17]. For a graph $G = (V, E)$ and a set $S \subseteq V(G)$ of at least two vertices, an S -Steiner tree or a Steiner tree connecting S (or simply, an S -tree) is a such subgraph $T = (V', E')$ of G that is a tree with $S \subseteq V'$. Let $\lambda(S)$ denote the maximum number of pairwise edge-disjoint Steiner

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trees T_1, T_2, \dots, T_ℓ in G such that if $E(T_i) \cap E(T_j) = \emptyset$ and $S \subseteq V(T_i) \cap V(T_j)$ for any pair of distinct integers i, j with $1 \leq i, j \leq \ell$. Then the *generalized k -edge-connectivity* $\lambda_k(G)$ of G is defined as $\lambda_k(G) = \min\{\lambda(S) \mid S \subseteq V(G), |S| = k\}$. Obviously, $\lambda_2(G) = \lambda(G)$. Set $\lambda_k(G) = 0$ if G is disconnected. Some results of the edge-connectivity and the generalized edge-connectivity can refer to [17, 19, 24] for details.

The generalized edge-connectivity is closely linked to an important problem *Steiner tree packing problem*, which asks for finding a set of maximum number of edge-disjoint S -trees in a given graph G where $S \subseteq V(G)$, see [12, 26, 7]. An extreme of Steiner tree packing problem is called the *Spanning tree packing problem* where $S = V(G)$. For any graph G , the *spanning tree packing number* or *STP number*, is the maximum number of edge-disjoint spanning trees contained in G . For the *STP number*, we refer to [20, 21, 1, 6, 10, 22, 25]. The difference between the Steiner tree packing problem and the generalized edge-connectivity is as follows: the former problem studies local properties of graphs since S is given beforehand, while the latter problem focuses on global properties of graphs since it first needs to compute the maximum number $\lambda(S)$ of S -trees and then S runs over all k -subsets of $V(G)$ to get the minimum value of $\lambda(S)$.

From a theoretical perspective, both extremes of the generalized edge-connectivity problem are fundamental theorems in combinatorics. One extreme is when we have two terminals. In this case edge-disjoint trees are just edge-disjoint paths between the two terminals, and so the problem becomes the well-known Menger theorem. The other extreme is when all the vertices are terminals. In this case edge-disjoint trees are just spanning trees of the graph, and so the problem becomes the classical Nash-Williams-Tutte theorem, see [18, 23].

Product graphs are an important method to construct large graphs from small ones, so it has many applications in the design and analysis of networks, see [7, 13, 14]. The lexicographic product, together with the Cartesian product, the strong product and the direct product, is the main four standard products of graphs. More information about the (edge) connectivity of these four product graphs can be found in [15, 9, 8, 11, 4, 5, 27].

In this paper, we study the generalized edge-connectivity of the lexicographic product graph and get the following theorems.

Theorem 1. *Let G and H be two non-trivial graphs and G is connected. Then $\lambda_3(G \circ H) \geq \lambda_3(H) + \lambda_3(G)|V(H)|$. Moreover, the lower bound is sharp.*

Theorem 2. *Let G and H be two non-trivial graphs and G is connected. Then*

$$\lambda_3(G \circ H) \leq \min \left\{ \left\lfloor \frac{4\lambda_3(G) + 2}{3} \right\rfloor |V(H)|^2, \delta(H) + \delta(G)|V(H)| \right\}.$$

Moreover, the upper bound is sharp.

2 Preliminaries

Let G be a graph and $S \subseteq V(G)$. Let $G[S]$ denote the induced subgraph of G on the vertex set S and let $d_G(v)$ denote the degree of v in G , where $v \in V(G)$. If u and v are two vertices on a path P , uPv will denote the segment of P from u to v . Given sets X, Y of vertices, we call a path P an XY -path if the end vertices of P are in X and Y , respectively, and all inner vertices are in neither X nor Y . Two distinct paths are *edge disjoint* if they have no edges in common. Two distinct paths are *internally disjoint* if they have no internal vertices in common. Two distinct paths are *vertex disjoint* if they have no vertices in common. For $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$, an XY -linkage is defined as a set Q of k vertex-disjoint paths $x_iP_iy_i$, $1 \leq i \leq k$.

Let $G = (V_1, E_1)$, $H = (V_2, E_2)$, the lexicographic product $G \circ H$ of G and H is defined as follows: $V(G \circ H) = V_1 \times V_2$, two vertices (u, v) and (u', v') are adjacent if and only if either $uu' \in E_1$ or $u = u'$, $vv' \in E_2$. On other words, $G \circ H$ is obtained by substituting a copy $H(u)$ of H for every vertex u of G and joining all vertices of $H(u)$ with all vertices of $H(u')$ if $uu' \in E_1$. Unlike the other product, the lexicographic product does not satisfy the commutative law, that is, $G \circ H$ need not be isomorphic to $H \circ G$. By a simple observation, $G \circ H$ is connected if and only if G is connected. Moreover, $\delta(G \circ H) = \delta(G)|V(H)| + \delta(H)$. The edge $(u, v)(u', v')$ is called one-type edge if $uu' \in E_1$ and $v = v'$; two-type edge if $vv' \in E_2$ and $u = u'$; three-type edge if $uu' \in E_1$ and $v \neq v'$.

The vertex set $G(v) = \{(u, v) | u \in V_1\}$ for some fixed vertex v of H is called a layer of graph G or simply a G -layer. Analogously we define the H -layer with respect to a vertex u of G and denote it by $H(u)$. It is not hard to see that any G -layer induces a subgraph of $G \circ H$ that is isomorphic to G and any H -layer induces a subgraph of $G \circ H$ that is isomorphic to H . For a subset W of $V(G)$ with $W = \{u_1, \dots, u_t\}$, $H(W) = H(u_1) \cup \dots \cup H(u_t)$. K_{u_1, \dots, u_t} denotes a subgraph of $G \circ H$, where $V(K_{u_1, \dots, u_t}) = V(W \circ H)$, $E(K_{u_1, \dots, u_t}) = E(G[u_1, \dots, u_t] \circ H) \setminus E(H(W))$, namely, the end vertices of an edge of K_{u_1, \dots, u_t} are in different H -layers.

Let G be a connected graph, $S = \{x, y, z\} \subseteq V(G)$, and T be an S -tree. By deleting some vertices and edges of T , it is easy to check that T has exactly two types, one is called type I if T is just a path whose two end vertices belong to $S = \{x, y, z\}$; the other is called type II if it is a tree with exactly three leaves x, y, z . Note that the vertices in a tree of type I have degree two except the two end vertices in S . If T is of type II , every vertex in $T \setminus S$ has degree two except one vertex of degree three. In this paper, we assume that each S -tree is of type I or II .

Proposition 1. Let G be a graph with $\lambda_3(G) = k \geq 2$, $S = \{x, y, z\} \subseteq V(G)$. Then there exist $k - 2$ edge-disjoint S -trees T_1, \dots, T_k such that $E(T_i) \cap E(G[S]) = \emptyset$.

Proof. By the definition of S -trees, we know that $|E(T_i) \cap E(G[S])| \leq 2$ and $|\{T_i | E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 3$. Let $\{T_1, \dots, T_k\}$ be k edge-disjoint S -trees. If $|\{T_i | E(T_i) \cap E(G[S]) \neq \emptyset\}| \leq 3$, then the result follows.

$\emptyset\} \leq 2$, we are done. Thus, suppose $|\{T_i \mid E(T_i) \cap E(G[S]) \neq \emptyset\}| = 3$. Without loss of generality, assume $E(T_i) \cap E(G[S]) \neq \emptyset$, where $i = 1, 2, 3$. Then T_1, T_2, T_3 have the structures F_1 or F_2 as shown in Figure 1. For these two cases, we can obtain T'_1, T'_2, T'_3 from T_1, T_2, T_3 such that $E(T'_i) \cap E(G[S]) = \emptyset$. See Figure F'_1 and F'_2 , where the tree T'_1 is shown by gray lines. Thus T'_1, T_4, \dots, T_k are our desired S -trees. \square

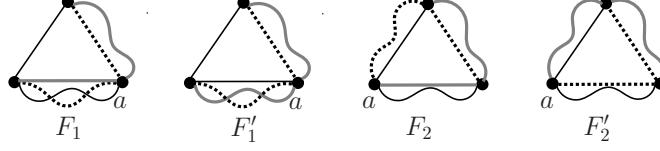


Figure 1. Three S -trees of type I .

Li et al. [17, 16] got the following results which are useful for our proof.

Proposition 2. [17] For any graph G of order n , $\lambda_k(G) \leq \lambda(G)$. Moreover, the upper bound is tight.

Observation 1. [17] If G be a connected graph, then $\lambda_k(G) \leq \delta(G)$. Moreover, the upper bound is tight.

Proposition 3. [16] Let G be a connected graph of order n with minimum degree δ . If there are two adjacent vertices of degree δ , then $\lambda_k(G) \leq \delta - 1$ for $3 \leq k \leq n$. Moreover, the upper bound is sharp.

From Proposition 3, it is easy to get the following observation.

Observation 2. Let G be a connected graph with $\lambda_3(G) = k$, x and y be two adjacent vertices of G . Then $d_G(x) \geq k + 1$ or $d_G(y) \geq k + 1$.

Before getting into our main results, we give an elementary observation.

Observation 3. (i) Let G and H be two non-trivial graphs and G is connected, let x, y, z be three distinct vertices of H and T_1, T_2, \dots, T_k be k edge-disjoint $\{x, y, z\}$ -trees in H . Then $G \circ \bigcup_{i=1}^k T_i = \bigcup_{i=1}^k (G \circ T_i)$. Moreover if $V(T_i) \cap V(T_j) = W$ for $i \neq j$, then $E(G \circ T_i) \cap E(G \circ T_j) = E(G \circ W) \setminus E(W(G))$.

(ii) Let G and H be two non-trivial graphs and G is connected, let x, y, z be three distinct vertices of G and T_1, T_2, \dots, T_k be k edge-disjoint $\{x, y, z\}$ -trees in G . Then $\bigcup_{i=1}^k T_i \circ H = \bigcup_{i=1}^k (T_i \circ H)$. Moreover if $V(T_i) \cap V(T_j) = W$ for $i \neq j$, $E(T_i \circ H) \cap E(T_j \circ H) = E(H(W))$.

For the above observation, we give two examples.

Example 1. Let G be a complete graph of order 4 and H be an arbitrary graph. The structure of $G \circ (T_1 \cup T_2)$ is shown as F_a in Figure 2.

Example 2. Let G be a path of length 2 and H be a complete graph of order 4. The structure of $(T_1 \cup T_2) \circ H$ is shown as F_b in Figure 2.

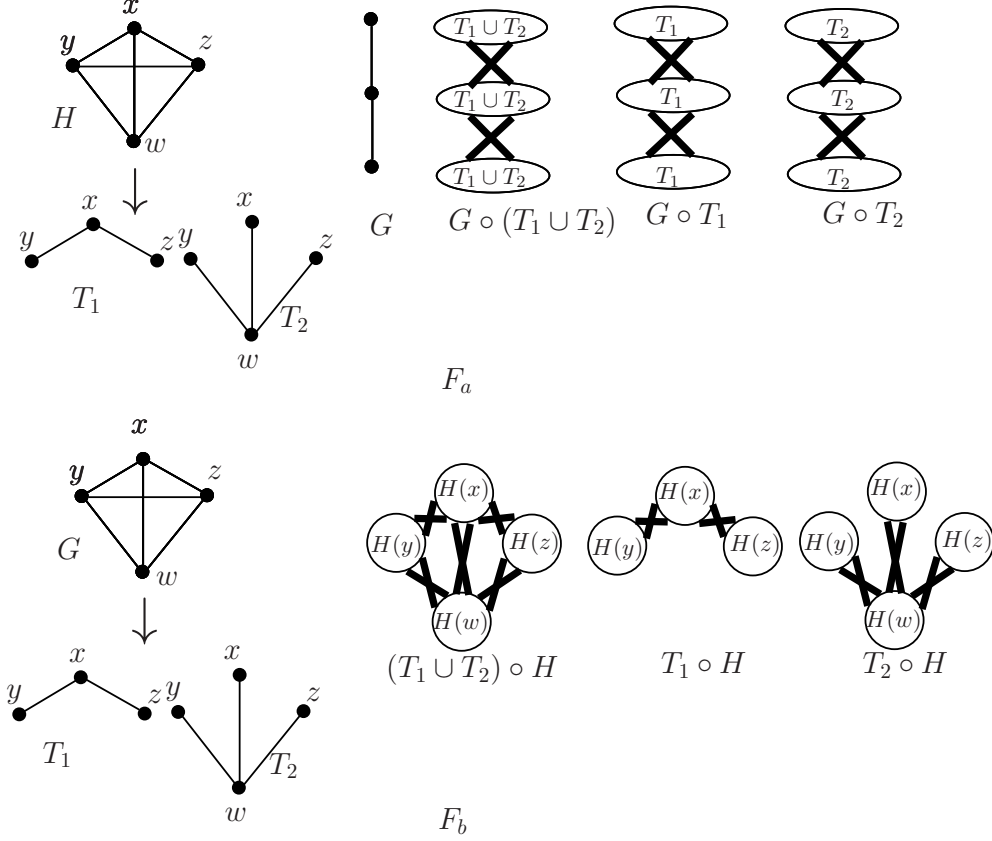


Figure 2. The structures of $G \circ (T_1 \cup T_2)$ and $(T_1 \cup T_2) \circ H$.

3 Lower bound of $\lambda_3(G \circ H)$

In this section, we give the lower bound of generalized 3-edge-connectivity of the lexicographic product of two graphs. Before proceeding, we give some notations and lemmas.

Set $V(G) = \{u_1, u_2, \dots, u_{n_1}\}$, $V(H) = \{v_1, v_2, \dots, v_{n_2}\}$ and set $\lambda_3(G) = \ell_1$, $\lambda_3(H) = \ell_2$ for simplicity. Let $S = \{x, y, z\} \subseteq V(G \circ H)$. In total, we construct our desired S -trees on two stages: ℓ_2 edge-disjoint S -trees by one-type and two-type edges on Stage *I* and $\ell_1 n_2$ edge-disjoint S -trees by one-type and three-type edges on Stage *II*. If H is disconnected, then $\lambda_3(H) = \ell_2 = 0$ as defined, thus we omit Stage *I* immediately. Next we always assume H is connected.

According to the position of x, y, z in $G \circ H$, we give some lemmas as follows.

Lemma 1. *If x, y, z belong to the same $H(u_i)$, $1 \leq i \leq n$, then there exist $\ell_2 + \ell_1 n_2$ edge-disjoint S -trees.*

Proof. Without loss of generality, assume $x, y, z \in H(u_1)$ and $x = (u_1, v_1)$, $y = (u_1, v_2)$, $z = (u_1, v_3)$. On Stage *I*, there are ℓ_2 edge-disjoint S -trees in $H(u_1)$, since $\lambda_3(H) = \ell_2$. On Stage *II*, since $\lambda_3(G) = \ell_1$ and Observation 1, there are ℓ_1 neighbors of u_1 in G , say $\beta_1, \beta_2, \dots, \beta_{\ell_1}$. Thus $T_{ij}^* = x(\beta_i, v_j) \cup y(\beta_i, v_j) \cup z(\beta_i, v_j)$ ($1 \leq i \leq \ell_1$ and $1 \leq j \leq n_2$) are $\ell_1 n_2$ S -trees. By Observation 3, it is easy to see that these $\ell_2 + \ell_1 n_2$ S -trees are edge-disjoint, as desired. \square

Lemma 2. *If only two of $\{x, y, z\}$ belong to the same $H(u_i)$, $1 \leq i \leq n$, then there exist $\ell_2 + \ell_1 n_2$ edge-disjoint S -trees.*

Proof. Suppose $x, y \in H(u_1)$, $z \in H(u_2)$. Let x'', y'' be the vertices in $H(u_2)$ corresponding to x, y , respectively, and z' be the vertex in $H(u_1)$ corresponding to z . Consider the following two cases.

Case 1. $z' \in \{x, y\}$.

Without loss of generality, assume $z' = x$ and $x = (u_1, v_1)$, $y = (u_1, v_2)$, $z = (u_2, v_1)$.

Since $\lambda(H) \geq \lambda_3(H) = \ell_2$, there are ℓ_2 edge-disjoint $v_1 v_2$ -paths $P_1, P_2, \dots, P_{\ell_2}$ in H such that $\ell(P_1) \leq \ell(P_2) \leq \dots \leq \ell(P_{\ell_2})$. For $1 \leq i \leq \ell_2$, denote the neighbor of v_1 in P_i by α_i . Notice that $\alpha_p \neq \alpha_q$ for $p \neq q$, $1 \leq p, q \leq \ell_2$.

Since $\lambda(G) \geq \lambda_3(G) = \ell_1$, there exist ℓ_1 edge-disjoint $u_1 u_2$ -paths $Q_1, Q_2, \dots, Q_{\ell_1}$ in G such that $\ell(Q_1) \leq \ell(Q_2) \leq \dots \leq \ell(Q_{\ell_1})$. For each i with $1 \leq i \leq \ell_1$, set $Q_i = u_1 \beta_{i,1} \beta_{i,2} \dots \beta_{i,t_i-1} u_2$ and $\ell(Q_i) = t_i$. Also, note that $\beta_{p,1} \neq \beta_{q,1}$ for $p \neq q$, $1 \leq p, q \leq \ell_1$.

Firstly, we come to Stage *I*. Choose the longest $u_1 u_2$ -path Q_{ℓ_1} and construct our desired ℓ_2 S -trees according to Q_{ℓ_1} . If v_1 and v_2 are not adjacent in H , then let $T_i^* = P_i(u_1) \cup Q_{\ell_1}(\alpha_i) \cup z(u_2, \alpha_i)$ for $1 \leq i \leq \ell_2$, where $P_i(u_1)$ is the path in $H(u_1)$ corresponding to P_i , $Q_{\ell_1}(\alpha_i)$ is the path in $G(\alpha_i)$ corresponding to Q_{ℓ_1} .

So suppose v_1 and v_2 are adjacent in H , that is, $P_1 = v_1 v_2$ and $(u_1, \alpha_1) = y$. Since $\lambda_3(H) = \ell_2$ and Observation 2, it follows that, $d_H(v_1) \geq \ell_2 + 1$ or $d_H(v_2) \geq \ell_2 + 1$. Let $d_H(v_1) \geq \ell_2 + 1$ (the case that $d_H(v_2) \geq \ell_2 + 1$ can be proved similarly). For P_1 , choose another neighbor α_{ℓ_2+1} of v_1 in H , which is not α_i and v_2 ($2 \leq i \leq \ell_2$). Let $T_1^* = xy \cup x(u_1, \alpha_{\ell_2+1}) \cup Q_{\ell_1}(\alpha_{\ell_2+1}) \cup z(u_2, \alpha_{\ell_2+1})$, where $Q_{\ell_1}(\alpha_{\ell_2+1})$ is the path in $G(\alpha_{\ell_2+1})$ corresponding to Q_{ℓ_1} . For P_i with $2 \leq i \leq \ell_2$, set $T_i^* = P_i(u_1) \cup Q_{\ell_1}(\alpha_i) \cup z(u_2, \alpha_i)$, where $P_i(u_1)$ is the path in $H(u_1)$ corresponding to P_i , $Q_{\ell_1}(\alpha_i)$ is the path in $G(\alpha_i)$ corresponding to Q_{ℓ_1} . Thus, by (i) of Observation 3, these ℓ_2 S -trees are edge-disjoint.

Note that, on Stage *I*, if v_1 and v_2 are adjacent in H , then we choose another neighbor of v_1 rather than v_2 (or another neighbor of v_2 rather than v_1) for the path P_1 . The aim is to make sure that the one-type edges incident to x in $G(v_1)$ and to y in $G(v_2)$ corresponding to each Q_i remain to be used on Stage *II*.

On Stage *II*, we construct $\ell_1 n_2$ S -trees corresponding to the length of Q_i in non-decreasing order. We distinguish two subcases by the length of Q_1 .

Subcase 1.1. $t_1 \geq 2$.

For Q_1 , we find n_2 internally disjoint xy -paths A_1, A_2, \dots, A_{n_2} in $K_{u_1, \beta_{1,1}}$ by using the remaining edges after Stage *I*, and get a $V(H(\beta_{1,1}))V(H(\beta_{1,t_1-1}))$ -linkage B_1, B_2, \dots, B_{n_2} by the three-type edges according to $\beta_{1,1}Q_1\beta_{1,t_1-1}$. Thus $T_i^* = A_i \cup B_i \cup (\beta_{1,t_1-1}, v_i)z$ are n_2 edge-disjoint S -trees, where $1 \leq i \leq n_2$ and the subscript i of v_i is expressed module n_2 as one of $1, 2, \dots, n_2$.

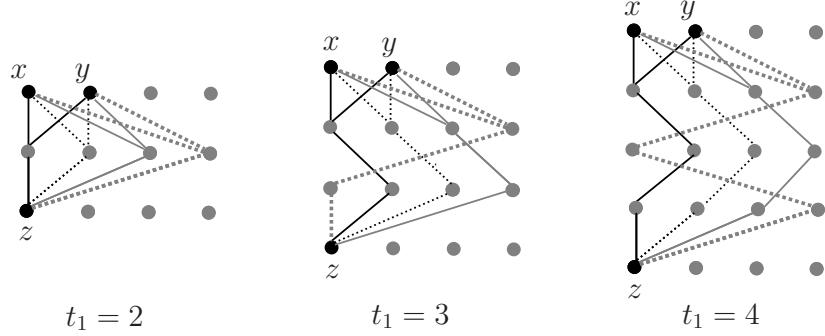


Figure 3. The 4 edge-disjoint S -trees corresponding to Q_1 when $n_2 = 4$
(The edges of a tree are shown by the same type of lines).

Indeed, this can always be done. Set $A_i = x(\beta_{1,1}, v_i)y$ for $1 \leq i \leq n_2$. If $t_1 = 2$, then $B_i = \emptyset$. If $t_1 \geq 3$, that is, $d_{Q_1}(\beta_{1,1}, \beta_{1,t_1-1}) = t_1 - 2 \geq 1$, then B_i has the structure as follows. If t_1 is even, then let $B_i = (\beta_{1,1}, v_i)(\beta_{1,2}, v_{i+1})(\beta_{1,3}, v_i)(\beta_{1,4}, v_{i+1}) \cdots (\beta_{1,t_1-1}, v_i)$ and $T_i^* = A_i \cup B_i \cup (\beta_{1,t_1-1}, v_i)z$. If t_1 is odd, then let $B_i = (\beta_{1,1}, v_i)(\beta_{1,2}, v_{i+1})(\beta_{1,3}, v_i)(\beta_{1,4}, v_{i+1}) \cdots (\beta_{1,t_1-1}, v_{i+1})$ and $T_i^* = A_i \cup B_i \cup (\beta_{1,t_1-1}, v_{i+1})z$. Take for example, let $n_2 = 4$, then 4 edge-disjoint S -trees are shown in Figure 3 when $t_1 = 2$, $t_1 = 3$ and $t_1 = 4$, respectively.

Similar to these n_2 S -trees corresponding to Q_1 , we continue to construct n_2 S -trees corresponding to Q_i by the edges in accord with $E(Q_i)$, since $\ell(Q_i) \geq 2$ for each i , where $2 \leq i \leq \ell_1$.

Subcase 1.2. $t_1 = 1$, that is, $Q_1 = u_1 u_2$.

Since $\lambda_3(G) = \ell_1$, it follows by Observation 2 that $d_G(u_1) \geq \ell_1 + 1$ or $d_G(u_2) \geq \ell_1 + 1$.

If $d_G(u_1) \geq \ell_1 + 1$, then denote another neighbor of u_1 in G by $\beta_{\ell_1+1,1}$ except u_2 and $\beta_{i,1}$ ($2 \leq i \leq \ell_1$). For Q_1 , we find out n_2 edge-disjoint S -trees as follows. Let $T_1^* = (\beta_{\ell_1+1,1}, v_1)x \cup (\beta_{\ell_1+1,1}, v_1)y \cup xz$, $T_2^* = (\beta_{\ell_1+1,1}, v_2)x \cup (\beta_{\ell_1+1,1}, v_2)y \cup yz$, $T_i^* = (u_2, v_i)x \cup (u_2, v_i)y \cup (u_2, v_i)(u_1, v_{i+1}) \cup (u_1, v_{i+1})z$ for $3 \leq i \leq n_2 - 1$, $T_{n_2}^* = (u_2, v_{n_2})x \cup (u_2, v_{n_2})y \cup (u_2, v_{n_2})(u_1, v_3) \cup (u_1, v_3)z$. See Figure 4(a).

If $d_G(u_2) \geq \ell_1 + 1$, then denote another neighbor of u_2 in G by γ_{ℓ_1+1} except u_1 and

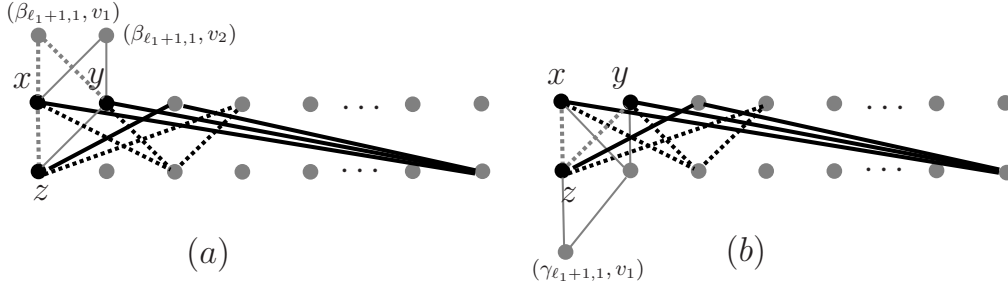


Figure 4. The n_2 edge-disjoint S -trees for Q_1 corresponding to $Q_1 = u_1 u_2$
(The edges of a tree are shown by the same type of lines).

β_{i,t_i-1} ($2 \leq i \leq \ell_1$). For Q_1 , set $T_1^* = xz \cup zy$, $T_2^* = xy'' \cup y''y \cup (\gamma_{\ell_1+1}, v_1)y'' \cup (\gamma_{\ell_1+1}, v_1)z$, $T_i^* = (u_2, v_i)x \cup (u_2, v_i)y \cup (u_2, v_i)(u_1, v_{i+1}) \cup (u_1, v_{i+1})z$ for $3 \leq i \leq n_2 - 1$, $T_{n_2}^* = (u_2, v_{n_2})x \cup (u_2, v_{n_2})y \cup (u_2, v_{n_2})(u_1, v_3) \cup (u_1, v_3)z$. See Figure 4(b).

Corresponding to Q_i with $2 \leq i \leq \ell_1$, construct n_2 edge-disjoint S -trees similar to that in Subcase 1.1 of Stage II.

Since the edges on Stage II are of three-type corresponding to each Q_i besides three one-type edges incident to x , y and z that are not used on Stage I, it follows that the edges used on Stage II are different from those used on Stage I. And by (ii) of Observation 3, these $\ell_1 n_2$ S -trees on Stage II are edge-disjoint, as desired.

Case 2. $z' \notin \{x, y\}$.

Assume $x = (u_1, v_1)$, $y = (u_1, v_2)$, $z = (u_2, v_3)$. Let $S' = \{v_1, v_2, v_3\}$, $S'' = \{x, y, z'\}$.

Since $\lambda(G) \geq \lambda_3(G) = \ell_1$, there exist ℓ_1 edge-disjoint $u_1 u_2$ -paths $Q_1, Q_2, \dots, Q_{\ell_1}$ in G such that $\ell(Q_1) \leq \ell(Q_2) \leq \dots \leq \ell(Q_{\ell_1})$.

Since $\lambda_3(H) = \ell_2$, there are ℓ_2 edge-disjoint S' -trees $T_1, T_2, \dots, T_{\ell_2}$ in H . Recall that $0 \leq |E(T_i) \cap E(G[S'])| \leq 2$. By Proposition 1, suppose $E(T_i) \cap E(G[S']) = \emptyset$ for $3 \leq i \leq \ell_2$. According to whether T_1 and T_2 have edges in $E(G[S'])$ or not, T_1 and T_2 have one of the following structures.

Subcase 2.1. $E(T_1) \cap E(G[S']) = \emptyset$ and $E(T_2) \cap E(G[S']) = \emptyset$.

For $1 \leq i \leq \ell_2$, denote the neighbor of v_3 in T_i by α_i .

On Stage I, let $T_i^* = T_i(u_1) \cup Q_{\ell_1}(\alpha_i) \cup z(u_2, \alpha_i)$ (where $1 \leq i \leq \ell_2$, and $T_i(u_1)$ is the path in $H(u_1)$ corresponding to T_i , $Q_{\ell_1}(\alpha_i)$ is the path in $G(\alpha_i)$ corresponding to Q_{ℓ_1}).

On Stage II, if $\ell(Q_i) \geq 2$ for each i with $1 \leq i \leq \ell_1$, construct n_2 S -trees similar to Case 1; otherwise $\ell(Q_1) = 1$, then either u_1 or u_2 has a neighbor which is not on each $u_1 u_2$ -paths Q_i in $G(v_1)$. Then n_2 S -trees corresponding to Q_1 are shown in Figure 5 and construct n_2 S -trees similar to Case 1 for each i with $2 \leq i \leq \ell_1$.

Subcase 2.2. $E(T_1) \cap E(G[S']) \neq \emptyset$ and $E(T_2) \cap E(G[S']) = \emptyset$.

Suppose $|E(T_1) \cap E(G[S'])| = 1$ and $E(T_2) \cap E(G[S']) = \emptyset$. Without loss of generality,

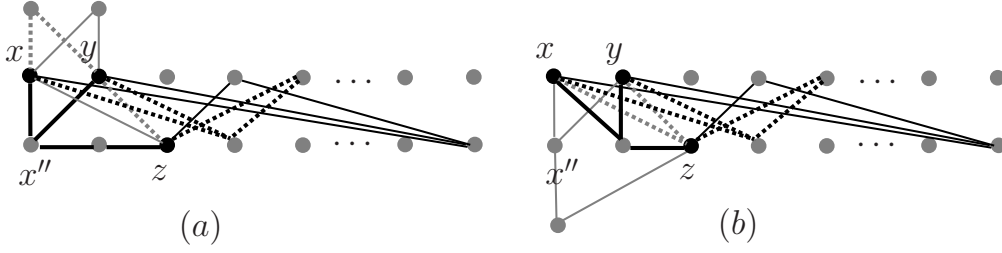


Figure 5. The n_2 edge-disjoint S -trees on Stage II for Q_1 (the edges of a tree are shown by the same type of lines).

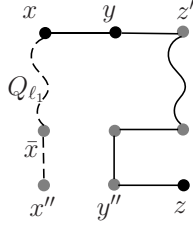


Figure 6. The solid lines stand for the edges of the S -tree.

suppose $E(T_1) \cap E(G[S']) = v_1v_2$. For $1 \leq i \leq \ell_2$, denote the neighbor of v_3 in T_i by α_i . Construct $\ell_2 + \ell_1 n_2$ S -trees similar to Subcase 2.1 by making use of α_i . It remains to consider $|E(T_1) \cap E(G[S'])| = 2$ and $E(T_2) \cap E(G[S']) = \emptyset$. Without loss of generality, suppose $E(T_1) \cap E(G[S]) = \{v_1v_2, v_2v_3\}$. For T_1 , if $d_{Q_{\ell_1}}(u_1, u_2) \geq 2$, then T_1^* has the structure as shown in Figure 6, where \bar{x} is the neighbor of x'' in $Q_{\ell_1}(v_1)$; if $d_{Q_{\ell_1}}(u_1, u_2) = 1$, set $T_1^* = xyz'z$. Construct other $\ell_2 + \ell_1 n_2 - 1$ S -trees similar to Subcase 2.1. Thus there exist $\ell_2 + \ell_1 n_2$ S -trees.

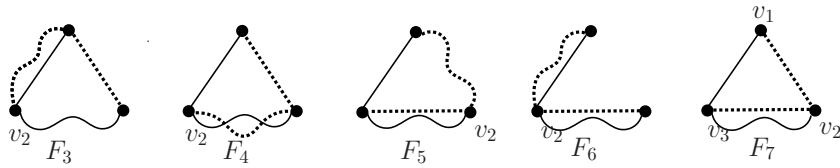


Figure 7. Two S' -trees of type I in Case 2 of Lemma 2.

Subcase 2.3. $E(T_1) \cap E(G[S']) \neq \emptyset$ and $E(T_2) \cap E(G[S']) \neq \emptyset$.

Without loss of generality, suppose $|E(T_2) \cap E(G[S])| = 1$. If $|E(T_1) \cap E(G[S'])| = 1$, then we may assume that the trees T_1 and T_2 have one of the structures F_3, F_4, F_5, F_6 as shown in Figure 7. For $1 \leq i \leq \ell_2$, denote the neighbor of v_2 in $T_i \setminus \{v_1, v_3\}$ by α_i . Construct $\ell_2 + \ell_1 n_2$ S -trees similar to Subcase 2.1.

If $|E(T_1) \cap E(G[S'])| = 2$, then the trees T_1 and T_2 have the structure F_7 as shown in Figure 7, where T_1 is shown by dotted lines. Construct T_1^* as shown in Figure 6. For $2 \leq i \leq \ell_2$, denote the neighbor of v_2 in $T_i \setminus \{v_1, v_3\}$ by α_i . Construct $\ell_2 + \ell_1 n_2 - 1$ S -trees similar to Subcase 2.1. Thus, there exist $\ell_2 + \ell_1 n_2$ S -trees.

By Observation 3, these $\ell_2 + \ell_1 n_2$ S -trees are edge-disjoint in each case, as desired. \square

Lemma 3. *If x, y, z belong to distinct $H(u_i)$ s, then there exist $\ell_2 + \ell_1 n_2$ edge-disjoint S -trees.*

Proof. Assume $x \in H(u_1)$, $y \in H(u_2)$, $z \in H(u_3)$. Let y', z' be the vertex corresponding to y, z in $H(u_1)$, x'', z'' be the vertex corresponding to x, z in $H(u_2)$, x''', y''' be the vertex corresponding to x, y in $H(u_3)$. We distinguish the following three cases.

Case 1. x, y', z' are the same vertex in $H(u_1)$.

We may assume $x = (u_1, v_1)$, $y = (u_2, v_1)$, $z = (u_3, v_1)$. Since $\lambda_3(H) = \ell_2$, there are ℓ_2 neighbors of v_1 in H , say $\alpha_1, \alpha_2, \dots, \alpha_{\ell_2}$. Since $\lambda_3(G) = \ell_1$, there are ℓ_1 edge-disjoint $\{u_1, u_2, u_3\}$ -trees $T_1, T_2, \dots, T_{\ell_1}$ in G . For a tree T_i in G , set by $T_i(\alpha_j)$ the tree in $G(\alpha_j)$ corresponding to T_i for $1 \leq i \leq \ell_1$, $1 \leq j \leq \ell_2$.

On Stage *I*, for $1 \leq j \leq \ell_2$, set $T_j^* = T_1(\alpha_j) \cup x(u_1, \alpha_j) \cup y(u_2, \alpha_j) \cup z(u_3, \alpha_j)$.

On Stage *II*, for each j with $1 \leq j \leq \ell_1$, if T_j is of type *I*, we may assume $d_{T_j}(u_2) = 2$. Denote the neighbor of u_1, u_3 in T_j by η_j, γ_j and the neighbor of u_2 by $\beta_j, \bar{\beta}_j$ (β_j is nearer to u_1 than $\bar{\beta}_j$), where β_j, η_j may be the same vertex, $\bar{\beta}_j, \gamma_j$ may be the same vertex. Corresponding to $u_1 T_j u_2$ and $u_2 T_j u_3$, we find n_2 edge-disjoint xy -paths $A = \{A_1, \dots, A_{n_2}\}$ and edge-disjoint yz -paths $B = \{B_1, \dots, B_{n_2}\}$ respectively. Then $T_{ij}^* = A_i \cup B_i$ ($1 \leq i \leq n_2$) are n_2 edge-disjoint S -trees. Since the construction of B is similar to that of A , we only provide the construction of A according to $d_{T_j}(u_1, u_2)$. If $d_{T_j}(u_1, u_2) = 1$, set $A_1 = xy$, $A_i = x(u_2, v_i) \cup (u_2, v_i)(u_1, v_{i+1}) \cup (u_1, v_{i+1})y$ for $2 \leq i \leq n_2 - 1$, $A_{n_2} = x(u_2, v_{n_2}) \cup (u_2, v_{n_2})(u_1, v_2) \cup (u_1, v_2)y$. If $d_{T_j}(u_1, u_2) = 2$, set $A_i = x(\eta_j, v_i) \cup (\eta_j, v_i)y$ for $1 \leq i \leq n_2$. It remains to consider the case that $d_{T_j}(u_1, u_2) \geq 3$. We first find out a $V(H(\eta_j))V(H(\beta_j))$ -linkage D_1, D_2, \dots, D_{n_2} by the three-type edges according to $\eta_j T_j \beta_j$. Thus $A_i = x(\eta_j, v_i) \cup D_i \cup (\beta_j, v_i)y$, where $1 \leq i \leq n_2$ and the subscript i of v_i is expressed module n_2 as one of $1, 2, \dots, n_2$. It remains to consider the case that T_j is of type *II*, denote the neighbor of u_1, u_2, u_3 in T_j by $\eta_j, \beta_j, \gamma_j$ and the only one three-degree vertex in T_j by $w_j(\eta_j, \beta_j, \gamma_j)$ and w_j may be the same vertex). We find a $V(H(\eta_j))V(H(\beta_j))$ -linkage and a $V(H(\gamma_j))V(H(w_j))$ -linkage respectively by three-type edges of $G \circ H$. And join x, y, z respectively to $H(\eta_j), H(\beta_j)$ and $H(\gamma_j)$. Thus, we are able to construct n_2 edge-disjoint S -trees corresponding to T_j . Since $1 \leq j \leq \ell_1$, thus $\ell_1 n_2$ edge-disjoint S -trees are constructed on Stage *II*.

Case 2. Only two of x, y', z' are the same vertex in $H(u_1)$.

We only consider the case of $x = y'$ (The other cases when $x = z'$ or $y' = z'$ can be proved with similar arguments). We may assume $x = (u_1, v_1)$, $y = (u_2, v_1)$, $z = (u_3, v_2)$.

Since $\lambda_3(H) = \ell_2$, there exist ℓ_2 edge-disjoint v_1v_2 -paths $P_1, P_2, \dots, P_{\ell_2}$ in H such that $\ell(P_1) \leq \ell(P_2) \leq \dots \leq \ell(P_{\ell_2})$. For $1 \leq i \leq \ell_2$, denote the vertex in P_i adjacent to v_1 by α_i , and the vertex in P_i adjacent to v_2 by β_i , and denote by $P_i(u_3)$ in $H(u_3)$ corresponding to P_i . Since $\lambda_3(G) = \ell_1$, there are ℓ_1 edge-disjoint $\{u_1, u_2, u_3\}$ -trees $T_1, T_2, \dots, T_{\ell_1}$ in G .

On Stage I, fix T_1 . If $\ell(P_i) \geq 2$ for each i with $1 \leq i \leq \ell_2$, let $T_i^* = x(u_1, \alpha_i) \cup y(u_2, \alpha_i) \cup zP_i(u_3)(u_3, \alpha_i) \cup T_1(\alpha_i)$. Otherwise, $\ell(P_1) = 1$, that is, v_1 is adjacent to v_2 , then $d_H(v_1) \geq \ell_2 + 1$ or $d_H(v_2) \geq \ell_2 + 1$. If $d_H(v_1) \geq \ell_2 + 1$, denote a neighbor of v_1 by α_{ℓ_2+1} which is not α_i ($1 \leq i \leq \ell_2$). Then $T_1^* = \{x(u_1, \alpha_{\ell_2+1}), y(u_2, \alpha_{\ell_2+1}), zx''', x'''(u_3, \alpha_{\ell_2+1})\} \cup T_1(\alpha_{\ell_2+1})$. If $d_H(v_2) \geq \ell_2 + 1$, denote the neighbor of v_2 by β_{ℓ_2+1} which is not β_i ($1 \leq i \leq \ell_2$). Then $T_1^* = \{xz', z'(u_1, \beta_{\ell_2+1}), yz'', z''(u_2, \beta_{\ell_2+1}), z(u_3, \beta_{\ell_2+1})\} \cup T_1(\beta_{\ell_2+1})$.

On Stage II, $\ell_1 n_2$ edge-disjoint S -trees are constructed with similar arguments as Case 1.

Case 3. x, y', z' are distinct vertices in $H(u_1)$.

Assume that $x = (u_1, v_1)$, $y = (u_2, v_2)$, $z = (u_3, v_3)$. Let $S' = \{v_1, v_2, v_3\}$ and $S'' = \{u_1, u_2, u_3\}$.

Since $\lambda_3(H) = \ell_2$, there are ℓ_2 edge-disjoint S' -trees $T_1, T_2, \dots, T_{\ell_2}$ in H . For $1 \leq i \leq \ell_2$, denote by α_i the vertex in T_i adjacent to a vertex v_1 in S' , and $\ell(T_i)$ denotes the number of edges in T_i , and denote by $T_i(u_2)$ ($T_i(u_3)$) in $H(u_2)$ ($H(u_3)$) the tree corresponding to T_i . Since $\lambda_3(G) = \ell_1$, there are ℓ_1 edge-disjoint S'' -trees $T'_1, T'_2, \dots, T'_{\ell_1}$ in G .

On Stage I, fix T'_1 . If $\ell(T_i) \geq 3$ for each i with $1 \leq i \leq \ell_2$, let $T_i^* = x(u_1, \alpha_i) \cup yT_i(u_2)(u_2, \alpha_i) \cup zT_i(u_3)(u_3, \alpha_i) \cup T'_1(\alpha_i)$. Otherwise, similar to Case 2 of Lemma 2, the difficult case is that there is an S' -tree of length 2. Suppose $\ell(T_1) = 2$ and assume $d_{T_1}(v_2) = 2$. Thus T_1^* has three structures as shown in Figure 7 where T'_1 is of type II in Figure 8(a); T'_1 is of type I and $d_{T'_1}(v_2) = 1$ in Figure 8(b); T'_1 is of type I and $d_{T'_1}(v_2) = 2$ in Figure 8(c).

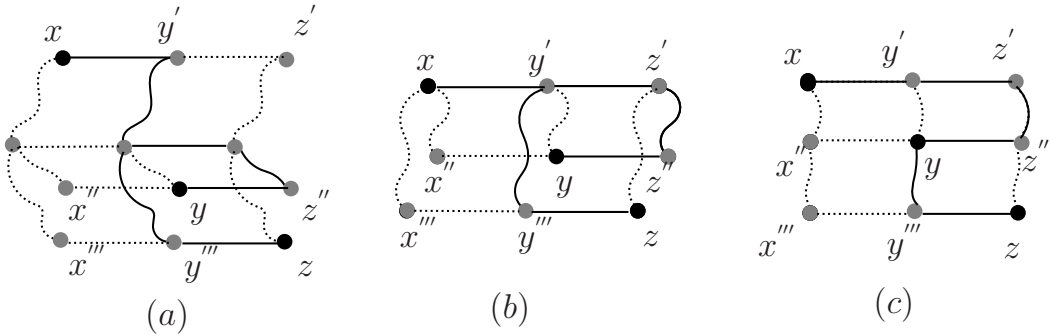


Figure 8. One S -tree corresponding to T'_1 , the solid lines stand for the edges of the S -tree.

On Stage II, $\ell_1 n_2$ edge-disjoint S -trees are constructed with similar arguments as Case 1.

By Observation 3, in each case, these $\ell_2 + \ell_1 n_2$ S -trees are edge-disjoint, as desired. \square

By combining the preceding three lemmas, we get the following result.

Theorem 3. *Let G and H be two non-trivial graphs and G is connected. Then $\lambda_3(G \circ H) \geq \lambda_3(H) + \lambda_3(G)|V(H)|$. Moreover, the lower bound is sharp.*

We know that the lower bounds of Theorem 3 is sharp by the following corollary.

Corollary 1. $\lambda_3(P_{n_1} \circ P_{n_2}) = n_2 + 1$.

Proof. By Theorem 3, $\lambda_3(P_{n_1} \circ P_{n_2}) \geq n_2 + 1$. On the other hand, by Observation 1, $\lambda_3(P_{n_1} \circ P_{n_2}) \leq \delta(P_{n_1} \circ P_{n_2}) = n_2 + 1$. Thus $\lambda_3(P_{n_1} \circ P_{n_2}) = n_2 + 1$. \square

4 Upper bound of $\lambda_3(G \circ H)$

In this section, we give the upper bound of generalized 3-edge-connectivity of the lexicographic product of two graphs.

Yang and Xu [28] investigated the classical edge-connectivity of the lexicographic product of two graphs.

Theorem 4. [28] *Let G and H be two non-trivial graphs and G is connected, then*

$$\lambda(G \circ H) = \min\{\lambda(G)|V(H)|^2, \delta(H) + \delta(G)|V(H)|\}.$$

In [17], the sharp lower bound of the generalized 3-edge-connectivity of a graph is given as follows.

Proposition 4. [17] *Let G be a connected graph with n vertices. For every two integers s and r with $s \geq 0$ and $r \in \{0, 1, 2, 3\}$, if $\lambda(G) = 4s + r$, then $\lambda_3(G) \geq 3s + \lceil \frac{r}{2} \rceil$. Moreover, the lower bound is sharp. We simply write $\lambda_3(G) \geq \frac{3\lambda - 2}{4}$.*

From the above two results, we get the upper bounds of $\lambda_3(G \circ H)$.

Theorem 5. *Let G and H be two non-trivial graphs and G is connected. Then*

$$\lambda_3(G \circ H) \leq \min \left\{ \left\lfloor \frac{4\lambda_3(G) + 2}{3} \right\rfloor |V(H)|^2, \delta(H) + \delta(G)|V(H)| \right\}.$$

Moreover, the upper bound is sharp.

Proof. By Proposition 4, $\lambda(G) \leq \lfloor \frac{4\lambda_3(G) + 2}{3} \rfloor$. By Proposition 2 and Theorem 4, we have $\lambda_3(G \circ H) \leq \lambda(G \circ H) = \min\{\lambda(G)|V(H)|^2, \delta(H) + \delta(G)|V(H)|\}$. It follows that $\lambda_3(G \circ H) \leq \min \left\{ \left\lfloor \frac{4\lambda_3(G) + 2}{3} \right\rfloor |V(H)|^2, \delta(H) + \delta(G)|V(H)| \right\}$. Moreover, the example in Corollary 1 shows that the upper bound is sharp. The proof is now complete. \square

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