Network Cournot Competition

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Abstract. Cournot competition, introduced in 1838 by Antoine Augustin Cournot, is a fundamental economic model that represents firms competing in a single market of a homogeneous good. Each firm tries to maximize its utility—naturally a function of the production cost as well as market price of the product—by deciding on the amount of production. This problem has been studied comprehensively in Economics and Game Theory; however, in today's dynamic and diverse economy, many firms often compete in more than one market simultaneously, i.e., each market might be shared among a subset of these firms. In this situation, a bipartite graph models the access restriction where firms are on one side, markets are on the other side, and edges demonstrate whether a firm has access to a market or not. We call this game *Network Cournot Competition* (NCC). Computation of equilibrium, taking into account a network of markets and firms and the different forms of cost and price functions, makes challenging and interesting new problems.

In this paper, we propose algorithms for finding pure Nash equilibria of NCC games in different situations. First, we carefully design a potential function for NCC, when the price functions for markets are linear functions of the production in that market. This result lets us leverage optimization techniques for a single function rather than multiple utility functions of many firms. However, for nonlinear price functions, this approach is not feasible—there is indeed no single potential function that captures the utilities of all firms for the case of nonlinear price functions. We model the problem as a nonlinear complementarity problem in this case, and design a polynomial-time algorithm that finds an equilibrium of the game for strongly convex cost functions and strongly monotone revenue functions. We also explore the class of price functions that ensures strong monotonicity of the revenue function, and show it consists of a broad class of functions. Moreover, we discuss the uniqueness of equilibria in both of these cases which means our algorithms find the unique equilibria of the games. Last but not least, when the cost of production in one market is independent from the cost of production in other markets for all firms, the problem can be separated into several independent classical *Cournot Oligopoly* problems in which the firms compete over a single market. We give the first combinatorial algorithm for this widely studied problem. Interestingly, our algorithm is much simpler and faster than previous optimization-based approaches.

1 Introduction

In this paper we study selling a utility with a distribution network, e.g., natural gas, water and electricity, in several markets when the clearing price of each market is de-

termined by its supply and demand. The distribution network fragments the market into different regional markets with their own prices. Therefore, the relations between suppliers and submarkets form a complex network [29, 22, 8, 36, 7, 15, 13]. For example, a market with access to only one supplier suffers a monopolistic price, while a market having access to multiple suppliers enjoys a lower price as a result of the price competition.

Antoine Augustin Cournot introduced the first model for studying the duopoly competition in 1838. He proposed a model where two individuals own different springs of water, and sell it independently. Each individual decides on the amount of water to supply, and then the aggregate water supply determines the market price through an inverse demand function. Cournot characterizes the unique equilibrium outcome of the market when both suppliers have the same marginal costs of production, and the inverse demand function is linear. He argued that in the unique equilibrium outcome, the market price is above the marginal cost.

Joseph Bertrand 1883 criticized the Cournot model, where the strategy of each player is the quantity to supply, and in turn suggested to consider prices, rather than quantities, as strategies. In the Bertrand model each firm chooses a price for a homogeneous good, and the firm announcing the lowest price gets all the market share. Since the firm with the lowest price receives all the demand, each firm has incentive to price below the current market price unless the market price matches its cost. Therefore, in an equilibrium outcome of the Bertrand model, assuming all marginal costs are the same and there are at least two competitors in the market, the market price will be equal to the marginal cost.

The Cournot and Bertrand models are two basic tools for investigating the competitive market price, and have attracted much interest for modeling real markets; see, e.g., [8, 36, 7, 15]. While these are two extreme models for analyzing the price competition, it is hard to say which one is essentially better than the other. In particular, the predictive power of each strongly depends on the nature of the market, and varies from application to application. For example, the Bertrand model explains the situation where firms literally set prices, e.g., the cellphone market, the laptop market, and the TV market. On the other hand, Cournot's approach would be suitable for modeling markets like those of crude oil, natural gas, and electricity, where firms decide about quantities rather than prices.

There are several attempts to find equilibrium outcomes of the Cournot or Bertrand competitions in the oligopolistic setting, where a small number of firms compete in only one market; see, e.g., [24, 33, 32, 18, 17, 37]. Nevertheless, it is not entirely clear what equilibrium outcomes of these games are when firms compete over more than one market. In this paper, we investigate the problem of finding equilibrium outcomes of the Cournot competition in a network setting where there are several markets for a homogeneous good and each market is accessible to a subset of firms.

1.1 Example

We start with the following warm-up example. This is a basic example for the Cournot competition in the network setting. It consists of three scenarios. We assume firm $i \in$

$\{A, B\}$ produces quantity q_{ij} of the good in market $j \in \{1, 2\}$. Let $\mathbf{q}$ be the vector of all quantities. The detailed computations are in the Appendix.

Scenario 1 Consider the Cournot competition in an oligopolistic setting with two firms and one market (see Figure 1). Let $p(\mathbf{q}) = 1 - q_{A1} - q_{B1}$ be the market price (the inverse demand function), and $c_i(\mathbf{q}) = \frac{1}{2}q_{i1}^2$ be the cost of production for firm $i \in \{A, B\}$. The profit of a firm is what it gets by selling all the quantities of good it produces in all markets minus its cost of production. Therefore, the profit of firm i denoted by $\pi_i(\mathbf{q})$ is $q_{i1}(1 - q_{A1} - q_{B1}) - \frac{1}{2}q_{i1}^2$. In a Nash equilibrium of the game, each firm maximizes its profit assuming its opponent does not change its strategy. Hence, the unique Nash equilibrium of the game can be found by solving the set of equations $\frac{\partial \pi_A}{\partial q_{A1}} = \frac{\partial \pi_B}{\partial q_{B1}} = 0$. So $q_{A1} = q_{B1} = \frac{1}{4}$ is the unique Nash equilibrium where $p(\mathbf{q}) = \frac{1}{2}$, and $\pi_A(\mathbf{q}) = \pi_B(\mathbf{q}) = 0.9375$.

Fig. 1. This figure represents the three scenarios of our example. Vector $\mathbf{q} = (\frac{1}{4}, \frac{1}{4})$ represents the unique equilibrium in the first scenario. Vector $\mathbf{q} = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ is the unique equilibrium of the second scenario. Finally, Vector $\mathbf{q} = (0.18, 0.1, 0.16)$ is the unique equilibrium in the third scenario.

Scenario 2 We construct the second scenario by splitting the market in the previous scenario into two identical markets such that both firms have access to both markets (see Figure 1). Since the demand is divided between two identical markets, the price for market j would be $p_j(\mathbf{q}) = 1 - 2q_{Aj} - 2q_{Bj}$, i.e., the clearance price of each market is the same as the clearance price of the market in Scenario 1, when the supply is half of the supply of the market in Scenario 1. In this scenario, the profit of firm $i \in \{A, B\}$ is $\pi_i(\mathbf{q}) = \sum_j q_{ij}(1 - 2q_{Aj} - 2q_{Bj}) - \frac{1}{2}(q_{i1} + q_{i2})^2$. Any Nash equilibrium of this game satisfies the set of equations $\frac{\partial \pi_A}{\partial q_{A1}} = \frac{\partial \pi_A}{\partial q_{A2}} = \frac{\partial \pi_B}{\partial q_{B1}} = \frac{\partial \pi_B}{\partial q_{B2}} = 0$. By finding the unique solution to this set of equations, one can verify that $\mathbf{q} = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})$ is the unique equilibrium of the game where $p_1(\mathbf{q}) = p_2(\mathbf{q}) = \frac{1}{2}$, and $\pi_A(\mathbf{q}) = \pi_B(\mathbf{q}) = 0.09375$. Since we artificially split the market into two

identical markets, this equilibrium is, not surprisingly, the same as the equilibrium in the previous scenario.

Scenario 3 Consider the previous scenario, and suppose firm 2 has no access to the first market (see Figure 1). Let the demand functions and the cost functions be the same as the previous scenario. The profits of firms 1 and 2 can be written as follows:

$$\pi_A(\mathbf{q}) = q_{A1}(1 - 2q_{A1}) + q_{A2}(1 - 2q_{A2} - 2q_{B2}) - \frac{1}{2}(q_{A1} + q_{A2})^2,$$

$$\pi_B(\mathbf{q}) = q_{B2}(1 - 2q_{A2} - 2q_{B2}) - \frac{1}{2}q_{B2}^2.$$

The unique equilibrium outcome of the game is found by solving the set of equations $\frac{\partial \pi_A}{\partial q_{A1}} = \frac{\partial \pi_A}{\partial q_{A2}} = \frac{\partial \pi_B}{\partial q_{B2}} = 0$. One can verify that vector $\mathbf{q} = (q_{A1}, q_{A2}, q_{B2}) = (0.18, 0.1, 0.16)$ is the unique equilibrium outcome of the game where $p_1(\mathbf{q}) = 0.64$, $p_2(\mathbf{q}) = 0.48$, $\pi_A(\mathbf{q}) = 0.124$, and $\pi_B(\mathbf{q}) = 0.064$. The following are a few observations worth mentioning.

– Firm A has more power in this scenario due to having a captive market[3].
– The equilibrium price of market 1 is higher than the equilibrium price in the previous scenarios.
– The position of firm B affects its profit. Since it has no access to market 1, it is not as powerful as firm A.
– The equilibrium price of market 2 is smaller than the equilibrium price in the previous scenarios.

1.2 Related Work

There are several papers that investigate the Cournot competition in an oligopolistic setting (see, e.g., [24, 33, 18, 17, 37]). In spite of these works, little is known about the Cournot competition in a network. Ilkılıç [19] studies the Cournot competition in a network setting, and considers a network of firms and markets where each firm chooses a quantity to supply in each accessible market. He studies the competition when the inverse demand functions are linear and the cost functions are quadratic (functions of the total production). In this study, we consider the same model when the cost functions and the demand functions may have quite general forms. We show the game with linear inverse demand functions is a potential game and therefore has a unique equilibrium outcome. Furthermore, we present two polynomial-time algorithms for finding an equilibrium outcome for a wide range of cost functions and demand functions. While we investigate the Cournot competition in networks, there is a recent paper which considers the Bertrand competition in network setting [3], albeit in a much more restricted case of only two firms competing in each market.

The final price of each market in the Cournot competition is a market clearing price; i.e, the final price is set such that the market becomes clear. Finding a market clearance equilibrium is a well-established problem, and there are several papers which propose polynomial-time algorithms for computing equilibriums of markets in which the price

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[3]A captive market is one in which consumers have limited options and the seller has a monopoly power.

of each good is defined as the price in which the market clears. Examples of such markets include Arrow-Debreu market and its special case Fisher market (see related work on these markets [12, 11, 10, 21, 9, 31, 16]). Devanur and Vazirani [10] design an approximation scheme which computes the market clearing prices for the Arrow-Debreu market, and Ghiyasvand and Orlin [16] improve the running time of the algorithm. The first polynomial-time algorithm for finding an Arrow-Debreu market equilibrium is proposed by Jain [21] for a special case with linear utilities. The Fisher market, a special case of the Arrow-Debreu market, attracted a lot of attention as well. Eisenberg and Gale [12] present the first polynomial-time algorithm by transferring the problem to a concave cost maximization problem. Devanur et al. [11] design the first combinatorial algorithm which runs in polynomial time and finds the market clearance equilibrium when the utility functions are linear. This result is later improved by Orlin [31].

For the sake of completeness, we refer to recent works in the computer science literature [20, 14], which investigate the Cournot competition in an oligopolistic setting. Immorlica et al. [20] study a coalition formation game in a Cournot oligopoly. In this setting, firms form coalitions, and the utility of each coalition, which is equally divided between its members, is determined by the equilibrium of a Cournot competition between coalitions. They prove the price of anarchy, which is the ratio between the social welfare of the worse stable partition and the social optimum, is $\Theta(n^{2/5})$ where n is the number of firms. Fiat et al. [14] consider a Cournot competition where agents may decide to be non-myopic. In particular, they define two principal strategies to maximize revenue and profit (revenue minus cost) respectively. Note that in the classic Cournot competition all agents want to maximize their profit. However, in their study each agent first chooses its principal strategy and then acts accordingly. The authors prove this game has a pure Nash equilibrium and the best response dynamics will converge to an equilibrium. They also show the equilibrium price in this game is lower than the equilibrium price in the standard Cournot competition.

1.3   Results and techniques

We consider the problem of Cournot competition on a network of markets and firms for different classes of cost and inverse demand functions. Adding these two dimensions to the classical Cournot competition which only involves a single market and basic cost and inverse demand functions yields an engaging but complicated problem which needs advanced techniques for analyzing. For simplicity of notation we model the competition by a bipartite graph rather than a hypergraph: vertices on one side denote the firms, and vertices on the other side denote the markets. An edge between a firm and a market demonstrates willingness of the firm to compete in that specific market. The complexity of finding the equilibrium, in addition to the number of markets and firms, depends on the classes that inverse demand and production cost functions belong to.

We summarize our results in the following table.

Cost functions	Inverse demand functions	Running time	Technique
Convex	Linear	$O(E^3)$	Convex optimization, formulation as an ordinal potential game
Convex	Strongly monotone marginal revenue function[4]	$\text{poly}(E)$	Reduction to a nonlinear complementarity problem
Convex, separable	Concave	$O(n \log^2 Q_{\max})$	Supermodular optimization, nested binary search

In the above table, E denotes the number of edges of the bipartite graph, n denotes the number of firms, and $Q_{\max}$ denotes the maximum possible total quantity in the oligopoly network at any equilibrium. In our results we assume the inverse demand functions are nonincreasing functions of total production in the market. This is the basic assumption in the classical Cournot Competition model: As the price in the market increases, it is reasonable to believe that the buyers drop out of the market and demand for the product decreases. The classical Cournot Competition model as well as many previous works on Cournot Competition model assumes linearity of the inverse demand function [19, 20]. In fact there is little work on generalizing the inverse demand function in this model. The second and third row of the above table shows we have developed efficient algorithms for more general inverse demand functions satisfying concavity rather than linearity. This can be accounted as a big achievement. The assumption of monotonicity of the inverse demand function is a standard assumption in Economics [2, 1, 27]. We assume cost functions to be convex which is the case in many works related to both Cournot Competition and Bertrand Network [25, 38]. In a previous work [19], the author considered Cournot Competition on a network of firms and markets; however, assumed that inverse demand functions are linear and all the cost functions are quadratic function of the total production by the firm in all markets which is quite restrictive. Most of the results in other related works in Cournot Competition and Bertrand Network require linearity of the cost functions [3, 20]. A brief summary of our results presented in three sections is given below.

Linear Inverse Demand Functions In case inverse demand functions are linear and production costs are convex, we present a fast and efficient algorithm to obtain the equilibrium. This approach works by showing that Network Cournot Competition belongs to a class of games called *ordinal potential games*. In such games, the collective strategy of the independent players is to maximize a single potential function. The potential function is carefully designed in such a way that changes made by one player reflects in the same way in the potential function as in their own utility function. We design a

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[4]Marginal revenue function is the vector function which maps production quantities for an edge to marginal revenue along that edge.

potential function for the game, which depends on the network structure, and show how it captures this property. Moreover, in the case where the cost functions are convex, we prove concavity of this designed potential function (Theorem 6) concluding convex optimization methods can be employed to find the optimum and hence, the equilibrium of the original Cournot competition. We also discuss uniqueness of equilibria in case the cost functions are strictly concave. Our result in this section is specifically interesting since we find the unique equilibrium of the game. We prove the following theorems in Section 3.

Theorem 1. *The Network Cournot Competition with linear inverse demand functions forms an ordinal potential game.*

Theorem 2. *Our designed potential function for the Network Cournot Competition with linear inverse demand functions is concave provided that the cost functions are convex. Furthermore, the potential function is strictly concave if the cost functions are strictly convex, and hence the equilibria for the game is unique. In addition, a polynomial-time algorithm finds the optimum of the potential function which describes the market clearance prices.*

The general case   Since the above approach does not work for nonlinear inverse demand functions, we design another interesting but more involved algorithm to capture more general forms of inverse demand functions. We show that an equilibrium of the game can be computed in polynomial time if the production cost functions are convex and the revenue function is monotone. Moreover, we show under strict monotonicity of the revenue function, the solution is unqiue, and therefore our results in this section is structural; i.e. we find the one and only equilibria. For convergence guarantee we also need Lipschitz condition on derivatives of inverse demand and cost functions. We start the section by modeling our problem as a complementarity problem. Then we prove how holding the aforementioned conditions for cost and revenue functions yields satisfying *Scaled Lipschitz Condition* (SLC) and semidefiniteness for matrices of derivatives of the profit function. SLC is a standard condition widely used in convergence analysis for scalar and vector optimization [39]. Finally , we present our algorithm, and show how meeting these new conditions by inverse demand and cost functions helps us to guarantee polynomial running time of our algorithm. We also give examples of classes of inverse demand functions satisfying the above conditions. These include many families of inverse demand functions including quadratic functions, cubic functions and entropy functions. The following theorem is the main result of Section 4 which summarizes the performance of our algorithm.

Theorem 3. *A solution to the Network Cournot Competition can be found in polynomial number of iterations under the following conditions:*

   1. The cost functions are (strongly) convex.
   2. The marginal revenue function is (strongly[5]) monotone.

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[5]For at least one of the first two conditions, strong version of condition should be satisfied, i.e., either cost functions should be strongly convex or the marginal revenue function should be strongly monotone.

3. Ther first derivative of cost functions and inverse demand functions and the second derivative of inverse demand functions are Lipschitz continuous.

Furthermore, the solution is unique assuming only the first condition. Therefore, our algorithm finds the unique equilibria of NCC.

Cournot oligopoly Another reasonable model for considering cost functions of the firms is the case where the cost of production in a market depends only on the quantity produced by the firm in that specific market (and not on quantities produced by this firm in other markets). In other words, the firms have completely independent sections for producing different goods in various markets, and there is no correlation between cost of production in separate markets. Interestingly, in this case the competitions are separable; i.e. equilibrium for Network Cournot Competition can be found by finding the quantities at equilibrium for each market individually. This motivates us for considering Cournot game where the firms compete over a single market. We present a new algorithm for computing equilibrium quantities produced by firms in a Cournot oligopoly, i.e., when the firms compete over a single market. Cournot Oligopoly is a well-known model in Economics, and computation of its Cournot Equilibrium has been subject to a lot of attention. It has been considered in many works including [34, 23, 30, 26, 6] to name a few. The earlier attempts for calculating equilibrium for a general class of inverse demand and cost functions are mainly based on solving a Linear Complementarity Problem or a Variational Inequality. These settings can be then turned into convex optimization problems of size $O(n)$ where n is the number of firms. This means the runtime of the earlier works cannot be better than $O(n^3)$ which is the best performance for convex optimization [5]. We give a novel combinatorial algorithm for this important problem when the quantities produced are integral. We limit our search to integral quantities for two reasons. First, in real-world all commodities and products are traded in integral units. Second, this algorithm can easily be adapted to compute approximate Cournot-Nash equilibrium for the continuous case and since the quantities at equilibrium may not be rational numbers, this is the best we can hope for. Our algorithm runs in time $O(n \log^2(Q_{\max}))$ where $Q_{\max}$ is an upper bound on total quantity produced at equilibrium. Our approach relies on the fact that profit functions are supermodular when the inverse demand function is nonincreasing and the cost functions are convex. We leverage the supermodularity of inverse demand functions and Topkis' Monotonicity Theorem [35] to design a nested binary search algorithm. The following is the main result of Section 5.

Theorem 4. *A polynomial-time algorithm successfully computes the quantities produced by each firm at an equilibrium of the Cournot oligopoly if the inverse demand function is non-increasing, and the cost functions are convex. In addition, the algorithm runs in $O(n \log^2(Q_{\max}))$ where $Q_{\max}$ is the maximum possible total quantity in the oligopoly network at any equilibrium.*

2   Notations

Suppose we have a set of n firms denoted by $\mathcal{F}$ and a set of m markets denoted by $\mathcal{M}$. A single good is produced in each of these markets. Each firm might or might

not be able to supply a particular market. A bipartite graph is used to demonstrate these relations. In this graph, the markets are denoted by the numbers $1, 2, \ldots, m$ on one side, and the firms are denoted by the numbers $1, 2, \ldots, n$ on the other side. For simplicity, throughout the paper we use the notation $i \in \mathcal{M}$ meaning the market i, and $j \in \mathcal{F}$ meaning firm j. For firm $j \in \mathcal{F}$ and market $i \in \mathcal{M}$ there exists an edge between the corresponding vertices in the bipartite graph if and only if firm j is able to produce the good in market i. This edge will be denoted (i, j). The set of edges of the graph is denoted by $\mathcal{E}$, and the number of edges in the graph is shown by E. For each market $i \in \mathcal{M}$, the set of vertices $N_{\mathcal{M}}(i)$ is the set of firms that this market is connected to in the graph. Similarly, $N_{\mathcal{F}}(j)$ denotes the set of neighbors of firms j among markets. The edges in $\mathcal{E}$ are sorted and numbered $1, \ldots, E$, first based on the number of their corresponding market and then based on the number of their corresponding firm. More formally, edge $(i, j) \in \mathcal{E}$ is ranked above edge $(l, k) \in \mathcal{E}$ if $i < l$ or $i = l$ and $j < k$. The quantity of the good that firm j produces in market i is denoted by q_{ij}. The vector $\mathbf{q}$ is an $E \times 1$ vector that contains all the quantities produced over the edges of the graph in the same order that the edges are numbered.

The demand for good i, denoted D_i, is the sum of the total quantity of this good produced by all firms, i.e., $D_i = \sum_{j \in N_{\mathcal{M}}(i)} q_{ij}$. The price of good i, denoted by the function $P_i(D_i)$, is only a decreasing function of total demand for this good and not the individual quantities produced by each firm in this market. For a firm j, the vector $\boldsymbol{s_j}$ denotes the strategy of firm j, which is the vector of all quantites produced by this firm in the markets $N_{\mathcal{F}}(j)$. Firm $j \in \mathcal{F}$ has a cost function related to its strategy denoted by $c_j(\boldsymbol{s_j})$. The profit that firm j makes is equal to the total money that it obtains by selling its production minus its cost of production. More formally, the profit of firm j, denoted by π_j, is

$$\pi_j = \sum_{i \in N_{\mathcal{F}}(j)} P_i(D_i) q_{ij} - c_j(\boldsymbol{s_j}). \tag{1}$$

3 Cournot competition and potential games

In this section, we design an efficient algorithm for the case where the price functions are linear. More specifically, we design an innovative *potential function* that captures the changes of all the utility functions simultaneously, and therefore, show how finding the quantities at the equilibrium would be equivalent to finding the set of quantities that maximizes this function. We use the notion of *potential games* as introduced in Monderer and Shapley [28]. In that paper, the authors introduce *ordinal potential games* as the set of games for which there exists a *potential function* P^* such that the pure strategy equilibrium set of the game coincides with the pure strategy equilibrium set of a game where every party's utility function is P^*.

In this section, we design a function for the Network Cournot Competition and show how this function is a potenial function for the problem if the price functions are linear. Interestingly, this holds for any cost function meaning Network Cournot Competition with arbitrary cost functions is an ordinal potential game as long as the price functions are linear. Furthermore, we show when the cost functions are convex, our designed potential function is concave, and hence any convex optimization method can find the

equilibrium of such a Network Cournot Competition. In case cost functions are strictly convex, the potential function is strictly concave. We restate a well known theorem in this section to conclude that the convex optimization in this case has a unique solution, and therefore the equilibria that we find in this case is the one and only equilibria of the game.

Definition 1. *A game is said to be an* ordinal potential game *if the incentive of all players to change their strategy can be expressed using a single global function called the potential function. More formally, a game with n players and utility function u_i for player $i \in \{1, \ldots, n\}$ is called ordinal potential with potential function P^* if for all the strategy profiles $q \in \mathbb{R}^n$ and every strategy x_i of player i the following holds:*

$$u_i(x_i, q_{-i}) - u_i(q_i, q_{-i}) > 0 \ \ iff \ \ P^*(x_i, q_{-i}) - P^*(q_i, q_{-i}) > 0.$$

An equivalent definition of an ordinal potential game is a game for which a potential funciton P^* exists such that the following holds for all strategy profiles $q \in \mathbb{R}^n$ and for each player i.

$$\frac{\partial u_i}{\partial q_i} = \frac{\partial P^*}{\partial q_i}.$$

In other words, for each strategy profile q, any change in the strategy of player i has the same impact on its utility function as on the game's potential function.

The pure strategy equilibrium set of any ordinal potential game coincides with the pure strategy equilibrium set of a game with the potential function P^* as all parties' utility function.

Theorem 5. *The Network Cournot Competition with linear price functions is an ordinal potential game.*

Proof. Let $P_i(D_i) = \alpha_i - \beta_i D_i$ be the linear price function for market $i \in \mathcal{M}$ where $\alpha_i \geq 0$ and $\beta_i \geq 0$ are constants determined by the properties of market i. Note that this function is decreasing with respect to D_i. Here we want to introduce a potential function P^*, and show that $\frac{\partial \pi_j}{\partial q_{ij}} = \frac{\partial P^*}{\partial q_{ij}}$ holds $\forall (i, j) \in \mathcal{E}$. The utility function of firm j is

$$\pi_j = \sum_{i \in N_{\mathcal{F}}(j)} \left(\alpha_i - \beta_i \sum_{k \in N_{\mathcal{F}}(j)} q_{kj} \right) q_{ij} - c_j(\boldsymbol{s_j}),$$

and taking partial derivative with respect to q_{ij} yields

$$\frac{\partial \pi_j}{\partial q_{ij}} = \alpha_i - \beta_i \sum_{k \in N_{\mathcal{F}}(j)} q_{kj} - \beta_i q_{ij} - \frac{\partial c_j(\boldsymbol{s_j})}{\partial q_{ij}}.$$

We define P^* to be

$$P^* = \sum_{i \in \mathcal{M}} \left[\alpha_i \sum_{j \in N_{\mathcal{M}}(i)} q_{ij} - \beta_i \sum_{j \in N_{\mathcal{M}}(i)} q_{ij}^2 - \beta_i \sum_{\substack{k \leq j \\ k, j \in N_{\mathcal{M}}(i)}} q_{ij} q_{ik} - \sum_{j \in N_{\mathcal{M}}(i)} \frac{c_j(\boldsymbol{s_j})}{|N_{\mathcal{F}}(j)|} \right],$$

whose partial derivative with respect to q_{ij} is

$$\frac{\partial P^*}{\partial q_{ij}} = \alpha_i - 2\beta_i q_{ij} - \frac{\partial}{\partial q_{ij}}\left(\beta_i \sum_{\substack{l \leq m \\ l,m \in N_{\mathcal{M}}(i)}} q_{il}q_{im}\right) - \frac{\partial c_j(\boldsymbol{s_j})}{\partial q_{ij}}$$

$$= \alpha_i - 2\beta_i q_{ij} - \beta_i(D_i - q_{ij}) - \frac{\partial c_j(\boldsymbol{s_j})}{\partial q_{ij}}$$

$$= \frac{\partial \pi_j}{\partial q_{ij}}.$$

Since this holds for each $i \in \mathcal{M}$ and each $j \in \mathcal{F}$, the Network Cournot Competition is an ordinal potential game.

We can efficiently compute the equilibrium of the game if the potential function P^* is easy to optimize. Below we prove that this function is concave.

Theorem 6. *The potential function P^* from the previous theorem is concave provided that the cost functions of the firms are convex. Moreover, if the cost functions are strictly convex then the potential function is strictly concave.*

Proof. The proof goes by decomposing P^* into pieces that are concave. We first define f for one specific market i as

$$f = \sum_{j \in N_{\mathcal{M}}(i)} q_{ij}^2 + \sum_{\substack{k \leq j \\ k,j \in N_{\mathcal{M}}(i)}} q_{ij}q_{ik},$$

and prove that it is convex.

Recall that $\mathbf{q}$ is an $E \times 1$ vector of all the quantities of good produced over the existing edges of the graph. We can write $f = \mathbf{q}^T M \mathbf{q}$ where M is an $E \times E$ matrix with all elements on its diagonal equal to 1 and all other elements equal to $\frac{1}{\sqrt{2}}$:

$$M = \begin{bmatrix} 1 & \frac{1}{\sqrt{2}} & \cdots & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 & \cdots & \frac{1}{\sqrt{2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \cdots & 1 \end{bmatrix}.$$

To show that f is convex, it suffices to prove that M is positive semidefinite, by finding a matrix R such that $M = R^T R$. Consider the following $(E+1) \times E$ matrix:

$$R = \begin{bmatrix} c & c & \cdots & c \\ a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{bmatrix},$$

where a, c are set below. Let R_i be the i-th column of R. We have $R_i \cdot R_i = a^2 + c^2$ and $R_i \cdot R_j = c^2$ for $i \neq j$.

Setting $c = 2^{-\frac{1}{4}}$ and $a = \sqrt{1-c}$ yields $M = R^T R$, showing that M is positive semidefinite, hence the convexity of f.

The following expression for a fixed market $i \in \mathcal{M}$, sum of three concave functions, is also concave.

$$\alpha_i \sum_{j \in N_{\mathcal{M}}(i)} q_{ij} - \beta_i \left(\sum_{j \in N_{\mathcal{M}}(i)} q_{ij}^2 + \sum_{\substack{k \leq j \\ k,j \in N_{\mathcal{M}}(i)}} q_{ij}q_{ik} \right) - \sum_{j \in N_{\mathcal{M}}(i)} \frac{c_j(s_j)}{|N_{\mathcal{F}}(j)|}.$$

Summing over all markets proves concavity of P^*. Note that if a function is the sum of a concave function and a strictly concave function, then it is strictly concave itself. Therefore, since f is concave, we can conclude strictly concavity of P^* under the assumption that the cost functions are strictly convex.

The following well-known theorem discusses the uniqueness of the solution to a convex optimization problem.

Theorem 7. *Let $F : \mathcal{K} \to \mathbb{R}^n$ be a strictly concave and continuous function for some finite convex space $\mathcal{K} \in \mathbb{R}^n$. Then the following convex optimization problem has a unique solution.*

$$\max f(x) \ \ s.t. \ \ x \in \mathcal{K}. \tag{2}$$

By Theorem 6, if the cost functions are strictly convex then the potential function is strictly concave and hence, by Theorem 7 the equilibrium of the game is unique.

Let $ConvexP(\mathcal{E}, (\alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_m), (c_1, \ldots, c_n))$ be the following convex optimization program:

$$\min \ -\sum_{i \in \mathcal{M}} \left[\alpha_i \sum_{j \in N_{\mathcal{M}}(i)} q_{ij} - \beta_i \sum_{j \in N_{\mathcal{M}}(i)} q_{ij}^2 - \beta_i \sum_{\substack{k \leq j \\ k,j \in N_{\mathcal{M}}(i)}} q_{ij}q_{ik} - \sum_{j \in N_{\mathcal{M}}(i)} \frac{c_j(s_j)}{|N_{\mathcal{F}}(j)|} \right] \tag{3}$$

$$\text{subject to} \quad\quad q_{ij} \geq 0 \ \ \forall (i,j) \in \mathcal{E}.$$

Note that in this optimization program we are trying to maximize P^* for a bipartite graph with set of edges $\mathcal{E}$, linear price functions characterized by the pair (α_i, β_i) for each market $i \in \mathcal{M}$, and cost functions c_j for each firm $j \in \mathcal{F}$.

Algorithm 1 Compute quantities at equilibrium for the Network Cournot Competition.

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procedure COURNOT-POTENTIAL($\mathcal{E}, c_j, (\alpha_i, \beta_i)$)   ▷ Set of edges, cost functions and price functions

Use a convex optimization algorithm to solve

$$ConvexP(\mathcal{E}, (\alpha_1, \ldots, \alpha_m), (\beta_1, \ldots, \beta_m), (c_1, \ldots, c_n)).$$

and return the vector **q** of equilibrium quantities.

end procedure

The above algorithm has a time complexity equal to the time complexity of a convex optimization algorithm with E variables. The best such algorithm has a running time $O(E^3)$[5].

4 Finding equilibrium for cournot game with general cost and inverse demand functions

In this section, we formulate an algorithm for a much more general class of price and cost functions. Our algorithm is based on reduction of Network Cournot Competition (NCC) to a polynomial time solvable class of Non-linear Complementarity Problem (NLCP). First in Subsection 4.1, we introduce our marginal profit function as the vector of partial derivatives of all firms with respect to the quantities that they produce. Then in Subsection 4.2, we show how this marginal profit function can help us to reduce NCC to a general NLCP. We also discuss uniqueness of equilibrium in this situation which yields the fact that solving NLCP would give us the one and only equilibrium of this problem. Unfortunately, in its most general form, NLCP is computationally intractable. However, for a large class of functions, these problems are polynomial time solvable. Most of the rest of this section is dedicated to proving the fact that NCC is polytime solvable on vast and important array of price and cost functions. In Subsection 4.3, we rigorously define the conditions under which NLCP is polynomial time solvable. We present our algorithm in this subsection along with a theorem which shows it converges in polynomial number of steps. To show the conditions that we introduce for convergence of our algorithm in polynomial time are not restrictive, we give a discussion in Subsection 4.4 on the functions satisfying these conditions, and show they hold for a wide range of price functions.

Assumptions Throughout the rest of this section we assume that the price functions are decreasing and concave and the cost functions are strongly convex (*The notion of strongly convex is to be defined later*). We also assume that for each firm there is a finite quantity at which extra production ceases to be profitable even if that is the only firm operating in the market. Thus, all production quantities and consequently quantities supplied to markets by firms are finite. In addition, we assume Lipschitz continuity and finiteness of the first and the second derivatives of price and cost functions. We note that these Lipschitz continuity assumptions are very common for convergence analysis in convex optimization [5] and finiteness assumptions are implied by Lipschitz continuity. In addition, they are not very restrictive as we don't expect unbounded fluctuation in costs and prices with change in supply. For sake of brevity, we use the terms inverse demand function and price function interchangeably.

4.1 Marginal profit function

For the rest of this section, we assume that P_i and c_i are twice differentiable functions of quantities. We define f_{ij} for a firm j and a market i such that $(i, j) \in \mathcal{E}$ as follows.

$$f_{ij} = -\frac{\partial \pi_j}{\partial q_{ij}} = -P_i(D_i) - \frac{\partial P_i(D_i)}{\partial q_{ij}} q_{ij} + \frac{\partial c_j}{\partial q_{ij}}. \tag{4}$$

Recall that the price function of a market is only a function of the total production in that market and not the individual quantities produced by individual firms. Thus $\frac{\partial P_i(D_i)}{\partial q_{ij}} = \frac{\partial P_i(D_i)}{\partial q_{ik}}$ $\forall j, k \in N_\mathcal{M}(i)$. Therefore, we replace these terms by $P_i'(D_i)$.

$$f_{ij} = -P_i(D_i) - P_i'(D_i)q_{ij} + \frac{\partial c_j}{\partial q_{ij}}. \tag{5}$$

Let vector F be the vector of all f_{ij}'s corresponding to the edges of the graph in the same format that we defined the vector q. That is f_{ij} corresponding to $(i,j) \in \mathcal{E}$ appears above f_{lk} corresponding to edge $(l,k) \in \mathcal{E}$ iff $i < l$ or $i = l$ and $j < k$. Note that F is a function of q.

Moreover, we separate the part representing marginal revenue from the part representing marginal cost in function F. More formally, we split F into two functions R and S such that $F = R + S$, and the element corresponding to the edge $(i,j) \in \mathcal{E}$ in the *marginal revenue fuction $R(q)$* is:

$$r_{ij} = -\frac{\partial \pi_j}{\partial q_{ij}} = -P_i(D_i) - P'(D_i)q_{ij},$$

whereas for the *marginal cost function $S(q)$*, it is:

$$s_{ij} = \frac{\partial c_j}{\partial q_{ij}}.$$

4.2   Non-linear complementarity problem

In this subsection we formally define the non-linear complementarity problem (NLCP), and prove our problem is a NLCP.

Definition 2. *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function on $\mathbb{R}_+^n$. The complementarity problem seeks a vector $x \in \mathbb{R}^n$ that satisfies the following constraints:*

$$\begin{aligned} x, F(x) &\geq 0, \\ x^T F(x) &= 0. \end{aligned} \tag{6}$$

Theorem 8. *The problem of finding the vector q at equilibrium in the Cournot game is a complementarity problem.*

Proof. Let q^* be the vector of the quantities at equilibrium. All quantities must be non-negative at all times; i.e., $q^* \geq 0$. It suffices to show $F(q^*) \geq 0$ and $q^{*T}F(q^*) = 0$. At equilibrium, no party benefits from changing its strategy, in particular, its production quantities. For each edge $(i,j) \in \mathcal{E}$, if the corresponding quantity q_{ij}^* is positive, then q_{ij}^* is a local maxima for π_j; i.e., $f_{ij}(q^*) = -\left.\frac{\partial \pi_j}{\partial q_{ij}}\right|_{q^*} = 0$. On the other hand, if $q_{ij}^* = 0$, then $\left.\frac{\partial \pi_j}{\partial q_{ij}}\right|_{q^*}$ cannot be positive, since, if it is, firm j would benefit by increasing the quantity q_{ij} to a small amount ϵ. Therefore, $\left.\frac{\partial \pi}{\partial q_{ij}}\right|_{q^*}$ is always nonpositive

or equivalently $f_{ij}(q^*) \geq 0$, i.e., $F(q^*) \geq 0$. Also, as we mentioned above, a nonzero q_{ij}^* is a local maximum for π_j; i.e., $f_{ij}(q^*) = -\frac{\partial \pi}{\partial q_{ij}}\big|_{q^*} = 0$. Hence, either $q_{ij}^* = 0$ or $f_{ij}(q^*) = 0$; thus, $q_{ij}^* f_{ij}(q^*) = 0$. This yields $\sum_{(i,j)\in\mathcal{E}} q_{ij}^* f_{ij}(q^*) = q^{*T} F(q^*) = 0$.

Definition 3. *$F : \mathcal{K} \to \mathbb{R}^n$ is said to be strictly monotone at x^* if*

$$\langle (F(x) - F(x^*))^T, x - x^* \rangle \geq 0, \forall x \in \mathcal{K}. \tag{7}$$

F is said to be strictly monotone if it is monotone at any $x^* \in \mathcal{K}$. Equivalently, F is strictly monotone if the jacobian matrix is positive definite.

The following theorem is a well known theorem for Complementarity Problems.

Theorem 9. *Let $F : \mathcal{K} \to \mathbb{R}^n$ be a continuous and strictly monotone function with a point $x \in \mathcal{K}$ such that $F(x) \geq 0$ (i.e. there exists a potential solution to the CP). Then the Complementarity Problem introduced in (6) characterized by function F has a unique solution.*

Hence, the Complementarity Problem characterized by function F introduced element by element in (4) has a unique solution under the assumption that the revenue function is strongly monotone (special case of strictly monotone). Note that the marginal profit function or F in our case is non-negative in at least one point. Otherwise, no firm has any incentive to produce in any market and the equilibrium is when all production quantities are equal to zero. In the next subsection, we aim to find this unique equilibrium of the NCC problem.

4.3 Designing a polynomial-time algorithm

In this subsection, we introduce Algorithm 2 for finding equilibrium of NCC, and show it converges in polynomial time by Theorem 10. This theorem requires the marginal profit function to satisfy Scaled Lipschitz Condition(SLC) and monotonicity. We first introduce SLC, and show how the marginal profit function satisfies SLC and montonicty by Lemmas 1 to 5. We argue the conditions that the cost and price functions should have in order for the marginal profit function to satisfy SLC and monotonicity in Lemma 5. Finally, in Theorem 10, we show convergence of our algorithm in polynomial time.

Before introducing the next theorem, we explain what the Jacobians ∇R, ∇S, and ∇F are for the Cournot game. First note that these are $E \times E$ matrices. Let $(i, j) \in \mathcal{E}$ and $(l, k) \in \mathcal{E}$ be two edges of the graph. Let e_1 denote the index of edge (i, j), and e_2 denote the index of edge (l, k) in the graph as we discussed in the first section. Then the element in row e_1 and column e_2 of matrix ∇R, denoted $\nabla R_{e_1 e_2}$, is equal to $\frac{\partial r_{ij}}{\partial q_{lk}}$. We name the corresponding elements in ∇F and ∇S similarly. We have $\nabla F = \nabla R + \nabla S$ as $F = R + S$.

Definition 4 (Scaled Lipschitz Condition (SLC)). *A function $G : D \mapsto \mathbb{R}^n$, $D \subseteq \mathbb{R}^n$ is said to satisfy Scaled Lipschitz Condition (SLC) if there exists a scalar $\lambda > 0$ such that $\forall h \in \mathbb{R}^n, \forall x \in D$, such that $\|X^{-1}h\| \leq 1$, we have:*

$$\|X[G(x + h) - G(x) - \nabla G(x)h]\|_\infty \leq \lambda |h^T \nabla G(x)h|, \tag{8}$$

where X is a diagonal matrix with diagonal entries equal to elements of the vector x in the same order, i.e., $X_{ii} = x_i$ for all $i \in \mathcal{M}$.

Satisfying SLC and monotonicity are essential for marginal profit function in Theorem 10. In Lemma 5 we discuss the assumptions for cost and revenue function under which these conditions hold for our marginal profit function. We use Lemmas 1 to 5 to show F satisfies SLC. More specifically, we demonstrate in Lemma 1, if we can derive an upperbound for LHS of SLC for R and S, then we can derive an upperbound for LHS of SLC for $F = R + S$ too. Then in Lemma 2 and Lemma 3 we show LHS of S and R in SLC definition can be upperbounded. Afterwards, we show monotonicity of S in Lemma 4. In Lemma 5 we aim to prove F satifies SLC under some assumptions for cost and revenue functions. We use the fact that LHS of SLC for F can be upperbounded using Lemma 3 and Lemma 2 combined with Lemma 1. Then we use the fact that RHS of SLC can be upperbounded using strong monotonicity of R and Lemma 4. Using these two facts, we conclude F satisfies SLC in Lemma 5.

Lemma 1. *Let F, R, S be three $\mathbb{R}^n \to \mathbb{R}^n$ functions such that $F(q) = R(q) + S(q), \quad \forall q \in \mathbb{R}^n$. Let R and S satisfy the following inequalities for some $C > 0$ and $\forall h$ such that $\|X^{-1}h\| \leq 1$:*

$$\|X[R(q+h) - R(q) - \nabla R(q)h]\|_\infty \leq C\|h\|^2,$$
$$\|X[S(q+h) - S(q) - \nabla S(q)h]\|_\infty \leq C\|h\|^2,$$

where X is the diagonal matrix with $X_{ii} = q_i$. Then we have:

$$\|X[F(q+h) - F(q) - \nabla F(q)h]\|_\infty \leq 2C\|h\|^2.$$

The following lemmas give upper bounds for LHS of the SLC for S and R respectively.

Lemma 2. *Assume X is the diagonal matrix with $X_{ii} = q_i$. $\forall h$ such that $\|X^{-1}h\| \leq 1$, there exists a constant $C > 0$ satisfying: $\|X[S(q+h) - S(q) - \nabla S(q)h]\|_\infty \leq C\|h\|^2$.*

Lemma 3. *Assume X is the diagonal matrix with $X_{ii} = q_i$. $\forall h$ such that $\|X^{-1}h\| \leq 1$, $\exists C > 0$ such that $\|X[R(q+h) - R(q) - \nabla R(q)h]\|_\infty \leq C\|h\|^2$.*

If R is assumed to be strongly monotone, we immediately have a lower bound on RHS of the SLC for R. The following lemma gives a lower bound on RHS of the SLC for S.

Lemma 4. *If cost functions are (strongly) convex S is (strongly) monotone*[6][7][8][9].

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[6] A matrix $M \in \mathbb{R}^{n \times n}$ is *strongly positive definite* iff $\forall x \in \mathbb{R}^n$ and some $\alpha > 0$ $x^T M x \geq \alpha\|x\|^2$.

[7] A differentiable function $f : D \mapsto \mathbb{R}^n$ is *monotone* iff its Jacobian ∇f is positive semidefinite over its domain D.

[8] A differentiable function $f : D \mapsto \mathbb{R}^n$ is *strongly monotone* iff its Jacobian ∇f is strongly positive definite over its domain, D.

[9] A twice differentiable function $f : D \mapsto \mathbb{R}$ is *strongly convex* iff its Hessian $\nabla^2 f$ is strongly positive definite over its domain, D.

The following lemma combines the results of Lemma 2 and Lemma 3 using Lemma 1 to derive an upper bound for LHS of the SLC for F. We bound RHS of the SLC from below by using strong montonicity of R and Lemma 4.

Lemma 5. *F satisfies SLC and is monotone if:*

1. *Cost functions are convex.*
2. *Marginal revenue function is monotone.*
3. *Cost functions are strongly convex or marginal revenue function is strongly monotone.*

We wrap up with the following theorem, which summarizes the main result of this section. Lemma 5 guarantees that our problem satisfies the two conditions mentioned in Zhao and Han 1999. Therefore, we can prove the following theorem.

Theorem 10. *Algorithm 2 converges to an equilibrium of Network Cournot Competition in time $O\big(E^2 \log(\mu_0/\epsilon)\big)$ under the following assumptions:*

1. *The cost functions are strongly convex.*
2. *The marginal revenue function is strongly monotone.*
3. *The first derivative of cost functions and price functions and the second derivative of price functions are Lipschitz continuous.*

This algorithm outputs an approximate solution $(F(q^*), q^*)$ satisfying $(q^*)^T F(q^*)/n \leq \epsilon$ where $\mu_0 = (q_0)^T F(q_0)/n$, and $(F(q_0), q_0)$ is the initial feasible point [10].

Algorithm 2 Compute quantities at equilibrium for the Cournot game.

1: **procedure** NETWORK-COURNOT(P_i, c_j, ϵ)   ▷ The price function P_i for each market $i \in \mathcal{M}$, the cost function c_j for each firm $j \in \mathcal{F}$, and ϵ as the desired tolerance
2:     Calculate vector F of length E as defined in (4).
3:     Find the initial feasible[11] solution $(F(x_0), x_0)$ for the complementarity problem. This solution should satisfy $x_0 \geq 0$ and $F(x_0) \geq 0$.
4:     Run Algorithm 3.1 from [39] to find the solution $(F(x^*), x^*)$ to the CP characterized by F.
5:     **return** x^* ▷ The vector q of quantities produced by firms at equilibrium
6: **end procedure**

4.4   Price Functions for Monotone Marginal Revenue Function

This section will be incomplete without a discussion of price functions that satisfy the convergence conditions for Algorithm 2. We will prove that a wide variety of price functions preserve monotonicity of the marginal revenue function. To this end, we prove the following lemma.

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[10]Initial feasible solution can be trivially found. E.g., it can be the same production quantity along each edge, large enough to ensure losses for all firms. Such quantity can easily be found by binary search between [0, Q].

Lemma 6. $\nabla R(q)$ *is a positive semidefinite matrix* $\forall\ q \geq 0$, *i.e.,* R *is monotone, provided that for all markets* $|P_i'(D_i)| \geq \frac{|P_i''(D_i)|D_i}{2}$.

While the above condition may seem somewhat restrictive, they allow the problem to be solved on a wide range of price functions. Intuitively, the condition implies that linear and quadratic terms dominate higher order terms. We present the following corollaries as examples of classes of functions that satisfy the above condition.

Corollary 1. *All decreasing concave quadratic price functions satisfy Lemma 6.*

Corollary 2. *All decreasing concave cubic price functions satisfy Lemma 6.*

Corollary 3. *Let* $a_i \in \mathbb{R}_{\geq 0}^n$ *for* $i \in \{1 \ldots k\}$ *be arbitrary positive vectors. Let* $f : \mathbb{R}_{\geq 0}^n \mapsto R$ *be the following function:* $f(x) = \sum_{i \in \{1 \ldots k\}} (a_i^T x) \log(a_i^T x)$. *Then* f *(and* $-f$) *satisfies Lemma 6.*

5   Algorithm for Cournot Oligopoly

In this section we present a new algorithm for computing equilibrium quantities produced by firms in a Cournot oligopoly, i.e., when the firms compete over a single market. Cournot Oligopoly is a standard model in Economics and computation of Cournot Equilibrium is an important problem in its own right. A considerable body of literature has been dedicated to this problem [34, 23, 30, 26, 6]. All of the earlier works that compute Cournot equilibrium for a general class of price and cost functions rely on solving a Linear Complementarity Problem or a Variational Inequality which in turn are set up as convex optimization problems of size $O(n)$ where n is the number of firms in oligopoly. Thus, the runtime guarantee of the earlier works is $O(n^3)$ at best. We give a novel combinatorial algorithm for this important problem when the quantities produced are integral. Our algorithm runs in time $n \log^2(Q_{max})$ where Q_{max} is an upper bound on total quantity produced at equilibrium. We note that, for two reasons, the restriction to integral quantities is practically no restriction at all. Firstly, in real-world all commodities and products are traded in integral units. Secondly, this algorithm can easily be adapted to compute approximate Cournot-Nash equilibrium for the continuous case and since the quantities at equilibrium may not be rational numbers, this is the best we can hope for.

As we have only a single market rather than a set of markets, we make a few changes to the notation. Let $[n] = \{1, \ldots, n\}$ be the set of firms competing over the single market. Let $\mathbf{q} = (q_1, q_2, \ldots, q_n)$ be the set of all quantities produced by the firms. Note that in this case, each firm is associated with only one quantity. Let $Q = \sum_{i \in [n]} q_i$ be the sum of the total quantity of good produced in the market. In this case, there is only a single inverse demand function $P : \mathbb{Z} \mapsto \mathbb{R}_{\geq 0}$, which maps total supply, Q, to market price. We assume that price decreases as the total quantity produced by the firms increases, i.e., P is a decreasing function of Q. For each firm $i \in [n]$, the function $c_i : \mathbb{Z} \mapsto \mathbb{R}_{\geq 0}$ denotes the cost to this firm when it produces quantity q_i of the good in the market. The profit of firm $i \in [n]$ as a function of q_i and Q, denoted $\pi_i(q_i, Q)$, is $P(Q)q_i - c_i(q_i)$. Also let $f_i(q_i, Q) = \pi_i(q_i + 1, Q + 1) - \pi_i(q_i, Q)$ be the marginal

profit for firm $i \in [n]$ of producing one extra unit of product. Although the quantities are nonnegative integers, for simplicity we assume the functions c_i, P, π_i and f_i are zero whenever any of their inputs are negative. Also, we refer to the forward difference $P(Q+1) - P(Q)$ by $P'(Q)$.

5.1 Polynomial time algorithm

We leverage the supermodularity of price functions and Topkis' Monotonicity Theorem [35] (Theorem 14) to design a nested binary search algorithm which finds the equilibrium quantity vector $\mathbf{q}$ when the price function is a decreasing function of Q and the cost functions of the firms are convex. Intuitively the algorithm works as follows. At each point we guess Q' to be the total quantity of good produced by all the firms. Then we check how good this guess is by computing for each firm the set of quantites that it can produce at equilibrium if we assume the total quantity is the fixed integer Q'. We prove that the set of possible quantities for each firm at equilibrium, assuming a fixed total production, is a consecutive set of integers. Let $I_i = \{q_i^l, q_i^l + 1, \ldots, q_i^u - 1, q_i^u\}$ be the range of all possible quantities for firm $i \in [n]$ assuming Q' is the total quantity produced in the market. We can conclude Q' was too low a guess if $\sum_{i \in [n]} q_i^l > Q'$. This implies our search should continue among total quantities above Q'. Similarly, if $\sum_{i \in [n]} q_i^u < Q'$, we can conclude our guess was too high, and the search should continues among total quantities below Q'. If neither case happens, then for each firm $i \in [n]$, there exists a $q_i' \in I_i$ such that $Q' = \sum_{i \in [n]} q_i'$ and firm i has no incentive to change this quantity if the total quantity is Q'. This means that the set $\mathbf{q}' = \{q_1', \ldots, q_n'\}$ forms an equilibrium of the game and the search is over.

The pseudocode for the algorithm is provided in Algorithm 3, whose correctness we prove next. The rest of this section is dedicated to proving Theorem 11. Here, we present a brief outline of the proof. To help with the proof we define the functions F_i and G_i as follows. Let $F_i(q_i, Q) = P(Q+1)q_i + \frac{P'(Q)}{2}(q_i - \frac{1}{2})^2 - c(q_i)$. We note that the first difference of $F(q_i, Q)$ is the marginal profit for firm i for producing one more quantity given that the total production quantity is Q and firm i is producing q_i. Let $G_i(q_i, Q) = F_i(q_i, Q-1)$. The first difference of $G_i(q_i, Q)$ is the marginal loss for firm i for producing one less quantity given that the total production quantity is Q and firm i is producing q_i. Maximizers of these functions are closely related to equilibrium quantities a firm can produce given that the total quantity in market is Q. We make this connection precise and prove the validity of binary search in Lines 8-12 of Algorithm 3 in Lemma 7. In Lemma 8, we prove that F_i and G_i are supermodular functions of q_i and $-Q$. In lemmas 9 and 10, we use Topkis' Monotonicity Theorem to prove the monotonicity of maximizers of F_i and G_i. This, along with lemmas 11 and 12 leads to the conclusion that the outer loop for finding total quatity at equilibrium is valid as well and hence the algorithm is correct.

5.2 Proof of correctness

Throughout this section *we assume that the price function is decreasing and concave and the cost functions are convex.*

Algorithm 3 Compute quantities produced by firms in a Cournot oligopoly.

1: **procedure** COURNOT-OLIGOPOLY(P, c_i)   ▷ The market price function P, the cost functions c_i for each firm $i \in [n]$

2:      Let $Q_{\min} := 1$

3:      Let Q_i^* be the optimal quantity that is produced by a firm when it is the only firm in the market

4:      Let $Q_{\max} := \sum_{i \in [n]} Q_i^*$

5:      **while** $Q_{\min} \leq Q_{\max}$ **do**

6:          $Q' := \lfloor \frac{Q_{\min} + Q_{\max}}{2} \rfloor$

7:          **for all** $i \in [n]$ **do**

8:              Binary search to find the minimum nonnegative integer q_i^l satisfying

9:              $f_i(q_i^l, Q') = \pi_i(q_i^l + 1, Q' + 1) - \pi_i(q_i^l, Q') \leq 0$

10:             Binary search to find the maximum integer $q_i^u \leq Q' + 1$ satisfying

11:             $f_i(q_i^u - 1, Q' - 1) = \pi_i(q_i^u, Q') - \pi_i(q_i^u - 1, Q' - 1) \geq 0$

12:             Let $I_i = \{q_i^l, \ldots, q_i^u\}$ be the set of all integers between q_i^l and q_i^u.

13:         **end for**

14:         **if** $\Sigma_{i \in [n]} q_i^l > Q'$ **then**

15:             $Q_{\min} := Q' + 1$

16:         **else if** $\Sigma_{i \in [n]} q_i^u < Q'$ **then**

17:             $Q_{\max} := Q' - 1$

18:         **else**

19:             Find a vector of quantities $\mathbf{q} = (q_1, \ldots, q_n)$ such that $q_i \in I_i$ and $\sum_{i \in [n]} q_i = Q'$

20:             **return q**

21:         **end if**

22:     **end while**

23: **end procedure**

Lemma 7. *Let $q_i^*(Q) = \{q_i^l \dots q_i^u\}$, where $q_i^l = \min argmax_{q_i \in \{0 \dots Q_{max}\}} F_i(q_i, Q)$ and $q_i^u = \max argmax_{q_i \in \{0 \dots Q_{max}\}} G_i(q_i, Q)$. Then $q_i^*(Q)$ is the set of consecutive integers I_i given by binary search in lines 8-12 of Algorithm 3. This is the set of quantities firm i can produce at equilibrium given that the total quantity produced is Q.*

Lemma 8. *Let $F_i^-(q_i, -Q) = F_i(q_i, Q)$ and $G_i^- = G_i(q_i, -Q)$. Then F_i^- and G_i^- are supermodular functions.*

Lemma 9. *Let $I = \{q_i^l, \dots, q_i^u\} = q_i^F(Q) = argmax_{q_i \in \{1 \dots Q_{max}\}} F_i(q_i, Q)$ and $I' = \{q_i'^l, \dots, q_i'^u\} = q_i^F(Q') = argmax_{q_i \in \{1 \dots Q_{max}\}} F_i(q_i, Q')$. Let $Q > Q'$. Then $q_i'^l \geq q_i^l$ and $q_i'^u \geq q_i^u$.*

Lemma 10. *Let $I = \{q_i^l, \dots, q_i^u\} = q_i^G(Q) = argmax_{q_i \in \{1 \dots Q_{max}\}} G_i(q_i, Q)$ and $I' = \{q_i'^l, \dots, q_i'^u\} = q_i^G(Q') = argmax_{q_i \in \{1 \dots Q_{max}\}} F_i(q_i, Q')$. Let $Q > Q'$. Then $q_i'^l \geq q_i^l$ and $q_i'^u \geq q_i^u$.*

Lemma 11. *Let Q be total production quantity guessed by Algorithm 3 at a step of outer binary search. Let $I = (I_1, \dots, I_n)$, where $I_i = \{q_i^l, \dots, q_i^u\}$, be the set of best reponse ranges of all firms if the total quantity is a fixed integer Q. Then, if $\sum_{i=1}^{n} q_i^u < Q$, there does not exist any equilibrium for which the total produced quantity is greater than or equal to Q.*

Lemma 12. *Let Q be total production quantity guessed by Algorithm 3 at a step of outer binary search. Let $I = (I_1, \dots, I_n)$, where $I_i = \{q_i^l, \dots, q_i^u\}$, be the set of best reponse ranges of all firms if the total quantity is a fixed integer Q. Then, if $\sum_{i=1}^{n} q_i^l > Q$, there does not exist any equilibrium for which the total produced quantity is less than or equal to Q.*

Finally, the results of this section culminate in the following theorem.

Theorem 11. *Algorithm 3 successfully computes the vector $\mathbf{q} = (q_1, q_2, \dots, q_n)$ of quantities at one equilibrium of the Cournot oligopoly if the price function is decreasing and concave and the cost function is convex. In addition, the algorithm runs in time $O(n \log^2(Q_{\max}))$ where $Q_{\max}$ is the maximum possible total quantity in the oligopoly network at any equilibrium.*

Proof. Lemma 7 guarantees that the inner binary search successfully finds the best response range for all firms. Lemmas 11 and 12 ensure that the algorithm always continues its search for the total quantity at equilibrium in the segment where all the equilibria are. Thus, when the search is over, it must be at an equilibrium of the game if one exists. If an equilibrium does not exist, then the algorithm will stop when it has eliminated all quantities in $\{1 \dots Q_{max}\}$ as possible total equilibrium production quantities. Let $Q_{\max}$ be the maximum total quantity possible at any equilibrium of the oligopoly network. Our algorithm performs a binary search over all possible quantities in $[1, Q_{\max}]$, and at each step finds a range of quantities for each firm $i \in [n]$ using another binary search. This means the algorithm runs in time $O(n \log^2(Q_{\max}))$. We can find an upper bound for $Q_{\max}$, noting that $Q_{\max}$ is at most the sum of the production quantites of

the firms when they are the only producer in the market; i.e, $Q_{\max} \leq \sum_{i \in [n]} Q_i^*$ where $Q_i^* = q_i^*(q_i)$ is the optimal quantity to be produced by firm i when there is no other firms to compete with.

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6 Example

Here we write the set of equations of our example in more details.

Scenario 1 The set of equations $\frac{\partial \pi_A}{\partial q_{A1}} = \frac{\partial \pi_B}{\partial q_{B1}} = 0$ can be written as:

$$\frac{\partial \pi_A}{\partial q_{A1}} = 1 - 3q_{A1} - q_{B1} = 0$$

$$\frac{\partial \pi_B}{\partial q_{B1}} = 1 - 3q_{B1} - q_{A1} = 0$$

Scenario 2 The set of equations $\frac{\partial \pi_A}{\partial q_{A1}} = \frac{\partial \pi_A}{\partial q_{A2}} = \frac{\partial \pi_B}{\partial q_{B1}} = \frac{\partial \pi_B}{\partial q_{B2}} = 0$ can be written as:

$$\frac{\partial \pi_A}{\partial q_{A1}} = 1 - 5q_{A1} - q_{A2} - 2q_{B1} = 0$$

$$\frac{\partial \pi_A}{\partial q_{A2}} = 1 - 5q_{A2} - q_{A1} - 2q_{B2} = 0$$

$$\frac{\partial \pi_B}{\partial q_{B1}} = 1 - 5q_{B1} - q_{B2} - 2q_{A1} = 0$$

$$\frac{\partial \pi_B}{\partial q_{B2}} = 1 - 5q_{B2} - q_{B1} - 2q_{A2} = 0.$$

Scenario 3 The set of equations $\frac{\partial \pi_A}{\partial q_{A1}} = \frac{\partial \pi_A}{\partial q_{A2}} = \frac{\partial \pi_B}{\partial q_{B2}} = 0$ can be written as:

$$\frac{\partial \pi_A}{\partial q_{A1}} = 1 - 5q_{A1} - q_{A2} = 0$$

$$\frac{\partial \pi_A}{\partial q_{A2}} = 1 - 5q_{A2} - q_{A1} - 2q_{B2} = 0$$

$$\frac{\partial \pi_B}{\partial q_{B2}} = 1 - 5q_{B2} - 2q_{A2} = 0.$$

7   Zhao and Han Convergence Theorem

The following theorem states the performance guarantee of the algorithm proposed by Zhao and Han 1999.

Theorem 12 (Zhao Han Convergence Theorem). *Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be the function associated with a complementarity problem satisfying the two following conditions:*

– *∇F is a positive semidefinite matrix for a constant scalar.*
– *F satisfies SLC; i.e., for some scalar $\lambda > 0$,*

$$\|X[F(x + h) - F(x) - \nabla F(x)h]\|_\infty \leq \lambda |h^T \nabla F(x)h|$$

holds $\forall x > 0$ and $\forall h$ satisfying $\|X^{-1}h\| \leq 1$.

Then the algorithm converges in time $O\left(n \max(1, \lambda) \, \log(\mu_0/\epsilon) \right)$ and outputs an approximate solution $(F(x^*), x^*)$ satisfying $(x^*)^T F(x^*)/n \leq \epsilon$ where $\mu_0 = (x_0)^T F(x_0)/n$, and $(F(x_0), x_0)$ is the initial feasible point.

8   Missing proofs

8.1   Missing proofs of Section 4

Proof (Proof of Theorem Lemma 1). Definition of function F implies

$$\|X[F(q + h) - F(q) - \nabla F(q)h]\|_\infty = \|X[R(q + h) - R(q) - \nabla R(q)h]$$
$$+ X[S(q + h) - S(q) - \nabla S(q)h]\|_\infty$$

applying triangle inequality gives

$$\|X[F(q + h) - F(q) - \nabla F(q)h]\|_\infty \leq \|X[R(q + h) - R(q) - \nabla R(q)h]\|_\infty$$
$$+ \|X[S(q + h) - S(q) - \nabla S(q)h]\|_\infty$$

Combining with assumptions of the lemma, we have the required inequality.

Proof (Proof of Theorem Lemma 3). Before we proceed, we state the following theorem from analysis and Lemma 13.

Theorem 13. *[5] Let $f : \mathbb{R}^n \mapsto \mathbb{R}$ be a continuously differentiable function with Lipschitz gradient, i.e., for some scaler $c > 0$,*

$$\|\nabla f(x) - \nabla f(y)\| \leq c\|x - y\| \ \ \forall \, x, y \in \mathbb{R}^n.$$

Then, we have $\forall \, x, y \in \mathbb{R}^n$,

$$f(y) \leq f(x) + \nabla f(x)^T (x - y) + \frac{c}{2}\|y - x\|^2 \tag{9}$$

Lemma 13. *For any vector $x \in \mathbb{R}^n$ and an arbitrary $S \subseteq [n]$, let $X = \sum_{i \in S} x_i$. Then we have $\sqrt{n}\|x\| \geq X$*

Proof. Let $Y = \sum_{i \in [n]} |x_i|$. Clearly, $|Y| \geq |X|$.

$$Y^2 = \sum_{i,j \in [n]} |x_i x_j| = \sum_{i < j} 2|x_i x_j| + \|x\|^2$$

Since, $s^2 + t^2 \geq 2st \; \forall \; s, t \in \mathbb{R}$, we have

$$X^2 \leq Y^2 \leq \sum_{i < j} (x_i^2 + x_j^2) + \|x\|^2 = n\|x\|^2$$

Now we are ready to prove Lemma 3. First note that $R(q + h) - R(q) - \nabla R(q)h$ is an $E \times 1$ vector. Let $H_i = \sum_{j \in N_{\mathcal{M}}(i)} h_{ij}$. The element corresponding to edge $(i, j) \in \mathcal{E}$ in vector $R(q+h)$ is $P_i(D_i + H_i) + P_i'(D_i + H_i)(q_{ij} + h_{ij})$. Similarly, the element corresponding to edge $(i, j) \in \mathcal{E}$ in $R(q)$ is $P_i(D_i) + P_i'(D_i)q_{ij}$ whereas the corresponding element in $\nabla R(q)h$ is $\sum_{k \in N_{\mathcal{M}}} h_{ik} \frac{\partial r_{ij}}{\partial q_{ik}} = -\sum_{k \in N_{\mathcal{M}}(i)} h_{ik}(P_i'(D_i) + P_i''(D_i)q_{ij})) + h_{ij}P_i'(D_i)$. Therefore, the element corresponding to edge $(i, j) \in \mathcal{E}$ in vector $R(q + h) - R(q) - \nabla R(q)h$ is:

$$-P_i(D_i + H_i) - P_i'(D_i + H_i)(q_{ij} - h_{ij}) - P_i(D_i) - P_i'(D_i)q_{ij}$$
$$+ \sum_{k \in N_{\mathcal{M}}(i)} h_{ik}(P_i'(D_i) + P_i''(D_i)q_{ij}) + h_{ij}P_i'(D_i).$$

Besides, X is the diagonal matrix of size $E \times E$ with diagonal entries equal to elements of q in the same order. Therefore, $X[R(q+h) - R(q) - \nabla R(q)h]$ is an $E \times 1$ vector where the element corresponding to edge $(i, j) \in \mathcal{E}$ is q_{ij} multiplied by the element corresponding to edge (i, j) in vector $R(q + h) - R(q) - \nabla R(q)h$:

$$- q_{ij} \Bigg(P_i(D_i + H_i) + P_i'(D_i + H_i)(q_{ij} + h_{ij}) - P_i(D_i) - P_i'(D_i)q_{ij}$$

$$- \sum_{k \in N_{\mathcal{M}}(i)} h_{ik}(P_i'(D_i) + P_i''(D_i)q_{ij}) - h_{ij}P_i'(D_i) \Bigg)$$

$$= - q_{ij} \Bigg([P_i(D_i + H_i) - P_i(D_i) - H_i P_i'(D_i)]$$

$$+ [P_i'(D_i + H_i) - P_i'(D_i) - H_i P_i''(D_i)](q_{ij} + h_{ij}) + h_{ij}H_i P_i''(D_i) \Bigg)$$

$$\leq q_{ij} \Bigg(|P_i(D_i + H_i) - P_i(D_i) - H_i P_i'(D_i)|$$

$$+ |P_i'(D_i + H_i) - P_i'(D_i) - H_i P_i''(D_i)||(q_{ij} + h_{ij})| + |h_{ij}H_i P_i''(D_i)| \Bigg).$$

Let P' and P'' be Lipschitz continuous functions with Lipschitz constants $2L_1$ and $2L_2$ respectively. To bound the last expression, we use Theorem 13 and Lemma 13

$$|P_i(D_i + H_i) - P_i(D_i) - H_i P_i'(D_i)| \leq L_1 H_i^2 \leq L_1 E \|h\|^2$$
$$|P_i'(D_i + H_i) - P_i'(D_i) - H_i P_i''(D_i)| \leq L_2 H_i^2 \leq L_2 E \|h\|^2$$
$$|h_{ij} H_i P_i''(D_i)| \leq E \|h\|^2 P_i''(D_i)$$

Then, from finiteness of derivatives, we have:

$$|h_{ij} H_i P_i''(D_i)| \leq E M_2 \|h\|^2$$

Thus, the LHS is bound from above by:

$$q_{ij} E \|h\|^2 (L_1 + L_2(q_{ij} + h_{ij}) + M_2)$$

Let Q be an upper bound on maximum profitable quantity for any producer in any market. Then the LHS is bound above by $C\|h\|^2$, where:

$$C = QE(L_1 + 2QL_2 + M_2) \tag{10}$$

Proof (Proof of Theorem Lemma 2). Let $m_{ij} = \frac{\partial c_i}{\partial q_{ij}}$. The element of vector $X(S(q + h) - S(q) - h\nabla S)$ corresponding to edge (i,j) is given by:

$$q_{ij}(m_{ij}(q + h) + m_{ij}(q) - h\nabla c_i(q))$$

Let $2L_3$ be an upper bound of Lipschitz constants for derivates of c_i's. Then, from Theorem 13 and upper bound Q on production quantities, we have:

$$|q_{ij}(m_{ij}(q + h) + m_{ij}(q) - h\nabla c_i(q))| \leq Q L_3 \|h\|^2$$

Proof (Proof of Theorem Lemma 4). Let $S' = \sum_{i \in \mathcal{F}} c_i$. Then S', being a sum of strongly convex functions, is a strongly convex function. Also, $S = \nabla S'$. Thus, $h^T \nabla^2 S' h = h^T \nabla S h$ is bounded from below by $\alpha_c \|h\|^2, \forall h \in \mathbb{R}^n$ for some $\alpha_c > 0$ if the cost functions are strongly convex and $alpha_c = 0$ is cost functions are convex.

Proof (Proof of Theorem Lemma 5). From lemmas 3, 2 and 1, RHS of SLC for F is $O(E\|h\|^2)$. If cost functions are strongly convex or marginal revenue function is strongly monotone, then from Lemma 4 and definition of strong monotonicity, the LHS of SLC for F is $\Omega(\|h\|^2)$. Thus, F satisfies SLC. We note that F is a sum of two monotone functions and hence is monotone.

Proof (Proof of Theorem Lemma 6). Let e_1 be the index of the edge (i,j) and e_2 be the index of edge (l,k). The elements of ∇R are as follows.

$$\nabla R_{e_1 e_1} = \begin{cases} \frac{\partial r_{ij}}{\partial q_{ij}} = -2P_i'(D_i) - P_i''(D_i) q_{ij} \text{ if } e_1 = e_2 \\ \frac{\partial r_{ij}}{\partial q_{ik}} = -P_i'(D_i) - P_i''(D_i) q_{ij} \text{ if } i = l, j \neq k \\ \frac{\partial r_{ij}}{\partial q_{lk}} = 0 \text{ if } i \neq l, j \neq k. \end{cases}$$

We note that since price functions are functions only of the total production in their corresponding markets and not the individual quantities produced by firms, $\frac{\partial P_i'(D_i)}{\partial q_{ij}} = \frac{\partial P_i'(D_i)}{\partial q_{ik}}$. Therefore, we have replaced the partial derivatives by $P_i''(D_i)$.

We must show $x^T \nabla R(D_i)x$ is nonnegative $\forall x \in \mathbb{R}^E$ and $\forall D_i \geq 0$.

$$
\begin{aligned}
x^T(\nabla R(D_i))x &= \sum_{(i,j)\in\mathcal{E}}\sum_{(k,l)\in\mathcal{E}} x_{ij}x_{lk}\frac{\partial r_{ij}}{\partial x_{lk}} = \sum_{i\in\mathcal{M}}\sum_{j,k\in N_\mathcal{M}(i)} x_{ij}x_{ik}\frac{\partial r_{ij}}{\partial x_{ik}} \\
&= \sum_{i\in\mathcal{M}}\left(\sum_{j\in N_\mathcal{M}(i)} x_{ij}^2\left[-2P_i'(D_i) - P_i''(D_i)x_{ij}\right]\right. \\
&\qquad\qquad \left. + \sum_{j,k\in N_\mathcal{M}(i),j\neq k} x_{ij}x_{ik}\left[-P_i'(D_i) - P_i''(D_i)q_{ij}\right]\right) \\
&= -\sum_{i\in\mathcal{M}}\left(\sum_{j\in N_\mathcal{M}(i)} x_{ij}^2 P_i'(D_i) + \sum_{j,k\in N_\mathcal{M}(i)} x_{ij}x_{ik}(P_i'(D_i) + P_i''(D_i)q_{ij})\right) \\
&= -\sum_{i\in\mathcal{M}}\left(P_i'(D_i)\sum_{j\in N_\mathcal{M}(i)} x_{ij}^2 + P_i'(D_i)\sum_{j,k\in N_\mathcal{M}(i)} x_{ij}x_{ik} + P_i''(D_i)\sum_{j,k\in N_\mathcal{M}(i)} x_{ij}q_{ij}x_{ik}\right) \\
&\geq -\sum_{i\in\mathcal{M}}\left(P_i'(D_i)\sum_{j\in N_\mathcal{M}(i)} x_{ij}^2 + P_i'(D_i)\sum_{j,k\in N_\mathcal{M}(i)} x_{ij}x_{ik} - |P_i''(D_i)||q||x||\sum_{j\in N_\mathcal{M}(i)} x_{ij}|\right) \\
&\geq \sum_{i\in\mathcal{M}}\left(-P_i'(D_i)|x|^2 - P_i'(D_i)\left(\sum_{j\in N_\mathcal{M}(i)} x_{ij}\right)^2 + |P_i''(D_i)||D_i|x||\sum_{j\in N_\mathcal{M}(i)} x_{ij}|\right)
\end{aligned}
$$

Since P_i's are decreasing functions, we have , $P_i'(D_i) \leq 0, \quad \forall i \in \mathcal{M}$. Thus, over domain of P_i's ($D_i \geq 0$), the above expression is non-negative if $|P_i''(D_i)|D_i \leq 2|P_i'(D_i)|$ Hence, $x^T(\nabla R(D_i))x \geq 0$ equivalently $\nabla R(D_i)$ is positive semidefinite.

8.2   Missing proofs of Section 5

Proof (Proof of Theorem Lemma 7). Again let $P'(Q) = P(Q + 1) - P(Q)$ be the forward difference of the price function, and let $c_i'(q_i) = c_i(q_i + 1) - c_i(q_i)$. From definition of profit function π_i and f_i, we have $f_i(q_i, Q) = P(Q+1)+P'(Q)q_i-c_i'(q_i)$. Assume Q is fixed. Suppose we have $q_i < \tilde{q}_i$. The marginal profit of firm at production quantity q_i is $P(Q + 1) + P'(Q)q_i - c_i'(q_i)$ whereas the marginal profit at production quantity $\tilde{q}_i$ is $P(Q+1) + P'(Q)\tilde{q}_i - c_i'(\tilde{q}_i)$. Thus, $P(Q+1) + P'(Q)q_i > P(Q+1) + P'(Q)\tilde{q}_i$ since $P'(Q)$ is negative (from concavity of P) and $q_i < \tilde{q}_i$. As the discrete cost functions are convex, we have $c_i'(q_i) < c_i'(\tilde{q}_i)$. This implies $f_i(q_i, Q) > f_i(\tilde{q}_i, Q)$ when $q_i < \tilde{q}_i$. Thus, for a fixed Q, $f_i(q_i, Q)$ is a non-increasing function of q_i. Similarly, we

can see that $f_i(q_i, Q)$ is a non-increasing function of Q. From definitions of F_i and G_i, we have:

$$F_i(q_i + 1, Q) - F_i(q_i, Q) = f_i(q_i, Q) \tag{11}$$

$$G_i(q_i + 1, Q) - G_i(q_i, Q) = F_i(q_i + 1, Q - 1) - F_i(q_i, Q - 1) = f_i(q_i, Q - 1) \tag{12}$$

For a fixed Q, Let q_l be the minimum maximizer of $F_i(q_i, Q)$. Then $f_i(q_l - 1, Q) > 0$. Let q_u be the maximum maximizer of $G_i(q_i, Q)$. Because f_i is non-increasing, we have $f_i(q_l - 1, Q - 1) \geq f_i(q_l - 1, Q) > 0$. Thus, any number smaller than q_l cannot be a maximizer of G_i and we have $q_l \leq q_u$. Let $q \in \{q_l \dots q_u\}$. Then, because $q \geq q_l$ we have $f_i(q, Q) \leq 0$ and from $q \leq q_u$, we have $f_i(q - 1, Q - 1) \geq 0$. Thus, q is an equilibrium quantity when total production quantity is Q. If $q < q_l$, then $f_i(q, Q) > 0$ and if $q > q_u$ then $f_i(q - 1, Q - 1) > 0$. Thus $\{q_l \dots q_u\}$ is the set of equilibrium quantities. In Line 9 of Algorithm 3, we are searching for the minimum maximizer of F_i and in Line 11 we are searching for maximum maximizer of G_i. Binary search for these quantities is valid because first differences for both functions (equations 11 and 12) are decreasing.

Proof (Proof of Theorem Lemma 8). We use the following definition from submodular optimization in the lemma.

Definition 5. *Given lattices $(X_1, \geq)$ and $(X_2, \geq)$, $f : X_1 \times X_2 \mapsto \mathbb{R}$ is supermodular iff for any $x_1, y_1 \in X_1; x_2, y_2 \in X_2$ such that $x_1 \geq y_1$ and $x_2 \geq y_2$, the following holds:*

$$f(x_1, y_2) - f(x_1, x_2) \geq f(y_1, y_2) - f(y_1, x_2)$$

We have, $F_i(q_i, Q) = P(Q + 1)q_i + \frac{P'(Q)}{2}(q_i - \frac{1}{2})^2 - c_i(q_i)$. Let $-Q_1 > -Q_2$. Let $q_i' > q_i$. Then, we have:

$$F_i(q_i, Q_1) - F_i(q_i, Q_2) = (P(Q_1 + 1) - P(Q_2 + 1))q_i + \frac{P'(Q_1) - P'(Q_2)}{2}(q_i - \frac{1}{2})^2$$

$$F_i(q_i', Q_1) - F_i(q_i', Q_2) = (P(Q_1 + 1) - P(Q_2 + 1))q_i' + \frac{P'(Q_1) - P'(Q_2)}{2}(q_i' - \frac{1}{2})^2$$

Since P and P' are a decreasing functions, we have $P(Q_1) \geq P(Q_2)$ and $P'(Q_1) \geq P'(Q_2)$. From this and the fact that $q_i' > q_i$, we have:

$$F_i(q_i', Q_1) - F_i(q_i', Q_2) \geq F_i(q_i, Q_1) - F_i(q_i, Q_2)$$

Therefore F_i^- is a supermodular function. Since $G_i(q_i, Q) = F_i(q_i, Q - 1)$, may a similar argument we can conclude that G_i^- is supermodular.

Proof (Proof of Theorem Lemma 9). We need the following definition and Topkis' Monotonicity Theorem for proving the lemma.

Definition 6. *Given a lattice $(X, \geq)$, we define* Strong Set Ordering *over $A, B \subseteq X$. We say $A \geq_s B$ iff $\forall a \in A, \forall b \in B, \max\{a, b\} \in A \wedge \min\{a, b\} \in B$.*

We note that the strong set ordering induces a natural ordering on sets of consecutive integers. Let $I_1 = \{l_1, \ldots, u_1\}$. Let $I_2 = \{l_2, \ldots, u_2\}$. Then $I_1 \geq_s I_2$ iff $l_1 \geq l_2$ and $u_1 \geq u_2$.

Theorem 14 (Topkis' Monotonicity Theorem[35]). *For any lattices $(X, \geq)$ and $(T, \geq)$, let $f : X \times T \mapsto \mathbb{R}$ be a supermodular function and let $x^*(t) = argmax_{x \in X} f(x, t)$. If $t \geq t'$, then $x^*(t) \geq_s x^*(t')$, i.e., $x^*(t)$ is non-decreasing in t.*

We note that in the theorem above, strong set ordering is used over x^* because $argmax$ returns a set of values from lattice X.

Now we are ready to prove Lemma 9. From Lemma 8, $F_i^-(q_i, -Q)$ is a supermodular function. Thus, from Theorem 14, $q_i^F(Q) = argmax F_i(q_i, Q)$ is a non-decreasing function of $-Q$, i.e., q_i^F is a non-increasing function of Q. Thus $Q > Q' \implies I' \geq_s I$. As noted above, strong set ordering on a set of consecutive integers implies that $q_i^{'l} \geq q_i^l$ and $q_i^{'u} \geq q_i^u$.

Proof (Proof of Theorem Lemma 10). From Lemma 8, $G_i^-(q_i, -Q)$ is a supermodular function. Thus, from Theorem 14, $q_i^G(Q) = argmax F_i(q_i, Q)$ is a non-decreasing function of $-Q$, i.e., q_i^G is a non-increasing function of Q. Thus $Q > Q' \implies I' \geq_s I$. As noted above, strong set ordering on a set of consecutive integers implies that $q_i^{'l} \geq q_i^l$ and $q_i^{'u} \geq q_i^u$.

Proof (Proof of Theorem Lemma 11). Assume for contradiction that such an equilibrium exists for total quantity $Q' > Q$. From Lemma 10, we have $q_i^G(Q) \geq_s q_i^G(Q') = \{q_i^{'l} \ldots q_i^{'u}\}$. Thus, we have $q_i^u \geq q_i^{'u}$. Since Q' is an equilibrium quantity, $\sum_{i=1}^n q_i^{'u} \geq Q'$. Thus, we have $Q' < Q$ and this is a contradiction.

Proof (Proof of Theorem Lemma 12). Assume for contradiction that such an equilibrium exists for total quantity $Q' < Q$. From Lemma 9, we have $q_i^F(Q) \leq_s q_i^F(Q') = \{q_i^{'l} \ldots q_i^{'u}\}$. Thus, we have $q_i^l \leq q_i^{'l}$. Since Q' is an equilibrium quantity, $\sum_{i=1}^n q_i^{'l} leq Q'$. Thus, we have $Q' > Q$ and this is a contradiction.