Partial Quantifier Elimination

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Abstract—We consider the problem of Partial Quantifier Elimination $(PQE)^1$. Given formula $\exists X[F(X,Y) \land G(X,Y)]$, where F, G are in conjunctive normal form, the PQE problem is to find a formula $F^*(Y)$ such that $F^* \land \exists X[G] \equiv \exists X[F \land G]$. We solve the PQE problem by generating and adding to F^* clauses over the free variables that make the clauses of F with quantified variables *redundant*. The traditional Quantifier Elimination problem (QE) is a special case of PQE where G is empty so *all* clauses of the input formula with quantified variables need to be made redundant. The importance of PQE is twofold. First, many problems are more naturally formulated in terms of PQE rather than QE. Second, in many cases PQE can be solved more efficiently than QE. We describe a PQE algorithm based on the machinery of dependency sequents and give experimental results showing the promise of PQE.

I. INTRODUCTION

The elimination of existential quantifiers is an important problem arising in many practical applications. We will refer to this problem as the Quantifier Elimination problem, or QE. Given a formula $\exists X[F]$ where F is a propositional formula, the **QE problem** is to find a quantifier free formula G such that $G \equiv \exists X[F]$. In this paper, we assume that all propositional formulas are represented in conjunctive normal form (CNF).

Unfortunately, the efficiency of current QE algorithms still leaves much to be desired. This is one reason that many successful theorem proving methods such as interpolation and IC3 avoid QE and use SAT-based reasoning instead. These methods can be viewed as solving specialized versions of the QE problem that can be solved efficiently. For example, finding an interpolant I(Y) of formula $A(X, Y) \wedge B(Y, Z)$ comes down to solving a special case of QE where $I \equiv \exists X[A]$ needs to hold only in subspaces where $B \equiv 1$. So it is important to perform a systematic study of the QE problem, looking for variants of the problem that can be solved efficiently. Such a study can help us better understand existing algorithms that sidestep the use of QE in favor for more limited, specialized methods. The study may also lead to the discovery of new applications of QE.

In this paper, we consider a variation of the QE problem called Partial QE (PQE). Let $\exists X[F(X,Y) \land G(X,Y)]$ be a formula where variables of X are quantified. The **PQE problem** is to find a formula $F^*(Y)$ such that $F^* \land \exists X[G] \equiv$ $\exists X[F \land G]$. We will say that F^* is obtained by **taking** F out of the scope of the quantifiers. Note that if $F^* \rightarrow \exists X[G]$ holds, then $F^* \equiv \exists X [F \land G]$. That is, in this case, a solution to the PQE problem is also a solution to the QE problem. We will say that in this case QE reduces to PQE.

Our motivation for solving the PQE problem is twofold. First, in many cases, a verification problem can be formulated as an instance of PQE rather than QE. Besides, even if the original problem is formulated in terms QE it can sometimes be reduced to PQE. Second, in many cases, the PQE problem can be solved much more efficiently than QE. We are especially interested in applying PQE when formula F is much smaller than G.

The relation between efficiency of solving PQE and QE can be better understood in terms of clause redundancy [9]. The PQE problem specified by $\exists X[F \land G]$ reduces to finding a set of clauses F^* that makes all X-clauses of F redundant in formula $\exists X[F \land G]$. (An X-clause is a clause that contains a variable from X.) Then every clause of F can be either dropped as redundant or removed from the scope of the quantifiers as it contains only free variables.

One can view the process of building F^* as follows. Xclauses of F are made redundant in $\exists X[F \land G]$ by adding to F resolvent clauses derived from $F \land G$. Notice that no clause obtained by resolving *only* clauses of G needs to be made redundant. Adding resolvents to F goes on until all Xclauses of the current formula F are redundant. At this point, the X-clauses of F can be dropped and the remaining clauses of F form F^* .

If F is much smaller than G, the process of solving PQE looks like wave propagation where F is the original "perturbation" and G is the "media" where this wave propagates. Such propagation can be efficient even if G is large. By contrast, when solving the QE problem for $\exists X[F \land G]$ one needs to make redundant the X-clauses of both F and G and *all* resolvent X-clauses including the ones obtained by resolving only clauses of G.

In this paper, we describe a PQE-algorithm called *DS*-*PQE* that is based on the machinery of D-Sequents [8], [9]. One needs this machinery for PQE for the same reason as for QE [8]. Every clause of $F^*(Y)$ can be obtained by resolving clauses of $F \wedge G$. However, the number of clauses that are implied by $F \wedge G$ and depend only on Y is, in general, exponential in |Y|. So it is crucial to identify the moment when the set of clauses derived so far that depend only on Y is sufficient to make the X-clauses of F redundant in $\exists X[F \wedge G]$. The machinery of D-sequents is used for such identification. Namely, one can stop generating new clauses when a D-sequent stating redundancy of the X-clauses of F is derived. We experimentally compare *DS-PQE* with our QE

¹The only difference of this technical report from the previous version [10] is as follows. The description of the algorithm given in [10] was missing a case. (The implementation that we tested in experiments was correct but the pseudo-code of the algorithm we gave missed a few lines addressing the case in question.) The missing part of the algorithm is described in Section VII of this report.

algorithm from [9] in the context of model checking.

The following exposition is structured as follows. In Sections II and III, we discuss some problems that can benefit from an efficient PQE-algorithm. A run of *DS-PQE* on a simple formula is described in Section IV. Sections V and VI give basic definitions and recall the notion of D-Sequents. In Section VII, *DS-PQE* is described. We discuss previous work in Section VIII. Experimental results are given in Section IX. Finally, we make conclusions in Section X.

II. USING PQE FOR MODEL CHECKING

In this section and the one that follows we describe some applications where using an efficient PQE solver can be very beneficial. A few more applications of PQE are listed in an extended abstract [11].

A. Computing pre-image in backward model checking

Let T(S, S') be a transition relation where S and S' specify the current and next state variables respectively. We will refer to complete assignments s and s' to variables S and S' as present and next states respectively. Let formula H(S') specify a set of next-states and G(S) specify the pre-image of H(S'). That is, a present state s satisfies G iff there exists a next state s' such that $H(s') \wedge T(s, s') = 1$.

Finding G reduces to QE that is to building a formula logically equivalent to $\exists S'[H \land T]$. However, one can construct the pre-image of H by PQE as follows. Let H^* be a formula such that $H^* \land \exists S'[T] \equiv \exists S'[H \land T]$ *i.e.*, H^* is a solution to the PQE problem. Notice that H^* implies $\exists S'[T]$ because $\exists S'[T] \equiv 1$. Indeed, for every present state s there always exists some next state s' such that T(s, s') = 1. So $H^* \equiv \exists S'[H \land T]$ and hence specifies the pre-image of H. In other words, here QE reduces to PQE.

B. State elimination in IC3-like model checkers

In this subsection, we discuss state elimination, a key problem for IC3-like model checkers [2]. Given a transition relation T(S, S'), the problem of eliminating a state s is to find a clause C falsified by s and inductive relative to a formula F. The latter means that $F \wedge C(S) \wedge T \rightarrow C(S')$.

The performance of IC3 strongly depends on the efficiency of solving the state elimination problem and the quality of inductive clauses generated to solve it. An IC3-like model checker would benefit from an efficient algorithm finding the pre-image of the state s to be eliminated [14]. Finding the pre-image of s can be useful when no inductive clause Celiminates s. In this case, IC3 removes some states that satisfy F and from which a direct transition to s is possible. This is done by adding new clauses to F, which eventually leads to appearance of a clause C that is inductive relative to F and eliminates s. Finding the best states to remove is crucial for the performance of IC3. The pre-image of s can be very useful to identify such states.

Finding the pre-image of s is a special case of the problem we discussed in Subsection II-A. Let H be the set of unit clauses specifying state s *i.e.*, s satisfies H. Let G(S) be a $SAT_by_PQE(G)$ {

```
1
   while (true) {
      C := GenClause(G):
2
      R := SolvePQE(\exists X[C \land G]);
3
      if (R \text{ is derived without using } C) {
4
5
         G := AddClause(G, R);
6
        if (R \equiv 0) return(UNSAT);
7
        continue;}
      if (R \equiv 1) {
8
9
        G := G \cup \{C\};
10
         continue; }
// the only possibility left is R \equiv 0
      if (G \rightarrow C) return(UNSAT);
11
      else return(SAT);
12
```

formula such that $F \wedge G \wedge \exists S'[T] \equiv F \wedge \exists S'[H(S') \wedge T]$. The complete assignments satisfying G specify the pre-image of s "relative" to F. Any clause C inductive relative to F has to be falsified by assignments satisfying $F \wedge G$. The PQE-algorithm we describe in this paper is not efficient enough to be used in the loop of IC3 right away, but this may change soon.

III. USING PQE FOR SAT-SOLVING

In this section, we describe a SAT-algorithm based on PQE. (We will refer to this algorithm as *PQE-SAT*.) We also contrast *PQE-SAT* with a SAT-solver based on Conflict Driven Clause Learning (CDCL).

A. High-level view of the algorithm

The pseudocode of PQE-SAT is shown in Figure 1. Let G(X) be a CNF formula to be checked for satisfiability. In the main loop, PQE-SAT performs the following actions. First, it generates a clause C that is not trivially subsumed by a clause of G (line 2). Then PQE-SAT solves an instance of the PQE problem (line 3). Namely, it calls procedure SolvePQE to find formula R such that $R \land \exists X[G] \equiv \exists X[C \land G]$. Depending on the type of formula R returned by SolvePQE, PQE-SAT either updates G by adding a clause or makes a final decision on whether G is satisfiable (lines 4-12).

SolvePQE returns three kinds of formula R. The actions PQE-SAT take for every kind of formula R are separated by the dotted lines in Figure 1. We will refer to a formula R returned by SolvePQE as a formula of the first kind if it is obtained by resolving only clauses of G (lines 4-7). In this case, R is just a clause that subsumes C. (In particular, R can be equal to C.) On the one hand, the fact that R is derived without using clause C means that R is implied by G. On the other hand, the fact that R subsumes C suggests that C is also implied by G. Thus C is trivially redundant in $\exists X[C \land G]$. PQE-SAT adds clause R to G. If clause R is empty, then G is obviously unsatisfiable.

If resolution derivation of the formula R returned by *SolvePQE* involves clause C we will refer to R as a formula of the second or third kind. In this case, R is a constant. That is R

either has no clauses (formula of the *second kind*, $R \equiv 1$) or it is an empty clause (formula of the *third find*, $R \equiv 0$). Indeed, just derivation of a clause A subsuming C does not mean that C is redundant in $\exists X[C \land G]$. The reason is that A is derived using clause C and so A may not be implied by G. On the other hand, if A is not empty (and hence contains variables of X), it cannot be taken out of the scope of quantifiers.

Actions of PQE-SAT when SolvePQE returns a formula R of the second kind are shown in lines 8-10. The fact that $R \equiv 1$ means that C is redundant in $\exists X[C \land G]$. That is either C is implied by G or C eliminates some (but not all) assignments satisfying G. In either case, $C \land G$ is equisatisfiable to G. For that reason PQE-SAT adds C to G.

What *PQE-SAT* does when *SolvePQE* returns a formula R of the third kind is shown in lines 11-12. The fact that $R \equiv 0$ means that either G is unsatisfiable or C is falsified by *every* assignment satisfying G. *PQE-SAT* tells these two cases apart by checking if C is implied by G.

B. Difference between PQE-SAT and a CDCL SAT-solver

The difference between PQE-SAT and a CDCL SAT-solver is twofold. First, PQE-SAT employs non-resolution derivation of clauses. This derivation occurs, when SolvePQE returns a formula R of the second kind (i.e. $R \equiv 1$). In contrast to a formula of the first kind, in this case, SolvePQE proves that C is redundant in $\exists X[C \land G]$ without generation of a clause subsuming C. A simple example of a clause obtained by nonresolution derivation is a blocked clause [19] (see Section IV). Adding clauses obtained by non-resolution derivation allows one to get proofs that are much shorter than those based on pure resolution. For example, in [18] it was shown that extending resolution with a rule allowing to add blocked clauses makes it exponentially more powerful.

The second difference between PQE-SAT and a CDCL SATsolver is in the way they generate a satisfying assignment. When SolvePQE returns an empty clause (a formula of the third kind) it checks if G implies C. A counterexample showing that $G \not\rightarrow C$ is also an assignment satisfying G. Checking if $G \rightarrow C$ holds reduces to testing the satisfiability of G in the subspace where C is falsified.

As far as finding a satisfying assignment is concerned, PQE-SAT potentially has three advantages over CDCL-solvers. The first advantage is that PQE-SAT can derive clauses that eliminate satisfying assignments of G. This is important because the ability of a CDCL-solver to efficiently find a satisfying assignment hinges on its ability to derive short clauses. For example, if a unit clause \overline{v} is derived by a CDCL-solver, it can immediately set v to 0. However, such a clause cannot be derived if formula G has satisfying assignments with v = 0 and v = 1. The ability of *POE-SAT* to add clauses removing satisfying assignments in general leads to enhancing the quality of learned clauses. Suppose, for example, that PQE-SAT adds to G a clause C that eliminates all satisfying assignments with v = 1 (but preserves at least one satisfying assignment with v = 0). Then formula G implies clause \overline{v} and hence the latter can be derived from G by resolution.

The second advantage of PQE-SAT is that if clause C is long (i.e. C has many literals), then checking $G \rightarrow C$ can be much simpler than just testing the satisfiability of G. The third advantage of PQE-SAT is that in case C is short PQE-SAT can exploit the resolution derivation of an empty clause it obtained. Let P denote such a derivation produced by PQE-SAT. The fact that G is satisfiable and $C \wedge G$ is not means that every assignment satisfying G falsifies C. This entails that every cut of P must contain either clause C itself or a descendant clause, it can have descendants that are very long. So if C is short, one can replace computationally hard check $G \rightarrow C$ with a sequence of checks $G \rightarrow A$ starting with the longest descendant clauses of C.

IV. EXAMPLE

In this section, we describe a run of a PQE algorithm called DS-PQE that is described in Section VII. DS-PQE is a modification of the QE algorithm called DCDS [9] based on the machinery of Dependency sequents (D-sequents). In this section, we use notions (*e.g.*, that of D-sequents) that will be formally defined in Section VI. Recall that an X-clause is a clause that contains at least one variable from a set X of Boolean variables.

Let $F = C_1 \wedge C_2$ where $C_1 = y \vee x_1$, $C_2 = \overline{y} \vee x_3$ Let $G = C_3 \wedge C_4 \wedge C_5 \wedge C_6$ where $C_3 = \overline{x}_1 \vee x_2$, $C_4 = \overline{x}_1 \vee \overline{x}_2$, $C_5 = \overline{x}_3 \vee x_4$, $C_6 = y \vee \overline{x}_4$. Let $X = \{x_1, x_2, x_3, x_4\}$ be the set of variables quantified in formula $\exists X[F \wedge G]$. So y is the only free variable of $F \wedge G$.

Problem formulation. Suppose one needs to solve the PQE problem of taking F out of the scope of the quantifiers in $\exists X[F \land G]$. That is one needs to find $F^*(y)$ such that $F^* \land \exists X[G] \equiv \exists X[F \land G]$. Below, we describe a run of DS-PQE when solving this problem.



Fig. 2. The search tree built by *DS*-*PQE*

Search tree. DS-PQE is a branching algorithm. It first proves redundancy of X-clauses of F in subspaces and then merges results of different branches. When DS-PQE returns to the root of the search tree, all the X-clauses of F are proved redundant in $\exists X[F \land G]$. The search tree built by DS-PQE is given in Figure 2. It also shows the nodes where new clauses C_7 and C_8 were derived.

DS-PQE assigns free variables *before* quantified. For that reason, variable y is assigned first. At every node of the search tree specified by assignment q, *DS-PQE* maintains a set of clauses denoted as PR(q). Here PR stands for "clauses to Prove Redundant". We will refer to a clause of PR(q) as a **PR-clause**. PR(q) includes all X-clauses of F plus some X-clauses of G. The latter are proved redundant to make proving redundancy of X-clauses of F easier. Sets PR(q) are

shown in Figure 4. For every non-leaf node of the search tree two sets of PR-clauses are shown. The set on the left side (respectively right side) of node q gives PR(q) when visiting node q for the first time (respectively when backtracking to the right branch of node q).



Fig. 3. Derived D-sequents

Using D-sequents. The main concern of DS-PQE is to prove redundancy of PR-Branching is used clauses. reach subspaces where to proving redundancy is easy. The redundancy of a PR-clause Cis expressed by a Dependency Sequent D-sequent. In short notation, a D-sequent is a record $s \to \{C\}$ saying that clause C is redundant in formula $\exists X[F \land G]$ in any subspace where assignment s is made. We will refer to *s* as the **conditional**

part of the D-sequent. The D-sequents S_1, \ldots, S_7 derived by *DS-PQE* are shown in Figure 3. They are numbered in the order they were generated. So-called atomic D-sequents record trivial cases of redundancy. More complex D-sequents are derived by a resolution-like operation called *join*. When *DS-PQE* returns to the root, it derives D-sequents stating the unconditional redundancy of the X-clauses of F.

Merging results of different branches. Let v be the current branching variable and v = 0 be the first branch explored by DS-PQE. After completing this branch, DS-PQE proves redundancy of all clauses that currently have the PR-status. (The only exception is the case when a PR-clause gets falsified in branch v = 0. We discuss this exception below.) Then DS-PQE explores branch v = 1 and derives D-sequents stating redundancy of clauses in this branch. Before backtracking from node v, DS-PQE uses operation join to produce Dsequents whose conditional part does not depend on v. For example, in branch y = 0, D-sequent S_1 equal to $(y = 0) \rightarrow$ $\{C_2\}$ was derived. In branch y = 1, D-sequent S_5 equal to $(y = 1) \rightarrow \{C_2\}$ was derived. By joining S_1 and S_5 at variable y, D-sequent S_7 equal to $\emptyset \rightarrow \{C_2\}$ was produced where the conditional part did not depend on y.



Fig. 4. Dynamics of the PR(q) set

Derivation of new clauses. Note that redundancy of the PR-clauses in subspace y =1 was proved without adding any new clauses. On the other hand, proving redundancy of PR-clauses in subspace y = 0required derivation of clauses $C_7 = \overline{x}_1$ and $C_8 = y$. For instance, clause C_7 was generated at node $(y = 0, x_1 = 1)$

by resolving C_3 and C_4 . Clause C_7 was *temporarily* added to F to make PR-clauses C_3 and C_4 redundant at the node above. However, C_7 was removed from formula F after derivation of clause C_8 because the former is subsumed by the latter in subspace y = 0. This is similar to conflict clause generation in SAT-solvers where the intermediate resolvents are discarded.

Derivation of atomic D-sequents. S_1, \ldots, S_5 are the atomic D-sequents derived by DS-PQE. They record trivial cases of redundancy. (Due to the simplicity of this example, the conditional part of all atomic D-sequents has only assignment to y *i.e.*, the free variable. In general, however, the conditional part of a D-sequent also contains assignments to quantified variables.) There are three kinds of atomic D-sequents. Dsequents of the first kind state redundancy of clauses satisfied in a subspace. For instance, D-sequent S_1 states redundancy of clause C_2 satisfied by assignment y = 0. D-sequents of the second kind record the fact that a clause is redundant because some other clause is falsified in the current subspace. For instance, D-sequent S_2 states that C_1 is redundant because clause $C_8 = y$ is falsified in subspace y = 0. D-sequents of the third kind record the fact that a clause is redundant in a subspace because it is blocked at a variable v. That is this clause cannot be resolved on v. For example, D-sequent S_4 states redundancy of C_5 that cannot be resolved on x_4 in subspace $(y = 1, x_3 = 1)$. Clause C_5 is resolvable on x_4 only with C_6 but C_6 is satisfied by assignment y = 1.

Computation of the set of PR-clauses. The original set of PR-clauses is equal to the the initial set of X-clauses of F. Denote this set as PR_{init} . In our example, $PR_{init} = \{C_1, C_2\}$. There are two situations where PR(q) is extended. The first situation occurs when a parent clause of a new resolvent is in PR(q) and this resolvent is an X-clause. Then this resolvent is added to PR(q). An example of that is clause $C_7 = \overline{x_1}$ obtained by resolving PR-clauses C_3 and C_4 .

The second situation occurs when a PR-clause becomes unit. Suppose a PR-clause C is unit at node q, v is the unassigned variable of C and $v \in X$. DS-PQE first makes the assignment falsifying C. Suppose that this is assignment v = 0. Note that all PR-clauses but C itself are obviously redundant at node $q \cup (v = 0)$. DS-PQE backtracks and explores the branch v =1 where clause C is satisfied. At this point DS-PQE extends the set $PR(q \cup (v = 1))$ by adding *every* clause of $F \wedge G$ that a) has literal \overline{v} ; b) is not satisfied; c) is not already in PR(q).

The extension of the set of PR-clauses above is done to guarantee that clause C will be proved redundant when backtracking off the node q. Let us consider the two possible cases. The first case is that formula $F \wedge G$ is *unsatisfiable* in branch v = 1. Then extension of the set of PR-clauses above guarantees that a clause falsified by $q \cup (v = 1)$ will be derived to make the new PR-clauses redundant. Most importantly, this clause will be resolved with C on v to produce a clause rendering C redundant in subspace q. The second case is that formula $F \wedge G$ is satisfiable in branch v = 0. Then the redundancy of the clauses with literal \overline{v} will be proved without derivation of a clause falsified by $q \cup (v = 1)$. When backtracking to node q, clause C will be blocked at variable v and hence redundant. Note that extension of the set PR(q) is *temporary*. When *DS-PQE* backtracks past node q, the clauses that became PR-clauses there lose their PR-status.

Let us get back to our example. The first case above occurs at node y = 0 where PR-clause C_1 becomes unit. DS-PQE falsifies C_1 in branch $x_1 = 0$, backtracks and explores branch $x_1 = 1$. In this branch, clauses C_3, C_4 of G are made PR-clauses. This branch is unsatisfiable. Making C_3, C_4 PRclauses forces DS-PQE to derive $C_7 = \overline{x_1}$ that makes C_3, C_4 redundant. But the real goal of obtaining C_7 is to resolve it with C_1 to produce clause $C_8 = y$ that makes C_1 redundant.

The second case above occurs at node y = 1 where clause C_2 becomes unit. Clause C_2 gets falsified in branch $x_3 = 0$. Then *DS-PQE* backtracks and explores branch $x_3 = 1$. In this branch, C_5 of G becomes a new PR-clause as containing literal \overline{x}_3 . This branch is satisfiable and C_5 is proved redundant without adding new clauses. Clause C_2 gets blocked at node y = 1 and hence redundant.

Forming a solution to the PQE problem. The D-sequents derived by DS-PQE at a node of the search tree are composable. This means that the clauses that are redundant individually are also redundant together. For example, on returning to the root node, D-sequents S_6 and S_7 equal to $\emptyset \to \{C_1\}$ and $\emptyset \to \{C_2\}$ respectively are derived. The composability of S_6 and S_7 means that D-sequent $\emptyset \to \{C_1, C_2\}$ holds as well. The only new clause added to F is $C_8 = y$ (clause C_7 was added temporarily). After dropping the X-clauses C_1, C_2 from F as proved redundant one concludes that $y \land \exists X[G] \equiv \exists X[F \land G]$ and $F^* = y$ is a solution to the PQE problem.

V. BASIC DEFINITIONS

In this section, we give relevant definitions.

Definition 1: An $\exists CNF \ formula$ is a formula of the form $\exists X[F]$ where F is a Boolean CNF formula, and X is a set of Boolean variables. Let q be an assignment, F be a CNF formula, and C be a clause. Vars(q) denotes the variables assigned in q; Vars(F) denotes the set of variables of F; Vars(C) denotes the variables of C; and $Vars(\exists X[F]) = Vars(F) \setminus X$.

We consider *true* and *false* as a special kind of clauses.

Definition 2: Let C be a clause, H be a CNF formula, and q be an assignment such that $Vars(q) \subseteq Vars(H)$. Denote by C_q the clause equal to *true* if C is satisfied by q; otherwise C_q is the clause obtained from C by removing all literals falsified by q. H_q denotes the formula obtained from H by replacing every clause C of H with C_q . In this paper, we assume that clause C_q equal to *true* remains in H_q . We treat such a clause as *redundant* in H_q . Let $\exists X[H]$ be an $\exists CNF$ and y be an assignment to $Vars(H) \setminus X$. Then $(\exists X[H])_y = \exists X[H_y]$.

Definition 3: Let S, Q be \exists CNF formulas. We say that S, Q are **equivalent**, written $S \equiv Q$, if for all assignments, y, such that $Vars(y) \supseteq (Vars(S) \cup Vars(Q))$, we have $S_y = Q_y$. Notice that S_y and Q_y have no free variables, so by $S_y = Q_y$ we mean semantic equivalence.

Definition 4: The Quantifier Elimination (QE) problem for \exists CNF formula \exists X[H] is to find a CNF formula H^{*} such that $H^* \equiv \exists$ X[H]. The **Partial QE (PQE) problem** for \exists CNF formula \exists X[F \land G] is to find a CNF formula F^{*} such that $F^* \land \exists$ X[G] $\equiv \exists$ X[F \land G]. *Definition 5:* Let X be a set of Boolean variables, H be a CNF formula and R be a subset of X-clauses of H. The clauses of R are **redundant** in CNF formula H if $H \equiv (H \setminus R)$. The clauses of R are **redundant** in \exists CNF formula $\exists X[H]$ if $\exists X[H] \equiv \exists X[H \setminus R]$. Note that $H \equiv (H \setminus R)$ implies $\exists X[H] \equiv \exists X[H \setminus R]$ but the opposite is not true.

VI. DEPENDENCY SEQUENTS

In this section, we recall clause Dependency sequents (D-sequents) introduced in [9], operation *join* and the notion of composability. In this paper, we will refer to clause D-sequents as just D-sequents.

Definition 6: Let $\exists X[H]$ be an $\exists CNF$ formula. Let s be an assignment to Vars(H) and R be a subset of X-clauses of H. A dependency sequent (**D-sequent**) has the form $(\exists X[H], s) \rightarrow R$. It states that the clauses of R_s are redundant in $\exists X[H_s]$. Alternatively, we will say that the clauses of R are redundant in $\exists X[H]$ in subspace s (and in any other subspace q such that $s \subseteq q$).

Definition 7: Let s' and s'' be assignments in which exactly one variable $v \in Vars(s') \cap Vars(s'')$ is assigned different values. The assignment s consisting of all the assignments of s' and s'' but those to v is called the *resolvent* of s',s'' on v. Assignments s',s'' are called *resolvable* on v.

Definition 8: Let $\exists X[H]$ be an $\exists CNF$ formula. Let Dsequents $(\exists X[H], s') \rightarrow R$ and $(\exists X[H], s'') \rightarrow R$ hold. We refer to these D-sequents as parent ones. Let s', s'' be resolvable on $v \in Vars(H)$ and s be the resolvent of s' and s''. We will say that D-sequent $(\exists X[H], s) \rightarrow R$ is obtained by **joining** the parents at v. The validity of this D-sequent is implied by that of its parents [9].

Definition 9: Let s' and s'' be assignments to a set of variables Z. We will say that s' and s'' are compatible if every variable of $Vars(s') \cap Vars(s'')$ is assigned the same value in s' and s''.

Definition 10: Let $(\exists X[H], s') \to R'$ and $(\exists X[H], s'') \to R''$ be two D-sequents where s' and s'' are compatible assignments to Vars(H). We will call these D-sequents **composable** if the D-sequent $(\exists X[H], s' \cup s'') \to R' \cup R''$ holds.

VII. ALGORITHM

In this section, we describe a PQE algorithm called **DS**-**PQE** where DS stands for Dependency Sequents. *DS*-*PQE* is based on our QE algorithm *DCDS* described in [9]. In this section, we will mostly focus on the features of *DS*-*PQE* that differentiate it from *DCDS*. The algorithm description given in the first version of this report [10] missed a case. We address this case in Subsections VII-B and VII-C.

DS-PQE derives D-sequents $(\exists X[F \land G], s) \rightarrow \{C\}$ stating the redundancy of X-clause C in any subspace q such that $s \subseteq q$. From now on, we will use a short notation of D-sequents writing $s \rightarrow \{C\}$ instead of $(\exists X[F \land G], s) \rightarrow \{C\}$. We will assume that the parameter $\exists X[F \land G]$ missing in $s \rightarrow \{C\}$ is the *current* \exists CNF formula (with all resolvents added to F). One can omit $\exists X[F \land G]$

- // q is an assignment to $Vars(F \land G)$
- // Ω denotes a set of active D-sequents
- // Φ denotes $\exists X[F \land G]$
- // W denotes PR(q)
- // If DS_PQE returns clause nil (respectively a non-nil clause),
- // $(F \wedge G)_q$ is satisfiable (respectively unsatisfiable)

 $DS_PQE (\Phi, W, q, \Omega) \{$

- 1 if $(\exists \text{ clause } C \in F \cup G \text{ falsif. by } q)$
- $\mathbf{2} \qquad \Omega := atomic_Dseqs1(\Omega, \boldsymbol{q}, C);$
- 3 return (Φ, Ω, C) ;
- 4 $\Omega := atomic_Dseqs2(\Phi, q, \Omega);$
- 5* if $(every_PR_clause_redund(W, \Omega))$ return (Φ, Ω, nil) ;
- 6 $v := pick_variable(F \land G, q, \Omega);$
- 7* $(\Phi, \Omega, C_b) := DS_PQE(\Phi, W, \boldsymbol{q} \cup (v = b), \Omega);$
- 8 $\Omega_{asym} := Dseqs_to_be_inactive(F, \Omega, v);$
- 9 if $(\tilde{\Omega}_{asym} = \emptyset)$ return (Φ, Ω, C_b) ;
- 10 $\Omega := \overline{\Omega} \setminus \Omega_{asym};$
- 11* if $(impl_assgn(v, \bar{b}))$ $W' := newPRclauses(W, F \land G, \bar{b});$ 12* else $W' := \emptyset;$
- 13* $(\Phi, \Omega, C_{\overline{b}}) := DS_PQE \ (\Phi, W \cup W', q \cup (v = \overline{b}), \Omega);$

14** if
$$((C_b = nil) \text{ and } (C_{\tau} \neq nil))$$

15**
$$F := F \wedge C_{\overline{b}}$$
:

17** return
$$(\Phi, \Omega, nil);$$
}

18 if $((C_b \neq nil) \text{ and } (C_{\overline{b}} \neq nil))$

- 19 $C := resolve_clauses(C_b, C_{\overline{b}}, v);$
- 20 $F := F \wedge C;$
- 21 $\Omega := atomic_Dseqs1(\Omega, q, C);$
- 22* if $((C_b \in W) \text{ or } (C_{\overline{b}} \in W))$
- $23^* \qquad W := W \cup \{C\};$
- 24 return $(\Phi, \Omega, C);$
- 25 $\Omega := merge(\Phi, \boldsymbol{q}, v, \Omega_{asym}, \Omega, C_b, C_{\overline{b}});$
- 26 return $(\Phi, \Omega, nil);$

Fig. 5. DS-PQE procedure

from D-sequents because $(\exists X[F \land G], s) \rightarrow \{C\}$ holds no matter how many resolvent clauses are added to F [9]. We will call D-sequent $s \rightarrow \{C\}$ active in subspace q if $s \subseteq q$. The fact that $s \rightarrow \{C\}$ is active in subspace q means that Cis redundant in $\exists X[F \land G]$ in subspace q.

A. Input and output of DS-PQE

Recall that a PR-clause is an X-clause of $F \wedge G$ whose redundancy needs to be proved in subspace q (see Section IV). A description of DS-PQE is given in Figure 5. DS-PQE accepts an $\exists CNF$ formula $\exists X[F \wedge G]$ (denoted as Φ), an assignment q to Vars(F), the set of PR-clauses (denoted as W) and a set Ω of D-sequents active in subspace q stating redundancy of *some* PR-clauses in $\exists X[F \wedge G]$ in subspace q.

Similarly to Section IV, we will assume that the resolvent clauses are added to formula F while formula G remains unchanged. *DS-PQE* returns a formula $\exists X[F \land G]$ modified by resolvent clauses added to F (if any), a set Ω of D-sequents active in subspace q that state redundancy of *all* PR-clauses in $\exists X[F \land G]$ in subspace q and a clause C. If $(F \land G)_q$ is unsatisfiable then C is a clause of $F \land G$ falsified by q.

Otherwise, C is equal to *nil* meaning that no clause implied by $F \wedge G$ is falsified by q.

The active D-sequents derived by DS-PQE are composable. That is if $s_1 \rightarrow \{C_1\}, \ldots, s_k \rightarrow \{C_k\}$ are the active D-sequents of subspace q, then the D-sequent $s^* \rightarrow \{C_1, \ldots, C_k\}$ holds where $s^* = s_1 \cup \ldots \cup s_k$ and $s^* \subseteq q$. Like *DCDS*, *DS-PQE* achieves composability of D-sequents by proving redundancy of PR-clauses in a particular order (that can be different for different paths). This guarantees that no circular reasoning is possible and hence the D-sequents derived at a node of the search tree are composable.

A solution to the PQE problem in subspace q is obtained by discarding the PR-clauses of subspace q (specified by W) from the CNF formula F returned by DS-PQE. To solve the original problem of taking F out of the scope of the quantifiers in $\exists X[F \land G]$, one needs to call DS-PQE with $q = \emptyset$, $\Omega =$ \emptyset , $W = PR_{init}$. Recall that PR_{init} is the set of X-clauses of the original formula F.

B. The big picture

DS-PQE consists of four parts separated in Figure 5 by the dotted lines. In the first part (lines 1-5), *DS-PQE* builds atomic D-sequents recording trivial cases of redundancy of X-clauses. If all the PR-clauses are proved redundant in $\exists X[F \land G]$ in subspace q, *DS-PQE* terminates at node q.

If some PR-clauses are not proved redundant yet, DS-PQE enters the second part of the code (lines 6-13). First, DS-PQE picks a branching variable v (line 6). Then it recursively calls itself (line 7) starting the left branch of v by adding to qassignment $v = b, b \in \{0, 1\}$. Once the left branch is finished, DS-PQE explores the right branch $v = \overline{b}$ (line 13).

The third part of DS-PQE (lines 14-17) takes care of the situation where

- the left branch is satisfiable
- the right branch is unsatisfiable
- assignment $v = \overline{b}$ was not derived from a unit clause

. (This situation was not mentioned in the first version of this report [10].) In this case, *DS-PQE* simply

- adds to formula F clause C_b derived in the right branch (and falsified by q ∪ (v = b̄)),
- discards D-sequents whose conditional part contains an assignment to variable v (derived in the left and right branches) and backtracks.

Note that after backtracking, value \overline{b} is derived from $C_{\overline{b}}$ and the third part of *DS-PQE* is not invoked again when branching on variable v. The reason for such a behavior of *DS-PQE* is explained in Subsection VII-C.

In the fourth part, DS-PQE merges the left and right branches (lines 18-26). This merging results in proving all PRclauses redundant in $\exists X[F \land G]$ in subspace q. For every PRclause C proved redundant in subspace q, the set Ω contains precisely one active D-sequent $s \rightarrow \{C\}$ where $s \subseteq q$. As soon as C is proved redundant, it is marked and ignored until DS-PQE enters a subspace q' where $s \not\subseteq q'$ *i.e.*, a subspace where D-sequent $s \rightarrow \{C\}$ becomes inactive. Then clause C gets unmarked signaling that DS-PQE does not have a proof of redundancy of C in subspace q' yet.

C. New features of DS-PQE with respect to DCDS

In this paper, we omit the description of functions of Figure 5 that operate identically to those of *DCDS*. What these functions do can be understood from the example of Section IV. If this is not enough, the detailed description of these functions can be found in [9]. In this subsection, we focus on the part of *DS-PQE* that is different from *DCDS*. The lines of code of this part are marked with asterisks in Figure 5. Lines 14-17 are marked with double asterisks to indicate that they are not present in the first version of this report [10].

The main difference between DS-PQE and DCDS is that at every node q of the search tree, DS-PQE maintains a set PR(q) of PR-clauses. PR(q) contains all the X-clauses of F and some X-clauses of G (if any). DS-PQE terminates its work at node q when all the current PR-clauses are proved redundant (line 5). In contrast to DS-PQE, DCDS terminates at node q, when all X-clauses are proved redundant. Line 7 is marked because DS-PQE uses an additional parameter Wwhen recursively calling itself to start the left branch of node q. Here W specifies the set of PR-clauses to prove redundant in the left branch.

Lines 11-12 show how PR(q) is extended. As we discussed in Section IV, this extension takes place when assignment $v = \overline{b}$ satisfies a unit PR-clause C. In this case, the set W' of new PR-clauses is computed. It consists of all the X-clauses that a) contain the literal of v falsified by assignment $v = \overline{b}$; b) are not PR-clauses and c) are not satisfied. As we explained in Section IV, this is done to facilitate proving redundancy of clause C at node q. The set W' is added to W before the right branch is explored (line 13). Notice that the clauses of W' have PR-status only in the subtree rooted at node q. Upon return to node q from the right branch, the clauses of W' lose their PR-status.

Lines 14-17 address the special situation described in Subsection VII-B: the left branch is satisfiable and clause $C_{\overline{b}}$ is derived in the right branch, the latter being unsatisfiable. The problem here is as follows. To prove redundancy of clause $C_{\overline{b}}$, one needs to show redundancy of clauses that can be resolved with $C_{\overline{b}}$ on variable v. The redundancy of such clauses is supposed to be proved in the left branch. However, the left branch was examined when clause $C_{\overline{b}}$ was not in formula F. So, DS-PQE could not compute the set of PR-clauses of the left branch correctly. To solve this problem, DS-PQE simply adds $C_{\overline{b}}$ to F and backtracks unassigning variable v. Note that now assignment $v = \overline{b}$ can be derived from clause $C_{\overline{b}}$. So DS-PQE knows that it needs to prove redundancy of clauses that can be resolved with $C_{\overline{b}}$ (line 11).

As we mentioned in Section IV, one more source of new PR-clauses are resolvents (lines 22-23). Let v = b and $v = \overline{b}$ be unsatisfiable branches and C_b and $C_{\overline{b}}$ be the clauses returned by *DS-PQE*. If C_b or $C_{\overline{b}}$ is currently a PR-clause, the resolvent C becomes a new PR-clause. One can think of a PR-clause as supplied with a tag indicating the level up to which this

clause preserves its PR-status. If only one of the clauses C_b and $C_{\overline{b}}$ is a PR-clause, then C inherits the tag of this clause. If both parents have the PR-status, the resolvent inherits the tag of the parent clause that preserves its PR-status longer.

D. Correctness of DS-PQE

The correctness of DS-PQE is proved similarly to that of DCDS [9]. DS-PQE is complete because it examines a finite search tree. Here is an informal explanation of why DS-PQE is sound. First, the clauses added to F are produced by resolution and so are correct in the sense they are implied by $F \wedge G$. Second, the atomic D-sequents built by DS-PQE are correct. Third, new D-sequents produced by operation *join* are correct. Fourth, the D-sequents of individual clauses are composable.

So when DS-PQE returns to the root node of the search tree, it derives the correct D-sequent $(\exists X[F \land G], \emptyset) \rightarrow F^X$. Here F^X denotes the set of all X-clauses of F. Thus, by removing the X-clauses from F one obtains formula F^* such that $\exists X[F^* \land G] \equiv \exists X[F \land G]$. Since F^* does not depend on variables of X it can be taken out of the scope of quantifiers.

VIII. BACKGROUND

QE has been studied by many researchers, due to its important role in verification *e.g.*, in model checking. QE methods are typically based on BDDs [3], [4] or SAT [20], [13], [22], [16], [7], [15], [17]. At the same time, we do not know of research where the PQE problem was solved or even formulated. Of course, identification and removal of redundant clauses is often used in preprocessing procedures of QBF-algorithms and SAT-solvers [6], [1]. However, these procedures typically exploit only situations where clause redundancies are obvious.

PQE is different from QE in at least two aspects. First, a PQE-algorithm has to have a significant degree of "structureawareness", since PQE is essentially based on the notion of redundancy. So it is not clear, for example, if a BDDbased algorithm would benefit from replacing QE with PQE. This also applies to many SAT-based algorithms of QE. For instance, in [8] we presented a QE algorithm called DDS that was arguably more structure aware than its SAT-based predecessors. DDS is based on the notion of D-sequents defined in terms of variable redundancy. DDS makes quantified variables redundant in subspaces and merges the results of different branches. Despite its structure-awareness, it is hard to adjust DDS to solving PQE: in PQE, one, in general, does not eliminate quantified variables (only some clauses with quantified variables are eliminated).

The second interesting aspect of PQE is as follows. QE can be solved by a trivial albeit inefficient algorithm. Namely, to find a quantifier-free formula equivalent to $\exists X[H]$ one can just resolve out all variables of X as it is done in the DP procedure [5]. However, the PQE problem does not have a counterpart of this algorithm *i.e.*, PQE does no have a "trivial" PQE-solver. Let C be a clause of H and v be a variable of C. One can always make C redundant by adding to H all resolvents of C with clauses of H on v [12], [21]. So one can always "resolve out" any clause of a CNF formula. It seems that one can take formula F out of the scope of the quantifiers in $\exists X[F \land G]$ using the following procedure. Keep resolving out clauses of F and their resolvents with G until all non-redundant resolvents depend only on free variables. Unfortunately, this procedure may loop *i.e.*, a previously seen set of clauses $F \land G$ may be reproduced later. *DS-PQE* does not have this problem due to branching.

IX. EXPERIMENTAL RESULTS

Since we are not aware of another tool performing PQE, in the experiments we focused on contrasting PQE and QE. Namely, we compared *DS-PQE* with our QE algorithm called *DCDS* [9]. The fact that *DS-PQE* and *DCDS* are close in terms of implementation techniques is beneficial: any difference in performance should be attributed to difference in algorithms rather than implementations.

In the experiments, we used DS-PQE and DCDS for backward model checking. We will refer to model checkers based on DS-PQE and DCDS as MC-PQE and MC-QE respectively. The difference between MC-PQE and MC-QE is as follows. Let F(S') and T(S,S') specify a set of next-states and transition relation respectively. The basic operation here is to find the pre-image H(S) of F where $H \equiv \exists S'[F \land T]$. So H is a solution to the QE problem. As we showed in Subsection II-A, one can also find H just by taking F out of the scope of the quantifiers in formula $\exists S'[F \land T]$. MC-QE computes H by making redundant all S'-clauses of $F \land T$ while MC-PQE finds H by making redundant only the S'-clauses of F.



The current implementations

DCDS of and DS-PQE lack Dsequent re-using: the parent D-sequents are discarded after a join operation. We believe that re-using **D**-sequents should boost performance like clause recording in SAT-solving. However, when working on a

Fig. 6. Performance of model checkers on 282 examples solved by *MC-QE* or *MC-PQE*

new version of *DCDS* we found out that re-using D-sequents indiscriminately may lead to circular reasoning. We have solved this problem theoretically and resumed our work on the new version of *DCDS*. However, here we report the results of implementations that do not re-use D-sequents.

We compared *MC-PQE* and *MC-QE* on the 758 benchmarks of HWMCC-10 competition [23]. With the time limit of 2,000s, *MC-QE* and *MC-PQE* solved 258 and 279 benchmarks respectively. On the set of 253 benchmarks solved by both model checkers, *MC-PQE* was about 2 times faster (the total time is 4,652s versus 8,528s). However, on the set of

 TABLE I

 Model checking results on some concrete examples

benchmark	#lat-	#gates	#ite-	bug	MC-	MC-
	ches		rati-		QE	PQE
			ons		(s.)	(s.)
bj08amba3g62	32	9,825	4	no	241	38
kenflashp03	51	3,738	2	no	33	104
pdtvishuffman2	55	831	6	yes	>2,000	296
pdtvisvsar05	82	2,097	4	no	1,368	7.7
pdtvisvsa16a01	188	6,162	2	no	>2,000	17
texaspimainp12	239	7,987	4	no	807	580
texasparsesysp1	312	11,860	10	yes	39	25
pj2002	1,175	15,384	3	no	254	47
mentorbm1and	4,344	31,684	2	no	1.4	1.7

282 benchmarks solved by at least one model checker *MC*-*PQE* was about 6 times faster (10,652s versus 60,528s). Here we charged 2,000s, *i.e.*, the time limit, for every unsolved benchmark.

Figure 6 gives the performance of MC-QE and MC-PQE on the 282 benchmarks solved by at least one model checker in terms of the number of problems finished in a given amount of time. Figure 6 shows that MC-PQE consistently outperformed MC-QE. Model checking results on some concrete benchmarks are given in Table I. The column *iterations* show the number of backward images computed by the algorithms before finding a bug or reaching a fixed point.

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X. CONCLUSION

We introduced the Partial Quantifier Elimination problem (PQE), a generalization of the Quantifier Elimination problem (QE). We presented a PQE-algorithm based on the machinery of D-sequents and gave experimental results showing that PQE can be much more efficient than QE. Efficient PQE-solver may lead to new methods of solving old problems like SAT-solving. In addition, many verification problems can be formulated and solved in terms of PQE rather than QE, a topic ripe for further exploration.

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