# Compositionality for Quantitative Specifications 

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#### Abstract

We provide a framework for compositional and iterative design and verification of systems with quantitative information, such as rewards, time or energy. It is based on disjunctive modal transition systems where we allow actions to bear various types of quantitative information. Throughout the design process the actions can be further refined and the information made more precise. We show how to compute the results of standard operations on the systems, including the quotient (residual), which has not been previously considered for quantitative non-deterministic systems. Our quantitative framework has close connections to the modal nu-calculus and is compositional with respect to general notions of distances between systems and the standard operations.


## 1 Introduction

Specifications of systems come in two main flavors. Logical specifications are formalized as formulae of modal or temporal logics, such as the modal $\mu$-calculus or LTL. A common way to verify them on a system is to translate them to automata and then analyze the composition of the system and the automaton. In contrast, in the behavioral approach, specifications are given, from the very beginning, in an automata-like formalism. Such properties can be verified using various equivalences and preorders, such as bisimilarity or refinement. Here

[^0]

Fig. 3 A simple real-time specification


Fig. 4 A disjunctive modal transition system

$$
\begin{aligned}
& \nu X \cdot([\text { grant }, \text { idle, work }] X \wedge \\
& {[\text { req }] \nu Y \cdot[\text { idle, req]ff } \wedge} \\
& \quad\left(\left\langle\left\langle\begin{array}{l}
\text { work } \\
{[2,4]}
\end{array}\right\rangle Y \vee\left\langle\begin{array}{c}
\text { grant } \\
{[0,5]}
\end{array}\right\rangle X\right)\right)
\end{aligned}
$$

Fig. 5 The $\nu$-calculus translation of the DMTS in Fig. 4

This low-level refinement process is, however, insufficient when the designer wants to get more specific about the implemented actions, such as going from the coarse specification just described to the more finegrained specification of Fig. 2 In order to relate such specifications, MTS with structured labels have been introduced [8. Given a preorder on labels, relating for instance coffee $\preccurlyeq$ beverage, we can refine a transition label into one which is below, for example implement "beverage" with its refinement "coffee".

This framework can be applied to various preorders. For example, one can use labels with a discrete component carrying the action information and an interval component to model time duration or energy consumption. As an example, consider the simple real-time property in Fig. 3. "after a req(uest), grant has to be executed within 5 time units without the process being idle meanwhile". The transition (grant, $[0,5]$ ) could be safely refined to (grant, $[l, r]$ ) for any $0 \leq l \leq r \leq 5$.

We proceed to identify several shortcomings of the current approaches.

Expressive power. The current theory of structured labels [8, 30] is available only for the basic MTS. Very often one needs to use richer structures such as disjunctive MTS (DMTS) [12,51] or acceptance automata 37, 55. While MTS generally cannot express disjunction of properties, DMTS and further related formalisms can and are, in fact, equivalent to the modal $\nu$-calculus [10, 33], i.e., the maximal-fixed point fragment of the modal $\mu$-calculus 44.

This allows, for instance, to prohibit deadlocks as in the example in Fig. 4. The disjunctive must, depicted


Fig. 6 Two implementations


Fig. 7 Another DMTS specification
as a branching arrow, requires at least one of the transitions to be present. Thus we allow the deadline for grant to be reset if additional work is generated. Note that specifying grant and work as two separate musttransitions would not allow postponing the deadline; and two separate may-transitions would not guarantee any progress, as none of them has to be implemented. We hence propose DMTS with structured labels and also extend the equivalence between DMTS and the modal $\nu$-calculus [10, 33] to our setting. Figure 5 shows a $\nu$ calculus translation of the DMTS in Fig. 4,

Robustness. Consider again the request-grant example in Fig. [4, together with the two labeled transition systems in Fig. 6. While $i_{1}$, issuing grant after precisely 5 time units, is a valid implementation of $x$, if there is but a small positive drift in the timing, like in $i_{2}$, it is not an implementation anymore. However, this drift might be easily mended or just might be due to measuring errors.

Therefore, when models and specifications contain such quantitative information, the standard Boolean notions of satisfaction and refinement are of limited utility 39,58 and should be replaced by notions which are more robust to perturbations. For another example, the DMTS of Fig. 7 is not a refinement of the one in Fig. (4, but for all practical purposes, it is rather close.

One approach to robustness is to employ metric distances instead of Boolean relations; this has been done for example in [19] $21,26,38,49,56,57,61,62]$ and many other papers. An advantage of behavioral specification formalisms is that models and specifications are closely related, hence distances between models can easily be extended to distances between specifications. We have developed a distance-based approach for MTS in [5-7, 30] and shown in [7, 29 32 that a good general
setting is given by recursively specified trace distances on an abstract quantale. Here we extend this to DMTS.

Compositionality. The framework should be compositional. In the quantitative setting, this in essence means that the operations we define on the systems should behave well with respect not only to satisfaction, but also to the distances. For instance, if $s_{1}$ is close to $t_{1}$ and $s_{2}$ close to $t_{2}$, then also the structural composition $s_{1} \| s_{2}$ should be close to $t_{1} \| t_{2}$. We prove this for the usual operations; in particular, we give a construction for such a well-behaved quotient.

The quotient of $s$ by $t$ is the most general specification that, when composed with $t$, refines $s$. This operation is thus useful for computing missing parts of a system to be implemented, when we already have several components at our disposal. The construction is complex already in the non-quantitative setting [10] and the extension of the algorithm to structured labels is non-trivial.

Our contribution. To sum up, we extend the framework of structured labels to DMTS and the modal $\nu$-calculus. To be able to reason about robustness, we equip this framework with distances and give constructions for the structured analogues of the standard operations, so that they behave compositionally with respect to the distances.

Further related work. Refinement of components is a frequently used design approach in various areas, ranging from subtyping [52] over the Java modeling language JML [41] or correct-by-design class diagram operations [27] to interface theories close to MTS such as interface automata [22] based on alternating simulation. A variant of alternating simulation called covariantcontravariant simulation has been compared to MTS modal refinement in [1]. The graphical representability of these variants was studied in [10, 14.

Quantitative specifications have been introduced in other settings. At first, the focus was on probabilities 40, 53, 54, but later, predicates with values in arbitrary metric spaces were also introduced [21. Robustness of probabilistic specifications is considered in [19| 21]. It is our hope that the close relationship between quantitative DMTS and the quantitative modal $\nu$-calculus which we expose in this paper will aid in the development of theory and tools also for probabilistic specifications.

There are a number of extensions of MTS specifically designed for coping with real-time properties: the
timed input-output specifications of [18], the timed interfaces of [23], and the modal event-clock specifications of [13]. Robustness for timed input-output specifications is considered in [47, 48, 59]. With only little extra work, our notions of distances and robustness can be applied to real-time specifications, see [28] for modal event-clock specifications.

Some other extensions of MTS have been developed for probabilistic properties: the constraint Markov chains of [15, 25, 42] and the abstract probabilistic automata of [15]. Distances for such specifications are used in [24], but no work on robustness is available.

## 2 Structured Labels

Let $\Sigma$ be a poset with partial order $\preccurlyeq$. We think of $\preccurlyeq$ as label refinement, so that if $a \preccurlyeq b$, then $a$ is less permissive (more restricted) than $b$.

Definition 1 A label $a \in \Sigma$ is an implementation label if $b \preccurlyeq a$ implies $b=a$ for all $b \in \Sigma$. The set of implementation labels is denoted $\Gamma$, and for $a \in \Sigma$, we let $\llbracket a \rrbracket=\{b \in \Gamma \mid b \preccurlyeq a\}$ denote the set of its implementations.

Hence $a$ is an implementation label iff $a$ cannot be further refined. Note that $a \preccurlyeq b$ implies $\llbracket a \rrbracket \subseteq \llbracket b \rrbracket$ for all $a, b \in \Sigma$.

Example 1 A trivial but important example of our label structure is the discrete one in which label refinement $\preccurlyeq$ is equality (and $\Gamma=\Sigma$ ). This is equivalent to the "standard" case of unstructured labels.

A typical label set in quantitative applications consists of a discrete component and real-valued weights. For specifications, weights are replaced by (closed) weight intervals, so that $\Sigma=U \times\{[l, r] \mid l \in \mathbb{R} \cup\{-\infty\}, r \in$ $\mathbb{R} \cup\{\infty\}, l \leq r\}$ for a finite set $U, c f$. [6] 8]. Label refinement is given by $\left(u_{1},\left[l_{1}, r_{1}\right]\right) \preccurlyeq\left(u_{2},\left[l_{2}, r_{2}\right]\right)$ iff $u_{1}=u_{2}$ and $\left[l_{1}, r_{1}\right] \subseteq\left[l_{2}, r_{2}\right]$, so that labels are more refined if they specify smaller intervals; thus, $\Gamma=U \times\{[x, x] \mid$ $x \in \mathbb{R}\} \approx U \times \mathbb{R}$.

For a quite general setting, we can instead start with an arbitrary set $\Gamma$ of implementation labels, let $\Sigma=2^{\Gamma}$, the powerset, and $\preccurlyeq=\subseteq$ be subset inclusion. Then $\llbracket a \rrbracket=a$ for all $a \in \Sigma$. (Hence we identify implementation labels with one-element subsets of $\Sigma$.)

### 2.1 Label operations

Specification theories come equipped with several standard operations that make compositional software design possible [4: conjunction for merging viewpoints
covering different system's aspects 960, structural composition for running components in parallel, and quotient to synthesize missing parts of systems 51]. In order to provide them for DMTS, we first need the respective atomic operations on their action labels.

We hence assume that $\Sigma$ comes equipped with a partial conjunction, i.e., an operator $\mathbb{Q}: \Sigma \times \Sigma \rightharpoonup \Sigma$ for which it holds that
(1) if $a_{1} \otimes a_{2}$ is defined, then $a_{1} \otimes a_{2} \preccurlyeq a_{1}$ and $a_{1} \otimes a_{2} \preccurlyeq$ $a_{2}$, and
(2) if $a_{3} \preccurlyeq a_{1}$ and $a_{3} \preccurlyeq a_{2}$, then $a_{1} \otimes a_{2}$ is defined and $a_{3} \preccurlyeq a_{1} \otimes a_{2}$.

Note that by these properties, any two partial conjunctions on $\Sigma$ have to agree on elements for which they are both defined.

Example 2 For discrete labels, the unique conjunction operator is given by
$a_{1} \otimes a_{2}= \begin{cases}a_{1} & \text { if } a_{1}=a_{2}, \\ \text { undef. } & \text { otherwise } .\end{cases}$
Indeed, by property (2), $a_{1} \otimes a_{2}$ must be defined for $a_{1}=a_{2}$, and by (1), if $a_{1} \otimes a_{2}=a_{3}$ is defined, then $a_{3}=a_{1}$ and $a_{3}=a_{2}$.

For labels in $U \times\{[l, r] \mid l, r \in \mathbb{R}, l \leq r\}$, the unique conjunction is

$$
\begin{aligned}
& \left(u_{1},\left[l_{1}, r_{1}\right]\right) \otimes\left(u_{2},\left[l_{2}, r_{2}\right]\right)= \\
& \begin{cases}\text { undef. } & \text { if } u_{1} \neq u_{2} \\
\left(u_{1},\left[l_{1}, r_{1}\right] \cap\left[l_{1}, r_{1}\right] \cap\left[l_{2}, r_{2}\right]=\emptyset,\right. & \text { otherwise } .\end{cases}
\end{aligned}
$$

To see uniqueness, let $a_{i}=\left(u_{i},\left[l_{i}, r_{i}\right]\right)$ for $i=1,2,3$. Using property (2), we see that $a_{1} \otimes a_{2}$ must be defined when $u_{1}=u_{2}$ and $\left[l_{1}, r_{1}\right] \cap\left[l_{2}, r_{2}\right] \neq \emptyset$, and by (2), if $a_{1} \otimes a_{2}=a_{3}$ is defined, then $u_{3}=u_{1}$ and $u_{3}=u_{2}$, and $\left[l_{3}, r_{3}\right] \subseteq\left[l_{1}, r_{1}\right],\left[l_{3}, r_{3}\right] \subseteq\left[l_{2}, r_{2}\right]$ imply $\left[l_{1}, r_{1}\right] \cap\left[l_{2}, r_{2}\right] \neq$ $\emptyset$.

Finally, for the case of specification labels as sets of implementation labels, the unique conjunction is $a_{1} \otimes$ $a_{2}=a_{1} \cap a_{2}$.

For structural composition and quotient of specifications, we assume a partial label synchronization operator $(1): \Sigma \times \Sigma \rightharpoonup \Sigma$ which specifies how to compose labels. We assume $(1)$ to be associative and commutative, with the following technical property which we shall need later: For all $a_{1}, a_{2}, b_{1}, b_{2} \in \Sigma$ with $a_{1} \preccurlyeq a_{2}$ and $b_{1} \preccurlyeq b_{2}, a_{1} \oplus b_{1}$ is defined iff $a_{2} \oplus b_{2}$ is, and if both are defined, then $a_{1} \oplus b_{1} \preccurlyeq a_{2} \oplus b_{2}$.

Example 3 For discrete labels, the conjunction of Example 2 is the same as CSP-style composition, i.e., $a$ ©
$b=a$ if $a=b$ and undefined otherwise, but other compositions can easily be defined.

For labels in $U \times\{[l, r] \mid l, r \in \mathbb{R}, l \leq r\}$, several useful label synchronization operators may be defined for different applications. One is given by addition of intervals, i.e.,

$$
\begin{aligned}
\left(u_{1},\left[l_{1}, r_{1}\right]\right) & \Phi\left(u_{2},\left[l_{2}, r_{2}\right]\right)= \\
& \left\{\begin{array}{l}
\text { undef. } \quad \text { if } u_{1} \neq u_{2}, \\
\left(u_{1},\left[l_{1}+l_{2}, r_{1}+r_{2}\right]\right)
\end{array} \quad\right. \text { otherwise }
\end{aligned}
$$

for example modeling computation time of actions on a single processor. Another operator, useful in scheduling, uses maximum instead of addition:

$$
\begin{aligned}
& \left(u_{1},\left[l_{1}, r_{1}\right]\right)^{\max }\left(u_{2},\left[l_{2}, r_{2}\right]\right)= \\
& \quad\left\{\begin{array}{l}
\text { undef. if } u_{1} \neq u_{2}, \\
\left(u_{1},\left[\max \left(l_{1}, l_{2}\right), \max \left(r_{1}, r_{2}\right)\right]\right)
\end{array} \quad\right. \text { otherwise. }
\end{aligned}
$$

For set-valued specification labels, we may take any synchronization operator $(1)$ given on implementation labels $\Gamma$ and lift it to one on $\Sigma$ by $a_{1} \oplus a_{2}=\left\{b_{1} \oplus b_{2} \mid\right.$ $\left.b_{1} \in \llbracket a_{1} \rrbracket, b_{2} \in \llbracket a_{2} \rrbracket\right\}$.

## 3 Specification Formalisms

In this section we introduce the specification formalisms which we use in the rest of the paper. The universe of models for our specifications is the one of standard labeled transition systems. For simplicity of exposition, we work only with finite specifications and implementations, but most of our results extend to the infinite (but finitely branching) case.

A labeled transition system (LTS) is a structure $\mathcal{I}=$ $\left(S, s^{0}, \longrightarrow\right)$ consisting of a finite set $S$ of states, an initial state $s^{0} \in S$, and a transition relation $\longrightarrow \subseteq S \times$ $\Gamma \times S$. We usually write $s \xrightarrow{a} t$ instead of $(s, a, t) \in \longrightarrow$. Note that transitions are labeled with implementation labels.

### 3.1 Disjunctive Modal Transition Systems

A disjunctive modal transition system (DMTS) is a structure $\mathcal{D}=\left(S, S^{0}, \rightarrow, \longrightarrow\right)$ consisting of finite sets $S \supseteq$ $S^{0}$ of states and initial states, respectively, may-transitions $\rightarrow \subseteq S \times \Sigma \times S$, and disjunctive must-transitions $\longrightarrow \subseteq S \times 2^{\Sigma \times S}$. It is assumed that for all $(s, N) \in \longrightarrow$ and $(a, t) \in N$ there is $(s, b, t) \in \rightarrow$ with $a \preccurlyeq b$.

Example 4 The specification $x$ in Fig. 5 has a maytransition to $y$; from there we have a disjunctive musttransition with identical underlying may-transitions. The
intuitive meaning of the transition, that either grant or work must be available, is formalized below using the modal refinement.

Note that we allow multiple (or zero) initial states. We write $s \xrightarrow{a} t$ instead of $(s, a, t) \in \longrightarrow$ and $s \longrightarrow N$ instead of $(s, N) \in \longrightarrow$.

A DMTS $\left(S, S^{0}, \rightarrow \rightarrow, \longrightarrow\right)$ is an implementation if $\rightarrow \subseteq S \times \Gamma \times S, \longrightarrow=\{(s,\{(a, t)\}) \mid s \xrightarrow{a} t\}$, and $S^{0}=\left\{s^{0}\right\}$ is a singleton; DMTS implementations are hence isomorphic to LTS.

DMTS were introduced in [51] in the context of equation solving, or quotient of specifications by processes and are used e.g., in [12] for LTL model checking. They are a natural extension of modal transition systems [50, which are DMTS in which all disjunctive must-transitions $s \longrightarrow N$ lead to singletons $N=$ $\{(a, t)\}$; in fact, DMTS are the closure of MTS under quotient 51.

We introduce a notion of modal refinement of DMTS with structured labels. For discrete labels, it coincides with the classical definition 51].

Definition 2 Let $\mathcal{D}_{1}=\left(S_{1}, S_{1}^{0}, \rightarrow \rightarrow_{1}, \longrightarrow_{1}\right)$ and $\mathcal{D}_{2}=$ $\left(S_{2}, S_{2}^{0}, \cdots \rightarrow_{2}, \longrightarrow_{2}\right)$ be DMTS. A relation $R \subseteq S_{1} \times S_{2}$ is a modal refinement if it holds for all $\left(s_{1}, s_{2}\right) \in R$ that

- for all $s_{1} \xrightarrow[-a_{1}]{a_{1}} t_{1}$ there is $s_{2} \xrightarrow{a_{2}}{ }_{2} t_{2}$ such that $a_{1} \preccurlyeq$ $a_{2}$ and $\left(t_{1}, t_{2}\right) \in R$, and
- for all $s_{2} \longrightarrow 2 N_{2}$ there is $s_{1} \longrightarrow_{1} N_{1}$ such that for all $\left(a_{1}, t_{1}\right) \in N_{1}$ there is $\left(a_{2}, t_{2}\right) \in N_{2}$ with $a_{1} \preccurlyeq a_{2}$ and $\left(t_{1}, t_{2}\right) \in R$.
$\mathcal{D}_{1}$ refines $\mathcal{D}_{2}$, denoted $\mathcal{D}_{1} \leq_{\mathrm{m}} \mathcal{D}_{2}$, if there exists an initialized modal refinement $R$, i.e., one for which it holds that for every $s_{1}^{0} \in S_{1}^{0}$ there is $s_{2}^{0} \in S_{2}^{0}$ for which $\left(s_{1}^{0}, s_{2}^{0}\right) \in R$.

Note that this definition reduces to the one of [12,51] for discrete labels ( $c f$. Example 11).

We write $\mathcal{D}_{1} \equiv_{\mathrm{m}} \mathcal{D}_{2}$ if $\mathcal{D}_{1} \leq_{\mathrm{m}} \mathcal{D}_{2}$ and $\mathcal{D}_{2} \leq_{\mathrm{m}} \mathcal{D}_{1}$. The implementation semantics of a DMTS $\mathcal{D}$ is $\llbracket \mathcal{D} \rrbracket=$ $\left\{\mathcal{I} \leq_{\mathrm{m}} \mathcal{D} \mid \mathcal{I}\right.$ implementation $\}$. This is, thus, the set of all LTS which satisfy the specification given by the DMTS $\mathcal{D}$. We say that $\mathcal{D}_{1}$ thoroughly refines $\mathcal{D}_{2}$, and write $\mathcal{D}_{1} \leq_{\text {th }} \mathcal{D}_{2}$, if $\llbracket \mathcal{D}_{1} \rrbracket \subseteq \llbracket \mathcal{D}_{2} \rrbracket$.

The below proposition, which follows directly from transitivity of modal refinement, shows that modal refinement is sound with respect to thorough refinement; in the context of specification theories, this is what one would expect. It can be shown that modal refinement is also complete for deterministic DMTS [11, but we will not need this here.

Proposition 1 For all $D M T S \mathcal{D}_{1}, \mathcal{D}_{2}, \mathcal{D}_{1} \leq_{\mathrm{m}} \mathcal{D}_{2}$ implies $\mathcal{D}_{1} \leq_{\text {th }} \mathcal{D}_{2}$.

### 3.2 Acceptance automata

A non-deterministic acceptance automaton (NAA) is a structure $\mathcal{A}=\left(S, S^{0}\right.$, Tran), with $S \supseteq S^{0}$ finite sets of states and initial states and Tran : $S \rightarrow 2^{2^{\Sigma \times S}}$ an assignment of transition constraints. The intuition is that a transition constraint $\operatorname{Tran}(s)=\left\{M_{1}, \ldots, M_{n}\right\}$ specifies a disjunction of $n$ choices $M_{1}, \ldots, M_{n}$ as to which transitions from $s$ have to be implemented.

An NAA is an implementation if $S^{0}=\left\{s^{0}\right\}$ is a singleton and it holds for all $s \in S$ that $\operatorname{Tran}(s)=\{M\} \subseteq$ $2^{\Gamma \times S}$ is a singleton; hence NAA implementations are isomorphic to LTS. Acceptance automata were first introduced in 555, based on the notion of acceptance trees in [37]; however, there they are restricted to be deterministic. We employ no such restriction here.

In [10], translations were discovered between DMTS and NAA. For a DMTS $\mathcal{D}=\left(S, S^{0},--\rightarrow, \longrightarrow\right)$ and $s \in$ $S$, let $\operatorname{Tran}(s)=\{M \subseteq \Sigma \times S \mid \forall(a, t) \in M: s \xrightarrow{a}$ $t, \forall s \longrightarrow N: N \cap M \neq \emptyset\}$ and define the NAA $d a(\mathcal{D})=$ $\left(S, S^{0}, \operatorname{Tran}\right)$. For an NAA $\mathcal{A}=\left(S, S^{0}\right.$, Tran), define the DMTS $\operatorname{ad}(\mathcal{A})=\left(D, D^{0},--\rightarrow, \longrightarrow\right)$ by
$D=\{M \in \operatorname{Tran}(s) \mid s \in S\}$,
$D^{0}=\left\{M^{0} \in \operatorname{Tran}\left(s^{0}\right) \mid s^{0} \in S^{0}\right\}$,
$\longrightarrow=\left\{\left(M,\left\{\left(a, M^{\prime}\right) \mid M^{\prime} \in \operatorname{Tran}(t)\right\}\right) \mid(a, t) \in M\right\}$,
$\rightarrow=\left\{\left(M, a, M^{\prime}\right) \mid \exists M \longrightarrow N:\left(a, M^{\prime}\right) \in N\right\}$.
Similarly to a theorem of [10,33, we can now show the following:

Theorem 1 For all DMTS $\mathcal{D}_{1}, \mathcal{D}_{2}$ and NAA $\mathcal{A}_{1}, \mathcal{A}_{2}$, $\mathcal{D}_{1} \leq_{\mathrm{m}} \mathcal{D}_{2}$ iff $d a\left(\mathcal{D}_{1}\right) \leq_{\mathrm{m}} d a\left(\mathcal{D}_{2}\right)$ and $\mathcal{A}_{1} \leq_{\mathrm{m}} \mathcal{A}_{2}$ iff $a d\left(\mathcal{A}_{1}\right) \leq_{\mathrm{m}} a d\left(\mathcal{A}_{2}\right)$.

This structural equivalence will allow us to freely translate forth and back between DMTS and NAA in the rest of the paper. Note, however, that the state spaces of $\mathcal{A}$ and $\operatorname{ad}(\mathcal{A})$ are not the same; the one of $a d(\mathcal{A})$ may be exponentially larger. [33] shows that this blow-up is unavoidable.

From a practical point of view, DMTS are a somewhat more useful specification formalism than NAA. This is because they are usually more compact and easily drawn and due to their close relation to the modal $\nu$-calculus, see below.

### 3.3 The Modal $\nu$-Calculus

The modal $\nu$-calculus [35] is the maximal-fixed point fragment of the modal $\mu$-calculus 44, i.e., the modal $\mu$-calculus without negation and without the minimal
fixed point operator. This is also sometimes called Henn-essy-Milner logic with maximal fixed points and represented using equation systems in Hennessy-Milner logic with variables, see [2,46]. We will use this representation below. In [10, translations were discovered between DMTS and the modal $\nu$-calculus, and refining the translations in [33], we could show that for discrete labels, these formalisms are structurally equivalent.

For a finite set $X$ of variables, let $\mathcal{H}(X)$ be the set of Hennessy-Milner formulae, generated by the abstract $\operatorname{syntax} \mathcal{H}(X) \ni \phi::=\mathbf{t t}|\mathbf{f f}| x|\langle a\rangle \phi|[a] \phi|\phi \wedge \phi|$ $\phi \vee \phi$, for $a \in \Sigma$ and $x \in X$. A $\nu$-calculus expression is a structure $\mathcal{N}=\left(X, X^{0}, \Delta\right)$, with $X^{0} \subseteq X$ sets of variables and $\Delta: X \rightarrow \mathcal{H}(X)$ a declaration.

We recall the greatest fixed point semantics of $\nu$ calculus expressions from [46, but extend it to structured labels. Let $\left(S, S^{0}, \longrightarrow\right)$ be an LTS, then an assignment is a mapping $\sigma: X \rightarrow 2^{S}$. The set of assignments forms a complete lattice with order $\sigma_{1} \sqsubseteq \sigma_{2}$ iff $\sigma_{1}(x) \subseteq \sigma_{2}(x)$ for all $x \in X$ and lowest upper bound $\left(\bigsqcup_{i \in I} \sigma_{i}\right)(x)=\bigcup_{i \in I} \sigma_{i}(x)$.

The semantics of a formula in $\mathcal{H}(X)$ is a function from assignments to subsets of $S$ defined as follows: $(\mathrm{tt}) \sigma=S,(\mathrm{ff}) \sigma=\emptyset,(x) \sigma=\sigma(x),(\phi \wedge \psi) \sigma=(\phi) \sigma \cap$ $(\psi) \sigma,(\phi \vee \psi) \sigma=(\langle \rangle) \sigma \cup(\psi\rangle) \sigma$, and
$(\langle a\rangle \phi \mid \sigma=\{s \in S \mid \exists s \xrightarrow{b} t: b \in \llbracket a \rrbracket, t \in(\phi \mid) \sigma\}$,
$([a] \phi) \sigma=\{s \in S \mid \forall s \xrightarrow{b} t: b \in \llbracket a \rrbracket \Longrightarrow t \in(\phi \mid \sigma\}$.
The semantics of a declaration $\Delta$ is then the assignment defined by $(\Delta)=\bigsqcup\left\{\sigma: X \rightarrow 2^{S} \mid \forall x \in X: \sigma(x) \subseteq\right.$ $(\Delta(x)) \sigma\}$; the greatest (pre)fixed point of $\Delta$.

An LTS $\mathcal{I}=\left(S, s^{0}, \longrightarrow\right)$ implements (or models) the expression $\mathcal{N}$, denoted $\mathcal{I} \models \mathcal{N}$, if there is $x^{0} \in X^{0}$ such that $s^{0} \in(\Delta)\left(x^{0}\right)$.

In [33] we have introduced another semantics for $\nu$-calculus expressions, which is given by a notion of refinement, like for DMTS and NAA. For this we need a normal form for $\nu$-calculus expressions:

Lemma 1 ([33]) For any $\nu$-calculus expression $\mathcal{N}_{1}=$ $\left(X_{1}, X_{1}^{0}, \Delta_{1}\right)$, there exists another $\mathcal{N}_{2}=\left(X_{2}, X_{2}^{0}, \Delta_{2}\right)$ with $\llbracket \mathcal{N}_{1} \rrbracket=\llbracket \mathcal{N}_{2} \rrbracket$ and such that for any $x \in X, \Delta_{2}(x)$ is of the form
$\Delta_{2}(x)=\bigwedge_{i \in I}\left(\bigvee_{j \in J_{i}}\left\langle a_{i j}\right\rangle x_{i j}\right) \wedge \bigwedge_{a \in \Sigma}[a]\left(\bigvee_{j \in J_{a}} y_{a, j}\right)$
for finite (possibly empty) index sets $I, J_{i}, J_{a}$ and all $x_{i j}, y_{a, j} \in X_{2}$.

As this is a type of conjunctive normal form, it is clear that translating a $\nu$-calculus expression into normal form may incur an exponential blow-up. We introduce some notation for $\nu$-calculus expressions in
normal form. Let $\mathcal{N}=\left(X, X^{0}, \Delta\right)$ be such an expression and $x \in X$, with $\Delta(x)=\bigwedge_{i \in I}\left(\bigvee_{j \in J_{i}}\left\langle a_{i j}\right\rangle x_{i j}\right) \wedge$ $\bigwedge_{a \in \Sigma}[a]\left(\bigvee_{j \in J_{a}} y_{a, j}\right)$ as in the lemma. Define $\diamond(x)=$ $\left\{\left\{\left(a_{i j}, x_{i j}\right) \mid j \in J_{i}\right\} \mid i \in I\right\}$ and, for each $a \in \Sigma$, $\square^{a}(x)=\left\{y_{a, j} \mid j \in J_{a}\right\}$. Intuitively, $\diamond(x)$ collects all $\langle a\rangle$-requirements from $x$, whereas $\square^{a}(x)$ specifies the disjunction of $[a]$-properties which must hold from $x$. Note that now,
$\Delta(x)=\bigwedge_{N \in \diamond(x)}\left(\bigvee_{(a, y) \in N}\langle a\rangle y\right) \wedge \bigwedge_{a \in \Sigma}[a]\left(\bigvee_{y \in \square^{a}(x)} y\right)$.
Let $\mathcal{N}_{1}=\left(X_{1}, X_{1}^{0}, \Delta_{1}\right), \mathcal{N}_{2}=\left(X_{2}, X_{2}^{0}, \Delta_{2}\right)$ be $\nu-$ calculus expressions in normal form and $R \subseteq X_{1} \times X_{2}$. The relation $R$ is a modal refinement if it holds for all $\left(x_{1}, x_{2}\right) \in R$ that

- for all $a_{1} \in \Sigma$ and $y_{1} \in \square_{1}^{a_{1}}\left(x_{1}\right)$ there is $a_{2} \in \Sigma$ and $y_{2} \in \square_{2}^{a_{2}}\left(x_{2}\right)$ with $a_{1} \preccurlyeq a_{2}$ and $\left(y_{1}, y_{2}\right) \in R$, and
- for all $N_{2} \in \diamond_{2}\left(x_{2}\right)$ there is $N_{1} \in \diamond_{1}\left(x_{1}\right)$ such that for all $\left(a_{1}, y_{1}\right) \in N_{1}$ there exists $\left(a_{2}, y_{2}\right) \in N_{2}$ with $a_{1} \preccurlyeq a_{2}$ and $\left(y_{1}, y_{2}\right) \in R$.

We say that a $\nu$-calculus expression $\left(X, X^{0}, \Delta\right)$ in normal form is an implementation if $X^{0}=\left\{x^{0}\right\}$ is a singleton, $\diamond(x)=\left\{\{(a, y)\} \mid y \in \square^{a}(x), a \in \Sigma\right\}$ and $\square^{a}(x)=\emptyset$ for all $a \notin \Gamma$, for all $x \in X$.

We can translate a $\operatorname{LTS}\left(S, S^{0}, \longrightarrow\right)$ to a $\nu$-calculus expression $\left(S, S^{0}, \Delta\right)$ in normal form by setting $\diamond(s)=$ $\{\{(a, t)\} \mid s \xrightarrow{a} t\}$ and $\square^{a}(s)=\{t \mid s \xrightarrow{a} t\}$ for all $s \in S, a \in \Sigma$. This defines a bijection between LTS and $\nu$-calculus implementations, hence, like for DMTS and NAA, an embedding of LTS into the modal $\nu$-calculus.

We have shown in [33, Thm. 15] that for discrete labels, the refinement semantics and the fixed point semantics of the modal $\nu$-calculus agree; the proof can easily be extended to our case of structured labels:

Theorem 2 For any LTS $\mathcal{I}$ and any $\nu$-calculus expression $\mathcal{N}$ in normal form, $\mathcal{I} \models \mathcal{N}$ iff $\mathcal{I} \leq_{\mathrm{m}} \mathcal{N}$.

For a DMTS $\mathcal{D}=\left(S, S^{0}, \rightarrow \rightarrow, \longrightarrow\right)$ and all $s \in S$, let $\diamond(s)=\{N \mid s \longrightarrow N\}$ and, for each $a \in \Sigma, \square^{a}(s)=$ $\{t \mid s \xrightarrow{a} t\}$. Define the (normal-form) $\nu$-calculus expression $\operatorname{dn}(\mathcal{D})=\left(S, S^{0}, \Delta\right)$, with $\Delta$ given as in (11). For a $\nu$-calculus expression $\mathcal{N}=\left(X, X^{0}, \Delta\right)$ in normal form, let $\rightarrow=\left\{(x, a, y) \in X \times \Sigma \times X \mid y \in \square^{a}(x)\right\}$, $\longrightarrow=\{(x, N) \mid x \in X, N \in \diamond(x)\}$ and define the DMTS $n d(\mathcal{N})=\left(X, X^{0},--\rightarrow, \longrightarrow\right)$. Given that these translations are entirely syntactic, the following theorem is not a surprise:

Theorem 3 For DMTS $\mathcal{D}_{1}, \mathcal{D}_{2}$ and $\nu$-calculus expressions $\mathcal{N}_{1}, \mathcal{N}_{2}, \mathcal{D}_{1} \leq_{m} \mathcal{D}_{2}$ iff $d n\left(\mathcal{D}_{1}\right) \leq_{m} d n\left(\mathcal{D}_{2}\right)$ and $\mathcal{N}_{1} \leq_{\mathrm{m}} \mathcal{N}_{2}$ iff $n d\left(\mathcal{N}_{1}\right) \leq_{\mathrm{m}} n d\left(\mathcal{N}_{2}\right)$.

## 4 Specification theory

Structural specifications typically come equipped with operations which permit compositional reasoning, viz. conjunction, structural composition, and quotient, $c f$. 4]. On deterministic MTS, these operations can be given easily using simple structural operational rules (for such semantics of weighted systems, see for instance [43]). For non-deterministic specifications this is significantly harder; in [10] it is shown that DMTS and NAA permit these operations and, additionally but trivially, disjunction. Here we show how to extend these operations on non-deterministic systems to our setting with structured labels.

We remark that structural composition and quotient operators are well-known from some logics, such as, e.g., linear [36] or spatial logic [16], see also [17] for a stochastic extension. However, whereas these operators are part of the formal syntax in those logics, for us they are simply operations on logical expressions (or DMTS, or NAA).

Given the equivalence of DMTS, NAA and the modal $\nu$-calculus exposed in the previous section, we will often state properties for all three types of specifications at the same time, letting $\mathcal{S}$ stand for any of the three types. For definitions and proofs, we are free to use the type of specification which is most well suited for the context; we will use DMTS for the logical operations (Section 4.1) and NAA for the structural operations (Sections 4.2 and 4.3).

### 4.1 Disjunction and conjunction

Disjunction of specifications is easily defined, as we allow for multiple initial states. For two DMTS $\mathcal{D}_{1}=$ $\left(S_{1}, S_{1}^{0}, \rightarrow \rightarrow_{1}, \longrightarrow_{1}\right)$ and $\mathcal{D}_{2}=\left(S_{2}, S_{2}^{0}, \rightarrow \rightarrow_{2}, \longrightarrow_{2}\right)$, we can hence define $\mathcal{D}_{1} \vee \mathcal{D}_{2}=\left(S_{1} \cup S_{2}, S_{1}^{0} \cup S_{2}^{0}, \rightarrow \rightarrow_{1} \cup\right.$ $-\rightarrow_{2}, \longrightarrow_{1} \cup \longrightarrow_{2}$ ) (with all unions disjoint).

For conjunction, we let $\mathcal{D}_{1} \wedge \mathcal{D}_{2}=\left(S_{1} \times S_{2}, S_{1}^{0} \times\right.$ $\left.S_{2}^{0}, \rightarrow \rightarrow, \longrightarrow\right)$, with
$-\left(s_{1}, s_{2}\right) \xrightarrow{a_{1} \otimes a_{2}}\left(t_{1}, t_{2}\right)$ whenever $s_{1} \xrightarrow{a_{1}}{ }_{1} t_{1}, s_{2} \xrightarrow{a_{2}}{ }_{2} t_{2}$ and $a_{1} \otimes a_{2}$ is defined,

- for all $s_{1} \longrightarrow N_{1_{1}},\left(s_{1}, s_{2}\right) \longrightarrow\left\{\left(a_{1} \otimes a_{2},\left(t_{1}, t_{2}\right)\right) \mid\right.$ $\left(a_{1}, t_{1}\right) \in N_{1}, s_{2} \xrightarrow{a_{2}}{ }_{2} t_{2}, a_{1} \otimes a_{2}$ defined $\}$,
- for all $s_{2} \longrightarrow N_{2},_{1}\left(s_{1}, s_{2}\right) \longrightarrow\left\{\left(a_{1} \otimes a_{2},\left(t_{1}, t_{2}\right)\right) \mid\right.$ $\left(a_{2}, t_{2}\right) \in N_{2}, s_{1} \xrightarrow{a_{1}} t_{1}, a_{1} \otimes a_{2}$ defined $\}$.

Theorem 4 For all specifications $\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3}$,
$-\mathcal{S}_{1} \vee \mathcal{S}_{2} \leq_{\mathrm{m}} \mathcal{S}_{3}$ iff $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{3}$ and $\mathcal{S}_{2} \leq_{\mathrm{m}} \mathcal{S}_{3}$,
$-\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{2} \wedge \mathcal{S}_{3}$ iff $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{2}$ and $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{3}$,
$-\llbracket \mathcal{S}_{1} \vee \mathcal{S}_{2} \rrbracket=\llbracket \mathcal{S}_{1} \rrbracket \cup \llbracket \mathcal{S}_{2} \rrbracket$, and $\llbracket \mathcal{S}_{1} \wedge \mathcal{S}_{2} \rrbracket=\llbracket \mathcal{S}_{1} \rrbracket \cap \llbracket \mathcal{S}_{2} \rrbracket$.

Proof The proof that $\mathcal{S}_{1} \vee \mathcal{S}_{2} \leq_{\mathrm{m}} \mathcal{S}_{3}$ iff $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{3}$ and $\mathcal{S}_{2} \leq_{\mathrm{m}} \mathcal{S}_{3}$ is trivial: any modal refinement $R \subseteq\left(S_{1} \cup\right.$ $\left.S_{2}\right) \times S_{3}$ splits into two refinements $R_{1} \subseteq S_{1} \times S_{3}$, $R_{2} \subseteq S_{2} \times S_{3}$ and vice versa.

For the proof of the second claim, which we show for DMTS, we prove the back direction first. Let $R_{2} \subseteq S_{1} \times$ $S_{2}, R_{3} \subseteq S_{1} \times S_{3}$ be initialized (DMTS) modal refinements which witness $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{2}$ and $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{3}$, respectively. Define $R=\left\{\left(s_{1},\left(s_{2}, s_{3}\right)\right) \mid\left(s_{1}, s_{2}\right) \in R_{2},\left(s_{1}, s_{3}\right) \in\right.$ $\left.R_{3}\right\} \subseteq S_{1} \times\left(S_{2} \times S_{3}\right)$, then $R$ is initialized.

Now let $\left(s_{1},\left(s_{2}, s_{3}\right)\right) \in R$, then $\left(s_{1}, s_{2}\right) \in R_{2}$ and $\left(s_{1}, s_{3}\right) \in R_{3}$. Assume that $s_{1} \xrightarrow{a_{1}}{ }_{1} t_{1}$, then by $\mathcal{S}_{1} \leq_{\mathrm{m}}$ $\mathcal{S}_{2}$, we have $s_{2} \xrightarrow{a_{2}}{ }_{2} t_{2}$ with $a_{1} \preccurlyeq a_{2}$ and $\left(t_{1}, t_{2}\right) \in R_{2}$. Similarly, by $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{3}$, we have $s_{3} \xrightarrow{a_{3}}{ }_{3} t_{3}$ with $a_{1} \preccurlyeq$ $a_{3}$ and $\left(t_{1}, t_{3}\right) \in R_{3}$. But then also $a_{1} \preccurlyeq a_{2} \otimes a_{3}$ and $\left(t_{1},\left(t_{2}, t_{3}\right)\right) \in R$, and $\left(s_{2}, s_{3}\right) \xrightarrow{a_{2} \otimes a_{3}}\left(t_{2}, t_{3}\right)$ by definition.

Assume that $\left(s_{2}, s_{3}\right) \longrightarrow N$. Without loss of generality we can assume that there is $s_{2} \longrightarrow{ }_{2} N_{2}$ such that $N=\left\{\left(a_{2} \otimes a_{3},\left(t_{2}, t_{3}\right)\right) \mid\left(a_{2}, t_{2}\right) \in N_{2}, s_{3} \xrightarrow{a_{3}} \rightarrow_{3} t_{3}\right\}$. By $S_{1} \leq_{\mathrm{m}} S_{2}$, we have $s_{1} \longrightarrow_{1} N_{1}$ such that $\forall\left(a_{1}, t_{1}\right) \in$ $N_{1}: \exists\left(a_{2}, t_{2}\right) \in N_{2}: a_{1} \preccurlyeq a_{2},\left(t_{1}, t_{2}\right) \in R_{2}$.

Let $\left(a_{1}, t_{1}\right) \in N_{1}$, then also $s_{1} \xrightarrow{a_{1}} \rightarrow_{1} t_{1}$, so by $S_{1} \leq_{\mathrm{m}}$ $S_{3}$, there is $s_{3}{ }_{-}^{a_{3}} \rightarrow_{3} t_{3}$ with $a_{1} \preccurlyeq a_{3}$ and $\left(t_{1}, t_{3}\right) \in R_{3}$. By the above, we also have $\left(a_{2}, t_{2}\right) \in N_{2}$ such that $a_{1} \preccurlyeq$ $a_{2}$ and $\left(t_{1}, t_{2}\right) \in R_{2}$, but then $\left(a_{2} \otimes a_{3},\left(t_{2}, t_{3}\right)\right) \in N$, $a_{1} \preccurlyeq a_{2} \wedge a_{3}$, and $\left(t_{1},\left(t_{2}, t_{3}\right)\right) \in R$.

For the other direction of the second claim, let $R \subseteq$ $S_{1} \times\left(S_{2} \times S_{3}\right)$ be an initialized (DMTS) modal refinement which witnesses $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{2} \wedge \mathcal{S}_{3}$. We show that $S_{1} \leq_{\mathrm{m}} S_{2}$, the proof of $S_{1} \leq_{\mathrm{m}} S_{3}$ being entirely analogous. Define $R_{2}=\left\{\left(s_{1}, s_{2}\right) \mid \exists s_{3} \in S_{3}:\left(s_{1},\left(s_{2}, s_{3}\right)\right) \in\right.$ $R\} \subseteq S_{1} \times S_{2}$, then $R_{2}$ is initialized.

Let $\left(s_{1}, s_{2}\right) \in R_{2}$, then we must have $s_{3} \in S_{3}$ such that $\left(s_{1},\left(s_{2}, s_{3}\right)\right) \in R$. Assume that $s_{1} \xrightarrow{a_{1}}{ }_{1} t_{1}$, then also $\left(s_{2}, s_{3}\right) \xrightarrow{a}\left(t_{2}, t_{3}\right)$ for some $a$ with $a_{1} \preccurlyeq a$ and $\left(t_{1},\left(t_{2}, t_{3}\right)\right) \in R$. By construction we have $s_{2} \xrightarrow{a_{2}}{ }_{2} t_{2}$ and $s_{3} \xrightarrow{a_{3}} \rightarrow_{3} t_{3}$ such that $a=a_{2} \otimes a_{3}$, but then $a_{1} \preccurlyeq$ $a_{2} \otimes a_{3} \preccurlyeq a_{2}$ and $\left(t_{1}, t_{2}\right) \in R_{2}$.

Assume that $s_{2} \longrightarrow_{2} N_{2}$, then by construction we have $\left(s_{2}, s_{3}\right) \longrightarrow N=\left\{\left(a_{2} \otimes a_{3},\left(t_{2}, t_{3}\right)\right) \mid\left(a_{2}, t_{2}\right) \in\right.$ $\left.N_{2}, s_{3}{ }_{-}^{a_{3}}{ }_{3} t_{3}\right\}$. By $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{2} \wedge \mathcal{S}_{3}$, there is $s_{1} \longrightarrow 1 N_{1}$ such that $\forall\left(a_{1}, t_{1}\right) \in N_{1}: \exists\left(a,\left(t_{2}, t_{3}\right)\right) \in N: a_{1} \preccurlyeq$ $a,\left(t_{1},\left(t_{2}, t_{3}\right)\right) \in R$.

Let $\left(a_{1}, t_{1}\right) \in N_{1}$, then we have $\left(a,\left(t_{2}, t_{3}\right)\right) \in N$ for which $a_{1} \preccurlyeq a$ and $\left(t_{1},\left(t_{2}, t_{3}\right)\right) \in R$. By construction of $N$, this implies that there are $\left(a_{2}, t_{2}\right) \in N_{2}$ and $s_{3} \xrightarrow{a_{3}}{ }_{3}$ $t_{3}$ such that $a=a_{2} \otimes a_{3}$, but then $a_{1} \preccurlyeq a_{2} \otimes a_{3} \preccurlyeq a_{2}$ and $\left(t_{1}, t_{2}\right) \in R$.

As to the last claims of the theorem, $\llbracket \mathcal{S}_{1} \wedge \mathcal{S}_{2} \rrbracket=$ $\llbracket \mathcal{S}_{1} \rrbracket \cap \llbracket \mathcal{S}_{2} \rrbracket$ is clear from what we just proved: for all implementations $\mathcal{I}, \mathcal{I} \leq_{\mathrm{m}} \mathcal{S}_{1} \wedge \mathcal{S}_{2}$ iff $\mathcal{I} \leq_{\mathrm{m}} \mathcal{S}_{1}$ and $\mathcal{I} \leq_{\mathrm{m}}$


Fig. 8 Two simple DMTS
$\mathcal{S}_{2}$. For the other part, it is clear by construction that for any implementation $\mathcal{I}$, any witness $R$ for $\mathcal{I} \leq_{m} \mathcal{S}_{1}$ is also a witness for $\mathcal{I} \leq_{\mathrm{m}} \mathcal{S}_{1} \vee \mathcal{S}_{2}$, and similarly for $\mathcal{S}_{2}$, hence $\llbracket \mathcal{S}_{1} \rrbracket \cup \llbracket \mathcal{S}_{2} \rrbracket \subseteq \llbracket \mathcal{S}_{1} \vee \mathcal{S}_{2} \rrbracket$.

To show that also $\llbracket \mathcal{S}_{1} \rrbracket \cup \llbracket \mathcal{S}_{2} \rrbracket \supseteq \llbracket \mathcal{S}_{1} \vee \mathcal{S}_{2} \rrbracket$, we note that an initialized refinement $R$ witnessing $\mathcal{I} \leq{ }_{\mathrm{m}} \mathcal{S}_{1} \vee \mathcal{S}_{2}$ must relate the initial state of $\mathcal{I}$ either to an initial state of $\mathcal{S}_{1}$ or to an initial state of $\mathcal{S}_{2}$. In the first case, and by disjointness, $R$ witnesses $\mathcal{I} \leq_{\mathrm{m}} \mathcal{S}_{1}$, in the second, $\mathcal{I} \leq{ }_{\mathrm{m}} \mathcal{S}_{2}$.

With bottom and top elements given by $\perp=(\emptyset, \emptyset, \emptyset)$ and $\top=\left(\{s\},\{s\}, \operatorname{Tran}_{\top}\right)$ with $\operatorname{Tran}_{\top}(s)=2^{2^{\Sigma \times\{s\}}}$, our classes of specifications form bounded distributive lattices up to $\equiv_{\mathrm{m}}$.

### 4.2 Structural composition

For NAA $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right), \mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right)$, their structural composition is $\mathcal{A}_{1} \| \mathcal{A}_{2}=\left(S_{1} \times S_{2}, S_{1}^{0} \times\right.$ $S_{2}^{0}$, Tran), with $\operatorname{Tran}\left(\left(s_{1}, s_{2}\right)\right)=\left\{M_{1} \oplus M_{2} \mid M_{1} \in\right.$ $\left.\operatorname{Tran}_{1}\left(s_{1}\right), M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)\right\}$ for all $s_{1} \in S_{1}, s_{2} \in S_{2}$, where $M_{1} \oplus M_{2}=\left\{\left(a_{1} \oplus a_{2},\left(t_{1}, t_{2}\right)\right) \mid\left(a_{1}, t_{1}\right) \in M_{1}\right.$, $\left(a_{2}, t_{2}\right) \in M_{2}, a_{1} \oplus a_{2}$ defined $\}$.

Remark a subtle difference between conjunction and structural composition, which we expose for discrete labels and CSP-style composition: for the DMTS $\mathcal{D}_{1}$, $\mathcal{D}_{2}$ shown in Fig. 8, both $\mathcal{D}_{1} \wedge \mathcal{D}_{2}$ and $\mathcal{D}_{1} \| \mathcal{D}_{2}$ have only one state, but $\operatorname{Tran}\left(s_{1} \wedge t_{1}\right)=\emptyset$ and $\operatorname{Tran}\left(s_{1} \| t_{1}\right)=\{\emptyset\}$, so that $\mathcal{D}_{1} \wedge \mathcal{D}_{2}$ is inconsistent, whereas $\mathcal{D}_{1} \| \mathcal{D}_{2}$ is not.

This definition extends the structural composition defined for modal transition systems, with structured labels, in [30]. For DMTS specifications (and hence also for $\nu$-calculus expressions), the back translation from NAA to DMTS entails an exponential explosion.
Theorem 5 Up to $\equiv_{\mathrm{m}}$, the operator $\|$ is associative, commutative and monotone.

Proof Associativity and commutativity are clear by associativity and commutativity of $(\mathbb{1}$. Monotonicity is equivalent to the assertion that (up to $\equiv_{\mathrm{m}}$ ) \| distributes over the least upper bound $\vee$; one easily sees that for all specifications $\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3}$, the identity is a two-sided modal refinement $\mathcal{S}_{1}\left\|\left(\mathcal{S}_{2} \vee \mathcal{S}_{3}\right) \equiv_{\mathrm{m}} \mathcal{S}_{1}\right\| \mathcal{S}_{2} \vee \mathcal{S}_{1} \| \mathcal{S}_{3}$.

Corollary 1 (Independent implementability) For all specifications $\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3}, \mathcal{S}_{4}, \mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{3}$ and $\mathcal{S}_{2} \leq_{\mathrm{m}}$ $\mathcal{S}_{4}$ imply $\mathcal{S}_{1}\left\|\mathcal{S}_{2} \leq_{\mathrm{m}} \mathcal{S}_{3}\right\| \mathcal{S}_{4}$.

### 4.3 Quotient

Because of non-determinism, we have to use a power set construction for the quotient, as opposed to conjunction and structural composition where product is sufficient. For NAA $\mathcal{A}_{3}=\left(S_{3}, S_{3}^{0}, \operatorname{Tran}_{3}\right), \mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right)$, the quotient is $\mathcal{A}_{3} / \mathcal{A}_{1}=\left(S,\left\{s^{0}\right\}\right.$, Tran), with $S=$ $2^{S_{3} \times S_{1}}$ and $s^{0}=\left\{\left(s_{3}^{0}, s_{1}^{0}\right) \mid s_{3}^{0} \in S_{3}^{0}, s_{1}^{0} \in S_{1}^{0}\right\}$. States in $S$ will be written $\left.\left\{s_{3}^{1} / s_{1}^{1}, \ldots, s_{3}^{n} / s_{1}^{n}\right)\right\}$. Intuitively, this denotes that such state when composed with $s_{1}^{i}$ conforms to $s_{3}^{i}$ for each $i$; we call this consistency here.

We now define Tran. First, $\operatorname{Tran}(\emptyset)=2^{\Sigma \times\{\emptyset\}}$, so $\emptyset$ is universal. For any other state $s=\left\{s_{3}^{1} / s_{1}^{1}, \ldots, s_{3}^{n} / s_{1}^{n}\right\} \in$ $S$, its set of permissible labels is defined by

$$
\begin{gathered}
p l(s)=\left\{a_{2} \in \Sigma \mid \forall i=1, \ldots, n: \forall\left(a_{1}, t_{1}\right) \in \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right):\right. \\
\left.\exists\left(a_{3}, t_{3}\right) \in \in \operatorname{Tran}_{3}\left(s_{3}^{i}\right): a_{1} \oplus a_{2} \preccurlyeq a_{3}\right\},
\end{gathered}
$$

that is, a label is permissible iff it cannot violate consistency. Here we use the notation $x \in \in z$ as a shortcut for $\exists y: x \in y \in z$.

Now for each $a \in p l(s)$ and each $i \in\{1, \ldots, n\}$, let $\left\{t_{1} \in S_{1} \mid\left(a, t_{1}\right) \in \in \operatorname{Tran}_{1}\left(t_{1}^{i}\right)\right\}=\left\{t_{1}^{i, 1}, \ldots, t_{1}^{i, m_{i}}\right\}$ be an enumeration of all the possible states in $S_{1}$ after an $a$-transition. Then we define the set of all sets of possible assignments of next- $a$ states from $s_{3}^{i}$ to next- $a$ states from $s_{1}^{i}$ :

$$
\begin{aligned}
& p t_{a}(s)=\left\{\left\{\left(t_{3}^{i, j}, t_{1}^{i, j}\right) \mid\right.\right.\left.i=1, \ldots, n, j=1, \ldots, m_{i}\right\} \\
&\left.\mid \forall i: \forall j:\left(a, t_{3}^{i, j}\right) \in \in \operatorname{Tran}_{3}\left(s_{3}^{i}\right)\right\}
\end{aligned}
$$

These are all possible next-state assignments which preserve consistency. Now let $p t(s)=\bigcup_{a \in p l(s)} p t_{a}(s)$ and define
$\operatorname{Tran}(s)=\{M \subseteq p t(s) \mid \forall i=1, \ldots, n:$

$$
\begin{aligned}
& \forall M_{1} \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right): \exists M_{3} \in \operatorname{Tran}_{3}\left(s_{3}^{i}\right): \\
&\left.M \triangleright M_{1} \preccurlyeq_{R} M_{3}\right\},
\end{aligned}
$$

where $M \triangleright M_{1}=\left\{\left(a_{1} \oplus a, t_{3}^{i}\right) \mid\left(a,\left\{t_{3}^{1} / t_{1}^{1}, \ldots, t_{3}^{k} / t_{1}^{k}\right)\right\}\right) \in$ $\left.M,\left(a_{1}, t_{1}^{i}\right) \in M_{1}\right\}$, to guarantee consistency no matter which element of $\operatorname{Tran}_{1}\left(s_{1}^{i}\right), s$ is composed with.

Example 5 Fig. 9 shows two simple specifications and their quotient under $\Phi$, i.e., using addition of intervals for label synchronization (see Example 3). During the construction and the translation back to DMTS, many states were eliminated as they were inconsistent (their Tran-set was empty). For instance, there is no may transition to state $\left\{s_{2} / t_{2}\right\}$, because when it is composed with $t_{2}$ there is no guarantee of a late-transition, hence no guarantee to refine $s_{2}$.

Note that in order to have a finite representation of the quotient, we have to extend the label set to allow


Fig. 9 Two DMTS (top and center) and their quotient (bottom)
intervals which are not closed; for instance, the maytransition (send, $] 1, \infty]$ ) from $\left\{s_{0} / t_{0}\right\}$ to $\emptyset$ comprises the fact that $p t_{a}\left(\left\{s_{0} / t_{0}\right\}\right)=\emptyset$ for all $a=($ send, $[x, \infty])$ with $x>1$. This can be formalized by introducing a (partial) label quotient operator $\oslash: \Sigma \times \Sigma \rightharpoonup \Sigma$ which is adjoint to label synchronization $\oplus$, see 30.

Theorem 6 For all specifications $\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3}, \mathcal{S}_{1} \| \mathcal{S}_{2} \leq_{\mathrm{m}}$ $\mathcal{S}_{3}$ iff $\mathcal{S}_{2} \leq_{\mathrm{m}} \mathcal{S}_{3} / \mathcal{S}_{1}$.

Proof We show the proof for NAA; for DMTS and $\nu$ calculus expressions it will follow through the translations. Let $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right), \mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right)$, $\mathcal{A}_{3}=\left(S_{3}, S_{3}^{0}\right.$, Tran $\left._{3}\right)$; we show that $\mathcal{A}_{1} \| \mathcal{A}_{2} \leq_{\mathrm{m}} \mathcal{A}_{3}$ iff $\mathcal{A}_{2} \leq_{\mathrm{m}} \mathcal{A}_{3} / \mathcal{A}_{1}$.

We assume that the elements of $\operatorname{Tran}_{1}\left(s_{1}\right)$ are pairwise disjoint for each $s_{1} \in S_{1}$; this can be achieved by, if necessary, splitting states.

First we note that by construction, $s \supseteq t$ implies $s \leq_{\mathrm{m}} t$ for all $s, t \in S$.

Assume that $\mathcal{A}_{2} \leq_{\mathrm{m}} \mathcal{A}_{3} / \mathcal{A}_{1}$ and let $R=\left\{\left(s_{2}, s_{3} / s_{1}\right) \mid\right.$ $\left.s_{2} \leq_{\mathrm{m}} s_{3} / s_{1}\right\}$ be the witnessing refinement relation. Let $R^{\prime}=\left\{\left(s_{1} \| s_{2}, s_{3}\right) \mid\left(s_{2}, s_{3} / s_{1}\right) \in R\right\}$ (for readability, we abuse notation here and write $\left(s_{1} \| s_{2}, s_{3}\right)$ instead of $\left.\left(s_{1}, s_{2}, s_{3}\right)\right)$; we show that $R^{\prime}$ is a witness for $\mathcal{A}_{1} \| \mathcal{A}_{2} \leq_{\mathrm{m}} \mathcal{A}_{3}$.

Let $\left(s_{1} \| s_{2}, s_{3}\right) \in R^{\prime}$ and $M_{\|} \in \operatorname{Tran}_{\|}\left(s_{1} \| s_{2}\right)$. Then $M_{\|}=M_{1} \| M_{2}$ with $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$ and $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$. As $s_{2} \leq_{\mathrm{m}} s_{3} / s_{1}$, we can pair $M_{2}$ with a set $M_{/} \in$ Tran/ $\left(s_{3} / s_{1}\right)$ such that $M_{2} \preccurlyeq_{R} M_{/}$.

Let $M_{3}=M_{/} \triangleright M_{1}$. We show that $M_{\|} \preccurlyeq R^{\prime} M_{3}$ :

- Let $\left(a, t_{1} \| t_{2}\right) \in M_{\|}$, then there are $a_{1}, a_{2} \in \Sigma$ with $a=a_{1} \oplus a_{2}$ and $\left(a_{1}, t_{1}\right) \in M_{1},\left(a_{2}, t_{2}\right) \in M_{2}$. By $M_{2} \preccurlyeq{ }_{R} M_{/}$, there is $\left(a_{2}^{\prime}, t\right) \in M_{/}$such that $a_{2} \preccurlyeq a_{2}^{\prime}$ and $t_{2} \leq_{\mathrm{m}} t$. Note that $a_{3}=a_{1} \oplus a_{2}^{\prime}$ is defined and $a \preccurlyeq a_{3}$. Write $t=\left\{t_{3}^{1} / t_{1}^{1}, \ldots, t_{3}^{n} / t_{1}^{n}\right\}$. By construction, there is an index $i$ for which $t_{1}^{i}=t_{1}$, hence $\left(a_{3}, t_{3}^{i}\right) \in M_{3}$. Also, $t \supseteq\left\{t_{3}^{i} / t_{1}^{i}\right\}$, hence $t_{2} \leq_{\mathrm{m}} t_{3}^{i} / t_{1}^{i}$ and consequently $\left(t_{1} \| t_{2}, t_{3}\right) \in R^{\prime}$.
- Let $\left(a_{3}, t_{3}\right) \in M_{3}$, then there are $\left(a_{2}^{\prime}, t\right) \in M_{/}$and $\left(a_{1}, t_{1}\right) \in M_{1}$ such that $a_{3}=a_{1} \oplus a_{2}^{\prime}$ and $t_{3} / t_{1} \in t$. By $M_{2} \preccurlyeq_{R} M_{/}$, there is $\left(a_{2}, t_{2}\right) \in M_{2}$ for which $a_{2} \preccurlyeq a_{2}^{\prime}$ and $t_{2} \leq_{\mathrm{m}} t$. Note that $a=a_{1} \oplus a_{2}$ is defined and $a \preccurlyeq a_{3}$. Thus $\left(a, t_{1} \| t_{2}\right) \in M_{\|}$, and by $t \supseteq\left\{t_{3} / t_{1}\right\}, t_{2} \leq_{m} t_{3} / t_{1}$.

Assume, for the other direction of the proof, that $\mathcal{A}_{1} \| \mathcal{A}_{2} \leq_{\mathrm{m}} \mathcal{A}_{3}$ and let $R=\left\{\left(s_{1} \| s_{2}, s_{3}\right) \mid s_{1} \| s_{2} \leq_{\mathrm{m}} s_{2}\right\}$ (again abusing notation) be the witnessing refinement relation. Define $R^{\prime} \subseteq S_{2} \times 2^{S_{3} \times S_{1}}$ by

$$
R=\left\{\left(s_{2},\left\{s_{3}^{1} / s_{1}^{1}, \ldots, s_{3}^{n} / s_{1}^{n}\right\}\right)\right.
$$

$$
\left.\mid \forall i=1, \ldots, n:\left(s_{1}^{i} \| s_{2}, s_{3}^{i}\right) \in R\right\} ;
$$

we show that $R^{\prime}$ is a witness for $\mathcal{A}_{2} \leq_{\mathrm{m}} \mathcal{A}_{3} / \mathcal{A}_{1}$. Let $\left(s_{2}, s\right) \in R^{\prime}$, with $s=\left\{s_{3}^{1} / s_{1}^{1}, \ldots, s_{3}^{n} / s_{1}^{n}\right\}$, and $M_{2} \in$ $\operatorname{Tran}_{2}\left(s_{2}\right)$.

For every $i=1, \ldots, n$, write the set $\operatorname{Tran}_{1}\left(s_{1}^{i}\right)=$ $\left\{M_{1}^{i, 1}, \ldots, M_{1}^{i, m_{i}}\right\}$. By assumption, $M_{1}^{i, j_{1}} \cap M_{1}^{i, j_{2}}=\emptyset$ for $j_{1} \neq j_{2}$, hence every $\left(a_{1}, t_{1}\right) \in \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right)$ is contained in a unique $M_{1}^{i, \delta_{i}\left(a_{1}, t_{1}\right)} \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right)$.

For every $j=1, \ldots, m_{i}$, let $M^{i, j}=M_{1}^{i, j} \| M_{2} \in$ $\operatorname{Tran}_{\|}\left(s_{1}^{i} \| s_{2}\right)$. By $s_{1}^{i} \| s_{2} \leq_{\mathrm{m}} s_{3}^{i}$, we have $M_{3}^{i, j} \in \operatorname{Tran}_{3}\left(s_{3}^{i}\right)$ such that $M^{i, j} \preccurlyeq{ }_{R} M_{3}^{i, j}$.

Now define

$$
\begin{align*}
M= & \left\{\left(a_{2}, t\right) \mid \exists\left(a_{2}, t_{2}\right) \in M_{2}: \forall t_{3} / t_{1} \in t:\right. \\
& \exists i, a_{1}, a_{3}:\left(a_{1}, t_{1}\right) \in \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right),  \tag{2}\\
& \left.\left(a_{3}, t_{3}\right) \in M_{3}^{i, \delta_{i}\left(a_{1}, t_{1}\right)}, a_{1} \oplus a_{2} \preccurlyeq a_{3}, t_{1} \| t_{2} \leq_{\mathrm{m}} t_{3}\right\} .
\end{align*}
$$

We need to show that $M \in \operatorname{Tran} /(s)$.
Let $i \in\{1, \ldots, n\}$ and $M_{1}^{i, j} \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right)$; we claim that $M \triangleright M_{1}^{i, j} \preccurlyeq_{R^{\prime}} M_{3}^{i, j}$. Let $\left(a_{3}, t_{3}\right) \in M \triangleright M_{1}^{i, j}$, then $a_{3}=a_{1} \oplus a_{2}$ for some $a_{1}, a_{2}$ such that $t_{3} / t_{1} \in t$, $\left(a_{1}, t_{1}\right) \in M_{1}^{i, j}$ and $\left(a_{2}, t\right) \in M$. By disjointness, $j=$ $\delta_{i}\left(a_{1}, t_{1}\right)$, hence by definition of $M,\left(a_{3}, t_{3}\right) \in M_{3}^{i, j}$ as was to be shown.

For the reverse inclusion, let $\left(a_{3}, t_{3}\right) \in M_{3}^{i, j}$. By $M^{i, j} \preccurlyeq{ }_{R} M_{3}^{i, j}$ and definition of $M^{i, j}$, there are $\left(a_{1}, t_{1}\right) \in$ $M_{1}^{i, j}$ and $\left(a_{2}, t_{2}\right) \in M_{2}$ for which $a_{1} \oplus a_{2} \preccurlyeq a_{3}$ and $t_{1} \| t_{2} \leq_{\mathrm{m}} t_{3}$. Thus $j=\delta_{i}\left(a_{1}, t_{1}\right)$, so that there must be $\left(a_{2}, t\right) \in M$ for which $t_{3} / t_{1} \in t$, but then also $\left(a_{1} \oplus a_{2}, t_{3}\right) \in M \triangleright M_{1}^{i, j}$.

We show that $M_{2} \preccurlyeq R^{\prime} M$.

- Let $\left(a_{2}, t_{2}\right) \in M_{2}$. For every $i=1, \ldots, n$ and every $\left(a_{1}, t_{1}\right) \in \in \operatorname{Tran}_{1}\left(t_{1}^{i}\right)$, we can use $M^{i, j} \preccurlyeq{ }_{R} M_{3}^{i, j}$ and choose an element $\left(\eta_{i}\left(a_{1}, t_{1}\right), \tau_{i}\left(a_{1}, t_{1}\right)\right) \in M_{3}^{i, \delta_{i}\left(a_{1}, t_{1}\right)}$ for which $t_{1} \| t_{2} \leq_{\mathrm{m}} \tau_{i}\left(a_{1}, t_{1}\right)$ and $a_{1} \oplus a_{2} \preccurlyeq \eta_{i}\left(a_{1}, t_{1}\right)$. Let $t=\left\{\tau_{i}\left(a_{1}, t_{1}\right) / t_{1} \mid i=1, \ldots, n,\left(a_{1}, t_{1}\right) \in \in\right.$ $\left.\operatorname{Tran}_{1}\left(t_{1}^{i}\right)\right\}$, then $\left(a_{2}, t\right) \in M$ and $\left(t_{2}, t\right) \in R^{\prime}$.
- Let $\left(a_{2}, t\right) \in M$, then we have $\left(a_{2}, t_{2}\right) \in M_{2}$ satisfying the conditions in (2). Hence $t_{1} \| t_{2} \leq_{\mathrm{m}} t_{3}$ for all $t_{3} / t_{1} \in t$, so that $\left(t_{2}, t\right) \in R^{\prime}$.


## 5 Robust Specification Theories

We proceed to lift the results of the previous sections to a quantitative setting, where the Boolean notions of modal and thorough refinement are replaced by refinement distances. We have shown in [7, 29-32] that a good setting for quantitative analysis is given by the one of recursively specified trace distances on an abstract commutative quantale as defined below; we refer to the above-cited papers for a detailed exposition of how this framework covers all common approaches to quantitative analysis.

Denote by $\Sigma^{\infty}=\Sigma^{*} \cup \Sigma^{\omega}$ the set of finite and infinite traces over $\Sigma$.

### 5.1 Recursively specified trace distances

Recall that a (commutative) quantale consists of a complete lattice $\left(\mathbb{L}, \sqsubseteq_{\mathbb{L}}\right)$ and a commutative, associative addition operation $\oplus_{\mathbb{L}}$ which distributes over arbitrary suprema; we denote by $\perp_{\mathbb{L}}, \mathrm{T}_{\mathbb{L}}$ the bottom and top elements of $\mathbb{L}$. We call a function $d: X \times X \rightarrow \mathbb{L}$, for a set $X$ and a quantale $\mathbb{L}$, an $\mathbb{L}$-hemimetric if it satisfies $d(x, x)=\perp_{\mathbb{L}}$ for all $x \in X$ and $d(x, z) \sqsubseteq_{\mathbb{L}}$ $d(x, y) \oplus_{\mathbb{L}} d(y, z)$ for all $x, y, z \in X$.
$\mathbb{L}$-hemimetrics are generalizations of distances: for $\mathbb{L}=\mathbb{R}_{\geq 0} \cup\{\infty\}$ the extended real line, an $\left(\mathbb{R}_{\geq 0} \cup\{\infty\}\right)$ hemimetric is simply an extended hemimetric, i.e., a function $d: X \times X \rightarrow \mathbb{R}_{\geq 0} \cup\{\infty\}$ which satisfies $d(x, x)=0$ for all $x \in X$ and the triangle inequality $d(x, z) \leq d(x, y)+d(y, z)$ for all $x, y, z \in X$. If $d$ also is symmetric, i.e., satisfies $d(x, y)=d(y, x)$ for all $x, y \in X$, then $d$ is usually called a pseudometric. If $d$ also satisfies the principle of separability, or indiscernibility of identicals, i.e., such that $d(x, y)=0$ implies $x=y$, it is called a metric.

A recursive trace distance specification $\left(\mathbb{L}\right.$, eval, $\left.d_{\mathrm{tr}}^{\mathbb{L}}, F\right)$ consists of a quantale $\mathbb{L}$, a quantale morphism eval : $\mathbb{L} \rightarrow \mathbb{R}_{\geq 0} \cup\{\infty\}$, an $\mathbb{L}$-hemimetric $d_{\mathrm{tr}}^{\mathbb{L}}: \Sigma^{\infty} \times \Sigma^{\infty} \rightarrow \mathbb{L}$ (called lifted trace distance), and a distance iterator function $F: \Sigma \times \Sigma \times \mathbb{L} \rightarrow \mathbb{L}$. For our purposes, $F$ must be monotone in the third and anti-monotone in
the second coordinate and satisfy an extended triangle inequality: for all $a, b, c \in \Sigma$ and $\alpha, \beta \in \mathbb{L}, F(a, b, \alpha) \oplus_{\mathbb{L}}$ $F(b, c, \beta) \sqsupseteq_{\mathbb{L}} F\left(a, c, \alpha \oplus_{\mathbb{L}} \beta\right)$.
$F$ is to specify $d_{\mathrm{tr}}^{\mathbb{L}}$ recursively in the sense that for all $a, b \in \Sigma$ and all $\sigma, \tau \in \Sigma^{\infty}$ (and with "." denoting concatenation),
$d_{\mathrm{tr}}^{\mathbb{L}}(a . \sigma, b . \tau)=F\left(a, b, d_{\mathrm{tr}}^{\mathbb{L}}(\sigma, \tau)\right)$.
The trace distance associated with such a distance specification is $d_{\mathrm{tr}}: \Sigma^{\infty} \times \Sigma^{\infty} \rightarrow \mathbb{R}_{\geq 0}$ given by $d_{\mathrm{tr}}=$ eval $\circ d_{\mathrm{tr}}^{\mathbb{U}}$.

Note that $d_{\mathrm{tr}}^{\mathbb{L}}$ specializes to a distance on labels (because $\Sigma \subseteq \Sigma^{\infty}$ ); we require that this is compatible with label refinement in the sense that $a \preccurlyeq b$ implies $d_{\mathrm{tr}}^{\mathbb{L}}(a, b)=\perp_{\mathbb{L}}$. Then (3) implies that whenever $a \preccurlyeq b$, then $F\left(a, b, \perp_{\mathbb{L}}\right)=d_{\mathrm{tr}}^{\mathbb{L}}(a, b)=\perp_{\mathbb{L}}$. As an inverse property, we say that $F$ is recursively separating if $F(a, b, \alpha)=$ $\perp_{\mathbb{L}}$ implies that $a \preccurlyeq b$ and $\alpha=\perp_{\mathbb{L}}$.

Example 6 We have shown in [7, 29-32 that all commonly used trace distances obey recursive characterizations as above. We give a few examples, all of which are recursively separating, and refer to [30, 31] for further details:

The point-wise distance from [20], for example, has $\mathbb{L}=\mathbb{R}_{\geq 0} \cup\{\infty\}$, eval $=$ id and
$d_{\mathrm{tr}}^{\mathbb{L}}(a . \sigma, b \cdot \tau)=\max \left(d(a, b), d_{\mathrm{tr}}^{\mathbb{L}}(\sigma, \tau)\right)$,
where $d: \Sigma \times \Sigma \rightarrow \mathbb{R}_{\geq 0} \cup\{\infty\}$ is a hemimetric on labels. For the label set $\Sigma=U \times\{[l, r] \mid l \in \mathbb{R} \cup\{-\infty\}, r \in$ $\mathbb{R} \cup\{\infty\}, l \leq r\}$ from Example 1, one useful example of such a hemimetric is $d\left(\left(u_{1},\left[l_{1}, r_{1}\right]\right),\left(u_{2},\left[l_{2}, r_{2}\right]\right)\right)=$ $\sup _{x_{1} \in\left[l_{1}, r_{1}\right]} \inf _{x_{2} \in\left[l_{2}, r_{2}\right]}\left|x_{1}-x_{2}\right|=\max \left(l_{2}-l_{1}, r_{1}-r_{2}, 0\right)$ if $u_{1}=u_{2}$ and $\infty$ otherwise, cf. 6].

The discounting distance, also used in 20, again uses $\mathbb{L}=\mathbb{R}_{\geq 0} \cup\{\infty\}$ and eval $=$ id, but
$d_{\mathrm{tr}}^{\mathbb{L}}(a . \sigma, b \cdot \tau)=d(a, b)+\lambda d_{\mathrm{tr}}^{\mathbb{L}}(\sigma, \tau)$
for a constant $\lambda \in[0,1[$.
For the limit-average distance used in [62] and other papers, $\mathbb{L}=\left(\mathbb{R}_{\geq 0} \cup\{\infty\}\right)^{\mathbb{N}}, \operatorname{eval}(\alpha)=\liminf _{j \in \mathbb{N}} \alpha(j)$, and
$d_{\mathrm{tr}}^{\mathbb{L}}(a . \sigma, b \cdot \tau)(j)=\frac{1}{j+1} d(a, b)+\frac{j}{j+1} d_{\mathrm{tr}}^{\mathbb{U}}(\sigma, \tau)(j-1)$.
It is clear that limit-average distance has no recursive specification which uses $\mathbb{L}=\mathbb{R}_{\geq 0} \cup\{\infty\}$ as for the other distances above. Intuitively, the quantale $\left(\mathbb{R}_{\geq 0} \cup\{\infty\}\right)^{\mathbb{N}}$ has to be used to memorize how many symbols one has seen in the sequences $\sigma, \tau$. This and other examples 30, 31] show that using general quantales in recursive trace distance specifications instead of simply $\mathbb{L}=\mathbb{R}_{\geq 0} \cup\{\infty\}$ is necessary.

The discrete trace distance is given by $d_{\mathrm{tr}}(\sigma, \tau)=0$ if $\sigma \preccurlyeq \tau$ and $\infty$ otherwise (here we have extended $\preccurlyeq$ to traces in the obvious way). It has a recursive characterization with $\mathbb{L}=\mathbb{R}_{\geq 0} \cup\{\infty\}$, eval $=\mathrm{id}$, and $d_{\mathrm{tr}}(a . \sigma, b . \tau)=d_{\mathrm{tr}}(\sigma, \tau)$ if $a \preccurlyeq b$ and $\infty$ otherwise.

For the rest of this paper, we fix a recursively specified trace distance.

### 5.2 Refinement distances

We lift the notions of modal refinement, for all our formalisms, to distances. Conceptually, this is done by replacing " $\forall$ " quantifiers by "sup" and " $\exists$ " by "inf" in the definitions, and then using the distance iterator to introduce a recursive functional whose least fixed point is the distance.

Definition 3 The lifted refinement distance on the states of DMTS $\mathcal{D}_{1}=\left(S_{1}, S_{1}^{0}, \cdots \rightarrow_{1}, \longrightarrow_{1}\right)$ and $\mathcal{D}_{2}=$ $\left(S_{2}, S_{2}^{0}, \rightarrow \rightarrow_{2}, \longrightarrow_{2}\right)$ is the least fixed point to the equations

$$
\begin{aligned}
& d_{\mathrm{m}}^{\mathbb{L}}\left(s_{1}, s_{2}\right)=
\end{aligned}
$$

for $s_{1} \in S_{1}, s_{2} \in S_{2}$. For NAA $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right)$, $\mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right)$, the right-hand side is replaced by

$$
\begin{array}{r}
\sup _{M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)} \inf _{M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)} \\
\max \left\{\begin{array}{l}
\sup _{\left(a_{1}, t_{1}\right) \in M_{1}} \inf _{\left(a_{2}, t_{2}\right) \in M_{2}} F\left(a_{1}, a_{2}, d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1}, t_{2}\right)\right), \\
\sup _{\left(a_{2}, t_{2}\right) \in M_{2}} \inf _{\left(a_{1}, t_{1}\right) \in M_{1}} F\left(a_{1}, a_{2}, d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1}, t_{2}\right)\right),
\end{array}\right.
\end{array}
$$

and for $\nu$-calculus expressions $\mathcal{N}_{1}=\left(X_{1}, X_{1}^{0}, \Delta_{1}\right), \mathcal{N}_{2}=$ $\left(X_{2}, X_{2}^{0}, \Delta_{2}\right)$ in normal form, it is
$\max \left\{\begin{array}{l}\sup _{a_{1} \in \Sigma, y_{1} \in \square_{1}^{a_{1}}\left(x_{1}\right)} \inf _{a_{2} \in \Sigma, y_{2} \in \square_{2}^{a_{2}}\left(x_{2}\right)} F\left(a_{1}, a_{2}, d_{\mathrm{m}}^{\mathbb{L}}\left(y_{1}, y_{2}\right)\right), \\ \sup _{N_{2} \in \overbrace{2}\left(x_{2}\right)} \inf _{\inf _{1} \in \nabla_{1}\left(x_{1}\right)} \sup _{\left(a_{1}, y_{1}\right) \in N_{1}} \inf _{\left(a_{2}, y_{2}\right) \in N_{2}} F\left(a_{1}, a_{2}, d_{\mathrm{m}}^{\mathbb{L}}\left(y_{1}, y_{2}\right)\right) .\end{array}\right.$
Using Tarski's fixed point theorem, one easily sees that the lifted refinement distances are indeed welldefined. (Here one needs monotonicity of $F$ in the third
coordinate, together with the fact that sup and inf are monotonic.)

Note that we define the distances using least fixed points, as opposed to the greatest fixed point definition of standard refinement. Informally, this is because our order is reversed: we are not interested in maximizing refinement relations, but in minimizing refinement distance.

The lifted refinement distance between specifications is defined by
$d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right)=\sup _{s_{1}^{0} \in S_{1}^{0}} \inf _{s_{2}^{0} \in S_{2}^{0}} d_{\mathrm{m}}^{\mathbb{L}}\left(s_{1}^{0}, s_{2}^{0}\right)$.
Analogously to thorough refinement, there is also a lifted thorough refinement distance, given by $d_{\mathrm{th}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right)=$ $\sup _{\mathcal{I}_{1} \in \llbracket \mathcal{S}_{1} \rrbracket} \inf _{\mathcal{I}_{2} \in \llbracket \mathcal{S}_{2} \rrbracket} d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{I}_{1}, \mathcal{I}_{2}\right)$.

Using the eval function, one gets distances $d_{\mathrm{m}}=$ eval $\circ d_{\mathrm{m}}^{\mathbb{L}}$ and $d_{\mathrm{th}}=$ eval $\circ d_{\mathrm{th}}^{\mathbb{L}}$, with values in $\mathbb{R}_{\geq 0} \cup\{\infty\}$, which will be the ones one is interested in for concrete applications.

Example 7 We compute the discounting refinement distance between the DMTS $x$ and $x^{\prime}$ in Figs. 4 and 7 on page 2, assuming sup-inf distance on quantitative labels (see Example 6). We have
$d_{\mathrm{m}}\left(x, x^{\prime}\right)=\max \left(0+\lambda d_{\mathrm{m}}\left(x, x^{\prime}\right), 0+\lambda d_{\mathrm{m}}\left(y, y^{\prime}\right)\right)$,
$d_{\mathrm{m}}\left(y, y^{\prime}\right)=\max \left(0+\lambda d_{\mathrm{m}}\left(x, x^{\prime}\right), 1+\lambda d_{\mathrm{m}}\left(y, y^{\prime}\right)\right.$,
the least fixed point of which is seen to be $d_{\mathrm{m}}\left(x, x^{\prime}\right)=$ $\frac{\lambda}{1-\lambda}$. Similarly, $d_{\mathrm{m}}\left(x^{\prime}, x\right)=\frac{\lambda}{1-\lambda}$. Note that $x \mathbb{Z}_{\mathrm{m}} x^{\prime}$ and $x^{\prime} \not \leq_{\mathrm{m}} x$.

We recall the notion of refinement family from 30, Def. 6] and extend it to specifications. We give the definition for NAA only; for DMTS and the modal $\nu$ calculus it is similar.

Definition 4 A refinement family from $\mathcal{A}_{1}$ to $\mathcal{A}_{2}$, for NAA $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right), \mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right)$, is an $\mathbb{L}$-indexed family of relations $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{2} \mid\right.$ $\alpha \in \mathbb{L}\}$ with the property that for all $\alpha \in \mathbb{L}$ with $\alpha \neq \top_{\mathbb{L}}$, all $\left(s_{1}, s_{2}\right) \in R_{\alpha}$, and all $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$, there is $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ such that
$-\forall\left(a_{1}, t_{1}\right) \in M_{1}: \exists\left(a_{2}, t_{2}\right) \in M_{2}, \beta \in \mathbb{L}:\left(t_{1}, t_{2}\right) \in$ $R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq \alpha$,
$-\forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{1}, t_{1}\right) \in M_{1}, \beta \in \mathbb{L}:\left(t_{1}, t_{2}\right) \in$ $R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq \alpha$.
Lemma 2 For all NAA $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right), \mathcal{A}_{2}=$ $\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right)$, there exists a refinement family $R$ from $\mathcal{A}_{1}$ to $\mathcal{A}_{2}$ such that for all $s_{1}^{0} \in S_{1}^{0}$, there is $s_{2}^{0} \in S_{2}^{0}$ for which $\left(s_{1}^{0}, s_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)$.

We say that a refinement family as in the lemma witnesses $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)$.

Proof Define $R$ by $R_{\alpha}=\left\{\left(s_{1}, s_{2}\right) \mid d_{\mathrm{m}}^{\mathbb{L}}\left(s_{1}, s_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha\right\}$. First, as $\left(s_{1}^{0}, s_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}\left(s_{1}^{0}, s_{2}^{0}\right)}$ for all $s_{1}^{0} \in S_{1}^{0}, s_{2}^{0} \in S_{2}^{0}$, it is indeed the case that for all $s_{1}^{0} \in S_{1}^{0}$, there is $s_{2}^{0} \in S_{2}^{0}$ for which
$\left(s_{1}^{0}, s_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{T}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}=R_{\max _{s_{1}^{0} \in S_{1}^{0}} \min _{s_{2}^{0} \in S_{2}^{0}} d_{\mathbf{m}}^{\mathbb{\pi}}\left(s_{1}^{0}, s_{2}^{0}\right)}$.
Now let $\alpha \in \mathbb{L}$ with $\alpha \neq \top_{\mathbb{L}}$ and $\left(s_{1}, s_{2}\right) \in R_{\alpha}$. Let $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$. We have $d_{\mathfrak{m}}^{\mathbb{L}}\left(s_{1}, s_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha$, hence there is $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ such that
$\alpha \sqsupseteq_{\mathbb{L}} \max \left\{\begin{array}{l}\sup _{\left(a_{1}, t_{1}\right) \in M_{1}} \inf _{\left(a_{2}, t_{2}\right) \in M_{2}} F\left(a_{1}, a_{2}, d_{\mathbf{m}}^{\mathbb{L}}\left(t_{1}, t_{2}\right)\right), \\ \sup _{\left(a_{2}, t_{2}\right) \in M_{2}} \inf _{\left(a_{1}, t_{1}\right) \in M_{1}} F\left(a_{1}, a_{2}, d_{\mathbf{m}}^{\mathbb{L}}\left(t_{1}, t_{2}\right)\right) .\end{array}\right.$
But this entails that for all $\left(a_{1}, t_{1}\right) \in M_{1}$, there is $\left(a_{2}, t_{2}\right) \in M_{2}$ and $\beta=d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1}, t_{2}\right)$ with $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}}$ $\alpha$, and that for all $\left(a_{2}, t_{2}\right) \in M_{2}$, there is $\left(a_{1}, t_{1}\right) \in M_{1}$ and $\beta=d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1}, t_{2}\right)$ such that $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$.

The following quantitative extension of Theorems 1 and 3 shows that our translations preserve and reflect refinement distances. Its proof is rather long and tedious, hence we present it in a separate appendix to this paper.

Theorem 7 For all DMTS $\mathcal{D}_{1}, \mathcal{D}_{2}$, all NAA $\mathcal{A}_{1}, \mathcal{A}_{2}$ and all $\nu$-calculus expressions $\mathcal{N}_{1}, \mathcal{N}_{2}$ :
$d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right)=d_{\mathrm{m}}^{\mathbb{L}}\left(d a\left(\mathcal{D}_{1}\right), d a\left(\mathcal{D}_{2}\right)\right)$
$d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=d_{\mathrm{m}}^{\mathbb{L}}\left(a d\left(\mathcal{A}_{1}\right), a d\left(\mathcal{A}_{2}\right)\right)$
$d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right)=d_{\mathrm{m}}^{\mathbb{L}}\left(d n\left(\mathcal{D}_{1}\right), d n\left(\mathcal{D}_{2}\right)\right)$
$d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{N}_{1}, \mathcal{N}_{2}\right)=d_{\mathrm{m}}^{\mathbb{L}}\left(n d\left(\mathcal{N}_{1}\right), n d\left(\mathcal{N}_{2}\right)\right)$

### 5.3 Properties

We sum up some important properties of our distances.
Proposition 2 For all specifications $\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{1} \leq_{\mathrm{m}}$ $\mathcal{S}_{2}$ implies $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right)=\perp_{\mathbb{L}}$, and $\mathcal{S}_{1} \leq_{\text {th }} \mathcal{S}_{2}$ implies $d_{\mathrm{th}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right)=\perp_{\mathbb{L}}$. If $F$ is recursively separating, then $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right)=\perp_{\mathbb{L}}$ implies $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{2}$.

Proof We show the proposition for NAA. First, if $\mathcal{A}_{1} \leq_{m}$ $\mathcal{A}_{2}$, with $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right), \mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right)$, then there is an initialized refinement relation $R \subseteq$ $S_{1} \times S_{2}$, i.e., such that for all $\left(s_{1}, s_{2}\right) \in R$ and all $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$, there is $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ for which
$-\forall\left(a_{1}, t_{1}\right) \in M_{1}: \exists\left(a_{2}, t_{2}\right) \in M_{2}: a_{1} \preccurlyeq a_{2},\left(t_{1}, t_{2}\right) \in$ $R$ and
$-\forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{1}, t_{1}\right) \in M_{1}: a_{1} \preccurlyeq a_{2},\left(t_{1}, t_{2}\right) \in$ $R$.

Defining $R^{\prime}=\left\{R_{\alpha}^{\prime} \mid \alpha \in \mathbb{L}\right\}$ by $R_{\alpha}^{\prime}=R$ for all $\alpha \in \mathbb{L}$, we see that $R^{\prime}$ is an initialized refinement family which witnesses $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=\perp_{\mathbb{L}}$.

We have shown that $\mathcal{A}_{1} \leq_{\mathrm{m}} \mathcal{A}_{2}$ implies $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=$ $\perp_{\mathbb{L}}$. Now if $\mathcal{A}_{1} \leq_{\text {th }} \mathcal{A}_{2}$ instead, then for all $\mathcal{I} \in \llbracket \mathcal{A}_{1} \rrbracket$, also $\mathcal{I} \in \llbracket \mathcal{A}_{2} \rrbracket$, hence $d_{\mathrm{th}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=\perp_{\mathbb{L}}$.

To show the last property, assume $F$ to be recursively separating. Define $R \subseteq S_{1} \times S_{2}$ by $R=\left\{\left(s_{1}, s_{2}\right) \mid\right.$ $\left.d_{\mathrm{m}}^{\mathbb{L}}\left(s_{1}, s_{2}\right)=\perp_{\mathbb{L}}\right\}$; we show that $R$ is a witness for $\mathcal{A}_{1} \leq_{\mathrm{m}} \mathcal{A}_{2}$. By $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=\perp_{\mathbb{L}}, R$ is initialized.

Let $\left(s_{1}, s_{2}\right) \in R$ and $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$, then there is $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ such that
$\forall\left(a_{1}, t_{1}\right) \in M_{1}: \exists\left(a_{2}, t_{2}\right) \in M_{2}, \beta_{1} \in \mathbb{L}:$

$$
d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1}, t_{2}\right) \sqsubseteq_{\mathbb{L}} \beta_{1}, F\left(a_{1}, a_{2}, \beta_{1}\right)=\perp_{\mathbb{L}}
$$

$\forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{1}, t_{1}\right) \in M_{1}, \beta_{1} \in \mathbb{L}:$

$$
d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1}, t_{2}\right) \sqsubseteq_{\mathbb{L}} \beta_{1}, F\left(a_{1}, a_{2}, \beta_{1}\right)=\perp_{\mathbb{L}} .
$$

As $F$ is recursively separating, we must have $a_{1} \preccurlyeq a_{2}$ in both these equations and $\beta_{1}=\beta_{2}=\perp_{\mathbb{L}}$. But then $\left(t_{1}, t_{2}\right) \in R$, hence $R$ is indeed a witness for $\mathcal{A}_{1} \leq_{\mathrm{m}} \mathcal{A}_{2}$.

Proposition 3 The functions $d_{\mathrm{m}}^{\mathbb{L}}$ and $d_{\mathrm{th}}^{\mathbb{L}}$ are $\mathbb{L}$-hemimetrics, and $d_{\mathrm{m}}$, $d_{\mathrm{th}}$ are hemimetrics.
Proof We show the proof for NAA. The properties that $d_{\mathrm{m}}^{\mathbb{L}}(\mathcal{A}, \mathcal{A})=\perp_{\mathbb{L}}$ and $d_{\mathrm{th}}^{\mathbb{U}}(\mathcal{A}, \mathcal{A})=\perp_{\mathbb{L}}$ follow from proposition (2)

We show the triangle inequality for $d_{\mathrm{m}}^{\mathbb{L}}$. The triangle inequality for $d_{\mathrm{th}}^{\mathbb{L}}$ will then follow from standard arguments used to show that the Hausdorff metric satisfies the triangle inequality, see e.g., [3, Lemma 3.72]. Let $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right), \mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right), \mathcal{A}_{3}=$ $\left(S_{3}, S_{3}^{0}, \operatorname{Tran}_{3}\right)$ be NAA and $R^{1}=\left\{R_{\alpha}^{1} \subseteq S_{1} \times S_{2} \mid \alpha \in\right.$ $\mathbb{L}\}, R^{2}=\left\{R_{\alpha}^{2} \subseteq S_{2} \times S_{3} \mid \alpha \in \mathbb{L}\right\}$ refinement families such that $\forall s_{1}^{0} \in S_{1}^{0}: \exists s_{2}^{0} \in S_{2}^{0}:\left(s_{1}^{0}, s_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}}^{1}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)$ and $\forall s_{2}^{0} \in S_{2}^{0}: \exists s_{3}^{0} \in S_{3}^{0}:\left(s_{2}^{0}, s_{3}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{L}}}^{2}\left(\mathcal{A}_{2}, \mathcal{A}_{3}\right)$.

Define $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{3} \mid \alpha \in \mathbb{L}\right\}$ by
$R_{\alpha}=\left\{\left(s_{1}, s_{3}\right) \mid \exists \alpha_{1}, \alpha_{2} \in \mathbb{L}, s_{2} \in S_{2}:\right.$

$$
\left.\left(s_{1}, s_{2}\right) \in R_{\alpha_{1}}^{1},\left(s_{2}, s_{3}\right) \in R_{\alpha_{2}}^{2}, \alpha_{1} \oplus_{\mathbb{L}} \alpha_{2}=\alpha\right\}
$$

We see that for all $s_{1}^{0} \in S_{1}^{0}$, there is $s_{3}^{0} \in S_{3}^{0}$ such that $\left(s_{1}^{0}, s_{3}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right) \oplus_{\mathbb{I}} d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{A}_{2}, \mathcal{A}_{3}\right)$; we show that $R$ is a refinement family from $\mathcal{A}_{1}$ to $\mathcal{A}_{2}$.

Let $\alpha \in \mathbb{L}$ and $\left(s_{1}, s_{3}\right) \in R_{\alpha}$, then we have $\alpha_{1}, \alpha_{2} \in$ $\mathbb{L}$ and $s_{2} \in S_{2}$ such that $\alpha_{1} \oplus_{\mathbb{L}} \alpha_{2}=\alpha,\left(s_{1}, s_{2}\right) \in R_{\alpha_{1}}^{1}$ and $\left(s_{2}, s_{3}\right) \in R_{\alpha_{2}}^{2}$. Let $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$, then we have $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ such that
$\forall\left(a_{1}, t_{1}\right) \in M_{1}: \exists\left(a_{2}, t_{2}\right) \in M_{2}, \beta_{1} \in \mathbb{L}:$

$$
\begin{equation*}
\left(t_{1}, t_{2}\right) \in R_{\beta_{1}}^{1}, F\left(a_{1}, a_{2}, \beta_{1}\right) \sqsubseteq_{\mathbb{L}} \alpha_{1}, \tag{4}
\end{equation*}
$$

$\forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{1}, t_{1}\right) \in M_{1}, \beta_{1} \in \mathbb{L}:$

$$
\begin{equation*}
\left(t_{1}, t_{2}\right) \in R_{\beta_{1}}^{1}, F\left(a_{1}, a_{2}, \beta_{1}\right) \sqsubseteq_{\mathbb{L}} \alpha_{1} . \tag{5}
\end{equation*}
$$

This in turn implies that there is $M_{3} \in \operatorname{Tran}_{3}\left(s_{3}\right)$ with

$$
\begin{align*}
& \forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{3}, t_{3}\right) \in M_{3}, \beta_{2} \in \mathbb{L}: \\
& \quad\left(t_{2}, t_{3}\right) \in R_{\beta_{2}}^{2}, F\left(a_{2}, a_{3}, \beta_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha_{2},  \tag{6}\\
& \forall\left(a_{3}, t_{3}\right) \in M_{3}: \exists\left(a_{2}, t_{2}\right) \in M_{2}, \beta_{2} \in \mathbb{L}: \\
& \quad\left(t_{2}, t_{3}\right) \in R_{\beta_{2}}^{2}, F\left(a_{2}, a_{3}, \beta_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha_{2} . \tag{7}
\end{align*}
$$

Now let $\left(a_{1}, t_{1}\right) \in M_{1}$, then we get $\left(a_{2}, t_{2}\right) \in M_{2}$, $\left(a_{3}, t_{3}\right) \in M_{3}$ and $\beta_{1}, \beta_{2} \in \mathbb{L}$ as in (4) and (6). Let $\beta=$ $\beta_{1} \oplus_{\mathbb{L}} \beta_{2}$, then $\left(t_{1}, t_{3}\right) \in R_{\beta}$, and by the extended triangle inequality for $F, F\left(a_{1}, a_{3}, \beta\right) \sqsubseteq_{\mathbb{L}} F\left(a_{1}, a_{2}, \beta_{1}\right) \oplus_{\mathbb{L}}$ $F\left(a_{2}, a_{3}, \beta_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha_{1} \oplus_{\mathbb{L}} \alpha_{2}=\alpha$.

Similarly, given $\left(a_{3}, t_{3}\right) \in M_{3}$, we can apply (7) and (5) to get $\left(a_{1}, t_{1}\right) \in M_{1}$ and $\beta \in \mathbb{L}$ such that $\left(t_{1}, t_{3}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{3}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$.

We have shown that $d_{\mathrm{m}}^{\mathbb{L}}$ and $d_{\mathrm{tr}}^{\mathbb{L}}$ are $\mathbb{L}$-hemimetrics. Using monotonicity of the eval function, it follows that $d_{\mathrm{m}}$ and $d_{\mathrm{tr}}$ are hemimetrics.

Proposition 4 For the discrete distances, $d_{\mathrm{m}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right)=$ 0 if $\mathcal{S}_{1} \leq_{\mathrm{m}} \mathcal{S}_{2}$ and $\infty$ otherwise. Similarly, $d_{\mathrm{th}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right)=$ 0 if $\mathcal{S}_{1} \leq_{\text {th }} \mathcal{S}_{2}$ and $\infty$ otherwise.

Proof We show the proposition for NAA. We already know that, also for the discrete distances, $\mathcal{A}_{1} \leq{ }_{\mathrm{m}} \mathcal{A}_{2}$ implies $d_{\mathrm{m}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=0$ and that $\mathcal{A}_{1} \leq_{\text {th }} \mathcal{A}_{2}$ implies $d_{\mathrm{th}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=0$. We show that $d_{\mathrm{m}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=0$ implies $\mathcal{A}_{1} \leq_{\mathrm{m}} \mathcal{A}_{2}$. Let $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{2} \mid \alpha \in \mathbb{L}\right\}$ be a refinement family such that $\forall s_{1}^{0} \in S_{1}^{0}: \exists s_{2}^{0} \in S_{2}^{0}$ : $\left(s_{1}^{0}, s_{2}^{0}\right) \in R_{0}$. We show that $R_{0}$ is a witness for $\mathcal{A}_{1} \leq_{\mathrm{m}}$ $\mathcal{A}_{2}$; it is clearly initialized.

Let $\left(s_{1}, s_{2}\right) \in R_{0}$ and $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$, then we have $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ such that

$$
\begin{align*}
\forall\left(a_{1}, t_{1}\right) \in M_{1}: & \exists\left(a_{2}, t_{2}\right) \in M_{2}, \beta \in \mathbb{L}: \\
& \left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right)=0, \\
\forall\left(a_{2}, t_{2}\right) \in M_{2}: & \exists\left(a_{1}, t_{1}\right) \in M_{1}, \beta \in \mathbb{L}:  \tag{8}\\
& \left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right)=0 .
\end{align*}
$$

Using the definition of the distance, we see that the condition $F\left(a_{1}, a_{2}, \beta\right)=0$ is equivalent to $a_{1} \preccurlyeq a_{2}$ and $\beta=0$, hence (8) degenerates to
$\forall\left(a_{1}, t_{1}\right) \in M_{1}: \exists\left(a_{2}, t_{2}\right) \in M_{2}:\left(t_{1}, t_{2}\right) \in R_{0}, a_{1} \preccurlyeq a_{2}$, $\forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{1}, t_{1}\right) \in M_{1}:\left(t_{1}, t_{2}\right) \in R_{0}, a_{1} \preccurlyeq a_{2}$,
which are exactly the conditions for $R_{0}$ to be a modal refinement.

Again by definition, we see that for any NAA $\mathcal{A}_{1}$, $\mathcal{A}_{2}$, either $d_{\mathrm{m}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=0$ or $d_{\mathrm{m}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=\infty$, hence $\mathcal{A}_{1} \not \geq_{\mathrm{m}} \mathcal{A}_{2}$ implies that $d_{\mathrm{m}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=\infty$.

To show the last part of the proposition, we notice that

$$
\begin{aligned}
& d_{\mathrm{th}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=\sup _{\mathcal{I}_{1} \in \llbracket \mathcal{A}_{1} \rrbracket \mathcal{I}_{2} \in \llbracket \mathcal{A}_{2} \rrbracket} \inf _{\mathrm{m}}\left(\mathcal{I}_{1}, \mathcal{I}_{2}\right) \\
& \\
& = \begin{cases}0 & \text { if } \forall \mathcal{I}_{1} \in \llbracket \mathcal{A}_{1} \rrbracket: \exists \mathcal{I}_{2} \in \llbracket \mathcal{A}_{2} \rrbracket: \mathcal{I}_{1} \leq \mathrm{m} \mathcal{I}_{2}, \\
\infty & \text { otherwise },\end{cases} \\
&
\end{aligned}= \begin{cases}0 & \text { if } \llbracket \mathcal{A}_{1} \rrbracket \subseteq \llbracket \mathcal{A}_{2} \rrbracket, \\
\infty & \text { otherwise } .\end{cases}
$$

Hence $d_{\mathrm{th}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=0$ if $\mathcal{A}_{1} \leq_{\text {th }} \mathcal{A}_{2}$ and $d_{\mathrm{th}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)=$ $\infty$ otherwise.

As a quantitative analogy to the implication from (Boolean) modal refinement to thorough refinement (see Proposition (1), the next theorem shows that thorough refinement distance is bounded above by modal refinement distance. Note that for the discrete trace distance (and using Proposition (4), this is equivalent to the Boolean statement.

Theorem 8 For all specifications $\mathcal{S}_{1}, \mathcal{S}_{2}, d_{\mathrm{th}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right) \sqsubseteq_{\mathbb{L}}$ $d_{\mathbf{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right)$.

Proof We prove the statement for NAA; for DMTS and $\nu$-calculus expressions it then follows from Theorem 7

Let $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right), \mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right)$. We have a refinement family $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{2} \mid \alpha \in \mathbb{L}\right\}$ such that for all $s_{1}^{0} \in S_{1}^{0}$, there is $s_{2}^{0} \in S_{2}^{0}$ with $\left(s_{1}^{0}, s_{2}^{0}\right) \in$ $R_{d_{\mathbf{m}}^{\mathbb{m}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}$. Let $\mathcal{I}=\left(S, S^{0}, T\right) \in \llbracket \mathcal{A}_{1} \rrbracket$, i.e., $\mathcal{I} \leq_{\mathrm{m}} \mathcal{A}_{1}$.

Let $R^{1} \subseteq S \times S_{1}$ be an initialized modal refinement, and define a relation family $R^{2}=\left\{R_{\alpha}^{2} \subseteq S \times S_{2} \mid \alpha \in\right.$ $\mathbb{L}\}$ by $R_{\alpha}^{2}=R^{1} \circ R_{\alpha}=\left\{\left(s, s_{2}\right) \mid \exists s_{1} \in S:\left(s, s_{1}\right) \in\right.$ $R^{1},\left(s_{1}, s_{2}\right) \in R_{\alpha}$. We define a $\operatorname{LTS} \mathcal{I}_{2}=\left(S_{2}, S_{2}^{0}, T_{2}\right)$ as follows:

For all $\alpha \in \mathbb{L}$ with $\alpha \neq \top_{\mathbb{L}}$ and $\left(s, s_{2}\right) \in R_{\alpha}^{2}$ : We must have $s_{1} \in S_{1}$ with $\left(s, s_{1}\right) \in R^{1}$ and $\left(s_{1}, s_{2}\right) \in R_{\alpha}$. Then there is $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$ such that

- for all $s \xrightarrow{a} t$, there is $\left(a, t_{1}\right) \in M_{1}$ with $\left(t, t_{1}\right) \in R_{1}$,
- for all $\left(a_{1}, t_{1}\right) \in M_{1}$, there is $s \xrightarrow{a} t$ with $\left(t, t_{1}\right) \in$ $R_{1}$.

This in turn implies that there is $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ satisfying the conditions in Definition 4. For all $\left(a_{2}, t_{2}\right) \in$ $M_{2}$ : add a transition $s_{2} \xrightarrow{a_{2}} t_{2}$ to $T_{2}$.

We show that the identity relation $\left\{\left(s_{2}, s_{2}\right) \mid s_{2} \in\right.$ $\left.S_{2}\right\}$ is a witness for $\mathcal{I}_{2} \leq_{\mathrm{m}} \mathcal{A}_{2}$. Let $s_{2} \in S_{2}$ and $s_{2} \xrightarrow{a_{2}}$ $t_{2}$. By construction, there is an $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ with $\left(a_{2}, t_{2}\right) \in M_{2}$, and for all $\left(a_{2}^{\prime}, t_{2}^{\prime}\right) \in M_{2}, s_{2} \xrightarrow{a_{2}^{\prime}} t_{2}^{\prime}$.

We show that $R^{2}$ is a witness for $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{I}, \mathcal{I}_{2}\right)$; clearly, $R^{2}$ is initialized. Let $\alpha \in \mathbb{L}$ with $\alpha \neq \top_{\mathbb{L}}$ and $\left(s, s_{2}\right) \in$ $R_{\alpha}^{2}$, then there is $s_{1} \in S_{1}$ with $\left(s, s_{1}\right) \in R^{1}$ and $\left(s_{1}, s_{2}\right) \in$ $R_{\alpha}$. We also have $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$ such that

- for all $s \xrightarrow{a} t$, there is $\left(a, t_{1}\right) \in M_{1}$ with $\left(t, t_{1}\right) \in R^{1}$,
- for all $\left(a, t_{1}\right) \in M_{1}$, there is $s \xrightarrow{a} t$ with $\left(t, t_{1}\right) \in R^{1}$
and thus $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ satisfying the conditions in Definition 4

Let $s \xrightarrow{a} t$, then there is $\left(a, t_{1}\right) \in M_{1}$ with $\left(t, t_{1}\right) \in$ $R^{1}$, hence also $\left(a_{2}, t_{2}\right) \in M_{2}$ and $\beta \in \mathbb{L}$ with $\left(t_{1}, t_{2}\right) \in$ $R_{\beta}$ and $F\left(a, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. But then $\left(t, t_{2}\right) \in R_{\beta}^{2}$, and $s_{2} \xrightarrow{a_{2}} t_{2}$ by construction.

Let $s_{2} \xrightarrow{a_{2}} t_{2}$. By construction, there is an $M_{2} \in$ $\operatorname{Tran}_{2}\left(s_{2}\right)$ with $\left(a_{2}, t_{2}\right) \in M_{2}$. This implies that there is $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right), \beta \in \mathbb{L}$ and $\left(a_{1}, t_{1}\right) \in M_{1}$ with $\left(t_{1}, t_{2}\right) \in$ $R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq \alpha$. But then there is also $s \xrightarrow{a_{1}} t$ with $\left(t, t_{1}\right) \in R^{1}$, hence $\left(t, t_{2}\right) \in R_{\beta}^{2}$.

### 5.4 Disjunction and conjunction

In order to generalize the properties of Theorem4 to our quantitative setting, we introduce a notion of relaxed implementation semantics:

Definition 5 The $\alpha$-relaxed implementation semantics of $\mathcal{S}$, for a specification $\mathcal{S}$ and $\alpha \in \mathbb{L}$, is
$\llbracket \mathcal{S} \rrbracket^{\alpha}=\left\{\mathcal{I}\right.$ implementation $\left.\mid d_{\mathrm{m}}^{\mathbb{L}}(\mathcal{I}, \mathcal{S}) \sqsubseteq \alpha\right\}$.
Hence, $\llbracket \mathcal{S} \rrbracket^{\alpha}$ comprises all labeled transition systems which are implementations of $\mathcal{S}$ up to $\alpha$. Note that by Proposition 2 and for $F$ recursively separating, $\llbracket \mathcal{S} \rrbracket^{\perp_{\mathbb{L}}}=$ $\llbracket \mathcal{S} \rrbracket$.

Theorem 9 For all specifications $\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3}$ and $\alpha \in$ $\mathbb{L}$,
$-d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1} \vee \mathcal{S}_{2}, \mathcal{S}_{3}\right)=\max \left(d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{3}\right), d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{2}, \mathcal{S}_{3}\right)\right)$,
$-d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{2} \wedge \mathcal{S}_{3}\right) \sqsupseteq_{\mathbb{L}} \max \left(d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{2}\right), d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{3}\right)\right)$,
$-\llbracket \mathcal{S}_{1} \vee \mathcal{S}_{2} \rrbracket^{\alpha}=\llbracket \mathcal{S}_{1} \rrbracket^{\alpha} \cup \llbracket \mathcal{S}_{2} \rrbracket^{\alpha}$, and
$-\llbracket \mathcal{S}_{1} \wedge \mathcal{S}_{2} \rrbracket^{\alpha} \subseteq \llbracket \mathcal{S}_{1} \rrbracket^{\alpha} \cap \llbracket \mathcal{S}_{2} \rrbracket^{\alpha}$.
Proof We show the proof for DMTS.
The proof that $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1} \vee \mathcal{D}_{2}, \mathcal{D}_{3}\right)=\max \left(d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{3}\right)\right.$, $\left.d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{2}, \mathcal{D}_{3}\right)\right)$ is trivial: any refinement family witnessing $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1} \vee \mathcal{D}_{2}, \mathcal{D}_{3}\right)$ splits into two families witnessing $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{3}\right)$ and $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{2}, \mathcal{D}_{3}\right)$ and vice versa.

To show that $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2} \wedge \mathcal{D}_{3}\right) \sqsupseteq_{\mathbb{L}} \max \left(d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right)\right.$, $\left.d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{3}\right)\right)$, let $R=\left\{R_{\alpha} \subseteq S_{1} \times\left(S_{2} \times S_{3}\right) \mid \alpha \in \mathbb{L}\right\}$ be a witness for $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2} \wedge \mathcal{D}_{3}\right)$ and define $R^{2}=\left\{R_{\alpha}^{2} \subseteq\right.$ $\left.S_{1} \times S_{2} \mid \alpha \in \mathbb{L}\right\}$ by $R_{\alpha}^{2}=\left\{\left(s_{1}, s_{2}\right) \mid \exists s_{3} \in S_{3}\right.$ : $\left.\left(s_{1},\left(s_{2}, s_{3}\right)\right) \in R_{\alpha}\right\}$ for all $\alpha \in \mathbb{L}$.

Let $s_{1}^{0} \in S_{1}^{0}$, then we have $\left(s_{2}^{0}, s_{3}^{0}\right) \in S_{2}^{0} \times S_{3}^{0}$ so that $\left(s_{1}^{0},\left(s_{2}^{0}, s_{3}^{0}\right)\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}}\left(\mathcal{D}_{1}, \mathcal{D}_{2} \wedge \mathcal{D}_{3}\right)$, hence also $\left(s_{1}^{0}, s_{2}^{0}\right) \in$ $R_{d_{\mathbf{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2} \wedge \mathcal{D}_{3}\right)}^{2}$.

Let $\alpha \in \mathbb{L}$ and $\left(s_{1}, s_{2}\right) \in R_{\alpha}^{2}$, then we have $s_{3} \in S_{3}$ for which $\left(s_{1},\left(s_{2}, s_{3}\right)\right) \in R_{\alpha}$. Assume first that $s_{1} \xrightarrow{a_{1}}$ $t_{1}$, then there is $\left(s_{2}, s_{3}\right) \xrightarrow{a}\left(t_{2}, t_{3}\right)$ and $\beta \in \mathbb{L}$ such that


Fig. 10 LTS $\mathcal{I}$ together with DMTS $\mathcal{D}_{1}, \mathcal{D}_{2}$ and their conjunction. For the point-wise or discounting distances, $d_{\mathrm{m}}\left(\mathcal{I}, \mathcal{D}_{1}\right)=$ $d_{\mathrm{m}}\left(\mathcal{I}, \mathcal{D}_{2}\right)=1$, but $d_{\mathrm{m}}\left(\mathcal{I}, \mathcal{D}_{1} \wedge \mathcal{D}_{2}\right)=\infty$
$F\left(a_{1}, a, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$ and $\left(t_{1},\left(t_{2}, t_{3}\right)\right) \in R_{\beta}$, hence $\left(t_{1}, t_{2}\right) \in$ $R_{\beta}^{2}$. By construction of $\mathcal{D}_{2} \wedge \mathcal{D}_{3}$, there are $s_{2} \xrightarrow{a_{2}} t_{2}$ and $s_{3} \xrightarrow{a_{3}} t_{3}$ such that $a=a_{2} \otimes a_{3}$, but then by antimonotonicity, $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} F\left(a_{1}, a, \beta\right) \sqsubseteq \alpha$.

Now assume $s_{2} \longrightarrow N_{2}$, then, by construction, $\left(s_{2}, s_{3}\right) \longrightarrow N=\left\{\left(a_{2} \otimes a_{3},\left(t_{2}, t_{3}\right)\right) \mid\left(a_{2}, t_{2}\right) \in N_{2}\right.$, $\left.s_{3} \xrightarrow[-]{a_{3}}{ }_{3} t_{3}\right\}$. Hence we have $s_{1} \longrightarrow 1 N_{1}$ such that $\forall\left(a_{1}, t_{1}\right) \in N_{1}: \exists\left(a,\left(t_{2}, t_{3}\right)\right) \in N, \beta \in \mathbb{L}: F\left(a_{1}, a, \beta\right) \sqsubseteq_{\mathbb{L}}$ $\alpha,\left(t_{1},\left(t_{2}, t_{3}\right)\right) \in R_{\beta}$.

Let $\left(a_{1}, t_{1}\right) \in N_{1}$, then we have $\left(a,\left(t_{2}, t_{3}\right)\right) \in N$ and $\beta \in \mathbb{L}$ for which $F\left(a_{1}, a, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$ and $\left(t_{1},\left(t_{2}, t_{3}\right)\right) \in R_{\beta}$, hence $\left(t_{1}, t_{2}\right) \in R_{\beta}^{2}$. By construction of $N$, this implies that there are $\left(a_{2}, t_{2}\right) \in N_{2}$ and $s_{3} \stackrel{a_{3}}{-\rightarrow_{3}} t_{3}$ such that $a=$ $a_{2} \otimes a_{3}$, but then by anti-monotonicity, $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}}$ $F\left(a_{1}, a, \beta\right) \sqsubseteq \alpha$.

We have shown that $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2} \wedge \mathcal{D}_{3}\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right)$; the proof of $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2} \wedge \mathcal{D}_{3}\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{3}\right)$ is entirely analogous.

The inclusion $\llbracket \mathcal{D}_{1} \wedge \mathcal{D}_{2} \rrbracket^{\alpha} \subseteq \llbracket \mathcal{D}_{1} \rrbracket^{\alpha} \cap \llbracket \mathcal{D}_{2} \rrbracket^{\alpha}$ is clear now: If $\mathcal{I} \in \llbracket \mathcal{D}_{1} \wedge \mathcal{D}_{2} \rrbracket^{\alpha}$, i.e., $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{I}, \mathcal{D}_{1} \wedge \mathcal{D}_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha$, then also $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{I}, \mathcal{D}_{1}\right) \sqsubseteq_{\mathbb{L}} \alpha$ and $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{I}, \mathcal{D}_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha$, thus $\mathcal{I} \in \llbracket \mathcal{D}_{1} \rrbracket^{\alpha} \cap \llbracket \mathcal{D}_{2} \rrbracket^{\alpha}$.

To show that $\llbracket \mathcal{D}_{1} \vee \mathcal{D}_{2} \rrbracket^{\alpha}=\llbracket \mathcal{D}_{1} \rrbracket^{\alpha} \cup \llbracket \mathcal{D}_{2} \rrbracket^{\alpha}$, one notices, like in the proof of Theorem (4. that for any LTS $\mathcal{I}$, any refinement family witnessing $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{I}, \mathcal{D}_{1}\right)$ or $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{I}, \mathcal{D}_{2}\right)$ is also a witness for $d_{\mathfrak{m}}^{\mathbb{L}}\left(\mathcal{I}, \mathcal{D}_{1} \vee \mathcal{D}_{2}\right)$ and vice versa.

The below example shows why the inclusions above cannot be replaced by equalities. To sum up, disjunction is quantitatively sound and complete, whereas conjunction is only quantitatively sound.

Example 8 For the point-wise or discounting distances, the DMTS in Fig. 10 are such that $d_{\mathrm{m}}\left(\mathcal{I}, \mathcal{D}_{1}\right)=1$ and $d_{\mathrm{m}}\left(\mathcal{I}, \mathcal{D}_{2}\right)=1$, but $d_{\mathrm{m}}\left(\mathcal{I}, \mathcal{D}_{1} \wedge \mathcal{D}_{2}\right)=\infty$. Hence $d_{\mathrm{m}}\left(\mathcal{I}, \mathcal{D}_{1} \wedge \mathcal{D}_{2}\right) \neq \max \left(d_{\mathrm{m}}\left(\mathcal{I}, \mathcal{D}_{1}\right), d_{\mathrm{m}}\left(\mathcal{I}, \mathcal{D}_{2}\right)\right)$, and $\mathcal{I} \in$ $\llbracket \mathcal{D}_{1} \rrbracket^{1} \cap \llbracket \mathcal{D}_{2} \rrbracket^{1}$, but $\mathcal{I} \notin \llbracket \mathcal{D}_{1} \wedge \mathcal{D}_{2} \rrbracket^{1}$.

### 5.5 Structural composition and quotient

We proceed to devise a quantitative generalization of the properties of structural composition and quotient
exposed in Section 4 To this end, we need to use a uniform composition bound on labels:

Let $P: \mathbb{L} \times \mathbb{L} \rightarrow \mathbb{L}$ be a function which is monotone in both coordinates, has $P\left(\alpha, \perp_{\mathbb{L}}\right)=P\left(\perp_{\mathbb{L}}, \alpha\right)=\alpha$ and $P\left(\alpha, \top_{\mathbb{L}}\right)=P\left(\top_{\mathbb{L}}, \alpha\right)=\top_{\mathbb{L}}$ for all $\alpha \in \mathbb{L}$. We require that for all $a_{1}, b_{1}, a_{2}, b_{2} \in \Sigma$ and $\alpha, \beta \in \mathbb{L}$ with $F\left(a_{1}, a_{2}, \alpha\right) \neq \top_{\mathbb{L}}$ and $F\left(b_{1}, b_{2}, \beta\right) \neq \top_{\mathbb{L}}, a_{1} \oplus b_{1}$ is defined iff $a_{2} \oplus b_{2}$ is, and if both are defined, then

$$
\begin{align*}
& F\left(a_{1} \oplus b_{1}, a_{2} \oplus b_{2}, P(\alpha, \beta)\right) \\
& \sqsubseteq_{\mathbb{L}} P\left(F\left(a_{1}, a_{2}, \alpha\right), F\left(b_{1}, b_{2}, \beta\right)\right) . \tag{9}
\end{align*}
$$

Note that (9) implies that
$d_{\mathrm{tr}}^{\mathbb{L}}\left(a_{1} \oplus a_{2}, b_{1} \oplus b_{2}\right) \sqsubseteq_{\mathbb{L}} P\left(d_{\mathrm{tr}}^{\mathbb{L}}\left(a_{1}, b_{1}\right), d_{\mathrm{tr}}^{\mathbb{L}}\left(a_{2}, b_{2}\right)\right)$.
Hence $P$ provides a uniform bound on distances between synchronized labels, and (9) extends this property so that it holds recursively. Also, this is a generalization of the condition that we imposed on $(1)$ in Section 2, it is shown in [30, p. 18] that it holds for all common label synchronizations.

Remark that $P$ can be understood as a (generalized) modulus of continuity [63] for the partial function $f$ : $\Sigma \times \Sigma \rightharpoonup \Sigma$ given by label synchronization $f(a, b)=$ $a \oplus b$ : with that notation, (10) asserts that the distance from $f\left(a_{1}, a_{2}\right)$ to $f\left(b_{1}, b_{2}\right)$ is bounded by $P$ applied to the distance from $\left(a_{1}, a_{2}\right)$ to $\left(b_{1}, b_{2}\right)$.

The following theorems show that composition is uniformly continuous (i.e., a quantitative generalization of independent implementability; Corollary (1) and that quotient preserves and reflects refinement distance (a quantitative generalization of Theorem (6).

Theorem 10 For all specifications $\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3}, \mathcal{S}_{4}$, $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}\left\|\mathcal{S}_{2}, \mathcal{S}_{3}\right\| \mathcal{S}_{4}\right) \sqsubseteq_{\mathbb{L}} P\left(d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1}, \mathcal{S}_{3}\right), d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{2}, \mathcal{S}_{4}\right)\right)$.

Proof We show the proof for NAA. For $i=1,2,3,4$, let $\mathcal{A}_{i}=\left(S_{i}, S_{i}^{0}, \operatorname{Tran}_{i}\right)$. Let $R^{1}=\left\{R_{\alpha}^{1} \subseteq S_{1} \times S_{3} \mid \alpha \in \mathbb{L}\right\}$, $R^{2}=\left\{R_{\alpha}^{2} \subseteq S_{2} \times S_{4} \mid \alpha \in \mathbb{L}\right\}$ be refinement families such that $\forall s_{1}^{0} \in S_{1}^{0}: \exists s_{3}^{0} \in S_{3}^{0}:\left(s_{1}^{0}, s_{3}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{A}_{1}, \mathcal{A}_{3}\right)}^{1}$ and $\forall s_{2}^{0} \in S_{2}^{0}: \exists s_{4}^{0} \in S_{4}^{0}:\left(s_{2}^{0}, s_{4}^{0}\right) \in R_{d_{m}^{\mathbb{L}}}^{2}\left(\mathcal{A}_{2}, \mathcal{A}_{4}\right)$. Define $R=\left\{R_{\alpha} \subseteq\left(S_{1} \times S_{2}\right) \times\left(S_{3} \times S_{4}\right) \mid \alpha \in\right\}$ by
$R_{\alpha}=\left\{\left(\left(s_{1}, s_{2}\right),\left(s_{3}, s_{4}\right)\right) \mid \exists \alpha_{1}, \alpha_{2} \in \mathbb{L}:\right.$

$$
\left.\left(s_{1}, s_{3}\right) \in R_{\alpha_{1}}^{1},\left(s_{2}, s_{4}\right) \in R_{\alpha_{2}}^{2}, P\left(\alpha_{1}, \alpha_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha\right\},
$$

then it is clear that $\forall\left(s_{1}^{0}, s_{2}^{0}\right) \in S_{1}^{0} \times S_{2}^{0}: \exists\left(s_{3}^{0}, s_{4}^{0}\right) \in$ $S_{3}^{0} \times S_{4}^{0}:\left(\left(s_{1}^{0}, s_{2}^{0}\right),\left(s_{3}^{0}, s_{4}^{0}\right)\right) \in R_{P\left(d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{A}_{1}, \mathcal{A}_{3}\right), d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{A}_{2}, \mathcal{A}_{4}\right)\right)}$. We show that $R$ is a refinement family from $\mathcal{A}_{1} \| \mathcal{A}_{2}$ to $\mathcal{A}_{3} \| \mathcal{A}_{4}$.

Let $\alpha \in \mathbb{L}$ and $\left(\left(s_{1}, s_{2}\right),\left(s_{3}, s_{4}\right)\right) \in R_{\alpha}$, then we have $\alpha_{1}, \alpha_{2} \in \mathbb{L}$ with $\left(s_{1}, s_{3}\right) \in R_{\alpha_{1}}^{1},\left(s_{2}, s_{4}\right) \in R_{\alpha_{2}}^{2}$ and $P\left(\alpha_{1}, \alpha_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha$. Let $M_{12} \in \operatorname{Tran}\left(\left(s_{1}, s_{2}\right)\right)$, then there must be $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right), M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ for which
$M_{12}=M_{1} \oplus M_{2}$. Thus we also have $M_{3} \in \operatorname{Tran}_{3}\left(s_{3}\right)$ and $M_{4} \in \operatorname{Tran}_{4}\left(s_{4}\right)$ such that
$\forall\left(a_{1}, t_{1}\right) \in M_{1}: \exists\left(a_{3}, t_{3}\right) \in M_{3}, \beta_{1} \in \mathbb{L}:$

$$
\begin{equation*}
\left(t_{1}, t_{3}\right) \in R_{\beta_{1}}^{1}, F\left(a_{1}, a_{3}, \beta_{1}\right) \sqsubseteq_{\mathbb{L}} \alpha_{1}, \tag{11}
\end{equation*}
$$

$\forall\left(a_{3}, t_{3}\right) \in M_{3}: \exists\left(a_{1}, t_{1}\right) \in M_{1}, \beta_{1} \in \mathbb{L}:$

$$
\begin{equation*}
\left(t_{1}, t_{3}\right) \in R_{\beta_{1}}^{1}, F\left(a_{1}, a_{3}, \beta_{1}\right) \sqsubseteq_{\mathbb{L}} \alpha_{1}, \tag{12}
\end{equation*}
$$

$\forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{4}, t_{4}\right) \in M_{4}, \beta_{2} \in \mathbb{L}:$

$$
\begin{equation*}
\left(t_{2}, t_{4}\right) \in R_{\beta_{2}}^{2}, F\left(a_{2}, a_{4}, \beta_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha_{2}, \tag{13}
\end{equation*}
$$

$\forall\left(a_{4}, t_{4}\right) \in M_{4}: \exists\left(a_{2}, t_{2}\right) \in M_{2}, \beta_{2} \in \mathbb{L}:$

$$
\begin{equation*}
\left(t_{2}, t_{4}\right) \in R_{\beta_{2}}^{2}, F\left(a_{2}, a_{4}, \beta_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha_{2} . \tag{14}
\end{equation*}
$$

Let $M_{34}=M_{3} \oplus M_{4}$, then $M_{34} \in \operatorname{Tran}\left(\left(s_{3}, s_{4}\right)\right)$. Let $\left(a_{12},\left(t_{1}, t_{2}\right)\right) \in M_{12}$, then there are $\left(a_{1}, t_{1}\right) \in M_{1}$ and $\left(a_{2}, t_{2}\right) \in M_{2}$ for which $a_{12}=a_{1} \oplus a_{2}$. Using (11) and (13), we get $\left(a_{3}, t_{3}\right) \in M_{3},\left(a_{4}, t_{4}\right) \in M_{4}$ and $\beta_{1}, \beta_{2} \in \mathbb{L}$ such that $\left(t_{1}, t_{3}\right) \in R_{\beta_{1}}^{1},\left(t_{2}, t_{4}\right) \in R_{\beta_{2}}^{2}$, $F\left(a_{1}, a_{3}, \beta_{1}\right) \sqsubseteq_{\mathbb{L}} \alpha_{1}$, and $F\left(a_{2}, a_{4}, \beta_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha_{2}$.

Let $a_{34}=a_{3} \oplus a_{4}$ and $\beta=P\left(\beta_{1}, \beta_{2}\right)$, then we have $\left(a_{34},\left(t_{3}, t_{4}\right)\right) \in M_{34}$. Also, $\left(t_{1}, t_{3}\right) \in R_{\beta_{1}}^{1}$ and $\left(t_{2}, t_{4}\right) \in$ $R_{\beta_{2}}^{2}$ imply that $\left(\left(t_{1}, t_{2}\right),\left(t_{3}, t_{4}\right)\right) \in R_{\beta}$, and

$$
\begin{aligned}
F\left(a_{12}, a_{34}, \beta\right) & =F\left(a_{1} \oplus a_{2}, a_{3} \oplus a_{4}, P\left(\beta_{1}, \beta_{2}\right)\right) \\
& \sqsubseteq P\left(F\left(a_{1}, a_{3}, \beta_{1}\right), F\left(a_{2}, a_{4}, \beta_{2}\right)\right) \\
& \sqsubseteq_{\mathbb{L}} P\left(\alpha_{1}, \alpha_{2}\right) \sqsubseteq_{\mathbb{L}} \alpha .
\end{aligned}
$$

We have shown that for all $\left(a_{12},\left(t_{1}, t_{2}\right)\right) \in M_{12}$, there exists $\left(a_{34},\left(t_{3}, t_{4}\right)\right) \in M_{34}$ and $\beta \in \mathbb{L}$ such that $\left(\left(t_{1}, t_{2}\right)\right.$, $\left.\left(t_{3}, t_{4}\right)\right) \in R_{\beta}$ and $F\left(a_{12}, a_{34}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. To show the reverse property, starting from an element $\left(a_{34},\left(t_{3}, t_{4}\right)\right) \in$ $M_{34}$, we can proceed entirely analogous, using (12) and (14).

Theorem 11 For all specifications $\mathcal{S}_{1}, \mathcal{S}_{2}, \mathcal{S}_{3}$, we have $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{1} \| \mathcal{S}_{2}, \mathcal{S}_{3}\right)=d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{S}_{2}, \mathcal{S}_{3} / \mathcal{S}_{1}\right)$.

Proof We show the proof for NAA. Let $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}\right.$, $\left.\operatorname{Tran}_{1}\right), \mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right), \mathcal{A}_{3}=\left(S_{3}, S_{3}^{0}, \operatorname{Tran}_{3}\right)$; we show that $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1} \| \mathcal{A}_{2}, \mathcal{A}_{3}\right)=d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{2}, \mathcal{A}_{3} / \mathcal{A}_{1}\right)$.

We assume that the elements of $\operatorname{Tran}_{1}\left(s_{1}\right)$ are pairwise disjoint for each $s_{1} \in S_{1}$; this can be achieved by, if necessary, splitting states.

Define $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{2} \times S_{3} \mid \alpha \in \mathbb{L}\right\}$ by $R_{\alpha}=$ $\left\{\left(s_{1} \| s_{2}, s_{3}\right) \mid d_{\mathrm{m}}^{\mathbb{L}}\left(s_{2}, s_{3} / s_{1}\right) \sqsubseteq_{\mathbb{L}} \alpha\right\}$. (We again abuse notation and write $\left(s_{1} \| s_{2}, s_{3}\right)$ instead of $\left(s_{1}, s_{2}, s_{3}\right)$.) We show that $R$ is a witness for $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1} \| \mathcal{A}_{2}, \mathcal{A}_{3}\right)$.

Let $s_{1}^{0} \| s_{2}^{0} \in S_{1}^{0} \times S_{2}^{0}$, then there is $s_{3}^{0} / s_{1}^{0} \in s^{0}$ for which it holds that $d_{\mathrm{m}}^{\mathbb{L}}\left(s_{2}^{0}, s_{3}^{0} / s_{1}^{0}\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{2}, \mathcal{A}_{3} / \mathcal{A}_{1}\right)$, hence $\left(s_{1}^{0} \| s_{1}^{0}, s_{3}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}}\left(\mathcal{A}_{2}, \mathcal{A}_{3} / \mathcal{A}_{1}\right)$.

Let $\alpha \in \mathbb{L} \backslash\left\{\top_{\mathbb{L}}\right\},\left(s_{1} \| s_{2}, s_{3}\right) \in R_{\alpha}$ and $M_{\|} \in$ $\operatorname{Tran}_{\|}\left(s_{1} \| s_{2}\right)$. Then $M_{\|}=M_{1} \| M_{2}$ with $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$ and $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$. As $d_{\mathfrak{m}}^{\mathbb{L}}\left(s_{2}, s_{3} / s_{1}\right) \sqsubseteq_{\mathbb{L}} \alpha$, we can
pair $M_{2}$ with an $M_{/} \in \operatorname{Tran} /\left(s_{3} / s_{1}\right)$, i.e., such that the conditions in Definition 4 are satisfied.

Let $M_{3}=M_{/} \triangleright M_{1}$. We show that the conditions in Definition 4 are satisfied for the pair $M_{\|}, M_{3}$ :

- Let $\left(a, t_{1} \| t_{2}\right) \in M_{\|}$, then there are $a_{1}, a_{2} \in \Sigma$ with $a=a_{1} \oplus a_{2}$ and $\left(a_{1}, t_{1}\right) \in M_{1},\left(a_{2}, t_{2}\right) \in M_{2}$. Hence there is $\left(a_{2}^{\prime}, t\right) \in M_{/}$and $\beta \in \mathbb{L}$ such that $F\left(a_{2}, a_{2}^{\prime}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$ and $d_{\mathrm{m}}^{\mathbb{L}}\left(t_{2}, t\right) \sqsubseteq_{\mathbb{L}} \beta$.
Note that $a_{3}=a_{1} \oplus a_{2}^{\prime}$ is defined and $F\left(a, a_{3}, \beta\right) \sqsubseteq$ $\alpha$. Write $t=\left\{t_{3}^{1} / t_{1}^{1}, \ldots, t_{3}^{n} / t_{1}^{n}\right\}$. By construction, there is an index $i$ for which $t_{1}^{i}=t_{1}$, hence $\left(a_{3}, t_{3}^{i}\right) \in$ $M_{3}$. Also, $t \supseteq\left\{t_{3}^{i} / t_{1}^{i}\right\}$, hence $d_{\mathrm{m}}^{\mathbb{L}}\left(t_{2}, t_{3}^{i} / t_{1}^{i}\right) \sqsubseteq \beta$ and consequently $\left(t_{1} \| t_{2}, t_{3}\right) \in R_{\beta}$.
- Let $\left(a_{3}, t_{3}\right) \in M_{3}$, then there are $\left(a_{2}^{\prime}, t\right) \in M_{/}$and $\left(a_{1}, t_{1}\right) \in M_{1}$ such that $a_{3}=a_{1} \oplus a_{2}^{\prime}$ and $t_{3} / t_{1} \in t$. Hence there are $\left(a_{2}, t_{2}\right) \in M_{2}$ and $\beta \in \mathbb{L}$ for which $F\left(a_{2}, a_{2}^{\prime}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$ and $d_{\mathrm{m}}^{\mathbb{L}}\left(t_{2}, t\right) \sqsubseteq_{\mathbb{L}} \beta$. Note that $a=a_{1} \oplus a_{2}$ is defined and $F\left(a, a_{3}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. Thus $\left(a, t_{1} \| t_{2}\right) \in M$, and by $t \supseteq\left\{t_{3} / t_{1}\right\}, d_{\mathbf{m}}^{\mathbb{L}}\left(t_{2}, t_{3} / t_{1}\right) \sqsubseteq$ $\beta$.
Assume, for the other direction of the proof, that $\mathcal{A}_{1} \| \mathcal{A}_{2} \leq_{\mathrm{m}} \mathcal{A}_{3}$. Define $R=\left\{R_{\alpha} \subseteq S_{2} \times 2^{S_{3} \times S_{1}} \mid \alpha \in \mathbb{L}\right\}$ by

$$
\begin{aligned}
R_{\alpha}=\left\{\left(s_{2},\left\{s_{3}^{1} / s_{1}^{1}, \ldots, s_{3}^{n} / s_{1}^{n}\right\}\right) \mid\right. & \forall i=1, \ldots, n: \\
& \left.d_{\mathrm{m}}^{\mathbb{L}}\left(s_{1}^{i} \| s_{2}, s_{3}^{i}\right) \sqsubseteq_{\mathbb{L}} \alpha\right\} ;
\end{aligned}
$$

we show that $R$ is a witness for $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{2}, \mathcal{A}_{3} / \mathcal{A}_{1}\right)$.
Let $s_{2}^{0} \in S_{2}^{0}$. We know that for every $s_{1}^{0} \in S_{1}^{0}$, there exists $\sigma\left(s_{1}^{0}\right) \in S_{3}^{0}$ such that $d_{\mathrm{m}}^{\mathbb{L}}\left(s_{1}^{0} \| s_{2}^{0}, s_{3}^{0}\right) \sqsubseteq_{\mathbb{L}}$ $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1} \| \mathcal{A}_{2}, \mathcal{A}_{3}\right)$. By $s^{0} \supseteq\left\{\sigma\left(s_{1}^{0}\right) / s_{1}^{0} \mid s_{1}^{0} \in S_{1}^{0}\right\}$, we see that $\left(s_{2}^{0}, s^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{m}}}\left(\mathcal{A}_{1} \| \mathcal{A}_{2}, \mathcal{A}_{3}\right)$.

Let $\alpha \in \mathbb{L} \backslash\left\{\top_{\mathbb{L}}\right\}$ and $\left(s_{2}, s\right) \in R_{\alpha}$, with $s=$ $\left\{s_{3}^{1} / s_{1}^{1}, \ldots, s_{3}^{n} / s_{1}^{n}\right\}$, and $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$.

For every $i=1, \ldots, n$, let us write $\operatorname{Tran}_{1}\left(s_{1}^{i}\right)=$ $\left\{M_{1}^{i, 1}, \ldots, M_{1}^{i, m_{i}}\right\}$. By assumption, $M_{1}^{i, j_{1}} \cap M_{1}^{i, j_{2}}=\emptyset$ for $j_{1} \neq j_{2}$, hence every $\left(a_{1}, t_{1}\right) \in \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right)$ is contained in a unique $M_{1}^{i, \delta_{i}\left(a_{1}, t_{1}\right)} \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right)$.

For every $j=1, \ldots, m_{i}$, let $M^{i, j}=M_{1}^{i, j} \| M_{2} \in$ $\operatorname{Tran}_{\|}\left(s_{1}^{i} \| s_{2}\right)$. By $d_{\mathrm{m}}^{\mathbb{L}}\left(s_{1}^{i} \| s_{2}, s_{3}^{i}\right) \sqsubseteq_{\mathbb{L}} \alpha$, we have $M_{3}^{i, j} \in$ $\operatorname{Tran}_{3}\left(s_{3}^{i}\right)$ such that the conditions in Definition 4 hold for the pair $M^{i, j}, M_{3}^{i, j}$.

Now define

$$
\begin{align*}
& M=\left\{\left(a_{2}, t\right) \mid \exists\left(a_{2}, t_{2}\right) \in M_{2}: \forall t_{3} / t_{1} \in t:\right. \\
& \exists i, a_{1}, a_{3}, \beta:\left(a_{1}, t_{1}\right) \in \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right), \\
&\left(a_{3}, t_{3}\right) \in M_{3}^{i, \delta_{i}\left(a_{1}, t_{1}\right)}, F\left(a_{1} \oplus a_{2}, a_{3}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha, \\
&\left.d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1} \| t_{2}, t_{3}\right) \sqsubseteq_{\mathbb{L}} \beta\right\} . \tag{15}
\end{align*}
$$

We need to show that $M \in \operatorname{Tran} /(s)$.
Let $i \in\{1, \ldots, n\}$ and $M_{1}^{i, j} \in \operatorname{Tran}_{1}\left(s_{1}^{i}\right)$; we claim that $M \triangleright M_{1}^{i, j} \preccurlyeq R M_{3}^{i, j}$. Let $\left(a_{3}, t_{3}\right) \in M \triangleright M_{1}^{i, j}$, then
$a_{3}=a_{1} \oplus a_{2}$ for some $a_{1}, a_{2}$ such that $t_{3} / t_{1} \in t$, $\left(a_{1}, t_{1}\right) \in M_{1}^{i, j}$ and $\left(a_{2}, t\right) \in M$. By disjointness, $j=$ $\delta_{i}\left(a_{1}, t_{1}\right)$, hence by definition of $M,\left(a_{3}, t_{3}\right) \in M_{3}^{i, j}$ as was to be shown.

For the reverse inclusion, let $\left(a_{3}, t_{3}\right) \in M_{3}^{i, j}$. By definition of $M^{i, j}$, there are $\left(a_{1}, t_{1}\right) \in M_{1}^{i, j},\left(a_{2}, t_{2}\right) \in$ $M_{2}$ and $\beta \in \mathbb{L}$ for which $F\left(a_{1} \oplus a_{2}, a_{3}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$ and $d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1} \| t_{2}, t_{3}\right) \sqsubseteq_{\mathbb{L}} \beta$. Thus $j=\delta_{i}\left(a_{1}, t_{1}\right)$, so that there must be $\left(a_{2}, t\right) \in M$ for which $t_{3} / t_{1} \in t$, but then also $\left(a_{1} \oplus a_{2}, t_{3}\right) \in M \triangleright M_{1}^{i, j}$.

We show that the pair $M_{2}, M$ satisfies the conditions of Definition 4

- Let $\left(a_{2}, t_{2}\right) \in M_{2}$. For every $i=1, \ldots, n$ and every $\left(a_{1}, t_{1}\right) \in \in \operatorname{Tran}_{1}\left(t_{1}^{i}\right)$, we can use Definition 4 applied to the pair $M_{1}^{i, \delta_{i}\left(a_{1}, t_{1}\right)} \| M_{2}, M_{3}^{i, \delta_{i}\left(a_{1}, t_{1}\right)}$ to choose an element $\left(\eta_{i}\left(a_{1}, t_{1}\right), \tau_{i}\left(a_{1}, t_{1}\right)\right) \in M_{3}^{i, \delta_{i}\left(a_{1}, t_{1}\right)}$ and $\beta_{i}\left(a_{1}, t_{1}\right) \in \mathbb{L}$ for which $d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1} \| t_{2}, \tau_{i}\left(a_{1}, t_{1}\right)\right) \sqsubseteq_{\mathbb{L}}$ $\beta_{i}\left(a_{1}, t_{1}\right)$ and $F\left(a_{1} \oplus a_{2}, \eta_{i}\left(a_{1}, t_{1}\right), \beta_{i}\left(a_{1}, t_{1}\right)\right) \sqsubseteq_{\mathbb{L}} \alpha$. Let $t=\left\{\tau_{i}\left(a_{1}, t_{1}\right) / t_{1} \mid i=1, \ldots, n,\left(a_{1}, t_{1}\right) \in \in\right.$ $\left.\operatorname{Tran}_{1}\left(t_{1}^{i}\right)\right\}$, then $\left(a_{2}, t\right) \in M$ and $\left(t_{2}, t\right) \in R_{\beta}$.
- Let $\left(a_{2}, t\right) \in M$, then we have $\left(a_{2}, t_{2}\right) \in M_{2}$ satisfying the conditions in (15). Hence for all $t_{3} / t_{1} \in t$, there are $i, a_{1}, a_{3}$, and $\beta\left(t_{3} / t_{1}\right)$ such that $\left(a_{3}, t_{3}\right) \in$ $M_{3}^{i, \delta_{i}\left(a_{1}, t_{1}\right)}, F\left(a_{1} \oplus a_{2}, a_{3}, \beta\left(t_{3} / t_{1}\right)\right) \sqsubseteq_{\mathbb{L}} \quad \alpha$ and $d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1} \| t_{2}, t_{3}\right) \sqsubseteq_{\mathbb{L}} \beta\left(t_{3} / t_{1}\right)$. Let $\beta=\sup \left\{\beta\left(t_{3} / t_{1}\right) \mid\right.$ $\left.t_{3} / t_{1} \in t\right\}$, then $d_{\mathrm{m}}^{\mathbb{L}}\left(t_{1} \| t_{2}, t_{3}\right) \sqsubseteq_{\mathbb{L}} \beta$ for all $t_{3} / t_{1} \in t$, hence $\left(t_{2}, t\right) \in R_{\beta}$.


## 6 Conclusion

We have presented a framework for compositional and iterative design and verification of systems which supports quantities and system and action refinement. Moreover, it is robust, in that it uses distances to measure quantitative refinement and the operations preserve distances.

The framework is very general. It can be applied to a large variety of quantities (energy, time, resource consumption etc.) and implement the robustness notions associated with them. It is also agnostic with respect to the type of specifications used, as it applies equally to behavioral and logical specifications. This means that logical and behavioral quantitative specifications can be freely combined in quantitative system development.

As to future work, we believe that that the close relationship between DMTS and the modal $\nu$-calculus which we expose here should be helpful for relating our robust semantics of the modal $\nu$-calculus to other quantitative logics $19,40,53$. We also plan to implement the operations detailed here within the graphical tool MoTraS [45].

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## Appendix: Proof of Theorem 7

$\frac{d_{\mathrm{m}}^{\mathbb{L}}\left(d a\left(\mathcal{D}_{1}\right), d a\left(\mathcal{D}_{2}\right)\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right):}{\left(S_{1}, S_{1}^{0},\right.}$
Let $\mathcal{D}_{1}=\left(S_{1}, S_{1}^{0}, \rightarrow \rightarrow_{1}, \longrightarrow_{1}\right)$ and $\mathcal{D}_{2}=\left(S_{2}, S_{2}^{0}\right.$, $--2_{2}, \longrightarrow_{2}$ ) be DMTS. There exists a DMTS refinement family $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{2} \mid \alpha \in \mathbb{L}\right\}$ such that for all $s_{1}^{0} \in S_{1}^{0}$, there is $s_{2}^{0} \in S_{2}^{0}$ with $\left(s_{1}^{0}, s_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right)}$. We show that $R$ is an NAA refinement family.

Let $\alpha \in \mathbb{L}$ and $\left(s_{1}, s_{2}\right) \in R_{\alpha}$. Let $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$ and define

$$
\begin{array}{r}
M_{2}=\left\{\left(a_{2}, t_{2}\right) \mid \stackrel{a_{2}}{s_{2}} t_{2}, \exists\left(a_{1}, t_{1}\right) \in M_{1}: \exists \beta \in \mathbb{L}:\right. \\
\\
\left.\left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha\right\} .
\end{array}
$$

The condition
$\forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{1}, t_{1}\right) \in M_{1}, \beta \in \mathbb{L}:$

$$
\left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq \alpha
$$

is satisfied by construction. For the inverse condition, let $\left(a_{1}, t_{1}\right) \in M_{1}$, then $s_{1} \xrightarrow{a_{1}}{ }_{1} t_{1}$, and as $R$ is a DMTS refinement family, this implies that there is $s_{2} \xrightarrow{a_{2}}{ }_{2} t_{2}$ and $\beta \in \mathbb{L}$ for which $\left(t_{1}, t_{2}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}}$ $\alpha$, so that $\left(a_{2}, t_{2}\right) \in M_{2}$ by construction.

We are left with showing that $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$. First we notice that by construction, indeed $s_{2} \xrightarrow{a_{2}}{ }_{2} t_{2}$ for all $\left(a_{2}, t_{2}\right) \in M_{2}$. Now let $s_{2} \longrightarrow N_{2}$; we need to show that $N_{2} \cap M_{2} \neq \emptyset$.

We have $s_{1} \longrightarrow N_{1}$ such that $\forall\left(a_{1}, t_{1}\right) \in N_{1}$ : $\exists\left(a_{2}, t_{2}\right) \in N_{2}, \beta \in \mathbb{L}:\left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. We know that $N_{1} \cap M_{1} \neq \emptyset$, so let $\left(a_{1}, t_{1}\right) \in N_{1} \cap$ $M_{1}$. Then there is $\left(a_{2}, t_{2}\right) \in N_{2}$ and $\beta \in \mathbb{L}$ such that $\left(t_{1}, t_{2}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. But $\left(a_{2}, t_{2}\right) \in N_{2}$ implies $s_{2} \xrightarrow{a_{2}} 2 t_{2}$, hence $\left(a_{2}, t_{2}\right) \in M_{2}$.
$d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(d a\left(\mathcal{D}_{1}\right), d a\left(\mathcal{D}_{2}\right)\right):$
Let $\mathcal{D}_{1}=\left(S_{1}, S_{1}^{0}, \cdots \rightarrow_{1}, \longrightarrow_{1}\right)$ and $\mathcal{D}_{2}=\left(S_{2}, S_{2}^{0}\right.$, $--\rightarrow_{2}, \longrightarrow_{2}$ ) be DMTS. There exists an NAA refinement family $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{2} \mid \alpha \in \mathbb{L}\right\}$ such that for all $s_{1}^{0} \in S_{1}^{0}$, there is $s_{2}^{0} \in S_{2}^{0}$ for which $\left(s_{1}^{0}, s_{2}^{0}\right) \in$ $R_{d_{\mathfrak{m}}^{\mathbb{m}}\left(d a\left(\mathcal{D}_{1}\right), d a\left(\mathcal{D}_{2}\right)\right)}$. We show that $R$ is a DMTS refinement family. Let $\alpha \in \mathbb{L}$ and $\left(s_{1}, s_{2}\right) \in R_{\alpha}$.

Let $s_{1} \xrightarrow{a_{1}}{ }_{1} t_{1}$, then we cannot have $s_{1} \longrightarrow \emptyset$. Let $M_{1}=\left\{\left(a_{1}, t_{1}\right)\right\} \cup \bigcup\left\{N_{1} \mid s_{1} \longrightarrow N_{1}\right\}$, then $M_{1} \in$ $\operatorname{Tran}_{1}\left(s_{1}\right)$ by construction. This implies that there is $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right),\left(a_{2}, t_{2}\right) \in M_{2}$ and $\beta \in \mathbb{L}$ such that $\left(t_{1}, t_{2}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $s_{2} \xrightarrow{a_{2}} t_{2}$ as was to be shown.

Let $s_{2} \longrightarrow N_{2}$ and assume, for the sake of contradiction, that there is no $s_{1} \longrightarrow N_{1}$ for which $\forall\left(a_{1}, t_{1}\right) \in$ $N_{1}: \exists\left(a_{2}, t_{2}\right) \in N_{2}, \beta \in \mathbb{L}:\left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}}$ $\alpha$ holds. Then for each $s_{1} \longrightarrow N_{1}$, there is an element $\left(a_{N_{1}}, t_{N_{1}}\right) \in N_{1}$ such that $\exists\left(a_{2}, t_{2}\right) \in N_{2}, \beta \in$ $\mathbb{L}:\left(t_{N_{1}}, t_{2}\right) \in R_{\beta}, F\left(a_{N_{1}}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$ does not hold.

Let $M_{1}=\left\{\left(a_{N_{1}}, t_{N_{1}}\right) \mid s_{1} \longrightarrow N_{1}\right\}$, then $M_{1} \in$ $\operatorname{Tran}_{1}\left(s_{1}\right)$ by construction. Hence we have $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ such that $\forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{1}, t_{2}\right) \in M_{1}, \beta \in \mathbb{L}$ : $\left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq \alpha$. Now $N_{2} \cap M_{2} \neq \emptyset$, so let $\left(a_{2}, t_{2}\right) \in N_{2} \cap M_{2}$, then there is $\left(a_{1}, t_{1}\right) \in M_{1}$ and $\beta \in \mathbb{L}$ such that $\left(t_{1}, t_{2}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$, in contradiction to how $M_{1}$ was constructed.
$\underline{d_{\mathrm{m}}^{\mathbb{L}}\left(a d\left(\mathcal{A}_{1}\right), a d\left(\mathcal{A}_{2}\right)\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right):}$
Let $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right), \mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right)$ be NAA, with DMTS translations $\operatorname{ad}\left(\mathcal{A}_{1}\right)=\left(D_{1}, D_{1}^{0}, \longrightarrow_{1}\right.$, $\left.--\rightarrow_{1}\right), \operatorname{ad}\left(\mathcal{A}_{2}\right)=\left(D_{2}, D_{2}^{0}, \longrightarrow_{2},-\rightarrow_{2}\right)$. There is an NAA refinement family $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{2} \mid \alpha \in \mathbb{L}\right\}$ such that for all $s_{1}^{0} \in S_{1}^{0}$, there is $s_{2}^{0} \in S_{2}^{0}$ with $\left(s_{1}^{0}, s_{2}^{0}\right) \in$ $R_{d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}$.

Define a relation family $R^{\prime}=\left\{R_{\alpha}^{\prime} \subseteq D_{1} \times D_{2} \mid \alpha \in\right.$ $\mathbb{L}\}$ by

$$
\begin{aligned}
& R_{\alpha}^{\prime}=\left\{\left(M_{1}, M_{2}\right) \mid \exists\left(s_{1}, s_{2}\right) \in R_{\alpha}:\right. \\
& \\
& M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right), M_{2} \in \operatorname{Tran}\left(s_{2}\right), \\
& \\
& \forall\left(a_{1}, t_{1}\right) \in M_{1}: \exists\left(a_{2}, t_{2}\right) \in M_{2}, \beta \in \mathbb{L}: \\
& \quad\left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha, \\
& \forall\left(a_{2}, t_{2}\right) \in M_{2}: \exists\left(a_{1}, t_{1}\right) \in M_{1}, \beta \in \mathbb{L}: \\
& \left.\quad\left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha\right\} .
\end{aligned}
$$

We show that $R^{\prime}$ is a witness for $d_{\mathrm{m}}^{\mathbb{L}}\left(a d\left(\mathcal{A}_{1}\right), a d\left(\mathcal{A}_{2}\right)\right) \sqsubseteq_{\mathbb{L}}$ $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)$. Let $\alpha \in \mathbb{L}$ and $\left(M_{1}, M_{2}\right) \in R_{\alpha}^{\prime}$.

Let $M_{2} \longrightarrow_{2} N_{2}$. By construction of $\longrightarrow$, there is $\left(a_{2}, t_{2}\right) \in M_{2}$ such that $N_{2}=\left\{\left(a_{2}, M_{2}^{\prime}\right) \mid M_{2}^{\prime} \in\right.$ $\left.\operatorname{Tran}_{2}\left(t_{2}\right)\right\}$. Then $\left(M_{1}, M_{2}\right) \in R_{\alpha}^{\prime}$ implies that there must be $\left(a_{1}, t_{1}\right) \in M_{1}$ and $\beta \in \mathbb{L}$ such that $\left(t_{1}, t_{2}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. Let $N_{1}=\left\{\left(a_{1}, M_{1}^{\prime}\right) \mid M_{1}^{\prime} \in\right.$ $\left.\operatorname{Tran}_{1}\left(t_{1}\right)\right\}$, then $M_{1} \longrightarrow N_{1}$.

We show that $\forall\left(a_{1}, M_{1}^{\prime}\right) \in N_{1}: \exists\left(a_{2}, M_{2}^{\prime}\right) \in N_{2}$ : $\left(M_{1}^{\prime}, M_{2}^{\prime}\right) \in R_{\beta}^{\prime}$ : Let $\left(a_{1}, M_{1}^{\prime}\right) \in N_{1}$, then $M_{1}^{\prime} \in \operatorname{Tran}_{1}\left(t_{1}\right)$. From $\left(t_{1}, t_{2}\right) \in R_{\beta}$ we get $M_{2}^{\prime} \in \operatorname{Tran}_{2}\left(t_{2}\right)$ such that
$\forall\left(b_{1}, u_{1}\right) \in M_{1}^{\prime}: \exists\left(b_{2}, u_{2}\right) \in M_{2}^{\prime}, \gamma \in \mathbb{L}:$

$$
\left(u_{1}, u_{2}\right) \in R_{\gamma}, F\left(b_{1}, b_{2}, \gamma\right) \sqsubseteq_{\mathbb{L}} \beta,
$$

$\forall\left(b_{2}, u_{2}\right) \in M_{2}^{\prime}: \exists\left(b_{1}, u_{1}\right) \in M_{1}^{\prime}, \gamma \in \mathbb{L}:$

$$
\left(u_{1}, u_{2}\right) \in R_{\gamma}, F\left(b_{1}, b_{2}, \gamma\right) \sqsubseteq_{\mathbb{L}} \beta,
$$

hence $\left(M_{1}^{\prime}, M_{2}^{\prime}\right) \in R_{\beta}^{\prime}$; also, $\left(a_{2}, M_{2}^{\prime}\right) \in N_{2}$ by construction of $N_{2}$.

Let $M_{1} \xrightarrow{a_{1}}{ }_{1} M_{1}^{\prime}$, then we have $M_{1} \longrightarrow 1 N_{1}$ for which $\left(a_{1}, M_{1}^{\prime}\right) \in N_{1}$ by construction of $\rightarrow \rightarrow_{1}$. This in turn implies that there must be $\left(a_{1}, t_{1}\right) \in M_{1}$ such that $N_{1}=\left\{\left(a_{1}, M_{1}^{\prime \prime}\right) \mid M_{1}^{\prime \prime} \in \operatorname{Tran}_{1}\left(t_{1}\right)\right\}$. By $\left(M_{1}, M_{2}\right) \in$ $R_{\alpha}^{\prime}$, we get $\left(a_{2}, t_{2}\right) \in M_{2}$ and $\beta \in \mathbb{L}$ such that $\left(t_{1}, t_{2}\right) \in$ $R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. Let $N_{2}=\left\{\left(a_{2}, M_{2}^{\prime}\right) \mid M_{2}^{\prime} \in\right.$ $\left.\operatorname{Tran}_{2}\left(t_{2}\right)\right\}$, then $M_{2} \longrightarrow 2 N_{2}$ and hence $M_{2} \stackrel{a_{2}}{-}{ }_{2} M_{2}^{\prime}$ for all $\left(a_{2}, M_{2}^{\prime}\right) \in N_{2}$. By the same arguments as above, there is $\left(a_{2}, M_{2}^{\prime}\right) \in N_{2}$ for which $\left(M_{1}^{\prime}, M_{2}^{\prime}\right) \in R_{\beta}^{\prime}$.

We miss to show that $R^{\prime}$ is initialized. Let $M_{1}^{0} \in D_{1}^{0}$, then we have $s_{1}^{0} \in S_{1}^{0}$ with $M_{1}^{0} \in \operatorname{Tran}_{1}\left(s_{1}^{0}\right)$. As $R$ is initialized, this entails that there is $s_{2}^{0} \in S_{2}^{0}$ with $\left(s_{1}^{0}, s_{2}^{0}\right) \in$ $R_{d_{\mathbf{m}}^{\mathbb{m}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}$, which gives us $M_{2}^{0} \in \operatorname{Tran}_{2}\left(s_{2}^{0}\right)$ which satisfies the conditions in the definition of $R_{d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}^{\prime}$, whence $\left(M_{1}^{0}, M_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right)}^{\prime}$.
$\underline{d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(a d\left(\mathcal{A}_{1}\right), a d\left(\mathcal{A}_{2}\right)\right):}$
Let $\mathcal{A}_{1}=\left(S_{1}, S_{1}^{0}, \operatorname{Tran}_{1}\right), \mathcal{A}_{2}=\left(S_{2}, S_{2}^{0}, \operatorname{Tran}_{2}\right)$ be NAA, with DMTS translations $\operatorname{ad}\left(\mathcal{A}_{1}\right)=\left(D_{1}, D_{1}^{0}, \longrightarrow_{1}\right.$, $\left.-\rightarrow_{1}\right), \operatorname{ad}\left(\mathcal{A}_{2}\right)=\left(D_{2}, D_{2}^{0}, \longrightarrow_{2},-\rightarrow_{2}\right)$. There is a DMTS refinement family $R=\left\{R_{\alpha} \subseteq D_{1} \times D_{2} \mid \alpha \in \mathbb{L}\right\}$ such that for all $M_{1}^{0} \in D_{1}^{0}$, there exists $M_{2}^{0} \in D_{2}^{0}$ with $\left(M_{1}^{0}, M_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}\left(a d\left(\mathcal{A}_{1}\right), \operatorname{ad}\left(\mathcal{A}_{2}\right)\right)}$.

Define a relation family $R^{\prime}=\left\{R_{\alpha}^{\prime} \subseteq S_{1} \times S_{2} \mid \alpha \in\right.$ $\mathbb{L}\}$ by

$$
\begin{aligned}
R_{\alpha}^{\prime}=\left\{\left(s_{1}, s_{2}\right) \mid\right. & \forall M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right): \\
& \left.\exists M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right):\left(M_{1}, M_{2}\right) \in R_{\alpha}\right\}
\end{aligned}
$$

we will show that $R^{\prime}$ is a witness for $d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{A}_{1}, \mathcal{A}_{2}\right) \sqsubseteq_{\mathbb{L}}$ $d_{\mathrm{m}}^{\mathbb{L}}\left(\operatorname{ad}\left(\mathcal{A}_{1}\right), a d\left(\mathcal{A}_{2}\right)\right)$.

Let $\alpha \in \mathbb{L},\left(s_{1}, s_{2}\right) \in R_{\alpha}^{\prime}$ and $M_{1} \in \operatorname{Tran}_{1}\left(s_{1}\right)$, then by construction of $R^{\prime}$, we have $M_{2} \in \operatorname{Tran}_{2}\left(s_{2}\right)$ with $\left(M_{1}, M_{2}\right) \in R_{\alpha}$.

Let $\left(a_{2}, t_{2}\right) \in M_{2}$ and define $N_{2}=\left\{\left(a_{2}, M_{2}^{\prime}\right) \mid\right.$ $\left.M_{2}^{\prime} \in \operatorname{Tran}_{2}\left(t_{2}\right)\right\}$, then $M_{2} \longrightarrow N_{2}$. Now $\left(M_{1}, M_{2}\right) \in$ $R_{\alpha}$ implies that there must be $M_{1} \longrightarrow_{1} N_{1}$ satisfying $\forall\left(a_{1}, M_{1}^{\prime}\right) \in N_{1}: \exists\left(a_{2}, M_{2}^{\prime}\right) \in N_{2}, \beta \in \mathbb{L}:\left(M_{1}^{\prime}, M_{2}^{\prime}\right) \in$ $R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. We have $\left(a_{1}, t_{1}\right) \in M_{1}$ such that $N_{1}=\left\{\left(a_{1}, M_{1}^{\prime}\right) \mid M_{1}^{\prime} \in \operatorname{Tran}_{1}\left(t_{1}\right)\right\} ;$ we only miss to show that $\left(t_{1}, t_{2}\right) \in R_{\beta}^{\prime}$ for some $\beta \in \mathbb{L}$ for which $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. Let $M_{1}^{\prime} \in \operatorname{Tran}_{1}\left(t_{1}\right)$, then $\left(a_{1}, M_{1}^{\prime}\right) \in$ $N_{1}$, hence there is $\left(a_{2}, M_{2}^{\prime}\right) \in N_{2}$ and $\beta \in \mathbb{L}$ such that $\left(M_{1}^{\prime}, M_{2}^{\prime}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq \alpha$, but $\left(a_{2}, M_{2}^{\prime}\right) \in$ $N_{2}$ also entails $M_{2}^{\prime} \in \operatorname{Tran}_{2}\left(t_{2}\right)$.

Let $\left(a_{1}, t_{1}\right) \in M_{1}$ and define $N_{1}=\left\{\left(a_{1}, M_{1}^{\prime}\right) \mid M_{1}^{\prime} \in\right.$ $\left.\operatorname{Tran}_{1}\left(t_{1}\right)\right\}$, then $M_{1} \longrightarrow_{1} N_{1}$. Now let $\left(a_{1}, M_{1}^{\prime}\right) \in N_{1}$, then $M_{1} \xrightarrow{a_{1}} M_{1}^{\prime}$, hence we have $M_{2} \xrightarrow{a_{2}}{ }_{2} M_{2}^{\prime}$ and $\beta \in$ $\mathbb{L}$ such that $\left(M_{1}^{\prime}, M_{2}^{\prime}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$. By construction of $-\rightarrow_{2}$, this implies that there is $M_{2} \longrightarrow_{2}$ $N_{2}$ with $\left(a_{2}, M_{2}^{\prime}\right) \in N_{2}$, and we have $\left(a_{2}, t_{2}\right) \in M_{2}$ for which $N_{2}=\left\{\left(a_{2}, M_{2}^{\prime \prime}\right) \mid M_{2}^{\prime \prime} \in \operatorname{Tran}_{2}\left(t_{2}\right)\right\}$. Now if $M_{1}^{\prime \prime} \in \operatorname{Tran}_{1}\left(t_{1}\right)$, then $\left(a_{1}, M_{1}^{\prime \prime}\right) \in N_{1}$, hence there is $\left(a_{2}, M_{2}^{\prime \prime}\right) \in N_{2}$ with $\left(M_{1}^{\prime \prime}, M_{2}^{\prime \prime}\right) \in R_{\beta}$, but $\left(a, M_{2}^{\prime \prime}\right) \in N_{2}$ also gives $M_{2}^{\prime \prime} \in \operatorname{Tran}_{2}\left(t_{2}\right)$.

We miss to show that $R^{\prime}$ is initialized. Let $s_{1}^{0} \in$ $S_{1}^{0}$ and $M_{1}^{0} \in \operatorname{Tran}_{1}\left(s_{1}^{0}\right)$. As $R$ is initialized, this gets us $M_{2}^{0} \in D_{2}$ with $\left(M_{1}^{0}, M_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}\left(a d\left(\mathcal{A}_{1}\right), a d\left(\mathcal{A}_{2}\right)\right)}$, but $M_{2}^{0} \in \operatorname{Tran}_{2}\left(s_{2}^{0}\right)$ for some $s_{2}^{0} \in S_{2}^{0}$, and then $\left(s_{1}^{0}, s_{2}^{0}\right) \in$ $R_{d_{\mathbf{m}}^{\mathbb{I}}\left(\operatorname{ad}\left(\mathcal{A}_{1}\right), \operatorname{ad}\left(\mathcal{A}_{2}\right)\right)}^{\prime}$. $\frac{d_{\mathrm{m}}^{\mathbb{L}}\left(d n\left(\mathcal{D}_{1}\right), d n\left(\mathcal{D}_{2}\right)\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right):}{\mathcal{D}_{1}}$

Let $\mathcal{D}_{1}=\left(S_{1}, S_{1}^{0}, \cdots \rightarrow_{1}, \longrightarrow_{1}\right)$ and $\mathcal{D}_{2}=\left(S_{2}, S_{2}^{0}\right.$, $-\rightarrow_{2}, \longrightarrow_{2}$ ) be DMTS, with $\nu$-calculus translations $\operatorname{dn}\left(\mathcal{D}_{1}\right)$ $\left(S_{1}, S_{1}^{0}, \Delta_{1}\right)$ and $d n\left(\mathcal{D}_{2}\right)=\left(S_{2}, S_{2}^{0}, \Delta_{2}\right)$. There is a DMTS refinement family $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{2} \mid \alpha \in \mathbb{L}\right\}$ such that for all $s_{1}^{0} \in S_{1}^{0}$, there exists $s_{2}^{0} \in S_{2}^{0}$ for which $\left(s_{1}^{0}, s_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right)}$.

Let $\alpha \in \mathbb{L},\left(s_{1}, s_{2}\right) \in R_{\alpha}, a_{1} \in \Sigma$, and $t_{1} \in \square_{1}^{a_{1}}\left(s_{1}\right)$. Then $s_{1} \xrightarrow{a_{1}}{ }_{1} t_{1}$, hence we have $s_{2} \xrightarrow{a_{2}}{ }_{2} t_{2}$ and $\beta \in \mathbb{L}$
with $\left(t_{1}, t_{2}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $t_{2} \in \square_{2}^{a_{2}}\left(s_{2}\right)$.

Let $N_{2} \in \diamond_{2}\left(s_{2}\right)$, then also $s_{2} \longrightarrow N_{2}$, so that there must be $s_{1} \longrightarrow_{1} N_{1}$ such that $\forall\left(a_{1}, t_{1}\right) \in N_{1}$ : $\exists\left(a_{2}, t_{2}\right) \in N_{2}, \beta \in \mathbb{L}:\left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $N_{1} \in \diamond_{1}\left(s_{1}\right)$.
$\underline{d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(d n\left(\mathcal{D}_{1}\right), d n\left(\mathcal{D}_{2}\right)\right):}$
Let $\mathcal{D}_{1}=\left(S_{1}, S_{1}^{0}, \cdots \rightarrow_{1}, \longrightarrow_{1}\right)$ and $\mathcal{D}_{2}=\left(S_{2}, S_{2}^{0}\right.$, $\rightarrow \rightarrow_{2}, \longrightarrow_{2}$ ) be DMTS, with $\nu$-calculus translations $d n\left(\mathcal{D}_{1}\right)=\left(S_{1}, S_{1}^{0}, \Delta_{1}\right)$ and $d n\left(\mathcal{D}_{2}\right)=\left(S_{2}, S_{2}^{0}, \Delta_{2}\right)$. There is a $\nu$-calculus refinement family $R=\left\{R_{\alpha} \subseteq S_{1} \times S_{2} \mid\right.$ $\alpha \in \mathbb{L}\}$ such that for all $s_{1}^{0} \in S_{1}^{0}$, there exists $s_{2}^{0} \in S_{2}^{0}$ for which $\left(s_{1}^{0}, s_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{D}_{1}, \mathcal{D}_{2}\right)}$.

Let $\alpha \in \mathbb{L}$ and $\left(s_{1}, s_{2}\right) \in R_{\alpha}$, and assume that $s_{1} \xrightarrow{a_{1}} \rightarrow_{1} t_{1}$. Then $t_{1} \in \square_{1}^{a_{1}}\left(s_{1}\right)$, so that there is $a_{2} \in \Sigma$, $t_{2} \in \square_{2}^{a_{2}}\left(s_{2}\right)$ and $\beta \in \mathbb{L}$ for which $\left(t_{1}, t_{2}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $s_{2} \xrightarrow{a_{2}}{ }_{2} t_{2}$.

Assume that $s_{2} \longrightarrow N_{2}$, then $N_{2} \in \diamond_{2}\left(s_{2}\right)$. Hence there is $N_{1} \in \nabla_{1}\left(s_{1}\right)$ so that $\forall\left(a_{1}, t_{1}\right) \in N_{1}: \exists\left(a_{2}, t_{2}\right) \in$ $N_{2}, \beta \in \mathbb{L}:\left(t_{1}, t_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $s_{1} \longrightarrow{ }_{1} N_{1}$.
$\underline{d_{\mathrm{m}}^{\mathbb{L}}\left(n d\left(\mathcal{N}_{1}\right), n d\left(\mathcal{N}_{2}\right)\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{N}_{1}, \mathcal{N}_{2}\right):}$
Let $\mathcal{N}_{1}=\left(X_{1}, X_{1}^{0}, \Delta_{1}\right), \mathcal{N}_{2}=\left(X_{2}, X_{2}^{0}, \Delta_{2}\right)$ be $\nu$ calculus expressions in normal form, with DMTS translations $n d\left(\mathcal{N}_{1}\right)=\left(X_{1}, X_{1}^{0}, \cdots \rightarrow_{1}, \longrightarrow_{1}\right)$ and $n d\left(\mathcal{N}_{2}\right)=$ $\left(X_{2}, X_{2}^{0},-\rightarrow_{2}, \longrightarrow_{2}\right)$. There is a $\nu$-calculus refinement family $R=\left\{R_{\alpha} \subseteq X_{1} \times X_{2} \mid \alpha \in \mathbb{L}\right\}$ such that for all $x_{1}^{0} \in X_{1}^{0}$, there is $x_{2}^{0} \in X_{2}^{0}$ for which $\left(x_{1}^{0}, x_{2}^{0}\right) \in$ $R_{d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{N}_{1}, \mathcal{N}_{2}\right)}$.

Let $\alpha \in \mathbb{L}$ and $\left(x_{1}, x_{2}\right) \in R_{\alpha}$, and assume that $x_{1} \xrightarrow{a_{1}}{ }_{1} y_{1}$. Then $y_{1} \in \square_{1}^{a_{1}}\left(x_{1}\right)$, hence there are $a_{2} \in$ $\Sigma, y_{2} \in \square_{2}^{a_{2}}$ and $\beta \in \mathbb{L}$ such that $\left(y_{1}, y_{2}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $x_{2} \stackrel{a_{2}}{-} 2 y_{2}$.

Assume that $x_{2} \longrightarrow_{2} N_{2}$, then $N_{2} \in \diamond_{2}\left(x_{2}\right)$. Hence there must be $N_{1} \in \nabla_{1}\left(x_{1}\right)$ such that $\forall\left(a_{1}, y_{1}\right) \in N_{1}$ : $\exists\left(a_{2}, y_{2}\right) \in N_{2}, \beta \in \mathbb{L}:\left(y_{1}, y_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}}$ $\alpha$, but then also $x_{1} \longrightarrow N_{1}$.
$\underline{d_{\mathrm{m}}^{\mathbb{L}}\left(\mathcal{N}_{1}, \mathcal{N}_{2}\right) \sqsubseteq_{\mathbb{L}} d_{\mathrm{m}}^{\mathbb{L}}\left(n d\left(\mathcal{N}_{1}\right), n d\left(\mathcal{N}_{2}\right)\right):}$
Let $\mathcal{N}_{1}=\left(X_{1}, X_{1}^{0}, \Delta_{1}\right), \mathcal{N}_{2}=\left(X_{2}, X_{2}^{0}, \Delta_{2}\right)$ be $\nu$ calculus expressions in normal form, with DMTS translations $n d\left(\mathcal{N}_{1}\right)=\left(X_{1}, X_{1}^{0}, \cdots \rightarrow_{1}, \longrightarrow_{1}\right)$ and $n d\left(\mathcal{N}_{2}\right)=$ $\left(X_{2}, X_{2}^{0}, \rightarrow \rightarrow_{2}, \longrightarrow_{2}\right)$. There is a DMTS refinement family $R=\left\{R_{\alpha} \subseteq X_{1} \times X_{2} \mid \alpha \in \mathbb{L}\right\}$ such that for all $x_{1}^{0} \in$ $=X_{1}^{0}$, there is $x_{2}^{0} \in X_{2}^{0}$ for which $\left(x_{1}^{0}, x_{2}^{0}\right) \in R_{d_{\mathbf{m}}^{\mathbb{I}}\left(\mathcal{N}_{1}, \mathcal{N}_{2}\right)}$.

Let $\alpha \in \mathbb{L},\left(x_{1}, x_{2}\right) \in R_{\alpha}, a_{1} \in \Sigma$, and $y_{1} \in \square_{1}^{a_{1}}\left(x_{1}\right)$. Then $x_{1} \xrightarrow{a_{1}} \longrightarrow_{1} y_{1}$, hence we have $x_{2} \xrightarrow{a_{2}}{ }_{2} y_{2}$ and $\beta \in \mathbb{L}$ so that $\left(y_{1}, y_{2}\right) \in R_{\beta}$ and $F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $y_{1} \in \square_{2}^{a_{2}}\left(x_{2}\right)$.

Let $N_{2} \in \diamond_{2}\left(x_{2}\right)$, then also $x_{2} \longrightarrow_{2} N_{2}$. Hence we must have $x_{1} \longrightarrow_{1} N_{1}$ with $\forall\left(a_{1}, y_{1}\right) \in N_{1}: \exists\left(a_{2}, y_{2}\right) \in$
$N_{2}, \beta \in \mathbb{L}:\left(y_{1}, y_{2}\right) \in R_{\beta}, F\left(a_{1}, a_{2}, \beta\right) \sqsubseteq_{\mathbb{L}} \alpha$, but then also $N_{1} \in \diamond_{1}\left(x_{1}\right) . \quad \square$


[^0]:    This paper is based on the conference contribution [34], which has been presented at the 11th International Symposium on Formal Aspects of Component Software in Bertinoro, Italy.

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