

Sharing Non-Anonymous Costs of Multiple Resources Optimally

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Abstract

In cost sharing games, the existence and efficiency of pure Nash equilibria fundamentally depends on the method that is used to share the resources' costs. We consider a general class of resource allocation problems in which a set of resources is used by a heterogeneous set of selfish users. The cost of a resource is a (non-decreasing) function of the *set* of its users. Under the assumption that the costs of the resources are shared by *uniform* cost sharing protocols, i.e., protocols that use only local information of the resource's cost structure and its users to determine the cost shares, we exactly quantify the inefficiency of the resulting pure Nash equilibria. Specifically, we show tight bounds on prices of stability and anarchy for games with only submodular and only supermodular cost functions, respectively, and an asymptotically tight bound for games with arbitrary set-functions. While all our upper bounds are attained for the well-known Shapley cost sharing protocol, our lower bounds hold for arbitrary uniform cost sharing protocols and are even valid for games with anonymous costs, i.e., games in which the cost of each resource only depends on the cardinality of the set of its users.

1 Introduction

Resource allocation problems are omnipresent in many areas of economics, computer science, and operations research with many applications, e.g., in routing, network design, and scheduling. Roughly speaking, when solving these problems the central question is how to allocate a given set of resources to a set of potential users so as to optimize a given social welfare function. In many applications, a major issue is that the users of the system are striving to optimize their own private objective instead of the overall performance of the system. Such systems of users with different objectives are analyzed using the theory of non-cooperative games.

A fundamental model of resource allocation problems with selfish users are congestion games [15]. In a congestion game, each user chooses a subset of a

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given set of resources out of a set of allowable subsets. The cost of each resource depends on the number of players using the particular resource. The private cost of each user equals the sum of costs given by the cost functions of the resources contained in the chosen subset. Congestion games can be interpreted as cost sharing games with fair cost allocation in which the cost of a resource is a function of the number of its users and each user pays the same cost share, see also Anshelevich et al. [1]. Rosenthal showed that every congestion game has a pure Nash equilibrium, i.e., a strategy vector such that no player can decrease its cost by a unilateral change of her strategy.

This existence result depends severely on the assumption that the players are identical in the sense that they contribute in equal terms to the congestion—and, thus, the cost—on the resources, which is unrealistic in many applications. As a better model for heterogeneous users, Milchtaich [13] introduced *weighted* congestion games, where each player is associated with a positive weight and resource costs are functions of the aggregated weights of their respective users. It is well known that these games may lack a pure Nash equilibrium [4, 6, 12]. A further generalization of weighted congestion games with even more modeling power are congestion games with set-dependent cost functions introduced by Fabrikant et al. [3]. Here, the cost of each resource is a (usually non-decreasing) function of the *set* of its users. Set-dependent cost function can be used to model multi-dimensional cost structures on the resources that may arise from different technologies at the resources required by different users, such as bandwidth, personal, machines, etc.

For the class of congestion games with set-dependent costs, Gopalakrishnan et al. [7] precisely characterized how the resources' costs can be distributed among its users such that the existence of a pure Nash equilibrium is guaranteed. Specifically, they showed that the class of generalized weighted Shapley protocols is the unique maximal class of cost sharing protocols that guarantees the existence of a pure Nash equilibrium in all induced cost sharing games. This class of protocols is quite rich as it contains, e.g., weighted versions of the Shapley protocol and ordered protocols as considered by Chen et al. [2]. Chen et al. examined which protocols give rise to good equilibria for cost sharing games where each resource has a fixed cost that has to be paid if the resource is used by at least one player. In subsequent work, von Falkenhausen and Harks [19] gave a complete characterization of the prices of anarchy and stability that is achievable by cost sharing protocols in games with weighted players and matroid strategy spaces. In this paper, we follow this line of research asking which cost sharing protocols guarantee the existence of good equilibria for *arbitrary* (non-decreasing) set-dependent cost functions and *arbitrary* strategy spaces.

Our Results. We study cost sharing protocols for a general resource allocation model with set-dependent costs that guarantee the existence of efficient pure Nash equilibria. Our results are summarized in Table 1. Specifically, we give tight and asymptotically tight bounds on the inefficiency of Nash equilibria in games that use the Shapley protocol both in terms of the price of anarchy and the price of stability and for games with submodular, supermodular and arbitrary non-decreasing cost functions, respectively. The lower bounds that we provide hold for *arbitrary* uniform cost sharing protocols and even in network games with anonymous costs in which the cost of a resource depends

Table 1: The inefficiency of n -player cost sharing games.

Cost Functions	Price of stability			Price of anarchy		
	Value	Lower bd.	Upper bd.	Value	Lower bd.	Upper bd.
submodular	H_n	[1]	Thm. 8	n	[2]	Thm. 9
supermodular	n	Pro. 10	Thm. 7	∞	[19], Thm. 14	—
arbitrary	$\Theta(nH_n)$	Thm. 13	Thm. 6	∞	[19], Thm. 14	—

only of the cardinality of the set of its users. Our upper and lower bounds are exactly matching except for the price of stability in games with arbitrary non-decreasing cost functions where they only match asymptotically. Nonetheless, the lower bound of $\Theta(nH_n) = \Theta(n \log n)$ for any uniform protocol is our technically most challenging result and relies on a combination of characterizations of stable protocols for constant cost functions taken from [2] and set-dependent cost functions from [7]. Our results imply that both for submodular and supermodular costs there is no other uniform protocol that gives rise to better pure Nash equilibria than the Shapley protocol, in the worst case. As another interesting side-product of our results, we obtain that moving from anonymous costs (that depend only on the number of users) to set-dependent costs does not deteriorate the quality of pure Nash equilibria in cost sharing games.

Related Work. To measure the inefficiency of equilibria, two notions have evolved. The price of anarchy [11, 14] is the worst case ratio of the social cost of an equilibrium and that of a social optimum. The price of stability [1, 17] is the ratio of the social cost of the most favorable equilibrium and that of the social optimum.

Chen et al. [2] initiated the study of cost sharing protocols that guarantee the existence of efficient pure Nash equilibria. They considered the case of constant resource costs and characterized the set of linear uniform protocols. For uniform protocols (that solely depend on local information) they showed that proportional cost sharing guarantees a price of stability equal to the n -th harmonic number and a price of anarchy of n for n -player games. Von Falkenhausen and Harks [19] studied cost sharing protocols for weighted congestion games with matroid strategy spaces. They gave various tight bounds for the prices of anarchy and stability that is achievable by uniform and more general protocols. Among other results, they showed that even for a bounded number of players, no uniform protocol can archive a constant price of anarchy. Kollias and Roughgarden [10] studied weighted congestion games with arbitrary strategy spaces. They recalled a result from Hart and Mas-Colell [8] to deduce that every weighted congestion game has a pure Nash equilibrium if the cost of each resource is distributed according to the (weighted) Shapley value. Furthermore, bounds on the price of stability for this cost sharing protocol are given. Gopalakrishnan et al. [7] considered congestion games with arbitrary set-dependent cost functions. They give a full characterization of the cost sharing protocols that guarantee the existence of a pure Nash equilibrium in all such games.

Independently of our work, Roughgarden and Schrijvers [16] showed that the price of stability of Shapley cost sharing games is H_n provided that the cost of each resource is a submodular function of its users. They further showed that the strong price of anarchy is H_n as well. In forthcoming work, Gkatzelis et al. [5] examined optimal cost sharing rules for weighted congestion games with polynomial cost functions. In particular, they showed that among the set of weighted Shapley cost sharing methods, the best price of anarchy can be guaranteed by unweighted Shapley cost sharing. Although in a similar vein, their result is independent from ours since our results hold even for unweighted players and arbitrary (submodular, supermodular, or arbitrarily non-decreasing) costs, while their result holds for weighted players and convex and polynomial costs. Yet, we believe that the combination of the results of Gkatzelis et al. and ours give strong evidence that in multiple scenarios of interest, Shapley cost sharing is the best way to share the cost among selfish users. Both papers thus contribute to the *quantitative* justification of Shapley cost sharing, as opposed to the *axiomatic* justification originally proposed by Shapley [18].

2 Preliminaries

We are given a finite set of *players* $N = \{1, \dots, n\}$ and a finite and non-empty set of *resources* R . For each player i , the set of *strategies* \mathcal{P}_i is a non-empty set of subsets of R . We set $\mathcal{P} = \mathcal{P}_1 \times \dots \times \mathcal{P}_n$ and call $P = (P_1, \dots, P_n) \in \mathcal{P}$ a *strategy vector*. For $P \in \mathcal{P}$ and $r \in R$, let $P^r = \{i : r \in P_i\}$ denote the set of players that use resource r in strategy vector P . For each resource r , we are given a non-negative cost function $C^r : 2^N \rightarrow \mathbb{R}_{\geq 0}$ mapping the set of its users to the respective cost value. We assume that all cost functions C^r are non-decreasing, in the sense that $C^r(T) \leq C^r(U)$ for all $T \subseteq U$ with $T, U \in 2^N$, and that $C^r(\emptyset) = 0$. The function C^r is *submodular* if $C^r(X \cup \{i\}) - C^r(X) \geq C^r(Y \cup \{i\}) - C^r(Y)$ for all $X \subseteq Y \subseteq N$, $i \in N \setminus Y$ and *supermodular* if $C^r(X \cup \{i\}) - C^r(X) \leq C^r(Y \cup \{i\}) - C^r(Y)$ for all $X \subseteq Y \subseteq N$, $i \in N \setminus Y$. We call C^r *anonymous* if $C^r(X) = C^r(Y)$ for all $X, Y \subseteq N$ with $|X| = |Y|$. For anonymous cost functions, submodularity and supermodularity are equivalent to concavity and convexity, respectively.

The tuple $\mathcal{M} = (N, R, \mathcal{P}, (C^r)_{r \in R})$ is called a *resource allocation model*. For the special case that R corresponds to the set of edges of a graph and for every i the set \mathcal{P}_i corresponds to the set of (s_i, t_i) -paths for two designated vertices s_i and t_i , we call \mathcal{M} a *network* resource allocation model.

The players' private costs are governed by local cost sharing protocols that decide how the cost of each resource is divided among its users. A cost sharing protocol is a family of functions $(c_i^r)_{i \in N, r \in R} : \mathcal{P} \rightarrow \mathbb{R}$ that determines for each resource r and each player i , the cost share $c_i^r(P)$ that player i has to pay for using resource r under strategy vector P . The private cost of player i under strategy vector P is then defined as $C_i(P) = \sum_{r \in P_i} c_i^r(P)$.

A resource allocation model $\mathcal{M} = (N, R, \mathcal{P}, (C^r)_{r \in R})$ together with a cost sharing protocol $(c_i^r)_{i \in N, r \in R}$ thus defines a strategic game G with player set N , strategy space \mathcal{P} and private cost functions $(C_i)_{i \in N}$. For a strategic game G , let $\text{PNE} \subseteq \mathcal{P}$ denote the set of pure Nash equilibria of G . The price of anarchy of G is then defined as $\max_{P \in \text{PNE}(G)} C(P)/C(\hat{P})$, and the price of stability is defined as $\min_{P \in \text{PNE}(G)} C(P)/C(\hat{P})$, where $C(P) = \sum_{r \in R} C^r(P^r)$ is the *social*

cost of a strategy vector P and \hat{P} is a strategy vector that minimizes C . The strategy vector \hat{P} is called *socially optimal*.

Throughout this paper, we impose the following assumptions on the cost sharing protocol that have become standard in the literature, cf. [2, 19]:

- *Stability*: There exists at least one pure Nash equilibrium, i.e., there is $P \in \mathcal{P}$ such that $C_i(P) \leq C_i(\tilde{P}_i, P_{-i})$ for all $i \in N$ and $\tilde{P}_i \in \mathcal{P}_i$.
- *Budget-balance*: The cost of all used resources is exactly covered by the cost shares of its users, i.e., $\sum_{i \in P^r} c_i^r(P) = C^r(P^r)$ and $c_i^r(P) = 0$ for all $i \notin P^r$ for all $r \in R$ and $P \in \mathcal{P}$.
- *Uniformity*: The cost shares depend only on the cost structure of the resource and the set of users, i.e., $c_i^r(P) = c_i^{\tilde{r}}(\tilde{P})$ for all $i \in N$ and all resource allocation models $(N, \mathcal{P}, (C^r)_{r \in R})$, $(N, \tilde{\mathcal{P}}, (\tilde{C}^{\tilde{r}})_{\tilde{r} \in \tilde{R}})$ with $C^r \equiv \tilde{C}^{\tilde{r}}$ and all strategy vectors $P \in \mathcal{P}$, $\tilde{P} \in \tilde{\mathcal{P}}$ with $P^r = \tilde{P}^{\tilde{r}}$.

With some abuse of notation, we henceforth write $c_i^r(P^r)$ instead of $c_i^r(P)$.

In this work, we only consider cost sharing protocols that satisfy these assumptions but it is worth noting that, except for the lower bound of $\Theta(nH_n)$ on the price of stability for arbitrary non-decreasing cost functions, all our results are valid for a slightly weaker notion of uniformity where the cost shares may depend on the identity of the resource.

A protocol that is stable, budget-balanced and uniform is the *Shapley cost sharing* protocol. To give a formal definition, for a set $S \subseteq N$ of players, let us denote by $\Pi(S)$ the set of permutations $\pi : S \rightarrow \{1, \dots, |S|\}$. Let $\pi \in \Pi(P^r)$ be a permutation of the players in P^r and let $P_{i,\pi}^r = \{j \in P^r : \pi(j) < \pi(i)\}$ be the set of players that precede player i in π . The *Shapley value* of player i at resource r with users P^r is then defined as

$$\phi_i^r(P^r) = \begin{cases} \frac{1}{|P^r|!} \sum_{\pi \in \Pi(P^r)} C^r(P_{i,\pi}^r \cup \{i\}) - C^r(P_{i,\pi}^r), & \text{if } i \in P^r, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

i.e., the Shapley cost share is the marginal increase in the cost due to player i averaged over all possible permutations of the players in P^r .

The following proposition is an immediate consequence of (1) and well known in the literature. For the sake of completeness, we give a proof here.

Proposition 1. *The Shapley cost sharing protocol is budget-balanced and uniform.*

Proof. Uniformity follows directly from (1), since the cost functions only depend on P^r and C^r . For budget-balance, it is easy to see that

$$\begin{aligned} \sum_{i \in P^r} \phi_i^r(P^r) &= \sum_{i \in P^r} \frac{1}{|P^r|!} \sum_{\pi \in \Pi(P^r)} C^r(P_{i,\pi}^r \cup \{i\}) - C^r(P_{i,\pi}^r) \\ &= \frac{1}{|P^r|!} \sum_{\pi \in \Pi(P^r)} \sum_{i \in P^r} C^r(P_{i,\pi}^r \cup \{i\}) - C^r(P_{i,\pi}^r) \\ &= C^r(P^r), \end{aligned}$$

which gives the claimed result. \square

To show that Shapley cost sharing is also stable, we follow the road taken by Kollias and Roughgarden [10], who gave a potential function for Shapley

cost sharing for weighted congestion games. The following result can be proven simply by verifying that each step in the proof by Kollias and Roughgarden applies to set-dependent costs, too. For the sake of completeness, we give a proof here.

Proposition 2. *Let $\pi \in \Pi(N)$ be arbitrary and let $\Phi : \mathcal{P} \rightarrow \mathbb{R}$ with $P \mapsto \sum_{r \in R} \Phi^r(P)$ and $\Phi^r(P) = \sum_{i \in P^r} \phi_i^r(P_{i,\pi}^r \cup \{i\})$. Then, Φ is an exact potential function for Shapley cost sharing games with set-dependent costs.*

Proof. We first show that the resource potential $\Phi^r(P)$ is independent of the chosen permutation π . Let us fix $r \in R$ and $\pi \in \Pi(P^r)$. We obtain

$$\begin{aligned} \Phi^r(P) &= \sum_{i \in P^r} \phi_i^r(P_{i,\pi}^r \cup \{i\}) \\ &= \sum_{i \in P^r} \frac{1}{|P_{i,\pi}^r \cup \{i\}|!} \sum_{p \in \Pi(P_{i,\pi}^r \cup \{i\})} (C^r(P_{i,p}^r \cup \{i\}) - C^r(P_{i,p}^r)) \\ &= \sum_{i \in P^r} \sum_{p \in \Pi(P_{i,\pi}^r \cup \{i\})} \frac{C^r(P_{i,p}^r \cup \{i\}) - C^r(P_{i,p}^r)}{|P_{i,\pi}^r \cup \{i\}|!}. \end{aligned} \quad (2)$$

We observe that (2) is a weighted sum of $C^r(T)$ for some sets $T \subseteq P^r$. We set

$$\sum_{i \in P^r} \sum_{p \in \Pi(P_{i,\pi}^r \cup \{i\})} \frac{C^r(P_{i,p}^r \cup \{i\}) - C^r(P_{i,p}^r)}{|P_{i,\pi}^r \cup \{i\}|!} = \sum_{T \subseteq P^r} \alpha_T \cdot C^r(T)$$

and proceed to calculate the coefficients α_T , $T \subseteq P^r$. To this end, for a fixed set T , we have to count how often $C^r(T)$ is in the sum of (2). Now let player $t \in T$ be the last player of T in π and let $l = \pi(t)$. We know that there are only positive contributions to α_T from the Shapley value of player t and a permutation p , where the players in $T \setminus \{t\}$ come first, are followed by t and all the other players come after t . There are $(|T| - 1)!(l - |T|)!$ many of these permutations and there are $l!$ permutations in the Shapley value of player t in total. That means we have a positive contribution to α_T with amount

$$\frac{(|T| - 1)!(l - |T|)!}{l!}. \quad (3)$$

Now we consider the negative contribution. There is a negative contribution to α_T from the Shapley value of all players $\pi^{-1}(j)$ that come after t in π_r , i.e., $j \in \{l + 1, \dots, |P^r|\}$. For a fixed j we have to count the permutations where all the players of T come first and are followed by $\pi^{-1}(j)$ immediately. We know that there are $|T|!(j - |T| - 1)!$ of these permutations in total, so the total negative contribution for all players $\pi^{-1}(j)$ is

$$\sum_{j=l+1}^{|P^r|} \frac{|T|!(j - |T| - 1)!}{j!}. \quad (4)$$

Now we can calculate α_T explicitly by subtracting (4) from (3):

$$\begin{aligned}
\alpha_T &= \frac{(|T|-1)!(l-|T|)!}{l!} - \sum_{j=l+1}^{|P^r|} \frac{|T|!(j-|T|-1)!}{j!} \\
&= (|T|-1)! \left(\frac{(l-|T|)!}{l!} - \sum_{j=l+1}^{|P^r|} \frac{|T|!(j-|T|-1)!}{j!} \right) \\
&= (|T|-1)! \left(\frac{(l-|T|)!}{l!} - \sum_{j=l+1}^{|P^r|} \left(\frac{(j-|T|-1)!}{(j-1)!} - \frac{(j-|T|)!}{j!} \right) \right) \\
&= (|T|-1)! \left(\frac{(l-|T|)!}{l!} - \frac{(l+1-|T|-1)!}{(l+1-1)!} + \frac{(|P^r|-|T|)!}{|P^r|!} \right) \\
&= (|T|-1)! \left(\frac{(|P^r|-|T|)!}{|P^r|!} \right). \tag{5}
\end{aligned}$$

We derive that α_T is independent of l and, thus, independent of the permutation π .

To finish the proof, let $P \in \mathcal{P}$, $i \in N$, and $\tilde{P}_i \in \mathcal{P}_i$ be arbitrary. We have shown that when calculating $\Phi = \sum_{r \in R} \sum_{i \in P^r} \phi_i^r(P_i^r \cup \{i\})$ it is without loss of generality to assume that player i appears last in π . We then obtain,

$$\begin{aligned}
\Phi(P) - \Phi(P_{-i}, P'_i) &= \sum_{r \in P_i \setminus P'_i} \phi_i^r(P^r) - \sum_{r' \in P'_i \setminus P_i} \phi_i^{r'}(P^{r'} \cup \{i\}) \\
&= C_i(P) - C_i(P_{-i}, P'_i),
\end{aligned}$$

which finishes the proof. \square

Using that potential games always have a pure Nash equilibrium, we obtain the following immediate corollary.

Corollary 3. *The Shapley cost sharing protocol is stable.*

As a side-product of the proof of Proposition 2, we obtain the following alternative representation of the exact potential function of a Shapley cost sharing game.

Corollary 4. *The exact potential function for Shapley cost sharing games can be written as $\Phi(P) = \sum_{r \in R} \sum_{T \subseteq P^r} \alpha_T \cdot C^r(T)$ where $\alpha_T = \frac{(|P^r|-|T|)! \cdot (|T|-1)!}{|P^r|!}$, and $\alpha_\emptyset = 0$.*

3 The Efficiency of Shapley Cost Sharing

Having established the existence of pure Nash equilibria in Shapley cost sharing games we proceed to analyze their efficiency. We start to consider the price of stability.

3.1 Price of Stability

We first need the following technical lemma that bounds the coefficients α_T , $T \subseteq N$ that occur when writing the potential function as in Corollary 4.

Lemma 5. *Let $P \in \mathcal{P}$ and $r \in R$. Then, $\sum_{T \subseteq P^r} \alpha_T = H_k$, where $k = |P^r|$ and H_k is the k -th harmonic number.*

Proof. Let us fix $P^r \subseteq N$ with $|P^r| = k$. Recall that $\alpha_T = \frac{(|P^r| - |T|)! (|T| - 1)!}{|P^r|!}$ and $\alpha_\emptyset = 0$, thus, α_T does not depend on the set T but only on its cardinality $|T|$. In particular, $\alpha_S = \alpha_T$ for all $S, T \in 2^N$ with $|S| = |T|$. Defining $\alpha_l = \alpha_T$ where $T \subseteq N$ with $|T| = l$ is arbitrary, we obtain $\alpha_l = \frac{(|P^r| - l)! (l - 1)!}{|P^r|!}$ and $\alpha_0 = 0$. We calculate

$$\begin{aligned} \sum_{T \subseteq P^r} \alpha_T &= \sum_{l=1}^k \sum_{\substack{T \subseteq P^r \\ |T|=l}} \alpha_T = \sum_{l=1}^k \sum_{\substack{T \subseteq P^r \\ |T|=l}} \alpha_l = \sum_{l=1}^k \alpha_l \binom{k}{l} = \sum_{l=1}^k \frac{(l-1)!(k-l)!}{k!} \binom{k}{l} \\ &= \sum_{l=1}^k \frac{(l-1)!(k-l)!}{k!} \cdot \frac{k!}{l!(k-l)!} = \sum_{l=1}^k \frac{1}{l} = H_k, \quad \square \end{aligned}$$

We obtain the following upper bound on the price of stability of n -player Shapley cost sharing games.

Theorem 6. *The price of stability for n -player Shapley cost sharing games with set-dependent cost functions is at most nH_n .*

Proof. Fix a Shapley cost sharing game and a socially optimal strategy vector \hat{P} . We proceed to show that a global minimum P of the potential function given in Proposition 2 has cost no larger than $nH_n \cdot C(\hat{P})$. We first calculate

$$C(P) = \sum_{r \in R} C^r(P^r) \leq \sum_{r \in R} |P^r| \cdot \left(\frac{1}{|P^r|} C^r(P^r) + \sum_{T \subseteq P^r} \alpha_T C^r(T) \right),$$

since $\alpha_T = \frac{(|T|-1)! (|P^r|-|T|)!}{|P^r|!} \geq 0$. We use $\alpha_{P^r} = \frac{1}{|P^r|}$ to obtain

$$\begin{aligned} C(P) &\leq \sum_{r \in R} |P^r| \cdot \left(\alpha_{P^r} C^r(P^r) + \sum_{T \subseteq P^r} \alpha_T C^r(T) \right) = \sum_{r \in R} |P^r| \cdot \sum_{T \subseteq P^r} \alpha_T C^r(T) \\ &\leq n \cdot \sum_{r \in R} \sum_{T \subseteq P^r} \alpha_T C^r(T) = n \cdot \Phi(P). \end{aligned} \quad (6)$$

As $\Phi(P) \leq \Phi(\hat{P})$ and cost functions C^r are non-decreasing, we obtain

$$\begin{aligned} C(P) &\leq n \cdot \Phi(\hat{P}) = n \cdot \sum_{r \in R} \sum_{T \subseteq \hat{P}^r} \alpha_T C^r(T) \leq n \cdot \sum_{r \in R} \sum_{T \subseteq \hat{P}^r} \alpha_T C^r(\hat{P}^r) \\ &= n \cdot \sum_{r \in R} C^r(\hat{P}^r) \sum_{T \subseteq \hat{P}^r} \alpha_T. \end{aligned}$$

Using Lemma 5, we obtain $C(P) \leq nH_n \cdot C(\hat{P})$, as claimed. \square

For n -player games with supermodular cost functions, we obtain a better upper bound for the price of stability of n .

Theorem 7. *The price of stability for n -player Shapley cost sharing games with supermodular cost functions is at most n .*

Proof. Let us again fix an arbitrary game and let us denote a socially optimal strategy vector and a potential minimum by \hat{P} and P , respectively. Using inequality (6) from the proof of Theorem 6, we obtain $C(P) \leq n\Phi(P) \leq n\Phi(\hat{P})$, so it suffices to show that $\Phi(\hat{P}) \leq C(\hat{P})$ for supermodular cost functions.

To this end, we first remove all players from the game and then add them iteratively to their strategy played in \hat{P} . Formally, for $i \in \{1, \dots, n\}$, let $\hat{P}_{(i)}$ denote the (partial) strategy vector in which only the players $j \in \{1, \dots, i\}$ play their strategies \hat{P}_j and all other players are removed from the game (and, thus, have costs 0).

For every i , we get $C(\hat{P}_{(i)}) - C(\hat{P}_{(i-1)}) = \sum_{r \in \hat{P}_i} C^r(\hat{P}_{(i-1)}^r \cup \{i\}) - C^r(\hat{P}_{(i-1)}^r)$, where $\hat{P}_{(i-1)}^r$ denotes the set of players using resource r under strategy vector $\hat{P}_{(i-1)}$. We write this expression as

$$C(\hat{P}_{(i)}) - C(\hat{P}_{(i-1)}) = \sum_{r \in \hat{P}_i} \sum_{\pi \in \Pi(\hat{P}_{(i-1)}^r \cup \{i\})} \frac{C^r(\hat{P}^r \cup \{i\}) - C^r(\hat{P}^r)}{|\hat{P}_{(i-1)}^r \cup \{i\}|!}.$$

As C^r is supermodular, the marginal cost $C^r(\hat{P}^r \cup \{i\}) - C^r(\hat{P}^r)$ of player i does not increase when considering only those players that appear before i in the permutation π . Hence,

$$\begin{aligned} C(\hat{P}_{(i)}) - C(\hat{P}_{(i-1)}) &\geq \sum_{r \in \hat{P}_i} \sum_{\pi \in \Pi(\hat{P}_{(i-1)}^r \cup \{i\})} \frac{C^r(\hat{P}_{i,\pi}^r \cup \{i\}) - C^r(\hat{P}_{i,\pi}^r)}{|\hat{P}_{(i-1)}^r \cup \{i\}|!} \\ &= \sum_{r \in \hat{P}_i} \phi_i^r(\hat{P}_{(i-1)}^r \cup \{i\}) \\ &= C_i(\hat{P}_{(i)}) - C_i(\hat{P}_{(i-1)}) = \Phi(\hat{P}_{(i)}) - \Phi(\hat{P}_{(i-1)}), \end{aligned}$$

where the last equation is due to the fact that Φ is an exact potential function. Thus, in each step, the cost increases at least as much as the potential, which finally implies $\Phi(\hat{P}) \leq C(\hat{P})$, as claimed. \square

We obtain an even better bound on the price of stability of Shapley cost sharing games with submodular costs. This result has been obtained independently by Roughgarden and Schrijvers [16].

Theorem 8. *The price of stability of n -player Shapley cost sharing games with submodular costs is at most H_n .*

Proof. The proof works very similar to the one of Theorem 7 with the difference that we now use the submodularity of the costs to show that $C(P) \leq \Phi(P)$ for the strategy vector P that minimizes Φ .

Let \hat{P} be a socially optimal strategy vector and let P be a strategy vector that minimizes Φ . The proof of Theorem 6 yields $\Phi(P) \leq \Phi(\hat{P}) \leq H_n C(\hat{P})$, so it suffices to show $C(P) \leq \Phi(P)$.

Analogously to the proof of Theorem 7, we first remove all players from the game and then insert them iteratively in order of their index. For $i \in \{1, \dots, n\}$ let $P_{(i)}$ denote the (partial) strategy vector, in which all players $j \in \{1, \dots, i\}$

play P_j . As in the proof of Theorem 7, we obtain

$$C(P_{(i)}) - C(P_{(i-1)}) = \sum_{r \in P_i} \sum_{\pi \in \Pi(P_{(i-1)}^r \cup \{i\})} \frac{C^r(P^r \cup \{i\}) - C^r(P^r)}{|\hat{P}_{(i-1)}^r \cup \{i\}|!}.$$

We use the submodularity of the cost functions to obtain $C(P_{(i)}) - C(P_{(i-1)}) \leq \Phi(P_{(i)}) - \Phi(P_{(i-1)})$, i.e., in each iteration, the social cost function decreases no more than the potential function. This yields $C(P) \leq \Phi(P)$. \square

3.2 Price of Anarchy

We proceed to analyze the price of anarchy of Shapley cost sharing games. It turns out that we obtain a finite bound on the price of anarchy (for a fixed number of players) only for submodular cost functions.

Theorem 9. *The price of anarchy of n -player Shapley cost sharing games with submodular costs is at most n .*

Proof. Let P be a pure Nash equilibrium and let \hat{P} be a socially optimal strategy vector. We calculate

$$C_i(P) \leq C_i(\hat{P}_i, P_{-i}) = \sum_{r \in \hat{P}_i} \sum_{\pi \in \Pi(P^r \cup \{i\})} \frac{C^r(P_{i,\pi}^r \cup \{i\}) - C^r(P_{i,\pi}^r)}{|\hat{P}^r \cup \{i\}|!},$$

and, by submodularity,

$$C_i(P) \leq \sum_{r \in \hat{P}_i} \sum_{\pi \in \Pi(P^r \cup \{i\})} \frac{C^r(\{i\}) - C^r(\emptyset)}{|\hat{P}_r \cup \{i\}|!} = \sum_{r \in \hat{P}_i} C^r(\{i\}) \leq C(\hat{P}). \square$$

4 General Uniform Cost Sharing

In Section 3, we showed upper bounds on the inefficiency of pure Nash equilibria for the Shapley cost sharing protocol. In this section, we show that these upper bounds are essentially tight, even for network games with anonymous resource costs that depend only on the cardinality of the set of the resource's users.

4.1 Price of Stability

It is well known [1, 2] that no uniform cost sharing protocol can guarantee a price of stability strictly below H_n for all n -player Shapley cost sharing games with submodular costs. In fact, this negative result holds even for anonymous and constant costs. We complement this result with the observation that even for anonymous and convex costs, no uniform cost sharing protocol can guarantee a price of stability strictly below n .

Proposition 10. *For all uniform cost sharing protocols and $\epsilon \in (0, 1)$ there is a n -player network resource allocation model with anonymous convex cost functions such that the price of stability is at least $n - \epsilon$.*

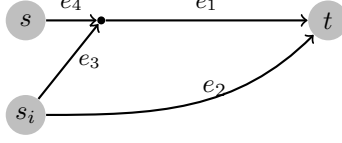


Figure 1: Cost sharing game with price of stability arbitrarily close to n .

Proof. Let us fix a uniform cost sharing protocol and a player set $N = \{1, \dots, n\}$. Because of the uniformity of the cost sharing protocol, the cost shares for a given edge are completely determined by the local information on that edge and do not depend on the structure of the network.

So let us consider an edge e_1 with the following convex cost function

$$C^{e_1}(P^{e_1}) = \begin{cases} 0, & \text{if } |P^{e_1}| \neq n, \\ n - \epsilon, & \text{if } |P^{e_1}| = n. \end{cases}$$

By budget-balance, $\sum_{i \in N} c_i^{e_1}(N) = n - \epsilon$, implying that there is a player $i \in N$ paying less than 1 on this edge when all users are using it.

Consider the network shown in Figure 1 where the cost of edge e_2 equals the number of its users. The other edges e_3 and e_4 are free. Player i has to route from node s_i to node t and all the other players route from s to t and must use the strategy $\{e_1, e_4\}$. If player i uses the strategy $\{e_1, e_3\}$, the costs equal $C(\{e_1, e_3\}, \{e_1, e_4\}, \dots, \{e_1, e_4\}) = C^{e_1}(N) = n - \epsilon$. If she chooses $\{e_2\}$ instead, the costs are 1. Clearly, the first strategy vector is the unique pure Nash equilibrium while the latter is the system optimum. Thus, the price of stability is $n - \epsilon$. \square

We proceed to show a lower bound on the price of stability of $\Theta(nH_n)$ for any uniform cost sharing protocol. To prove this, we use a result of Gopalakrishnan et al. [7, Theorem 1] who showed that the set of generalized weighted Shapley cost sharing protocols is the unique maximal set of uniform cost sharing protocols that is stable. To state their result, we first recall the definition of the generalized weighted Shapley cost sharing protocols [9].

Definition 11 (Generalized weighted Shapley cost sharing). *Let $N = \{1, \dots, n\}$ be a set of players. A tuple $w = (\lambda, \Sigma)$ with $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\Sigma = (S_1, \dots, S_m)$ is called a weight system if $\lambda_i > 0$ for all $i \in N$ and Σ is a partition of N , i.e., $S_1 \cup \dots \cup S_m = N$ and $S_i \cap S_j = \emptyset$ for all $i \neq j$.*

Given a resource allocation model $\mathcal{M} = (N, R, \mathcal{P}, (C^r)_{r \in R})$ and a weight system $w = (\lambda, \Sigma)$, the generalized weighted Shapley cost sharing protocol assigns the cost shares $(\psi_i^r)_{i \in N, r \in R}$ defined as

$$\psi_i^r(P^r) = \begin{cases} \sum_{T \subseteq P^r : i \in \overline{T}} \frac{\lambda_i}{\sum_{j \in \overline{T}} \lambda_j} \left(\sum_{U \subseteq T} (-1)^{|T|-|U|} C^r(U) \right), & \text{if } i \in P^r \\ 0, & \text{otherwise,} \end{cases}$$

where $\overline{T} = T \cap S_k$ and $k = \min\{j : S_j \cap T \neq \emptyset\}$.

The following proposition is a special case of [7, Theorem 1].

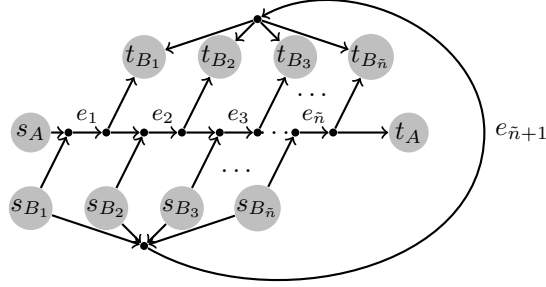


Figure 2: Cost sharing game with price of stability equal to $(\frac{n}{2} + 1) H_{n/2}$

Proposition 12 (Gopalakrishnan et al. [7]). *For every budget-balanced, uniform and stable cost sharing protocol ξ there is a weight system w such that ξ is equivalent to the generalized weighted Shapley cost sharing protocol with weight system w .*

We use this characterization to obtain the following lower bound.

Theorem 13. *For all uniform cost sharing protocols, $\epsilon \in (0, \frac{1}{2})$ and n even, there is a network n -player resource allocation model with anonymous costs for which the price of stability is equal to $\frac{n/2+1}{1+\epsilon} H_{n/2} \in \Theta(nH_n)$.*

Proof. Referring to Proposition 12, it is sufficient to show the claimed result for an arbitrary generalized weighted Shapley cost sharing protocol with weight system w only.

To this end, consider the network resource allocation model with anonymous costs shown in Figure 2. There are n players that we partition into two sets A and B of cardinality $\tilde{n} := n/2$, i.e., $N = A \cup B$, $A \cap B = \emptyset$, and $|A| = |B| = \frac{n}{2} = \tilde{n}$. For edges $e_1, e_2, \dots, e_{\tilde{n}}, e_{\tilde{n}+1}$ we assume the following cost functions:

$$C^{e_i}(P^{e_i}) = \begin{cases} 0, & \text{if } |P^{e_i}| < \tilde{n} + 1, \\ \frac{\tilde{n}+1}{i}, & \text{if } |P^{e_i}| = \tilde{n} + 1, \end{cases} \quad \text{for all } i = 1, \dots, \tilde{n}.$$

$$C^{e_{\tilde{n}+1}}(P^{e_{\tilde{n}+1}}) = \begin{cases} 0, & \text{if } P^{e_{\tilde{n}+1}} = \emptyset, \\ 1 + \epsilon, & \text{if } P^{e_{\tilde{n}+1}} \neq \emptyset. \end{cases}$$

All other edges are for free.

Note that the cost function of edge $e_{\tilde{n}+1}$ is constant as the cost is equal for every non-empty set of users. In previous work [2, Lemma 5.2] it was shown that in games where all cost functions are constant, all uniform cost sharing protocols have to be monotone in the sense that a player does not pay less if the cost of an edge has to be divided among less players, i.e.,

$$c_i^{e_{\tilde{n}+1}}(P^{e_{\tilde{n}+1}}) \leq c_i^{e_{\tilde{n}+1}}(P^{e_{\tilde{n}+1}} \setminus \{j\}) \quad \forall i \neq j \in P^{e_{\tilde{n}+1}} \subseteq N.$$

This result applies to our setting as well due to the uniformity of the protocol. It is worth noting, however, that it does not necessarily hold on the other edges in our game, since their cost is not constant.

For a given generalized weighted Shapley cost sharing protocol with weight system $w = (\lambda, \Sigma)$ we assign the players to the sets and start nodes as follows.

First, we sort the players ascending to their sets S_i where ties are broken in favor of larger λ_j , i.e., for $a \in S_i$ and $b \in S_j$ we have the property

$$a < b \Leftrightarrow S_i < S_j \vee (S_i = S_j \wedge \lambda_a \geq \lambda_b).$$

. Up to renaming we can assume that the obtained order is $(1, \dots, n)$. We then put the first $n/2$ players into A and the remaining $n/2$ into B .

All players in A have start node s_A and target node t_A . The start and target nodes of the players in B are assigned in a similar fashion as in the proof of Proposition 10. We first pick the player, say $i_{\tilde{n}}$ with largest cost share on edge $e_{\tilde{n}+1}$ among the players in B using $e_{\tilde{n}+1}$. We let $i_{\tilde{n}}$ route from node $s_{\tilde{n}}$ to node $t_{\tilde{n}}$. Then, we consider the remaining players $B \setminus \{i_{\tilde{n}}\}$ and choose a player

$$i_{\tilde{n}-1} \in \operatorname{argmax}_{b \in B \setminus \{i_{\tilde{n}}\}} c_b^{e_{\tilde{n}+1}}(B \setminus \{i_{\tilde{n}}\}).$$

and let $i_{\tilde{n}-1}$ route from $s_{\tilde{n}-1}$ to $t_{\tilde{n}-1}$. We iterate this process until all players in B are assigned to their respective start and target nodes.

We proceed to show that in the unique pure Nash equilibrium no player uses edge $e_{\tilde{n}+1}$. For a contradiction, suppose P is a pure Nash equilibrium and there is a non-empty set of players $\tilde{B} \subseteq B$ using edge $e_{\tilde{n}+1}$. Let $j = \max\{k \in \{1, \dots, \tilde{n}\} : i_k \in \tilde{B}\}$ be the player chosen first among the players in \tilde{B} in the assignment procedure described above. First, we show that i_j pays more than $1/j$ for edge $e_{\tilde{n}+1}$ in P . To this end, note that $c_{i_j}^{e_{\tilde{n}+1}}(\tilde{B}) \geq c_{i_j}^{e_{\tilde{n}+1}}(\cup_{k=1}^j \{i_k\}) \geq (1 + \epsilon)/j > 1/j$, where for the first inequality we used the monotonicity of the cost shares, and for the second inequality, we used that player i_j pays most among the players in $\cup_{k=1}^j \{i_k\}$ by construction.

We proceed to argue that player $i_j \in B$ would not pay more than $1/j$ on edge e_j when deviating to her other path, which is then a contradiction to P being a pure Nash equilibrium with a non-empty set of players using edge $e_{\tilde{n}+1}$. We calculate the cost share of player i_j on e_j by the definition of the generalized weighted Shapley cost sharing protocol.

$$\begin{aligned} c_{i_j}^{e_j}(A \cup \{i_j\}) &= \sum_{T \subseteq A \cup \{i_j\} : i_j \in T} \frac{\lambda_{i_j}}{\sum_{k \in T} \lambda_k} \left(\sum_{R \subseteq T} (-1)^{|T|-|R|} C^{e_j}(R) \right) \\ &= \sum_{T = A \cup \{i_j\} : i_j \in T} \frac{\lambda_{i_j}}{\sum_{k \in T} \lambda_k} (C^{e_j}(T)), \end{aligned}$$

where we used that $C^{e_i}(R) = 0$ for $R \neq A \cup \{i_j\}$. We obtain

$$c_{i_j}^{e_j}(A \cup \{i_j\}) = \begin{cases} \frac{\lambda_{i_j}}{\sum_{k \in S_1} \lambda_k} (C^{e_j}(A \cup \{i_j\})), & \text{if } i_j \in S_1, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, only the players in S_1 have to pay for edge e_j . If $i_j \notin S_1$ we immediately obtain $c_{i_j}^{e_j}(A \cup \{i_j\}) = 0 \leq 1/j$. If, on the other hand, player $i_j \in S_1$ we know from the construction of A and B that $A \subseteq S_1$. Player i_j has the smallest weight of all these players by construction, so i_j pays not more than all the other players. We then obtain $c_{i_j}^{e_j}(A \cup \{i_j\}) \leq \frac{(\tilde{n}+1)/j}{\tilde{n}+1} = 1/j$.

We have shown that in no pure Nash equilibrium there is a player using edge $e_{\tilde{n}+1}$. This implies that in all pure Nash equilibria, each player i_j uses edge e_j , so the players in N pay in total $\sum_{j=1}^{\tilde{n}} C^{e_j}(A \cup i_j) = \sum_{j=1}^{\tilde{n}} \frac{\tilde{n}+1}{j} = (\tilde{n}+1)H_{\tilde{n}}$. The price of stability thus amounts to $\frac{1}{1+\epsilon}(n/2+1)H_{n/2}$, as claimed. \square

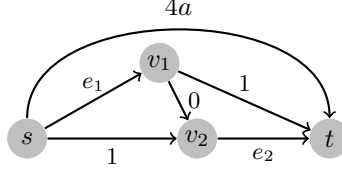


Figure 3: Cost sharing game used in case 1 of the proof of Theorem 14.

4.2 Price of Anarchy

Von Falkenhausen and Harks [19] showed that any uniform cost sharing protocol leads to unbounded price of anarchy for cost sharing games with three (weighted) players and three parallel arcs.

We complement their result by showing that also for anonymous costs but more complicated networks, no constant price of anarchy can be obtained.

Theorem 14. *For all uniform cost sharing protocols there is a 2-player network resource allocation model with anonymous convex cost functions such that the price of anarchy is unbounded.*

Proof. Let us fix a uniform cost sharing protocol and let $a \geq 1$ be arbitrary. In the following, we will show that there is a 2-player network resource allocation model with anonymous convex cost such that the resulting price of anarchy is at least a . As a was chosen arbitrarily, this implies the claimed result.

First, consider the two-player game shown in Figure 3. For arbitrary $q \geq 2$, we assume the following supermodular cost functions for the edges e_1 and e_2 .

$$\begin{array}{ll} C^{e_1}(\emptyset) = 0 & C^{e_2}(\emptyset) = 0 \\ C^{e_1}(\{1\}) = 1 & C^{e_2}(\{1\}) = 1 \\ C^{e_1}(\{2\}) = 1 & C^{e_2}(\{2\}) = 1 \\ C^{e_1}(\{1, 2\}) = q & C^{e_2}(\{1, 2\}) = q. \end{array}$$

The per-player costs of all the other edges are given in Figure 3. Both players have to route from s to t . Observe that $\min_{i,j \in \{1,2\}} \{c_i^{e_j}(\{1,2\})\}$ may depend on q as the costs of edges e_1 and e_2 when used by both players are q . We distinguish the following two cases.

First case: $\min_{i,j \in \{1,2\}} \{c_i^{e_j}(\{1,2\})\}$ is unbounded for $q \in [2, \infty)$. Fix q such that

$$\min_{i,j \in \{1,2\}} \{c_i^{e_j}(\{1,2\})\} \geq 4a > 2.$$

This implies that if both players choose the direct edge, one of them has to pay at least 2 for this edge. Up to labeling let this be player 2. In a socially optimal strategy vector, one player routes from s via v_1 to t and the other player from s via v_2 to t . Because of budget-balance each player pays 2 for her strategy, so this strategy vector has a total cost of 4.

However, player 1 using the direct edge from s to t and player 2 using the path $s \rightarrow v_1 \rightarrow v_2 \rightarrow t$ is a pure Nash equilibrium. Player 1 cannot improve her strategy because she can not pay less on any of the paths containing edge e_1 or

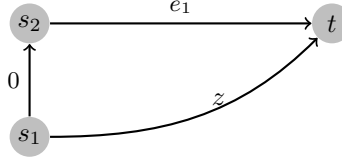


Figure 4: Cost sharing game used in case 2 of the proof of Theorem 14.

edge e_2 , because of the choice of the cost function of the direct edge. Player 2 would not deviate to the direct edge by construction and not to the other edges due to budget-balance. So this game has a price of anarchy of at least

$$\frac{\min_{i,j \in \{1,2\}} \{c_i^{e_j}(\{1,2\})\} + 2}{4} \geq \frac{4a+2}{4} \geq a.$$

This finishes the proof for the first case.

Second case: $\min_{i,j \in \{1,2\}} \{c_i^{e_j}(\{1,2\})\}$ is bounded for all $q \in [2, \infty)$. Let $z \in \mathbb{N}$ be such that

$$\min_{i,j \in \{1,2\}} \{c_i^{e_j}(\{1,2\})\} \leq z \quad \text{for all } q \in [2, \infty). \quad (7)$$

Let us fix $q \geq \min\{a(z+1), 2\}$. Using (7), there is a player $i \in \{1, 2\}$ and an edge $e \in \{e_1, e_2\}$ with

$$c_i^e(\{1, 2\}) \leq z.$$

Up to labeling let this be player 1 at edge e_1 . Now consider the network shown in Figure 4. Player 2 has to route from s_2 to t , so she has to use edge e_1 . Player 1 starts at s_1 , so she can choose the direct edge from s_1 to t or path $s_1 \rightarrow s_2 \rightarrow t$ using e_1 . In a socially optimal strategy vector she would choose the direct edge and we have

$$C((s_1, t), e_1) = C^{(s_1, t)}(\{1\}) + C^{e_1}(\{2\}) = z + 1,$$

whereas we know by construction of the second case that $P = (e_1, e_1)$ is a pure Nash equilibrium. So we immediately get a price of anarchy of

$$\frac{C(e_1, e_1)}{C((s_1, t), e_1)} = \frac{C^{e_1}(\{1, 2\})}{C^{(s_1, t)}(\{1\}) + C^{e_1}(\{2\})} = \frac{q}{z+1} \geq \frac{a(z+1)}{z+1} \geq a.$$

This finishes the proof of the second case.

We conclude that there is a game with anonymous convex costs and arbitrarily large price. \square

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