Commuting Quantum Circuits with Few Outputs are Unlikely to be Classically Simulatable

Yasuhiro Takahashi
NTT Communication Science Laboratories,
NTT Corporation
takahashi.yasuhiro@lab.ntt.co.jp

Seiichiro Tani
NTT Communication Science Laboratories,
NTT Corporation
tani.seiichiro@lab.ntt.co.jp

Takeshi Yamazaki
Mathematical Institute, Tohoku University
yamazaki@math.tohoku.ac.jp

Kazuyuki Tanaka
Mathematical Institute, Tohoku University
tanaka@math.tohoku.ac.jp

Abstract

We study the classical simulatability of commuting quantum circuits with n input qubits and $O(\log n)$ output qubits, where a quantum circuit is classically simulatable if its output probability distribution can be sampled up to an exponentially small additive error in classical polynomial time. First, we show that there exists a commuting quantum circuit that is not classically simulatable unless the polynomial hierarchy collapses to the third level. This is the first formal evidence that a commuting quantum circuit is not classically simulatable even when the number of output qubits is exponentially small. Then, we consider a generalized version of the circuit and clarify the condition under which it is classically simulatable. Lastly, we apply the argument for the above evidence to Clifford circuits in a similar setting and provide evidence that such a circuit augmented by a depth-1 non-Clifford layer is not classically simulatable. These results reveal subtle differences between quantum and classical computation.

1 Introduction and Summary of Results

One of the most important challenges in quantum information processing is to understand the difference between quantum and classical computation. An approach to meeting this challenge is to study the classical simulatability of quantum computation. Previous studies have shown that restricted models of quantum computation, such as commuting quantum circuits, are useful for this purpose [20, 5, 17, 16, 2, 3, 12, 8, 19, 11]. Because of the simplicity of such restricted models, they are also useful for identifying the source of the computational power of quantum computers. It is therefore of great interest to study their classical simulatability.

In this paper, we study the classical simulatability of commuting quantum circuits with n input qubits and $O(\mathrm{poly}(n))$ ancillary qubits initialized to $|0\rangle$, where a commuting quantum circuit is a quantum circuit consisting of pairwise commuting gates, each of which acts on a constant number of qubits. When all commuting gates in a commuting quantum circuit act on at most c qubits for some constant $c \geq 2$, the circuit is said to be c-local. For considering the classical simulatability, we adopt strong and weak simulations. The strong simulation of a quantum circuit is to compute its output probability up to an exponentially small additive error in classical polynomial time and the weak one is to sample its output probability distribution similarly. Any strongly simulatable quantum circuit is weakly simulatable. Our main focus is on the hardness of classically simulating quantum circuits and thus we mainly deal with the weak simulatability, which yields a stronger result than that the strong simulatability yields. Previous hardness results on the weak simulatability are usually obtained with respect to multiplicative error [20, 3, 8], but such an error seems to be too strong an assumption as discussed in [2]. Our results are obtained with respect to additive error.

In 2011, Bremner et al. showed that there exists a 2-local IQP circuit with $O(\mathrm{poly}(n))$ output qubits such that it is not weakly simulatable (under a plausible assumption) [3], where an IQP circuit is a quantum circuit consisting of pairwise commuting gates that are diagonal in the X-basis $\{(|0\rangle \pm |1\rangle)/\sqrt{2}\}$. Roughly speaking, this result means that when the number of output qubits is large, even a simple commuting quantum circuit is powerful. On the other hand, in 2013, Ni et al. showed that any 2-local commuting quantum circuit with $O(\log n)$ output qubits is strongly simulatable and that there exists a 3-local commuting quantum circuit with only one output qubit such that it is not strongly simulatable (under a plausible assumption) [12]. Thus, when the number of output qubits is $O(\log n)$, the classical simulatability of commuting quantum circuits depends on the number of qubits affected by each commuting gate. A natural question is whether there exists a commuting quantum circuit with $O(\log n)$ output qubits such that it is not weakly simulatable.

There are two previous results related to this question. The first one is that any (constant-local) IQP circuit with $O(\log n)$ output qubits is weakly simulatable [3]. Thus, if we want to answer the above question affirmatively, we need to consider commuting quantum circuits other than IQP circuits. The second one is that, if any commuting quantum circuit with only one output qubit is weakly simulatable, there exists a polynomial-time classical algorithm for the problem of estimating the matrix element $|\langle 0|U|0\rangle|$ (up to a polynomially small additive error) for any unitary matrix U that is implemented by a constant-depth quantum circuit [12]. This suggests an affirmative answer to the above question since the matrix element estimation problem seems to be hard for a classical computer. However, the hardness has not been formally understood yet.

We provide the first formal evidence for answering the above question affirmatively:

Theorem 1. *There exists a 5-local commuting quantum circuit with $O(\log n)$ output qubits such that it is not weakly simulatable unless the polynomial hierarchy* PH *collapses to the third level.*

It is widely believed that PH does not collapse to any level [15]. Thus, the circuit in Theorem 1 is the desired evidence. To construct the circuit, we first show the existence of a depth-3 quantum circuit A_n that is not weakly simulatable with respect to additive error (under a plausible assumption), where it has n input qubits, $O(\mathrm{poly}(n))$ ancillary qubits, and $O(\mathrm{poly}(n))$ output qubits. This is shown by our new analysis of the weak simulatability (with respect to additive error) of a depth-3 quantum circuit that is not weakly simulatable with respect to multiplicative error (under a plausible assumption) [3, 5]. Our idea for constructing the circuit in Theorem 1 is to combine A_n with the OR reduction circuit [7], which reduces the computation of the OR function on k bits to that on $O(\log k)$ bits. The resulting circuit has $O(\log n)$ output qubits and is not weakly simulatable (under a plausible assumption). It is of course not a commuting quantum circuit, but an important observation is that the OR reduction circuit can be transformed into a 2-local commuting quantum circuit. We consider a quantum circuit consisting gates of the form $A_n^{\dagger} g A_n$ for any commuting gate g in the commuting OR reduction circuit and analyze it rigorously, which implies Theorem 1.

Then, in order to generalize the above-mentioned result that any IQP circuit with $O(\log n)$ output qubits is weakly simulatable [3], we consider the weak simulatability of a generalized version of the circuit in Theorem 1. We assume that we are given two quantum circuits: F_n is a quantum circuit with n input qubits, $O(\mathrm{poly}(n))$ ancillary qubits, and $O(\mathrm{poly}(n))$ output qubits and D is a quantum circuit on $O(\mathrm{poly}(n))$ qubits consisting of pairwise commuting gates that are diagonal in the Z-basis $\{|0\rangle, |1\rangle\}$. The generalized version is the circuit $(F_n^{\dagger} \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})$, where $l = O(\log n)$. The input qubits and output qubits of the circuit are the input qubits of F_n and the ancillary qubits on which $H^{\otimes l}$ is applied, respectively. In particular, when $F_n = A_n$ and D is a quantum circuit consisting of controlled phase-shift gates, the whole circuit becomes the circuit in Theorem 1. We show that the weak simulatability of F_n implies that of the whole circuit:

Theorem 2. *If F_n is weakly simulatable, then $(F_n^\dagger \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})$ with $l = O(\log n)$ output qubits is also weakly simulatable.*

The above-mentioned result in [3] corresponds to the case when F_n is a tensor product of H. Theorem 2 implies an interesting suggestion on how to improve Theorem 1. As described above, the 5-local commuting quantum circuit in Theorem 1 is constructed by choosing a depth-3 quantum circuit as F_n. A possible way to improve Theorem 1, or more concretely, a possible way to construct a 3- or 4-local commuting quantum circuit that is not weakly simulatable would be to somehow choose a depth-2 quantum circuit as F_n. Theorem 2 implies that such a construction is impossible. This is because, since any depth-2 quantum circuit is weakly simulatable [20, 10], choosing a depth-2 quantum circuit as F_n yields only a weakly simulatable quantum circuit.

We show Theorem 2 by simply generalizing the proof of the above-mentioned result in [3]. More precisely, we fix the states of the qubits other than the $O(\log n)$ output qubits on the basis of the assumption in Theorem 2 and then follow the change of the states of the output qubits. This yields a polynomial-time classical algorithm for weakly simulating $(F_n^\dagger \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})$.

Lastly, we apply the argument for proving Theorem 1 to Clifford circuits with n input qubits, $O(\mathrm{poly}(n))$ ancillary qubits in a product state, and $O(\log n)$ output qubits. A simple extension of the proof in [4, 8] implies that any Clifford circuit in the setting is strongly simulatable. We provide evidence that a slightly extended circuit is not weakly simulatable:

Theorem 3. *There exists a Clifford circuit augmented by a depth-1 non-Clifford layer with $O(\mathrm{poly}(n))$ ancillary qubits in a particular product state and with $O(\log n)$ output qubits such that it is not weakly simulatable unless* PH *collapses to the third level.*

Similar to Theorems 1 and 2, Theorem 3 contributes to understanding a subtle difference between quantum and classical computation. As in the proof of Theorem 1, using the result in [8], we show the existence of a Clifford circuit that is not weakly simulatable with respect to additive error (under a plausible assumption), where it has n input qubits, $O(\mathrm{poly}(n))$ ancillary qubits in a particular product state, and $O(\mathrm{poly}(n))$ output qubits. Then, we combine the Clifford circuit with a constant-depth OR reduction circuit with unbounded fan-out gates [7]. The resulting circuit has $O(\log n)$ output qubits and is not weakly simulatable (under a plausible assumption). By decomposing the unbounded fan-out gates into CNOT gates, we transform the combination of the Clifford circuit and OR reduction circuit into a Clifford circuit augmented by a depth-1 non-Clifford layer, which implies Theorem 3. A similar argument with a constant-depth quantum circuit for the OR function with unbounded fan-out gates [18] implies that the number of output qubits can further be decreased to one at the cost of adding one more depth-1 non-Clifford layer.

2 Preliminaries

2.1 Quantum Circuits

We use the standard notation for quantum states and the standard diagrams for quantum circuits [13]. The elementary gates in this paper are a Hadamard gate H, a phase-shift gate $R(\theta)$ with angle $\theta = \pm 2\pi/2^k$ for any $k \in \mathbb{N}$, and a controlled-Z gate ΛZ, where

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ R(\theta) = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix},\ \Lambda Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

We denote $R(\pi)$, $R(\pi/2)$, and $HR(\pi)H$ as Z, P, and X, respectively, where Z and X (with $Y = iXZ$ and identity I) are called Pauli gates. We also denote $H\Lambda ZH$ as ΛX, which is a CNOT gate, where H acts on the target qubit. A quantum circuit consists of the elementary gates. In particular, when a quantum circuit consists only of H, P, and ΛZ, it is called a Clifford circuit. A commuting quantum circuit is a quantum circuit consisting of pairwise commuting gates, where we do not require that each commuting gate be one of the elementary gates. In other words, when we think of a quantum circuit as a commuting quantum circuit, we are allowed to regard a group of elementary gates in the circuit as a single gate and we require that such gates, which are not necessarily elementary gates, be pairwise commuting.

The complexity measures of a quantum circuit are its size and depth. The size is the number of elementary gates in the circuit. To define the depth, we consider the circuit as a set of layers $1, \ldots, d$ consisting of one-qubit and two-qubit gates, where gates in the same layer act on pairwise disjoint sets of qubits and any gate in layer j is applied before any gate in layer $j+1$. The depth of the circuit is the smallest possible value of d [5]. It seems to be natural to require that each gate in a layer be one of the elementary gates, but we do not require this for simplicity and we consider one-qubit and two-qubit gates determined from the context. In other words, when we count the depth, we are allowed to consider one-qubit and two-qubit gates generated by elementary gates in the circuit. Regardless of whether we adopt the requirement or not, the depth of the circuit we are interested in is a constant. A quantum circuit can use ancillary qubits initialized to $|0\rangle$. We do not require that the states of the ancillary qubits be reset to $|0\rangle$ at the end of the computation.

We deal with a uniform family of polynomial-size quantum circuits $\{C_n\}_{n\geq 1}$, where each C_n is a quantum circuit with n input qubits and $O(\mathrm{poly}(n))$ ancillary qubits, and can use phase-shift gates with angles $\theta = \pm 2\pi/2^k$ for any $k = O(\mathrm{poly}(n))$. Some of the input and ancillary qubits are called output qubits. At the end of the computation, Z-measurements, i.e., measurements in the Z-basis, are performed on the output qubits. The uniformity means that there exists a polynomial-time deterministic classical algorithm for computing the function $1^n \mapsto \overline{C_n}$, where $\overline{C_n}$ is the classical description of C_n. A symbol denoting a quantum circuit, such as C_n, also denotes its matrix representation in some fixed basis. Any quantum circuit in this paper is understood to be an element of a uniform family of polynomial-size quantum circuits and thus, for simplicity, we deal with a quantum circuit C_n in place of a family $\{C_n\}_{n\geq 1}$. We require that each commuting gate in a commuting quantum circuit act on a constant number of qubits. When all commuting gates act on at most c qubits for some constant $c \geq 2$, the circuit is said to be c-local [12].

2.2 Classical Simulatability and Complexity Classes

We deal with a uniform family of polynomial-size classical circuits to model a polynomial-time deterministic classical algorithm. Similarly, to model its probabilistic version, we deal with a uniform family of polynomial-size randomized classical circuits, each of which has a register initialized with random bits for each run of the computation [3]. As in the case of quantum circuits, for simplicity, we consider a classical circuit in place of a family of classical circuits.

Let C_n be a polynomial-size quantum circuit with n input qubits, $O(\mathrm{poly}(n))$ ancillary qubits, and m output qubits. For any $x \in \{0,1\}^n$, there exists an output probability distribution $\{(y, \Pr[C_n(x) = y])\}_{y\in\{0,1\}^m}$, where $\Pr[C_n(x) = y]$ is the probability of obtaining $y \in \{0,1\}^m$ by Z-measurements on the output qubits of C_n with the input state $|x\rangle$. The classical simulatability of C_n is defined as follows [20, 21, 3, 22, 12, 8, 19]:

Definition 1.

- *C_n is strongly simulatable if the output probability $\Pr[C_n(x) = y]$ and its marginal output probability can be computed up to an exponentially small additive error in classical $O(\mathrm{poly}(n))$ time. More precisely, for any polynomial p, there exists a polynomial-size classical circuit D_n such that, for any $x \in \{0,1\}^n$ and $y \in \{0,1\}^m$,*

$$|D_n(x, y) - \Pr[C_n(x) = y]| \leq \frac{1}{2^{p(n)}},$$

and, when we choose arbitrary m' output qubits from the m output qubits of C_n for any $m' < m$, the output probability $\Pr[C_n(x) = y']$ *can be computed similarly for any $x \in \{0,1\}^n$ and $y' \in \{0,1\}^{m'}$.*

- *C_n is weakly simulatable if the output probability distribution $\{(y, \Pr[C_n(x) = y])\}_{y\in\{0,1\}^m}$ can be sampled up to an exponentially small additive error in classical $O(\mathrm{poly}(n))$ time. More precisely, for any polynomial p, there exists a polynomial-size randomized classical circuit R_n such that, for any $x \in \{0,1\}^n$ and $y \in \{0,1\}^m$,*

$$|\Pr[R_n(x) = y] - \Pr[C_n(x) = y]| \leq \frac{1}{2^{p(n)}}.$$

Any strongly simulatable quantum circuit is weakly simulatable [20, 3].

The following two complexity classes are important for our discussion [1, 3, 6]:

Definition 2. *Let $L \subseteq \{0,1\}^*$.*

- *$L \in \mathsf{PostBQP}$ if there exists a polynomial-size quantum circuit C_n with n input qubits, $O(\mathrm{poly}(n))$ ancillary qubits, one output qubit, and one particular qubit (other than the output qubit) called the postselection qubit such that, for any $x \in \{0,1\}^n$,*
  - $\Pr[\mathrm{post}_n(x) = 0] > 0$,
  - *if $x \in L$,* $\Pr[C_n(x) = 1|\mathrm{post}_n(x) = 0] \geq 2/3$,
  - *if $x \notin L$,* $\Pr[C_n(x) = 1|\mathrm{post}_n(x) = 0] \leq 1/3$,

  where the event "$\mathrm{post}_n(x) = 0$" means that the classical outcome of the Z-measurement on the postselection qubit is 0.

- *$L \in \mathsf{PostBPP}$ if there exists a polynomial-size randomized classical circuit R_n with n input bits that, for any $x \in \{0,1\}^n$, outputs $R_n(x), \mathrm{post}_n(x) \in \{0,1\}$ such that*
  - $\Pr[\mathrm{post}_n(x) = 0] > 0$,
  - *if $x \in L$,* $\Pr[R_n(x) = 1|\mathrm{post}_n(x) = 0] \geq 2/3$,
  - *if $x \notin L$,* $\Pr[R_n(x) = 1|\mathrm{post}_n(x) = 0] \leq 1/3$.

We use the notation $\mathrm{post}_n(x) = 0$ both in the quantum and classical settings, but the meaning will be clear from the context. Another important class is the polynomial hierarchy $\mathsf{PH} = \bigcup_{j\geq 1} \Delta_j^p$. Here, $\Delta_1^p = \mathsf{P}$ and $\Delta_{j+1}^p = \mathsf{P}^{\mathsf{N}\Delta_j^p}$ for any $j \geq 1$, where P is the class of languages decided by polynomial-time classical algorithms and $\mathsf{N}\Delta_j^p$ is the non-deterministic class associated to Δ_j^p [15, 3]. It is widely believed that $\mathsf{PH} \neq \Delta_j^p$ for any $j \geq 1$ [15]. As shown in [3], if $\mathsf{PostBQP} \subseteq \mathsf{PostBPP}$, then $\mathsf{PH} = \Delta_3^p$, i.e., PH collapses to the third level. It can be shown that, in our setting of elementary gates and quantum circuits, this relationship also holds when the condition $\Pr[\mathrm{post}_n(x) = 0] > 0$ in the definition of $\mathsf{PostBQP}$ is replaced with the condition that, for some polynomial q (depending only on C_n), $\Pr[\mathrm{post}_n(x) = 0] \geq 1/2^{q(n)}$. In the following, we adopt the latter condition.

3 Commuting Quantum Circuits

3.1 Hardness of the Weak Simulation

It is known that there exists a depth-3 quantum circuit with n input qubits, $O(\mathrm{poly}(n))$ ancillary qubits, and $O(\mathrm{poly}(n))$ output qubits such that it is not weakly simulatable with respect to multiplicative error unless PH collapses to the third level [3]. We first analyze its weak simulatability with respect to additive error and show the following lemma:

Lemma 1. *There exists a depth-3 polynomial-size quantum circuit with $O(\mathrm{poly}(n))$ output qubits such that it is not weakly simulatable (with respect to additive error) unless* PH *collapses to the third level.*

Proof. We assume that PH does not collapse to the third level. Then, as described above, $\mathsf{PostBQP} \not\subseteq \mathsf{PostBPP}$. Let $L \in \mathsf{PostBQP} \setminus \mathsf{PostBPP}$. Then, there exists a polynomial-size quantum circuit C_n with n input qubits, $a = O(\mathrm{poly}(n))$ ancillary qubits, one output qubit, and one postselection qubit (and some polynomial q) such that, for any $x \in \{0,1\}^n$,

- $\Pr[\mathrm{post}_n(x) = 0] \geq 1/2^{q(n)}$,
- if $x \in L$, $\Pr[C_n(x) = 1|\mathrm{post}_n(x) = 0] \geq 2/3$,
- if $x \notin L$, $\Pr[C_n(x) = 1|\mathrm{post}_n(x) = 0] \leq 1/3$.

As shown in [5], there exists a depth-3 polynomial-size quantum circuit A_n with n input qubits, $a + b$ ancillary qubits, and one output qubit such that, for any $x \in \{0,1\}^n$,

- if $x \in L$, $\Pr[A_n(x) = 1|\mathrm{qpost}_n(x) = 0^{b+1}] \geq 2/3$,
- if $x \notin L$, $\Pr[A_n(x) = 1|\mathrm{qpost}_n(x) = 0^{b+1}] \leq 1/3$,

where $b = O(\mathrm{poly}(n))$, the event "$\mathrm{qpost}_n(x) = 0^{b+1}$" means that all classical outcomes of Z-measurements on the qubit corresponding to the postselection qubit of C_n and particular b qubits (other than the output qubit) are 0. We call these $b+1$ qubits the postselection qubits of A_n. Since the probability of obtaining 0^b by Z-measurements on the b qubits is $1/2^b$ [5], it holds that

$$\Pr[\mathrm{qpost}_n(x) = 0^{b+1}] = \frac{1}{2^b} \cdot \Pr[\mathrm{post}_n(x) = 0] \geq \frac{1}{2^{b+q}}.$$

We regard A_n, which has only one output qubit, as a new circuit with $b+2$ output qubits, where one of the output qubits is the original output qubit q_{out} of A_n and the others are the $b+1$ postselection qubits of A_n. We also denote this circuit as A_n. Thus, A_n is a depth-3 polynomial-size quantum circuit with $O(\mathrm{poly}(n))$ output qubits. For any $x \in \{0,1\}^n$,

- $\Pr[A_n(x) = 0^{b+1}1] = \Pr[A_n(x) = 1\&\mathrm{qpost}_n(x) = 0^{b+1}]$,
- $\Pr[A_n(x) = 0^{b+1}0] = \Pr[A_n(x) = 0\&\mathrm{qpost}_n(x) = 0^{b+1}]$,

where, for simplicity, we assume that the last output qubit of A_n is q_{out}. Thus, for any $x \in \{0,1\}^n$,

- if $x \in L$, $\Pr[A_n(x) = 0^{b+1}1] \geq 2 \cdot \Pr[\mathrm{qpost}_n(x) = 0^{b+1}]/3$,
- if $x \notin L$, $\Pr[A_n(x) = 0^{b+1}1] \leq \Pr[\mathrm{qpost}_n(x) = 0^{b+1}]/3$.

We can show that, if A_n is weakly simulatable, then $L \in \mathsf{PostBPP}$. This contradicts the assumption that $L \notin \mathsf{PostBPP}$ and completes the proof. The details can be found in Appendix A.1. $\square$

The proof method of Lemma 1 can be considered as an elaborated version of the one in [19]. As pointed out by Nishimura and Morimae [14], we note that their proof method in [11] based on the complexity class SBQP [9] can also be used to show the lemma.

The OR reduction circuit reduces the computation of the OR function on b bits to that on $O(\log b)$ bits [7]: for any b-qubit input state $|x_1\rangle \cdots |x_b\rangle$ with $x_j \in \{0,1\}$, the circuit outputs $|0\rangle^{\otimes m}$ if $x_j = 0$ for every j and an m-qubit state orthogonal to $|0\rangle^{\otimes m}$ if $x_j = 1$ for some j, where $m = \lceil \log(b+1) \rceil$. Besides the b input qubits, the circuit has m ancillary qubits as output qubits. The first part of the circuit is a layer consisting of H gates on the ancillary qubits. The middle

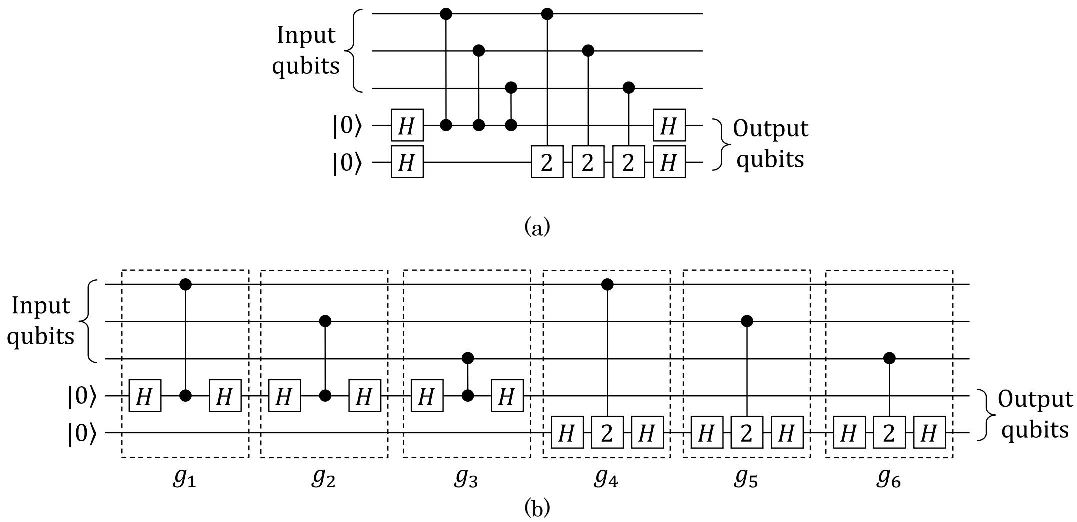


Figure 1: (a) The non-commuting OR reduction circuit, where $b = 3$, the gate represented by two black circles connected by a vertical line is a ΛZ gate, i.e., a controlled-$R(2\pi/2^1)$ gate, and the gate represented by "2" is an $R(2\pi/2^2)$ gate. (b) The commuting OR reduction circuit, where $b = 3$.

part is a quantum circuit consisting of b controlled-$R(2\pi/2^k)$ gates over all $1 \leq k \leq m$, where each gate uses an input qubit as the control qubit and an ancillary qubit as the target qubit. Such a gate is not an elementary gate, but it can be decomposed into a sequence of elementary gates. The last part is the same as the first one. We call the circuit the non-commuting OR reduction circuit. It is depicted in Fig. 1(a), where $b = 3$.

An important observation is that the non-commuting OR reduction circuit can be transformed into a 2-local commuting quantum circuit. This is shown by considering a quantum circuit consisting of gates g_j on two qubits, where each g_j is a controlled-$R(2\pi/2^k)$ gate, which is in the non-commuting OR reduction circuit, sandwiched between Hadamard gates on the target qubit. Since $HH = I$ and controlled-$R(2\pi/2^k)$ gates are pairwise commuting gates on two qubits, the operation performed by the circuit is the same as that performed by the non-commuting OR reduction circuit and the gates g_j are pairwise commuting gates on two qubits. We call the circuit the commuting OR reduction circuit. It is depicted in Fig. 1(b), where $b = 3$. Combining this commuting OR reduction circuit with A_n in the above proof implies the following lemma:

Lemma 2. *There exists a commuting quantum circuit with $O(\log n)$ output qubits such that it is not weakly simulatable unless* PH *collapses to the third level.*

Proof. As in the proof of Lemma 1, we can take $L \in \mathsf{PostBQP} \setminus \mathsf{PostBPP}$ and obtain a depth-3 polynomial-size quantum circuit A_n with n input qubits, $a + b$ ancillary qubits, and $b + 2$ output qubits such that, for any $x \in \{0, 1\}^n$,

- if $x \in L$, $\Pr[A_n(x) = 0^{b+1}1] \geq 2 \cdot \Pr[\mathrm{qpost}_n(x) = 0^{b+1}]/3$,
- if $x \notin L$, $\Pr[A_n(x) = 0^{b+1}1] \leq \Pr[\mathrm{qpost}_n(x) = 0^{b+1}]/3$.

We construct a quantum circuit E_n with n input qubits, $a + b + m + 1$ ancillary qubits, and $m + 1$ output qubits as follows, where $m = \lceil \log(b+2) \rceil$. As an example, E_n is depicted in Fig. 2(a), where $n = 5$, $a = 0$, and $b = 2$ (and thus $m = 2$).

1. Apply A_n on n input qubits and $a+b$ ancillary qubits, where the input qubits of E_n are those of A_n.

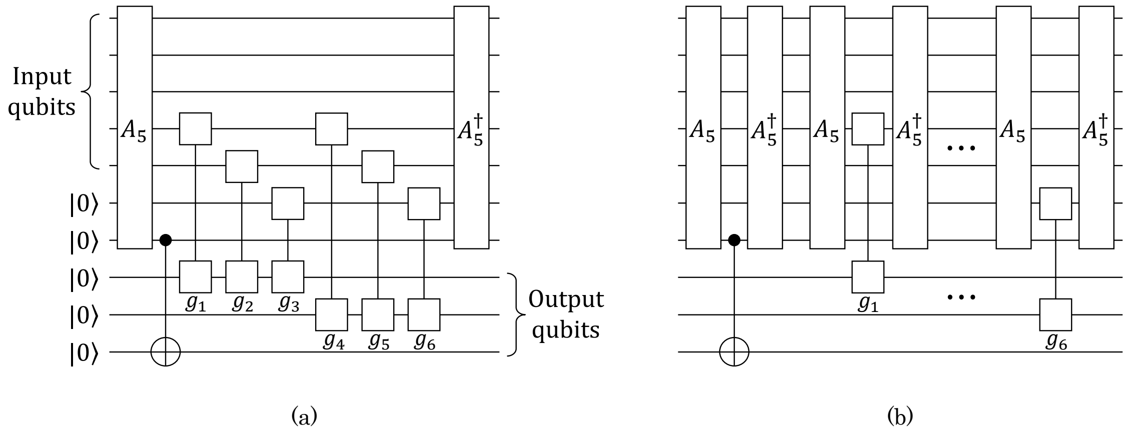


Figure 2: (a) Circuit E_n, where $n = 5$, $a = 0$, and $b = 2$ (and thus $m = 2$). The gate represented by a black circle and $\oplus$ connected by a vertical line is a ΛX gate. The gates g_j are the ones in Fig. 1. (b) The commuting quantum circuit based on E_n in (a).

2. Apply a ΛX gate on the last output qubit of A_n and on an ancillary qubit (other than the ancillary qubits in Step 1), where the output qubit is the control qubit.

3. Apply the commuting OR reduction circuit on the other output qubits of A_n, i.e., the $b+1$ postselection qubits of A_n, and m ancillary qubits (other than the ancillary qubits in Steps 1 and 2), where the postselection qubits are the input qubits of the OR reduction circuit.

4. Apply $A_n^\dagger$ as in Step 1.

The $m+1$ ancillary qubits used in Steps 2 and 3 are the output qubits of E_n. Step 4 does not affect the output probability distribution of E_n, but it allows us to construct the commuting quantum circuit described below. By the construction of E_n, for any $x \in \{0,1\}^n$,

$$\Pr[A_n(x) = 0^{b+1}1] = \Pr[E_n(x) = 0^m 1],\ \Pr[A_n(x) = 0^{b+1}0] = \Pr[E_n(x) = 0^m 0].$$

This implies that E_n is not weakly simulatable. The proof is the same as that of Lemma 1 except that the number of output qubits we need to consider is only $m+1 = O(\log n)$.

We show that there exists a commuting quantum circuit with $m+1$ output qubits such that its output probability distribution is the same as that of E_n. We consider a quantum circuit consisting of gates $A_n^\dagger g A_n$ for any gate g that is either a ΛX gate in Step 2 of E_n or g_j in the commuting OR reduction circuit. The input qubits and output qubits of E_n are naturally considered as the input qubits and output qubits of the new circuit, respectively. The circuit based on E_n in Fig. 2(a) is depicted in Fig. 2(b). Since these gates g in E_n are pairwise commuting, so are the gates $A_n^\dagger g A_n$. Moreover, $A_n^\dagger g A_n$ acts on a constant number of qubits (in fact, on at most $2^3+1 = 9$ qubits) since the depth of A_n is three, g is on two qubits, and the number of qubits on which both g and A_n are applied is one. By the construction of the circuit, its output probability distribution is the same as that of E_n. $\square$

To complete the proof of Theorem 1, it suffices to show that the commuting quantum circuit in the proof of Lemma 2 is 5-local. To show this, we give the details of the depth-3 quantum circuit constructed by the method in [5]. The circuit is based on a one-qubit teleportation circuit. We adopt the teleportation circuit depicted in Fig. 3(a), which is obtained from the standard one by decomposing it into the elementary gates. If the classical outcomes of Z-measurements on the two qubits other than the output qubit are 0, the output state is the same as the input state. We call

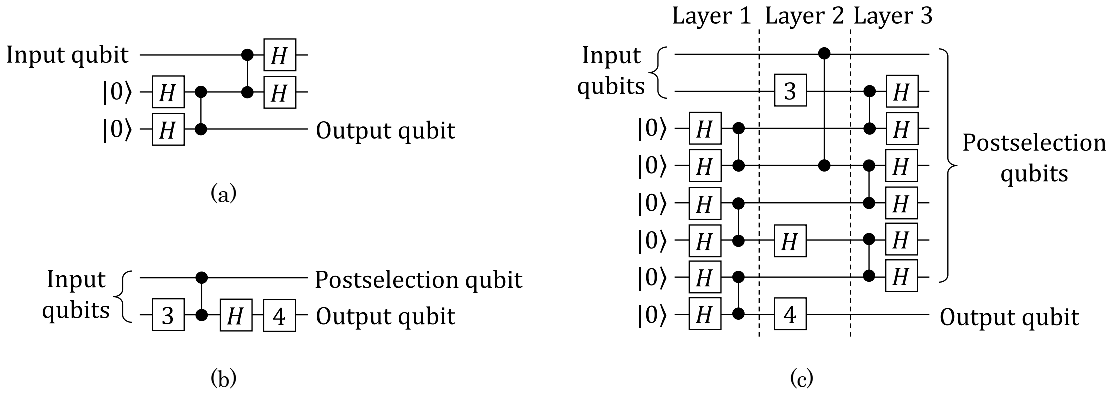


Figure 3: (a) The teleportation circuit. (b) An example of circuit C_n, where $n = 2$ and $a = 0$. The gate represented by $k \in \mathbb{N}$ is an $R(2\pi/2^k)$ gate. (c) Depth-3 circuit A_n constructed from C_n in (b) by the method in [5], where $b = 6$ and thus the total number of postselection qubits is seven.

the first measured qubit, which is the input qubit, "the first teleportation qubit", and the second one "the second teleportation qubit".

For example, we consider the circuit depicted in Fig. 3(b) as C_n in the proof of Lemma 1, where $n = 2$ and $a = 0$. The depth-3 circuit A_n constructed from C_n by the method in [5] is depicted in Fig. 3(c), where $b = 6$ and thus the total number of postselection qubits is seven. The first layer consists of the first halves of the teleportation circuits and the third consists of the last halves. The second layer consists of the gates in C_n. The teleportation qubits are the postselection qubits. If all classical outcomes of Z-measurements on the teleportation qubits are 0, all teleportation circuits teleport their input states successfully and thus the output state is the same as that of C_n.

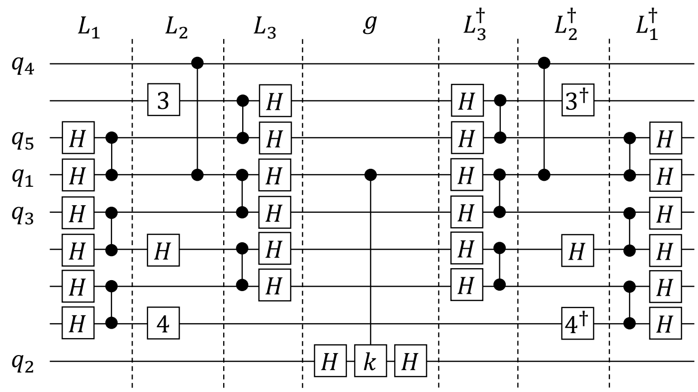
We will analyze $A_n^\dagger g A_n$ in the proof of Lemma 2, which implies the following lemma:

Lemma 3. *For any gate $A_n^\dagger g A_n$ in the proof of Lemma 2, there exists a quantum circuit on at most five qubits that implements the gate.*

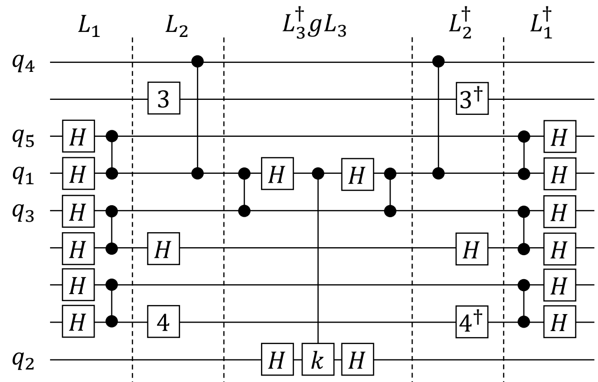
Proof. We first consider the case when $g = g_j$ in the commuting OR reduction circuit. We divide this case into the following three cases, where we assume that g is applied on a postselection qubit q_1 and an output qubit q_2 of E_n:

- Case 1: q_1 is the first teleportation qubit (of a teleportation circuit).
- Case 2: q_1 is the second teleportation qubit (of a teleportation circuit).
- Case 3: q_1 is the postselection qubit corresponding to the one of C_n.

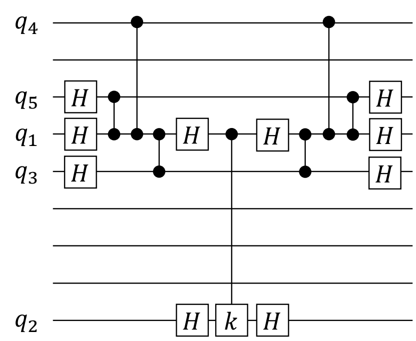
We obtain the desired circuit on at most five qubits by simplifying $A_n^\dagger g A_n$, where we represent A_n as $L_3 L_2 L_1$, each of which is a layer of A_n. We consider Case 1 using an example of $A_n^\dagger g A_n$ depicted in Fig. 4(a), where A_n is the circuit in Fig. 3(c), g is a controlled-$R(2\pi/2^k)$ gate sandwiched between H gates, and q_1 is the fourth qubit of A_n from the top, which is the first teleportation qubit. By simplifying $L_3^\dagger g L_3$, we obtain the circuit depicted in Fig 4(b). We can further simplify the circuit and obtain the desired circuit on five qubits $q_1, \ldots, q_5$ depicted in Fig. 4(c). In general, we can similarly simplify $A_n^\dagger g A_n$ and a similar analysis works for Cases 2 and 3 and the case when $g = \Lambda X$. The details can be found in Appendix A.2. $\square$



(a)



(b)



(c)

Figure 4: (a) Gate $A_n^\dagger g A_n$, where A_n is the circuit in Fig. 3(c), g is a controlled-$R(2\pi/2^k)$ gate sandwiched between H gates, and q_1 is the fourth qubit of A_n from the top. (b) The circuit obtained from $A_n^\dagger g A_n$ in (a) by simplifying $L_3^\dagger g L_3$. (c) The circuit on five qubits obtained from (b).

3.2 Weak Simulatability of a Generalized Version

The non-commuting OR reduction circuit with $b+1$ input qubits can be represented as $H^{\otimes m} D' H^{\otimes m}$, where $m = \lceil \log(b+2) \rceil$ and D' is a quantum circuit consisting only of controlled-$R(2\pi/2^k)$ gates. Since ΛX is $H \Lambda Z H$, we can represent the circuit in Theorem 1 as $(A_n^\dagger \otimes H^{\otimes(m+1)}) D'' (A_n \otimes H^{\otimes(m+1)})$, where D'' consists of D' and ΛZ, and A_n is a depth-3 quantum circuit with n input qubits, $a+b$ ancillary qubits, and $b+2$ output qubits. The output qubits of the whole circuit are the ancillary qubits on which $H^{\otimes(m+1)}$ is applied.

We generalize the circuit in Theorem 1. We assume that we are given two quantum circuits: F_n is a quantum circuit with n input qubits, $s = O(\mathrm{poly}(n))$ ancillary qubits, and t $(\leq n+s)$ output qubits and D is a quantum circuit on $t+l$ qubits consisting of pairwise commuting gates that are diagonal in the Z-basis and act on a constant number of qubits, where $l = O(\log n)$. We consider the following quantum circuit, which can be represented as $(F_n^\dagger \otimes H^{\otimes l}) D (F_n \otimes H^{\otimes l})$, with n input qubits, $s+l$ ancillary qubits, and l output qubits:

1. Apply F_n on n input qubits and s ancillary qubits, where the input qubits of the whole circuit are those of F_n.

2. Apply $H^{\otimes l}$ on l ancillary qubits (other than the ancillary qubits in Step 1).

3. Apply D on $t+l$ qubits, which are the output qubits of F_n and the ancillary qubits in Step 2.

4. Apply $H^{\otimes l}$ as in Step 2 and then apply $F_n^\dagger$ as in Step 1.

The output qubits are the ancillary qubits on which $H^{\otimes l}$ is applied. The circuit in Theorem 1 corresponds to the case when $F_n = A_n$, $D = D''$, $s = a + b$, $t = b + 2$, and $l = m + 1$.

When $F_n = H^{\otimes(n+s)}$ with arbitrary s and t, $(F_n^\dagger \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})$ is weakly simulatable [3]. A simple generalization of the proof in [3] implies Theorem 2. In fact, we fix the state of the qubits other than the $O(\log n)$ output qubits on the basis of the assumption in Theorem 2 and then follow the change of the states of the output qubits. The details of the proof can be found in Appendix A.3. As described in Section 1, Theorem 2 implies an interesting suggestion on how to improve Theorem 1. Concretely speaking, a possible way to construct a 3- or 4-local commuting quantum circuit that is not weakly simulatable would be to somehow choose a depth-2 quantum circuit as F_n, but such a construction is impossible.

4 Clifford Circuits

As an application of the construction method for the circuit in Theorem 1, we consider Clifford circuits with n input qubits, $O(\mathrm{poly}(n))$ ancillary qubits, and $O(\log n)$ output qubits. In this section, the ancillary qubits are allowed to be in a general product state (not restricted to a tensor product of $|0\rangle$). As shown in [4, 8], such a Clifford circuit with only one output qubit is strongly simulatable. We first show that a simple extension of the proof in [4, 8] implies the strong simulatability of a Clifford circuit with $O(\log n)$ output qubits:

Lemma 4. *Any Clifford circuit with $O(\mathrm{poly}(n))$ ancillary qubits in a general product state and with $O(\log n)$ output qubits is strongly simulatable.*

The proof can be found in Appendix A.4.

In contrast to Lemma 4, it is known that there exists a Clifford circuit with n input qubits, $O(\mathrm{poly}(n))$ ancillary qubits in a particular product state, and $O(\mathrm{poly}(n))$ output qubits such that it is not weakly simulatable with respect to multiplicative error unless PH collapses to the third level [8]. This is shown by using the fact that any $\mathsf{PostBQP}$ circuit can be simulated (in some sense) by a Clifford circuit. More precisely, let $L \in \mathsf{PostBQP}$ and C_n be a polynomial-size quantum circuit with n input qubits, $a = O(\mathrm{poly}(n))$ ancillary qubits initialized to $|0\rangle$, one output qubit, and one postselection qubit (and some polynomial q) such that, for any $x \in \{0,1\}^n$,

- $\Pr[\mathrm{post}_n(x) = 0] \geq 1/2^{q(n)}$,
- if $x \in L$, $\Pr[C_n(x) = 1|\mathrm{post}_n(x) = 0] \geq 2/3$,
- if $x \notin L$, $\Pr[C_n(x) = 1|\mathrm{post}_n(x) = 0] \leq 1/3$.

Then, there exists a Clifford circuit A_n with n input qubits, a ancillary qubits initialized to $|0\rangle$, $b = O(\mathrm{poly}(n))$ ancillary qubits in a product state $|\varphi\rangle^{\otimes b}$, and one output qubit, where $|\varphi\rangle = R(\pi/4)H|0\rangle = (|0\rangle + e^{i\pi/4}|1\rangle)/\sqrt{2}$, such that, for any $x \in \{0,1\}^n$,

- if $x \in L$, $\Pr[A_n(x) = 1|\mathrm{qpost}_n(x) = 0^{b+1}] \geq 2/3$,
- if $x \notin L$, $\Pr[A_n(x) = 1|\mathrm{qpost}_n(x) = 0^{b+1}] \leq 1/3$,

where the event "$\mathrm{qpost}_n(x) = 0^{b+1}$" means that all classical outcomes of Z-measurements on the qubit corresponding to the postselection qubit of C_n and particular b qubits (other than the output qubit) are 0. We call these $b+1$ qubits the postselection qubits of A_n. We can show that $\Pr[\mathrm{qpost}_n(x) = 0^{b+1}] \geq 1/2^{b+q}$. By using this property and A_n obtained from $L \in \mathsf{PostBQP} \setminus \mathsf{PostBPP}$ as in the proof of Lemma 1, we can show the following lemma, where the classical simulatability is defined with respect to additive error:

Lemma 5. *There exists a Clifford circuit with $O(\mathrm{poly}(n))$ ancillary qubits in a particular product state and with $O(\mathrm{poly}(n))$ output qubits such that it is not weakly simulatable unless* PH *collapses to the third level.*

As in the proof of Lemma 2, we construct a quantum circuit E'_n with n input qubits and $a + b + m + 1$ ancillary qubits by combining A_n with the non-commuting OR reduction circuit as follows, where $m = \lceil \log(b + 2) \rceil$ and the $m + 1$ ancillary qubits are the output qubits of E'_n. As an example, E'_n is depicted in Fig. 5(a), where $n = 5$, $a = 0$, and $b = 2$.

1. Apply A_n on n input qubits, a ancillary qubits initialized to $|0\rangle$, and b ancillary qubits initialized to $|\varphi\rangle$, where the input qubits of E'_n are those of A_n.

2. Apply a ΛX gate on the (original) output qubit of A_n and an ancillary qubit (other than the ancillary qubits in Step 1), where the output qubit is the control qubit.

3. Apply the non-commuting OR reduction circuit on the $b + 1$ postselection qubits of A_n and m ancillary qubits (other than the ancillary qubits in Steps 1 and 2), where the postselection qubits are the input qubits of the OR reduction circuit.

A direct application of the proof of Lemma 2 implies the following lemma:

Lemma 6. *There exists a Clifford circuit combined with the OR reduction circuit as described above with $O(\mathrm{poly}(n))$ ancillary qubits in a particular product state and with $O(\log n)$ output qubits such that it is not weakly simulatable unless* PH *collapses to the third level.*

We replace the non-commuting OR reduction circuit in Step 3 with a constant-depth OR reduction circuit with unbounded fan-out gates [7], where an unbounded fan-out gate can be considered as a sequence of CNOT gates with the same control qubit. It is easy to show that decomposing the unbounded fan-out gates into CNOT gates in the constant-depth OR reduction circuit yields a Clifford-1 circuit, which is a Clifford circuit augmented by a depth-1 non-Clifford layer. In particular, this procedure transforms the middle part of the non-commuting OR reduction circuit in Step 3, which is the only part that includes non-Clifford gates, into a quantum circuit that has CNOT gates and a depth-1 layer consisting of all gates in the middle part. The circuit obtained from the middle part in Fig. 5(a) is depicted in Fig. 5(b). This transformation with Lemma 6 implies Theorem 3.

A similar argument implies that there exists a Clifford-2 circuit with $O(\mathrm{poly}(n))$ ancillary qubits in a particular product state and with only one output qubit such that it is not weakly simulatable unless PH collapses to the third level, where a Clifford-2 circuit has two depth-1 non-Clifford layers. Let $L \in \mathsf{PostBQP} \setminus \mathsf{PostBPP}$. We obtain A_n as described above and combine it with a constant-depth quantum circuit for the OR function with unbounded fan-out gates [18]. By decomposing the unbounded fan-out gates into CNOT gates, the OR circuit can be transformed into a Clifford-2 circuit. Unfortunately, a combination of the circuits similar to the above construction has two output qubits. Thus, we construct two circuits with one output qubit. One circuit consists of A_n and the OR circuit, where the input qubits of the OR circuit are the output qubit of A_n and $b + 1$ postselection qubits, and the output qubit of the OR circuit is the output qubit of the whole circuit. The other similarly consists of XA_n and the OR circuit, where X is applied on the output qubit of A_n. By a similar argument in [19], we can show that, if these two Clifford-2 circuits are weakly simulatable, then $L \in \mathsf{PostBPP}$. Thus, at least one of the circuits is not weakly simulatable.

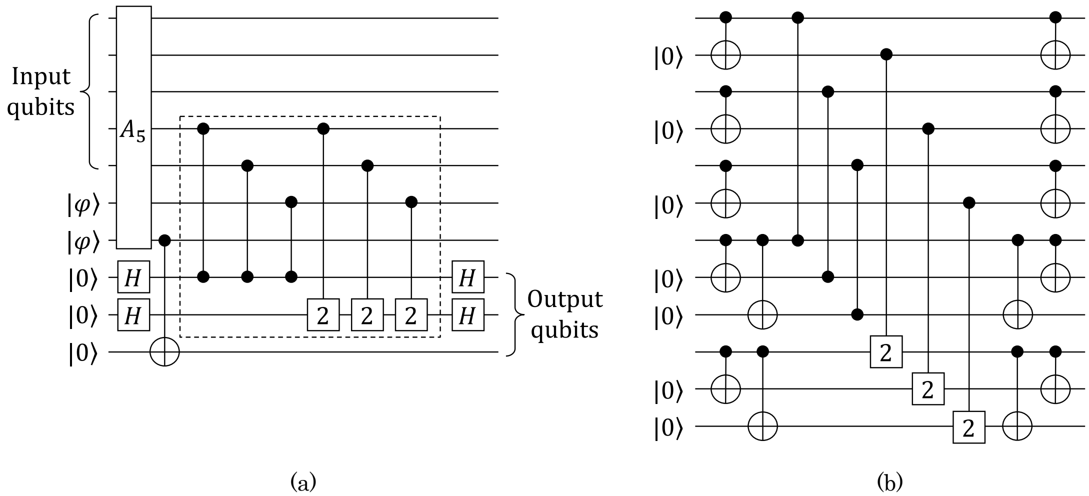


Figure 5: (a) Circuit E'_n, where $n = 5$, $a = 0$, and $b = 2$ (and thus $m = 2$). The dashed box represents the middle part of the non-commuting OR reduction circuit. (b) The circuit that has CNOT gates and a depth-1 layer consisting of all gates in the middle part in (a). The qubits in state $|0\rangle$ are new ancillary qubits, which are not depicted in (a).

5 Open Problems

Interesting challenges would be to further investigate commuting quantum circuits and to consider closely related computational models. Some examples are as follows:

- Does there exist a 3- or 4-local commuting quantum circuit with $O(\log n)$ output qubits such that it is not weakly simulatable (under a plausible assumption)?
- Do the theorems in this paper hold when exponentially small error $1/2^{p(n)}$ is replaced with polynomially small error $1/p(n)$ in the definitions of the classical simulatability?
- Can we apply the results on commuting quantum circuits to investigating the computational power of constant-depth quantum circuits?

Acknowledgment

We thank Harumichi Nishimura and Tomoyuki Morimae for pointing out to us the applicability of their proof method [11], which inspired us to realize that a slight modification of our proof method in the previous version of the present paper yields the stronger results described in this version.

References

[1] S. Aaronson. Quantum computing, postselection, and probabilistic polynomial-time. *Proceedings of the Royal Society A*, 461:3473–3482, 2005.

[2] S. Aaronson and A. Arkhipov. The computational complexity of linear optics. In *Proceedings of the 43rd ACM Symposium on Theory of Computing (STOC)*, pages 333–342, 2011.

[3] M. J. Bremner, R. Jozsa, and D. J. Shepherd. Classical simulation of commuting quantum computations implies collapse of the polynomial hierarchy. *Proceedings of the Royal Society A*, 467:459–472, 2011.

[4] S. Clark, R. Jozsa, and N. Linden. Generalized Clifford groups and simulation of associated quantum circuits. *Quantum Information and Computation*, 8(1&2):106–126, 2008.

[5] S. Fenner, F. Green, S. Homer, and Y. Zhang. Bounds on the power of constant-depth quantum circuits. In *Proceedings of Fundamentals of Computation Theory (FCT)*, volume 3623 of *Lecture Notes in Computer Science*, pages 44–55, 2005.

[6] Y. Han, L. A. Hemaspaandra, and T. Thierauf. Threshold computation and cryptographic security. *SIAM Journal on Computing*, 26(1):59–78, 1997.

[7] P. Høyer and R. Špalek. Quantum fan-out is powerful. *Theory of Computing*, 1(5):81–103, 2005.

[8] R. Jozsa and M. van den Nest. Classical simulation complexity of extended Clifford circuits. *Quantum Information and Computation*, 14(7&8):633–648, 2014.

[9] G. Kuperberg. How hard is it to approximate the Jones polynomial?, 2009. arXiv:quant-ph/0908.0512.

[10] I. L. Markov and Y. Shi. Simulating quantum computation by contracting tensor networks. *SIAM Journal on Computing*, 38(3):963–981, 2008.

[11] T. Morimae, H. Nishimura, K. Fujii, and S. Tamate. Classical simulation of $\mathrm{DQC1}_2$ or $\mathrm{DQC2}_1$ implies collapse of the polynomial hierarchy, 2014. arXiv:quant-ph/1409.6777.

[12] X. Ni and M. van den Nest. Commuting quantum circuits: efficient classical simulations versus hardness results. *Quantum Information and Computation*, 13(1&2):54–72, 2013.

[13] M. A. Nielsen and I. L. Chuang. *Quantum Computation and Quantum Information*. Cambridge University Press, 2000.

[14] H. Nishimura and T. Morimae. Private communication, 2014.

[15] C. H. Papadimitriou. *Computational Complexity*. Addison Wesley, 1994.

[16] D. Shepherd. Binary matroids and quantum probability distributions, 2010. arXiv:quant-ph/1005.1744.

[17] D. Shepherd and M. J. Bremner. Temporally unstructured quantum computation. *Proceedings of the Royal Society A*, 465:1413–1439, 2009.

[18] Y. Takahashi and S. Tani. Collapse of the hierarchy of constant-depth exact quantum circuits. In *Proceedings of the 28th IEEE Conference on Computational Complexity (CCC)*, pages 168–178, 2013.

[19] Y. Takahashi, T. Yamazaki, and K. Tanaka. Hardness of classically simulating quantum circuits with unbounded Toffoli and fan-out gates. *Quantum Information and Computation*, 14(13&14):1149–1164, 2014.

[20] B. M. Terhal and D. P. DiVincenzo. Adaptive quantum computation, constant-depth quantum circuits and Arthur-Merlin games. *Quantum Information and Computation*, 4(2):134–145, 2004.

[21] M. van den Nest. Classical simulation of quantum computation, the Gottesman-Knill theorem, and slightly beyond. *Quantum Information and Computation*, 10(3&4):258–271, 2010.

[22] M. van den Nest. Simulating quantum computers with probabilistic methods. *Quantum Information and Computation*, 11(9&10):784–812, 2011.

A Proofs

A.1 Proof of Lemma 1

We assume that A_n is weakly simulatable. Then, there exists a polynomial-size randomized classical circuit R_n such that, for any $x \in \{0,1\}^n$ and $y \in \{0,1\}^{b+2}$,

$$|\Pr[R_n(x) = y] - \Pr[A_n(x) = y]| \leq \frac{1}{2^{b+q+10}}.$$

This implies that

$$\Pr[A_n(x) = 0^{b+1}1] - \frac{1}{2^{b+q+10}} \leq \Pr[R_n(x) = 0^{b+1}1] \leq \Pr[A_n(x) = 0^{b+1}1] + \frac{1}{2^{b+q+10}},$$

$$\Pr[A_n(x) = 0^{b+1}0] - \frac{1}{2^{b+q+10}} \leq \Pr[R_n(x) = 0^{b+1}0] \leq \Pr[A_n(x) = 0^{b+1}0] + \frac{1}{2^{b+q+10}}.$$

Since $\Pr[A_n(x) = 0^{b+1}1] + \Pr[A_n(x) = 0^{b+1}0] = \Pr[\mathrm{qpost}_n(x) = 0^{b+1}]$, it holds that

$$\begin{aligned}\Pr[\mathrm{qpost}_n(x) = 0^{b+1}] - \frac{1}{2^{b+q+9}} &\leq \Pr[R_n(x) = 0^{b+1}1] + \Pr[R_n(x) = 0^{b+1}0] \\ &\leq \Pr[\mathrm{qpost}_n(x) = 0^{b+1}] + \frac{1}{2^{b+q+9}}.\end{aligned}$$

We construct a polynomial-size randomized classical circuit S_n that implements the following classical algorithm with input $x \in \{0,1\}^n$:

1. Compute $R_n(x)$.
2. (a) If $R_n(x) = 0^{b+1}1$, set $\mathrm{post}_n(x) = 0$ and $S_n(x) = 1$.
   (b) If $R_n(x) = 0^{b+1}0$, set $\mathrm{post}_n(x) = 0$ and $S_n(x) = 0$.
   (c) Otherwise, set $\mathrm{post}_n(x) = 1$ and $S_n(x) = 1$.

By the definition of S_n,

$$\begin{aligned}\Pr[\mathrm{post}_n(x) = 0] &= \Pr[R_n(x) = 0^{b+1}1] + \Pr[R_n(x) = 0^{b+1}0] \\ &\geq \Pr[\mathrm{qpost}_n(x) = 0^{b+1}] - \frac{1}{2^{b+q+9}} \\ &\geq \frac{1}{2^{b+q}} - \frac{1}{2^{b+q+9}} > 0.\end{aligned}$$

Moreover, for any $x \in \{0,1\}^n$,

$$\Pr[S_n(x) = 1|\mathrm{post}_n(x) = 0] = \frac{\Pr[R_n(x) = 0^{b+1}1]}{\Pr[R_n(x) = 0^{b+1}1] + \Pr[R_n(x) = 0^{b+1}0]}.$$

If $x \in L$,

$$\begin{aligned}\Pr[S_n(x) = 1|\mathrm{post}_n(x) = 0] &\geq \frac{\Pr[A_n(x) = 0^{b+1}1] - \frac{1}{2^{b+q+10}}}{\Pr[\mathrm{qpost}_n(x) = 0^{b+1}] + \frac{1}{2^{b+q+9}}} \\ &\geq \frac{\frac{2}{3} \cdot \Pr[\mathrm{qpost}_n(x) = 0^{b+1}] - \frac{1}{2^{b+q+10}}}{\Pr[\mathrm{qpost}_n(x) = 0^{b+1}] + \frac{1}{2^{b+q+9}}} \\ &= \frac{2}{3} - \frac{7\varepsilon}{3(1+2\varepsilon)} > \frac{2}{3} - \frac{7}{3}\varepsilon > \frac{3}{5},\end{aligned}$$

where $\varepsilon = 1/(2^{b+q+10} \cdot \Pr[\text{qpost}_n(x) = 0^{b+1}])$ and it holds that

$$0 < \varepsilon \leq \frac{1}{2^{b+q+10} \cdot \frac{1}{2^{b+q}}} = \frac{1}{2^{10}}.$$

If $x \notin L$,

$$\begin{aligned}
\Pr[S_n(x) = 1 | \text{post}_n(x) = 0] &\leq \frac{\Pr[A_n(x) = 0^{b+1}1] + \frac{1}{2^{b+q+10}}}{\Pr[\text{qpost}_n(x) = 0^{b+1}] - \frac{1}{2^{b+q+9}}} \\
&\leq \frac{\frac{1}{3} \cdot \Pr[\text{qpost}_n(x) = 0^{b+1}] + \frac{1}{2^{b+q+10}}}{\Pr[\text{qpost}_n(x) = 0^{b+1}] - \frac{1}{2^{b+q+9}}} \\
&= \frac{1}{3} + \frac{5\varepsilon}{3(1-2\varepsilon)} < \frac{2}{5}.
\end{aligned}$$

The constants 2/3 and 1/3 in the definition of PostBPP can be replaced with $1/2 + \delta$ and $1/2 - \delta$, respectively, for any constant $0 < \delta < 1/2$ [3]. Thus, $L \in \mathsf{PostBPP}$.

A.2 Proof of Lemma 3

- Case 1: q_1 is the first teleportation qubit (of a teleportation circuit).

  We note that g is on the set of qubits $\{q_1, q_2\}$ and that there is no gate on q_2 in each layer. All ΛZ gates other than the one on q_1 and qubit q_3 in layer 3 are cancelled out in $L_3^\dagger g L_3$. Only the ΛZ gate, which is not cancelled out, increases the number of qubits involved with $\{q_1, q_2\}$ by one. Thus, $L_3^\dagger g L_3$ is on $\{q_1, q_2, q_3\}$. By the construction of the teleportation circuit, there is no gate on q_3 in layer 2. Only one ΛZ gate on q_1 and qubit q_4 in layer 2 increases the number of qubits involved with $\{q_1, q_2, q_3\}$ by one. Thus, $L_2^\dagger L_3^\dagger g L_3 L_2$ is on at most four qubits. If a ΛZ gate is on q_3 or q_4 and on another qubit, it is cancelled out in $L_1^\dagger L_2^\dagger L_3^\dagger g L_3 L_2 L_1$. Only one ΛZ gate on q_1 and qubit q_5 in layer 1 increases the number of qubits involved with $\{q_1, q_2, q_3, q_4\}$ by one. Thus, $L_1^\dagger L_2^\dagger L_3^\dagger g L_3 L_2 L_1$ is on at most five qubits.

- Case 2: q_1 is the second teleportation qubit (of a teleportation circuit).

  As an example, $A_n^\dagger g A_n$ is depicted in Fig. 6(a), where A_n is the circuit in Fig. 3(c), g is a controlled-$R(2\pi/2^k)$ gate sandwiched between H gates, and q_1 is the second qubit of A_n from the bottom, which is the second teleportation qubit. As in Case 1, there is no gate on q_2 in each layer and $L_3^\dagger g L_3$ is on $\{q_1, q_2, q_3\}$. The circuit obtained from $A_n^\dagger g A_n$ in Fig. 6(a) by simplifying $L_3^\dagger g L_3$ is depicted in Fig. 6(b). By the construction of the teleportation circuit, there is no gate on q_1 in layer 2. If a ΛZ gate is on q_3 and a qubit in layer 2, it is cancelled out in $L_2^\dagger L_3^\dagger g L_3 L_2$. Thus, gates in layer 2 do not increase the number of qubits involved with $\{q_1, q_2, q_3\}$. In layer 1, a ΛZ gate on q_1 and qubit q_4 increases the number of qubits involved with $\{q_1, q_2, q_3\}$ by one, and so does a ΛZ gate on q_3 and qubit q_5. In particular, the latter happens only when an H gate is on q_3 in layer 2. This is because, when any other gate, i.e., a ΛZ or $R(\pm 2\pi/2^k)$ gate, is on q_3 in layer 2, the gate is cancelled out in $L_2^\dagger L_3^\dagger g L_3 L_2$ and thus a ΛZ gate on q_3 and qubit q_5 is also cancelled out in $L_1^\dagger L_2^\dagger L_3^\dagger g L_3 L_2 L_1$. Thus, $L_1^\dagger L_2^\dagger L_3^\dagger g L_3 L_2 L_1$ is on at most five qubits. The circuit obtained from $A_n^\dagger g A_n$ in Fig. 6(b) is depicted in Fig. 6(c).

- Case 3: q_1 is the postselection qubit corresponding to the one of C_n.

  Similar to the above cases, there is no gate on q_2 in each layer. By the construction of A_n, there is no gate on q_1 in layer 3. Thus, it suffices to consider only $L_2 L_1$. Since g is on two qubits and the number of qubits on which both g and $L_2 L_1$ are applied is one, $L_1^\dagger L_2^\dagger g L_2 L_1$ is on at most $2^2 + 1 = 5$ qubits.

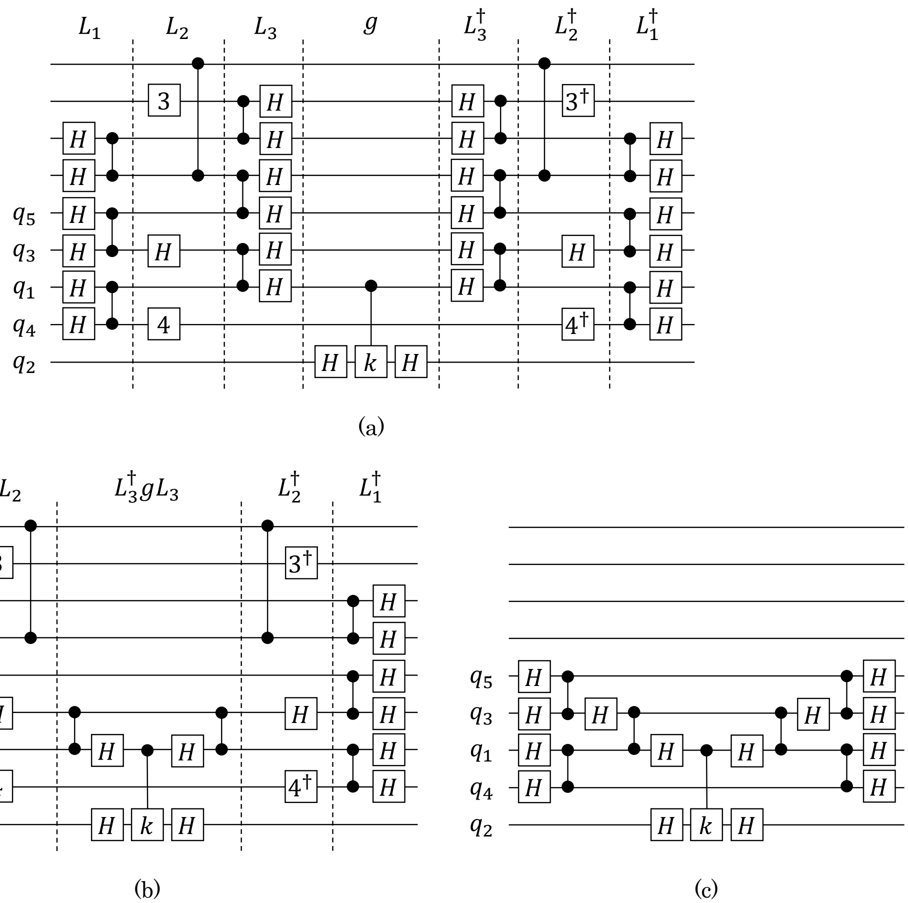


Figure 6: (a) Gate $A_n^\dagger g A_n$, where A_n is the circuit in Fig. 3(c), g is a controlled-$R(2\pi/2^k)$ gate sandwiched between H gates, and q_1 is the second qubit of A_n from the bottom. (b) The circuit obtained from $A_n^\dagger g A_n$ in (a) by simplifying $L_3^\dagger g L_3$. (c) The circuit on five qubits obtained from (b).

The analysis for Case 3 also works for the case when $g = \Lambda X$ in Step 2 of E_n.

A.3 Proof of Theorem 2

Let $|x\rangle$ be an n-qubit input state, where $x \in \{0,1\}^n$. Moreover, let

$$F_n|x\rangle|0\rangle^{\otimes s} = \sum_{z\in\{0,1\}^t} \alpha_{x,z}|z\rangle|\psi_{x,z}\rangle,$$

where $\alpha_{x,z} \in \mathbb{C}$ and $|\psi_{x,z}\rangle$ is an $(n+s-t)$-qubit state. Then,

$$(F_n^\dagger \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})|x\rangle|0\rangle^{\otimes(s+l)} = \frac{1}{\sqrt{2^l}}(F_n^\dagger \otimes H^{\otimes l}) \sum_{z\in\{0,1\}^t, w\in\{0,1\}^l} \alpha_{x,z}D|z\rangle|\psi_{x,z}\rangle|w\rangle.$$

Since D consists only of gates that are diagonal in the Z-basis, $D|z\rangle|w\rangle = e^{if(z,w)}|z\rangle|w\rangle$ for some value $f(z,w)$ computed from the diagonal elements of D. Thus, the above state is

$$\frac{1}{\sqrt{2^l}}(F_n^\dagger \otimes H^{\otimes l}) \sum_{z\in\{0,1\}^t, w\in\{0,1\}^l} \alpha_{x,z}e^{if(z,w)}|z\rangle|\psi_{x,z}\rangle|w\rangle.$$

Thus, for any $y \in \{0,1\}^l$, the probability that $(F_n^\dagger \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})$ outputs y, which is represented as

$$\Pr[(F_n^\dagger \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})(x) = y],$$

is computed as

$$\begin{aligned}
&\frac{1}{2^l} \sum_{z,z' \in \{0,1\}^t, w,w' \in \{0,1\}^l} \alpha_{x,z'}^\dagger \alpha_{x,z} e^{-if(z',w')+if(z,w)} \langle z'|z\rangle \langle \psi_{x,z'}|\psi_{x,z}\rangle \langle w'|H^{\otimes l}|y\rangle\langle y|H^{\otimes l}|w\rangle \\
&= \sum_{z \in \{0,1\}^t} |\alpha_{x,z}|^2 \cdot \frac{1}{2^l} \sum_{w,w' \in \{0,1\}^l} e^{-if(z,w')+if(z,w)} \langle w'|H^{\otimes l}|y\rangle\langle y|H^{\otimes l}|w\rangle.
\end{aligned}$$

Let $p(n)$ be an arbitrary polynomial. By the assumption, there exists a polynomial-size randomized classical circuit R_n such that, for any $x \in \{0,1\}^n$ and $z \in \{0,1\}^t$,

$$|\Pr[R_n(x) = z] - \Pr[F_n(x) = z]| = |\Pr[R_n(x) = z] - |\alpha_{x,z}|^2| \leq \frac{1}{2^{p(n)+t}}.$$

We consider a polynomial-size randomized classical circuit T_n that implements the following classical algorithm for generating the probability distribution

$$\{(y, \Pr[(F_n^\dagger \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})(x) = y])\}_{y \in \{0,1\}^l},$$

where the input is $x \in \{0,1\}^n$:

1. Compute $z_0 = R_n(x) \in \{0,1\}^t$.
2. Compute the probability that Z-measurements on the state
$$\frac{1}{\sqrt{2^l}} \sum_{w \in \{0,1\}^l} e^{if(z_0,w)} H^{\otimes l}|w\rangle$$
output y for any $y \in \{0,1\}^l$.
3. Output $y \in \{0,1\}^l$ according to the probability distribution computed in Step 2.

The probability in Step 2 is represented as

$$\frac{1}{2^l} \sum_{w,w' \in \{0,1\}^l} e^{-if(z_0,w')+if(z_0,w)} \langle w'|H^{\otimes l}|y\rangle\langle y|H^{\otimes l}|w\rangle.$$

We can compute $f(z_0, w)$ using a polynomial-size classical circuit since D has polynomially many gates g and it is easy to classically compute $\gamma_g \in \mathbb{C}$ such that $g|z_0\rangle|w\rangle = \gamma_g|z_0\rangle|w\rangle$ by using the classical description of D, which includes information about the complex numbers defining g and the qubit numbers on which g is applied. Moreover, since the state in Step 2 is only on $l = O(\log n)$ qubits, we can compute the probability in Step 2 up to an exponentially small additive error using a polynomial-size classical circuit. In the following, for simplicity, we assume that we can compute the probability exactly. Then, for any $y \in \{0,1\}^l$,

$$\Pr[T_n(x) = y] = \sum_{z_0 \in \{0,1\}^t} \Pr[R_n(x) = z_0] \cdot \frac{1}{2^l} \sum_{w,w' \in \{0,1\}^l} e^{-if(z_0,w')+if(z_0,w)} \langle w'|H^{\otimes l}|y\rangle\langle y|H^{\otimes l}|w\rangle.$$

This implies that, for any $x \in \{0,1\}^n$ and $y \in \{0,1\}^l$,

$$\begin{aligned}
|\Pr[T_n(x) = y] - \Pr[(F_n^\dagger \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})(x) = y]| &\leq \sum_{z_0 \in \{0,1\}^t} |\Pr[R_n(x) = z_0] - |\alpha_{x,z_0}|^2| \\
&\leq \frac{2^t}{2^{p(n)+t}} = \frac{1}{2^{p(n)}}.
\end{aligned}$$

A similar argument works when we compute the probability in Step 2 up to an exponentially small additive error. Thus, $(F_n^\dagger \otimes H^{\otimes l})D(F_n \otimes H^{\otimes l})$ is weakly simulatable.

A.4 Proof of Lemma 4

Let $C_n = G_N \cdots G_1$ be a Clifford circuit with n input qubits, $a = O(\mathrm{poly}(n))$ ancillary qubits, and $l = O(\log n)$ output qubits, where $N = O(\mathrm{poly}(n))$ and G_j is H, P, or ΛZ. For any $x = x_1 \cdots x_n \in \{0,1\}^n$, let $|\psi_x\rangle = |x_1\rangle \cdots |x_n\rangle|\psi_1\rangle \cdots |\psi_a\rangle$ be an input state, where $|\psi_j\rangle$ is a one-qubit state. For any $y \in \{0,1\}^l$,

$$\Pr[C_n(x) = y] = \langle\psi_x|C_n^\dagger|y\rangle\langle y|C_n|\psi_x\rangle = \langle\psi_x|C_n^\dagger X_y|0\rangle^{\otimes l}\langle 0|^{\otimes l} X_y C_n|\psi_x\rangle,$$

where X_y is the tensor product of X and I such that $X_y|0\rangle^{\otimes l} = |y\rangle$. As described in [12], it can be shown by induction on l that

$$|0\rangle^{\otimes l}\langle 0|^{\otimes l} = \frac{1}{2^l}\sum_{S\subseteq\{1,\ldots,l\}} Z(S),$$

where $Z(S)$ is the tensor product of Z and I such that Z is only on qubit $j \in S$. Thus,

$$\Pr[C_n(x) = y] = \frac{1}{2^l}\sum_{S\subseteq\{1,\ldots,l\}} \langle\psi_x|G_1^\dagger \cdots G_N^\dagger X_y Z(S) X_y G_N \cdots G_1|\psi_x\rangle.$$

We can represent $G_N^\dagger X_y Z(S) X_y G_N$ as a tensor product of Pauli gates with some coefficient ± 1 since G_N is a Clifford gate and $X_y Z(S) X_y$ is a tensor product of Pauli gates (in fact, Z and I gates with some coefficient ± 1). We repeat this transformation N times and obtain

$$\begin{aligned}\Pr[C_n(x) = y] &= \frac{1}{2^l}\sum_{S\subseteq\{1,\ldots,l\}} \gamma^S \langle\psi_x|P_1^S \otimes \cdots \otimes P_{n+a}^S|\psi_x\rangle \\ &= \frac{1}{2^l}\sum_{S\subseteq\{1,\ldots,l\}} \gamma^S \langle x_1|P_1^S|x_1\rangle \cdots \langle x_n|P_n^S|x_n\rangle\langle\psi_1|P_{n+1}^S|\psi_1\rangle \cdots \langle\psi_a|P_{n+a}^S|\psi_a\rangle\end{aligned}$$

for some coefficient γ^S and Pauli gates P_j^S. It is easy to construct a polynomial-time classical algorithm for obtaining γ^S and P_j^S for any $S \subseteq \{1,\ldots,l\}$. Moreover, since $l = O(\log n)$, it suffices to consider only polynomially many S. Thus, the above representation immediately implies a polynomial-time classical algorithm for computing $\Pr[C_n(x) = y]$. The marginal output probability can also be computed similarly and thus C_n is strongly simulatable.