# Reconfiguration on sparse graphs 

Daniel Lokshtanov ${ }^{1}$, Amer E. Mouawad ${ }^{2}$, Fahad Panolan ${ }^{3}$, M.S. Ramanujan ${ }^{1}$, and Saket Saurabh ${ }^{3}$<br>${ }^{1}$ University of Bergen, Norway.<br>daniello,ramanujan.sridharan@ii.uib.no<br>2 David R. Cheriton School of Computer Science<br>University of Waterloo, Ontario, Canada.<br>aabdomou@uwaterloo.ca<br>${ }^{3}$ Institute of Mathematical Sciences<br>Chennai, India.<br>fahad,saket@imsc.res.in


#### Abstract

A vertex-subset graph problem $\mathcal{Q}$ defines which subsets of the vertices of an input graph are feasible solutions. A reconfiguration variant of a vertex-subset problem asks, given two feasible solutions $S_{s}$ and $S_{t}$ of size $k$, whether it is possible to transform $S_{s}$ into $S_{t}$ by a sequence of vertex additions and deletions such that each intermediate set is also a feasible solution of size bounded by $k$. We study reconfiguration variants of two classical vertex-subset problems, namely Independent Set and Dominating Set. We denote the former by ISR and the latter by DSR. Both ISR and DSR are PSPACE-complete on graphs of bounded bandwidth and W[1]-hard parameterized by $k$ on general graphs. We show that ISR is fixed-parameter tractable parameterized by $k$ when the input graph is of bounded degeneracy or nowhere-dense. As a corollary, we answer positively an open question concerning the parameterized complexity of the problem on graphs of bounded treewidth. Moreover, our techniques generalize recent results showing that ISR is fixed-parameter tractable on planar graphs and graphs of bounded degree. For DSR, we show the problem fixed-parameter tractable parameterized by $k$ when the input graph does not contain large bicliques, a class of graphs which includes graphs of bounded degeneracy and nowhere-dense graphs.


## 1 Introduction

Given an $n$-vertex graph $G$ and two vertices $s$ and $t$ in $G$, determining whether there exists a path and computing the length of the shortest path between $s$ and $t$ are two of the most fundamental graph problems. In the classical battle of P versus NP or "easy" versus "hard", both of these problems are on the easy side. That is, they can be solved in $\operatorname{poly}(n)$ time, where poly is any polynomial function. But what if our input consisted of a $2^{n}$-vertex graph? Of course, we can no longer assume $G$ to be part of the input, as reading the input alone requires more than poly $(n)$ time. Instead, we are given an oracle encoded using poly $(n)$ bits and that can, in constant or $\operatorname{poly}(n)$ time, answer queries of the form "is $u$ a vertex in $G$ " or "is there an edge between $u$ and $v$ ?". Given such an oracle and two vertices of the $2^{n}$-vertex graph, can we still determine if there is a path or compute the length of the shortest path between $s$ and $t$ in $\operatorname{poly}(n)$ time?

A slightly different, but equally insightful, formulation of the question above is as follows. Given a set $S$ of $n$ objects, consider the graph $R(S)$ which contains one node for each set in the power set of $S, 2^{S}$, and two nodes are adjacent in $R(S)$ whenever the size of their symmetric difference is equal to one. Clearly, this graph contains $2^{n}$ nodes and can be easily encoded in $\operatorname{poly}(n)$ bits using the oracle described above. It is not hard to see that there exists a path between any two nodes of $R(S)$. Moreover, computing the length of a shortest path can be accomplished in constant time; it is equal to the size of the symmetric difference of the two underlying sets. If the node set of $R(S)$ were instead restricted to a subset of $2^{S}$, both of our problems can become NP-complete or even PSPACE-complete. Therefore, another interesting question is whether we can determine what types of "restriction" on the node set of $R(S)$ induce such variations in the complexity of the two problems.

These two seemingly artificial questions are in fact quite natural and appear in many practical and theoretical problems. In particular, these are exactly the types of questions asked under the reconfiguration
framework, the main subject of this work. Under the reconfiguration framework, instead of finding a feasible solution to some instance $\mathcal{I}$ of a search problem $\mathcal{Q}$, we are interested in structural and algorithmic questions related to the solution space of $\mathcal{Q}$. Naturally, given some adjacency relation $\mathcal{A}$ defined over feasible solutions of $\mathcal{Q}$, size of the symmetric difference being one such relation, the solution space can be represented using a graph $R_{\mathcal{Q}}(\mathcal{I}) . R_{\mathcal{Q}}(\mathcal{I})$ contains one node for each feasible solution of $\mathcal{Q}$ on instance $\mathcal{I}$ and two nodes share an edge whenever their corresponding solutions are adjacent under $\mathcal{A}$. An edge in $R_{\mathcal{Q}}(\mathcal{I})$ corresponds to a reconfiguration step, a walk in $R_{\mathcal{Q}}(\mathcal{I})$ is a sequence of such steps, a reconfiguration sequence, and $R_{\mathcal{Q}}(\mathcal{I})$ is a reconfiguration graph.

Studying problems related to reconfiguration graphs has received considerable attention in recent literature $[4,22,25,26,30,34]$, the most popular problem being to determine whether there exists a reconfiguration sequence between two given feasible solution. In most cases, this problem was shown PSPACE-hard in general, although some polynomial-time solvable restricted cases have been identified. For PSPACE-hard cases, it is not surprising that shortest paths between solutions can have exponential length. More surprising is that for most known polynomial-time solvable cases the diameter of the reconfiguration graph has been shown to be polynomial. Some of the problems that have been studied under the reconfiguration framework include Independent Set [31], Vertex Cover [33], Shortest Path [5, 30], Coloring [3, 6, 7, 9-11, 29], and Boolean Satisfiability [22]. We refer the reader to the recent survey by Van den Heuvel [43] for a detailed overview. Recently, a systematic study of the parameterized complexity of reconfiguration problems was initiated by Mouawad et al. [34]; various problems were identified where the problem was not only NP-hard (or PSPACE-hard), but also W-hard under various parameterizations.

Overview of our results. In this work, we focus on reconfiguration variants of the Independent Set (IS) and Dominating Set (DS) problems. Given two independent sets $I_{s}$ and $I_{t}$ of a graph $G$ such that $\left|I_{s}\right|=\left|I_{t}\right|=k$, the Independent Set Reconfiguration problem asks whether there exists a sequence of independents sets $\sigma=\left\langle I_{0}, I_{1}, \ldots, I_{\ell}\right\rangle$, for some $\ell$, such that:
(1) $I_{0}=I_{s}$ and $I_{\ell}=I_{t}$,
(2) $I_{i}$ is an independent set of $G$ for all $0 \leq i \leq \ell$,
(3) $\left|I_{i} \Delta I_{i+1}\right|=1$ for all $0 \leq i<\ell$, and
(4) $k-1 \leq\left|S_{i}\right| \leq k$ for all $0 \leq i \leq \ell$.

Alternatively, given a graph $G$ and integer $k$, the reconfiguration graph $R_{\mathrm{IS}}(G, k-1, k)$ has a node for each independent set of $G$ of size $k$ or $k-1$ and two nodes are adjacent in $R_{\mathrm{IS}}(G, k-1, k)$ whenever the corresponding independent sets can be obtained from one another by either the addition or the deletion of a single vertex. The reconfiguration graph $R_{\mathrm{DS}}(G, k, k+1)$ is defined similarly for dominating sets. Hence, ISR and DSR can be formally stated as follows:

Independent Set Reconfiguration (ISR)
Input: $\quad$ Graph $G$, positive integer $k$, and two $k$-independent sets $I_{s}$ and $I_{t}$
Question: Is there a path from $I_{s}$ to $I_{t}$ in $R_{\mathrm{IS}}(G, k-1, k)$ ?

Dominating Set Reconfiguration (DSR)
Input: $\quad$ Graph $G$, positive integer $k$, and two $k$-dominating sets $D_{s}$ and $D_{t}$
Question: Is there a path from $D_{s}$ to $D_{t}$ in $R_{\mathrm{DS}}(G, k, k+1)$ ?
Note that since we only allow independent sets of size $k$ and $k-1$ the ISR problem is equivalent to reconfiguration under the token jumping model considered by Ito et al. [27, 28]. ISR is known to be PSPACEcomplete on graphs of bounded bandwidth [35,44] (hence pathwidth and treewidth) and W[1]-hard on general graphs [28]. On the positive side, the problem was shown fixed-parameter tractable, with parameter $k$, for graphs of bounded degree, planar graphs, and graphs excluding $K_{3, d}$ as a (not necessarily induced) subgraph, for any constant $d[27,28]$. We push this boundary further by showing that the problem remains fixed-parameter tractable for graphs of bounded degeneracy and nowhere-dense graphs (Figure 1). As a
corollary, we answer positively an open question concerning the parameterized complexity of the problem (parameterized by $k$ ) on graphs of bounded treewidth.

For DSR, we first show that the problem is W[1]-hard on general graphs by adapting the well-known (parameter-preserving) reduction from Independent Set to Dominating Set. Then, we show that the problem is fixed-parameter tractable, with parameter $k$, for graphs excluding $K_{d, d}$ as a (not necessarily induced) subgraph, for any constant $d$. Note that this class of graphs includes both nowhere-dense and bounded degeneracy graphs and is the "largest" class on which the Dominating Set problem is known to be in FPT [40, 42].

Clearly, our main open question is whether ISR remains fixed-parameter tractable on graphs excluding $K_{d, d}$ as a subgraph. Intuitively, all of the classes we consider fall under the category of "sparse" graph classes. Hence, in some sense, one would not expect a sparse graph to have "too many" dominating sets of fixed small size $k$ as $n$ becomes larger and larger. For independent sets, the situation is reversed. As $n$ grows larger, so does the number of independent sets of fixed size $k$. So it remains to be seen whether some structural properties of graphs excluding $K_{d, d}$ as a subgraph can be used to settle our open question or whether the problem becomes $\mathrm{W}[1]$-hard. In the latter case, this would be the first example of a $\mathrm{W}[1]$-hard problem (in general), which is in FPT on a class $\mathcal{C}$ of graphs but where the reconfiguration version is not; finding such a problem, we believe, is interesting in its own right. Another open question is whether we can adapt our results for ISR to find shortest reconfiguration sequences. Our algorithm for DSR does in fact guarantee shortest reconfiguration sequences but, as we shall see, the same does not hold for both ISR algorithms.

## 2 Preliminaries

For an in-depth review of general graph theoretic definitions we refer the reader to the book of Diestel [16]. Unless otherwise stated, we assume that each graph $G$ is a simple, undirected graph with vertex set $V(G)$ and edge set $E(G)$, where $|V(G)|=n$ and $|E(G)|=m$. The open neighborhood, or simply neighborhood, of a vertex $v$ is denoted by $N_{G}(v)=\{u \mid u v \in E(G)\}$, the closed neighborhood by $N_{G}[v]=N_{G}(v) \cup\{v\}$. Similarly, for a set of vertices $S \subseteq V(G)$, we define $N_{G}(S)=\{v \mid u v \in E(G), u \in S, v \notin S\}$ and $N_{G}[S]=N_{G}(S) \cup S$. The degree of a vertex is $\left|N_{G}(v)\right|$. We drop the subscript $G$ when clear from context. A subgraph of $G$ is a graph $G^{\prime}$ such that $V\left(G^{\prime}\right) \subseteq V(G)$ and $E\left(G^{\prime}\right) \subseteq E(G)$. The induced subgraph of $G$ with respect to $S \subseteq V(G)$ is denoted by $G[S] ; G[S]$ has vertex set $S$ and edge set $\{u v \in E(G[S]) \mid u, v \in S, u v \in E(G)\}$. We denote by $\Delta(G)$ and $\delta(G)$ the maximum and minimum degree of $G$, respectively.

A walk of length $\ell$ from $v_{0}$ to $v_{\ell}$ in $G$ is a vertex sequence $v_{0}, \ldots, v_{\ell}$, such that for all $i \in\{0, \ldots, \ell-1\}$, $v_{i} v_{i+1} \in E(G)$. It is a path if all vertices are distinct. It is a cycle if $\ell \geq 3, v_{0}=v_{\ell}$, and $v_{0}, \ldots, v_{\ell-1}$ is a path. A path from vertex $u$ to vertex $v$ is also called a uv-path. The distance between two vertices $u$ and $v$ of $G$, $\operatorname{dist}_{G}(u, v)$, is the length of a shortest $u v$-path in $G$ (positive infinity if no such path exists). The eccentricity of a vertex $v \in V(G), \operatorname{ecc}(v)$, is equal to $\max _{u \in V(G)}\left(\operatorname{dist}_{G}(u, v)\right)$. The radius of $G, \operatorname{rad}(G)$, is equal to $\min _{v \in V(G)}(\operatorname{ecc}(v))$. The diameter of $G$, $\operatorname{diam}(G)$, is equal to $\max _{v \in V(G)}(\operatorname{ecc}(v))$. For $r \geq 0$, the $r$-neighborhood of a vertex $v \in V(G)$ is defined as $N_{G}^{r}[v]=\left\{u \mid \operatorname{dist}_{G}(u, v) \leq r\right\}$. We write $B(v, r)=N_{G}^{r}[v]$ and call it a ball of radius $r$ around $v$; for $S \subseteq V(G), B(S, r)=\bigcup_{v \in S} N_{G}^{r}[v]$.

Contracting an edge $u v$ of $G$ results in a new graph $H$ in which the vertices $u$ and $v$ are deleted and replaced by a new vertex $w$ that is adjacent to $N_{G}(u) \cup N_{G}(v) \backslash\{u, v\}$. If a graph $H$ can be obtained from $G$ by repeatedly contracting edges, $H$ is said to be a contraction of $G$. If $H$ is a subgraph of a contraction of $G$, then $H$ is said to be a minor of $G$, denoted by $H \preceq_{m} G$. An equivalent characterization of minors states that $H$ is a minor of $G$ if there is a map that associates to each vertex $v$ of $H$ a non-empty connected subgraph $G_{v}$ of $G$ such that $G_{u}$ and $G_{v}$ are disjoint for $u \neq v$ and whenever there is an edge between $u$ and $v$ in $H$ there is an edge in $G$ between some node in $G_{u}$ and some node in $G_{v}$. The subgraphs $G_{v}$ are called branch sets. $H$ is a minor at depth $r$ of $G, H \preceq_{m}^{r} G$, if $H$ is a minor of $G$ which is witnessed by a collection of branch sets $\left\{G_{v} \mid v \in V(H)\right\}$, each of which induces a graph of radius at most $r$. That is, for each $v \in V(H)$, there is a $w \in V\left(G_{v}\right)$ such that $V\left(G_{v}\right) \subseteq N_{G_{v}}^{r}[w]$.

Sparse graph classes. We define the three main classes we consider. Figure 1 illustrates the relationship between these classes and some other well-known classes of sparse graphs. We refer the reader to $[8,38,36]$ for more details.

Definition $1([38,36])$. A class of graphs $\mathcal{C}$ is said to be nowhere-dense if for every $d \geq 0$ there exists $a$ graph $H_{d}$ such that $H_{d} \not \nwarrow_{m}^{d} G$ for all $G \in \mathcal{C}$. $\mathcal{C}$ is effectively nowhere-dense if the $\operatorname{map} d \mapsto H_{d}$ is computable. Otherwise, $\mathcal{C}$ is said to be somewhere-dense.

Nowhere-dense classes of graphs were introduced by Nesetril and Ossona de Mendez [38, 36] and "nowheredensity" turns out to be a very robust concept with several natural characterizations [23]. We use one such characterization in Section 3.2. It follows from the definition that planar graphs, graphs of bounded treewidth, graphs of bounded degree, $H$-minor-free graphs, and $H$-topological-minor-free graphs are nowhere-dense [38, 36]. As in the work of Dawar and Kreutzer [14], we are only interested in effectively nowhere-dense classes; all natural nowhere-dense classes are effectively nowhere-dense, but it is possible to construct artificial classes that are nowhere-dense, but not effectively so.

Definition 2. A class of graphs $\mathcal{C}$ is said to be $d$-degenerate if there is an integer $d$ such that every induced subgraph of any graph $G \in \mathcal{C}$ has a vertex of degree at most $d$.

Graphs of bounded degeneracy and nowhere-dense graphs are incomparable [24]. In other words, graphs of bounded degeneracy are somewhere-dense.

Proposition 1 ([32]). The number of edges in a d-degenerate graph is at most $d n$ and hence its average degree is at most $2 d$.

Degeneracy is a hereditary property, hence any induced subgraph of a $d$-degenerate graph is also $d$ degenerate. It is well-known that graphs of treewidth at most $d$ are also $d$-degenerate. Moreover a $d$-degenerate graph cannot contain $K_{d+1, d+1}$ as a subgraph, which brings us to the class of biclique-free graphs. The relationship between bounded degeneracy, nowhere-dense, and $K_{d, d}$-free graphs was shown by Philip et al. and Telle and Villanger [40,42].

Definition 3. A class of graphs $\mathcal{C}$ is said to be $d$-biclique-free, for some $d>0$, if $K_{d, d}$ is not a subgraph of any $G \in \mathcal{C}$, and it is said to be biclique-free if it is d-biclique-free for some $d$.

Proposition $2([40,42])$. Any degenerate or nowhere-dense class of graphs is biclique-free, but not viceversa.


Fig. 1. Sparse graph classes $[8,38,36]$. Arrows indicate inclusion.

Parameterized complexity. Using the framework developed by Downey and Fellows [17], a parameterized problem includes in the input a parameter $p$. For a parameterized problem $\mathcal{Q}$ with inputs of the form $(x, p)$, $|x|=n$ and $p$ a positive integer, $\mathcal{Q}$ is fixed-parameter tractable (or in FPT) if it can be decided in $f(p) n^{c}$ time, where $f$ is an arbitrary function and $c$ is a constant independent of both $n$ and $p . \mathcal{Q}$ is in the class XP if it can be decided in $n^{f(p)}$ time. $\mathcal{Q}$ has a kernel of size $f(p)$ if there is an algorithm that transforms the input $(x, p)$ to $\left(x^{\prime}, p^{\prime}\right)$ in polynomial time (with respect to $|x|$ and $p$ ) such that $(x, p)$ is a yes-instance if and only if $\left(x^{\prime}, p^{\prime}\right)$ is a yes-instance, $p^{\prime} \leq g(p)$, and $\left|x^{\prime}\right| \leq f(p)$. Each problem in FPT has a kernel, possibly of exponential (or worse) size [17].

In order to distinguish between parameterized problems solvable in $n^{f(p)}$ time and parameterized problems solvable in $f(p) n^{c}$ time, Downey and Fellows [17] introduced the W-hierarchy. The hierarchy consists of a complexity class $\mathrm{W}[\mathrm{t}]$ for every integer $t \geq 1$ such that $\mathrm{W}[\mathrm{t}] \subseteq \mathrm{W}[\mathrm{t}+1]$ for all $t$. They proved that FPT $\subseteq$ $\mathrm{W}[1] \subseteq \mathrm{W}[2] \subseteq \ldots \subseteq \mathrm{W}[\mathrm{t}]$ and conjectured that strict containment holds. In particular, the assumption FPT $\subset \mathrm{W}[1]$ is a natural parameterized analogue of the conjecture that $\mathrm{P} \neq \mathrm{NP}$. Moreover, Downey and Fellows showed that the Independent Set problem parameterized by solution size is $W$ [1]-complete and the Dominating Set problem parameterized by solution size is W[2]-complete. Showing hardness in the parameterized setting is usually accomplished using FPT reductions. The reader is referred to the books of Niedermeier, Flum, and Grohe for more on parameterized complexity [21, 39].

Reconfiguration. For any vertex-subset problem $\mathcal{Q}$, graph $G$, and positive integer $k$, we consider the reconfiguration graph $R_{\mathcal{Q}}(G, k, k+1)$ when $\mathcal{Q}$ is a minimization problem (e.g. Dominating $\operatorname{Set}$ ) and the reconfiguration graph $R_{\mathcal{Q}}(G, k-1, k)$ when $\mathcal{Q}$ is a maximization problem (e.g. Independent Set). A set $S \subseteq V(G)$ has a corresponding node in $V\left(R_{\mathcal{Q}}\left(G, r_{l}, r_{u}\right)\right), r_{l} \in\{k-1, k\}$ and $r_{u} \in\{k, k+1\}$, if and only if $S$ is a feasible solution for $\mathcal{Q}$ and $r_{l} \leq|S| \leq r_{u}$. We refer to vertices in $G$ using lower case letters (e.g. $u, v$ ) and to the nodes in $R_{\mathcal{Q}}\left(G, r_{l}, r_{u}\right)$, and by extension their associated feasible solutions, using upper case letters (e.g. $A, B)$. If $A, B \in V\left(R_{\mathcal{Q}}\left(G, r_{l}, r_{u}\right)\right)$ then there exists an edge between $A$ and $B$ in $R_{\mathcal{Q}}\left(G, r_{l}, r_{u}\right)$ if and only if there exists a vertex $u \in V(G)$ such that $\{A \backslash B\} \cup\{B \backslash A\}=\{u\}$. Equivalently, for $A \Delta B=\{A \backslash B\} \cup\{B \backslash A\}$ the symmetric difference of $A$ and $B, A$ and $B$ share an edge in $R_{\mathcal{Q}}\left(G, r_{l}, r_{u}\right)$ if and only if $|A \Delta B|=1$.

We write $A \leftrightarrow B$ if there exists a path in $R_{\mathcal{Q}}\left(G, r_{l}, r_{u}\right)$, a reconfiguration sequence, joining $A$ and $B$. Any reconfiguration sequence from source feasible solution $S_{s}$ to target feasible solution $S_{t}$, which we sometimes denote by $\sigma=\left\langle S_{0}, S_{1}, \ldots, S_{\ell}\right\rangle$, for some $\ell$, has the following properties:

- $S_{0}=S_{s}$ and $S_{\ell}=S_{t}$,
- $S_{i}$ is a feasible solution for $\mathcal{Q}$ for all $0 \leq i \leq \ell$,
- $\left|S_{i} \Delta S_{i+1}\right|=1$ for all $0 \leq i<\ell$, and
- $r_{l} \leq\left|S_{i}\right| \leq r_{u}$ for all $0 \leq i \leq \ell$.

We denote the length of $\sigma$ by $|\sigma|$. For $0<i \leq \ell$, we say vertex $v \in V(G)$ is added at step/index/position/slot $i$ if $v \notin S_{i-1}$ and $v \in S_{i}$. Similarly, a vertex $v$ is removed at step/index/position/slot $i$ if $v \in S_{i-1}$ and $v \notin S_{i}$. A vertex $v \in V(G)$ is touched in the course of a reconfiguration sequence if $v$ is either added or removed at least once; it is untouched otherwise. A vertex is removable (addable) from feasible solution $S$ if $S \backslash\{v\}(S \cup\{v\})$ is also a feasible solution for $\mathcal{Q}$. For any pair of consecutive solutions ( $S_{i-1}, S_{i}$ ) in $\sigma$, we say $S_{i}\left(S_{i-1}\right)$ is the successor (predecessor) of $S_{i-1}\left(S_{i}\right)$. A reconfiguration sequence $\sigma^{\prime}=\left\langle S_{0}, S_{1}, \ldots, S_{\ell^{\prime}}\right\rangle$ is a prefix of $\sigma=\left\langle S_{0}, S_{1}, \ldots, S_{\ell}\right\rangle$ if $\ell^{\prime}<\ell$.

We adapt the concept of irrelevant vertices from parameterized complexity to introduce the notions of irrelevant and strongly irrelevant vertices for reconfiguration. Since these notions apply to almost any reconfiguration problem, we give general definitions.

Definition 4. For any vertex-subset problem $\mathcal{Q}$, n-vertex graph $G$, positive integers $r_{l}$ and $r_{u}$, and $S_{s}, S_{t} \in$ $V\left(R_{\mathcal{Q}}\left(G, r_{l}, r_{u}\right)\right)$ such that there exists a reconfiguration sequence from $S_{s}$ to $S_{t}$ in $R_{\mathcal{Q}}\left(G, r_{l}, r_{u}\right)$, we say a vertex $v \in V(G)$ is irrelevant (with respect to $S_{s}$ and $S_{t}$ ) if and only if $v \notin S_{s} \cup S_{t}$ and there exists a reconfiguration sequence from $S_{s}$ to $S_{t}$ in $R_{\mathcal{Q}}\left(G, r_{l}, r_{u}\right)$ which does not touch $v$. We say $v$ is strongly irrelevant (with respect to $S_{s}$ and $S_{t}$ ) if it is irrelevant and the length of a shortest reconfiguration sequence
from $S_{s}$ to $S_{t}$ which does not touch $v$ is no greater than the length of a shortest reconfiguration sequence which does (if the latter sequence exists).

At a high level, it is enough to consider irrelevant vertices when trying to find any reconfiguration sequence between two feasible solutions, but strongly irrelevant vertices must be considered if we wish to find a shortest reconfiguration sequence. As we shall see, our algorithm for DSR does in fact find strongly irrelevant vertices and can therefore be used to find shortest reconfiguration sequences. For ISR, we are only able to find irrelevant vertices and reconfiguration sequences are not guaranteed to be of shortest possible length.

## 3 Independent set reconfiguration

### 3.1 Graphs of bounded degeneracy

To show that the ISR problem is fixed-parameter tractable on $d$-degenerate graphs, for some integer $d$, we will proceed in two stages. In the first stage, we will show, for an instance ( $G, I_{s}, I_{t}, k$ ), that as long as the number of low-degree vertices in $G$ is "large enough" we can find an irrelevant vertex (Definition 4). Once the number of low-degree vertices is bounded, a simple counting argument (Proposition 3) shows that the size of the remaining graph is also bounded and hence we can solve the instance by exhaustive enumeration.

Proposition 3. Let $G$ be an n-vertex d-degenerate graph, $S_{1} \subseteq V(G)$ be the set of vertices of degree at most $2 d$, and $S_{2}=V(G) \backslash S_{1}$. If $\left|S_{1}\right|<s$, then $|V(G)| \leq(2 d+1) s$.

Proof. The number of edges in a $d$-degenerate graph is at most $d n$ and hence its average degree is at most $2 d$ (Proposition 1). If $|V(G)|=(2 d+1) s+c$, for $c \geq 1$, then $\left|S_{2}\right|=\left|V(G) \backslash S_{1}\right|>2 d s+c, \sum_{v \in S_{2}}\left|N_{G}(v)\right|>$ $(2 d s+c)(2 d+1)$, and we obtain the following contradiction:

$$
\begin{aligned}
\frac{\sum_{v \in S_{1}}\left|N_{G}(v)\right|+\sum_{v \in S_{2}}\left|N_{G}(v)\right|}{|V(G)|} & >\frac{(2 d s+c)(2 d+1)}{(2 d+1) s+c} \\
& =\frac{4 d^{2} s+2 d s+2 d c+c}{(2 d+1) s+c} \\
& =\frac{2 d(2 d s+s+c)+c}{2 d s+s+c}>2 d .
\end{aligned}
$$

To find irrelevant vertices, we make use of the following classical result of Erdõs and Rado [20], also known in the literature as the sunflower lemma. We first define the terminology used in the statement of the theorem. A sunflower with $k$ petals and a core $Y$ is a collection of sets $S_{1}, \ldots, S_{k}$ such that $S_{i} \cap S_{j}=Y$ for all $i \neq j$; the sets $S_{i} \backslash Y$ are petals and we require none of them to be empty. Note that a family of pairwise disjoint sets is a sunflower (with an empty core).

Theorem 1 (Sunflower Lemma [20]). Let $\mathcal{A}$ be a family of sets (without duplicates) over a universe $\mathcal{U}$, such that each set in $\mathcal{A}$ has cardinality at most $d$. If $|A|>d!(k-1)^{d}$, then $\mathcal{A}$ contains a sunflower with $k$ petals and such a sunflower can be computed in time polynomial in $|\mathcal{A}|,|\mathcal{U}|$, and $k$.

Lemma 1. Let $\left(G, I_{s}, I_{t}, k\right)$ be an instance of ISR where $G$ is d-degenerate and let $B$ be the set of vertices in $V(G) \backslash\left\{I_{s} \cup I_{t}\right\}$ of degree at most $2 d$. If $|B|>(2 d+1)!(2 k-1)^{2 d+1}$, then there exists an irrelevant vertex $v \in V(G) \backslash\left\{I_{s} \cup I_{t}\right\}$ such that $\left(G, I_{s}, I_{t}, k\right)$ is a yes-instance if and only if $\left(G^{\prime}, I_{s}, I_{t}, k\right)$ is a yes-instance, where $G^{\prime}$ is obtained from $G$ by deleting $v$ and all edges incident on $v$.

Proof. Let $b_{1}, b_{2}, \ldots, b_{|B|}$ denote the vertices in $B$ and let $\mathcal{A}=\left\{N_{G}\left[b_{1}\right], N_{G}\left[b_{2}\right], \ldots, N_{G}\left[b_{|B|}\right]\right\}$ denote the family of sets corresponding to the closed neighborhoods of each vertex in $B$ and set $\mathcal{U}=\bigcup_{b \in B} N[b]$. Since
$|B|$ is greater than $(2 d+1)!(2 k-1)^{2 d+1}$, we know from Theorem 1 that $\mathcal{A}$ contains a sunflower with $2 k$ petals and such a sunflower can be computed in time polynomial in $|\mathcal{A}|$ and $k$. Note that we assume, without loss of generality, that there are no two vertices $u$ and $v$ in $V(G) \backslash\left\{I_{s} \cup I_{t}\right\}$ such that $N_{G}[u]=N_{G}[v]$, as we can safely delete one of them from the input graph otherwise, i.e. one of the two is (strongly) irrelevant. Let $v_{i r}$ be a vertex whose closed neighborhood corresponds to one of those $2 k$ petals. We claim that $v_{i r}$ is irrelevant and can therefore be deleted from $G$ to obtain $G^{\prime}$.

To see why, consider any reconfiguration sequence $\sigma=\left\langle I_{s}=I_{0}, I_{1}, \ldots, I_{t}=I_{\ell}\right\rangle$ from $I_{s}$ to $I_{t}$ in $R_{\mathrm{IS}}(G, k-$ $1, k)$. Since $v_{i r} \notin I_{s} \cup I_{t}$, we let $p, 0<p<\ell$, be the first index in $\sigma$ at which $v_{i r}$ is added, i.e. $v_{i r} \in I_{p}$ and $v_{i r} \notin I_{i}$ for all $i<p$. Moreover, we let $q+1, p<q+1 \leq \ell$ be the first index after $p$ at which $v_{i r}$ is removed, i.e. $v_{i r} \in I_{q}$ and $v_{i r} \notin I_{q+1}$. We will consider the subsequence $\sigma_{s}=\left\langle I_{p}, \ldots, I_{q}\right\rangle$ and show how to modify it so that it does not touch $v_{i r}$. Applying the same procedure to every such subsequence in $\sigma$ suffices to prove the lemma.

Since the sunflower constructed to obtain $v_{i r}$ has $2 k$ petals and the size of any independent set in $\sigma$ (or any reconfiguration sequence in general) is at most $k$, there must exist another free vertex $v_{f r}$ whose closed neighborhood corresponds to one of the remaining $2 k-1$ petals which we can add at index $p$ instead of $v_{i r}$, i.e. $v_{f r} \notin N_{G}\left[I_{p}\right]$. We say $v_{f r}$ represents $v_{i r}$. Assume that no such vertex exists. Then we know that either some vertex in the core of the sunflower is in $I_{p}$ contradicting the fact that we are adding $v_{i r}$, or every petal of the sunflower contains a vertex in $I_{p}$, which is not possible since the size of any independent set is at most $k$ and the number of petals is larger. Hence, we first modify the subsequence $\sigma_{s}$ by adding $v_{f r}$ instead of $v_{i r}$. Formally, we have $\sigma_{s}^{\prime}=\left\langle\left(I_{p} \backslash\left\{v_{i r}\right\}\right) \cup\left\{v_{f r}\right\}, \ldots,\left(I_{q} \backslash\left\{v_{i r}\right\}\right) \cup\left\{v_{f r}\right\}\right\rangle$.

To be able to replace $\sigma_{s}$ by $\sigma_{s}^{\prime}$ in $\sigma$ and obtain a reconfiguration sequence from $I_{s}$ to $I_{t}$, then all of the following conditions must hold:
(1) $\left|\left(I_{q} \backslash\left\{v_{i r}\right\}\right) \cup\left\{v_{f r}\right\}\right|=k$.
(2) $\left(I_{i} \backslash\left\{v_{i r}\right\}\right) \cup\left\{v_{f r}\right\}$ is an independent set of $G$ for all $p \leq i \leq q$,
(3) $\left|\left(I_{i} \backslash\left\{v_{i r}\right\}\right) \cup\left\{v_{f r}\right\} \Delta\left(I_{i+1} \backslash\left\{v_{i r}\right\}\right) \cup\left\{v_{f r}\right\}\right|=1$ for all $p \leq i<q$, and
(4) $k-1 \leq\left|\left(I_{i} \backslash\left\{v_{i r}\right\}\right) \cup\left\{v_{f r}\right\}\right| \leq k$ for all $p \leq i \leq q$.

It is not hard to see that if there exists no $i, p<i \leq q$, such that $\sigma_{s}^{\prime}$ adds a vertex in $N\left[v_{f r}\right]$ at position $i$, then all four conditions hold. If there exists such a position, we will modify $\sigma_{s}^{\prime}$ into yet another subsequence $\sigma_{s}^{\prime \prime}$ by finding a new vertex to represent $v_{i r}$. The length of $\sigma_{s}^{\prime \prime}$ will be one greater than the length of $\sigma_{s}^{\prime}$.

We let $i, p<i \leq q$, be the first position in $\sigma_{s}^{\prime}$ at which a vertex in $u \in N\left[v_{f r}\right]$ (possibly equal to $v_{f r}$ ) is added. Using the same arguments discussed to find $v_{f r}$, and since we constructed a sunflower with $2 k$ petals, we can find another vertex $v_{f r}^{\prime}$ such that $N\left[v_{f r}\right] \cap I_{i-1}=\emptyset$. This new vertex will represent $v_{i r}$ instead of $v_{f r}$. We construct $\sigma_{s}^{\prime \prime}$ from $\sigma_{s}^{\prime}$ as follows: $\sigma_{s}^{\prime \prime}=\left\langle I_{p} \backslash\left\{v_{i r}\right\} \cup\left\{v_{f r}\right\}, \ldots, I_{i-1} \backslash\left\{v_{i r}\right\} \cup\left\{v_{f r}\right\}, I_{i-1} \backslash\left\{v_{i r}\right\} \cup\left\{v_{f r}^{\prime}\right\}, I_{i} \backslash\right.$ $\left.\left\{v_{i r}\right\} \cup\left\{v_{f r}^{\prime}\right\}, \ldots, I_{q} \backslash\left\{v_{i r}\right\} \cup\left\{v_{f r}^{\prime}\right\}\right\rangle$. If $\sigma_{s}^{\prime \prime}$ now satisfies all four conditions then we are done. Otherwise, we repeat the same process (which can occur at most $q-p$ times) until we reach such a subsequence.

Theorem 2. ISR on d-degenerate graphs is fixed-parameter tractable parameterized by $k+d$.
Proof. For an instance $\left(G, I_{s}, I_{t}, k\right)$ of ISR, we know from Lemma 1 that as long as $V(G) \backslash\left\{I_{s} \cup I_{t}\right\}$ contains more than $(2 d+1)!(2 k-1)^{2 d+1}$ vertices of degree at most $2 d$ we can find an irrelevant vertex and reduce the size of the graph. After exhaustively reducing the graph to obtain $G^{\prime}$, we known that $G^{\prime}\left[V\left(G^{\prime}\right) \backslash\left\{I_{s} \cup I_{t}\right\}\right]$, which is also $d$-degenerate, has at most $(2 d+1)!(2 k-1)^{2 d+1}$ vertices of degree at most $2 d$. Hence, applying Proposition 3 , we know that $\left|V\left(G^{\prime}\right) \backslash\left\{I_{s} \cup I_{t}\right\}\right| \leq(2 d+1)(2 d+1)!(2 k-1)^{2 d+1}$ and $\left|V\left(G^{\prime}\right)\right| \leq(2 d+1)(2 d+$ $1)!(2 k-1)^{2 d+1}+2 k$.

### 3.2 Nowhere-dense graphs

Nesetril and Ossona de Mendez [37] showed an interesting relationship between nowhere-dense classes and a property of classes of structures introduced by Dawar $[12,13]$ called quasi-wideness. We will use quasiwideness and show a rather interesting relationship between ISR on graphs of bounded degeneracy and nowhere-dense graphs. That is, our algorithm for nowhere-dense graphs will closely mimic the previous
algorithm in the following sense. Instead of using the sunflower lemma to find a large sunflower, we will use quasi-wideness to find a "large enough almost sunflower" with an initially "unknown" core and then use structural properties of the graph to find this core and complete the sunflower. We first state some of the results that we need. Given a graph $G$, a set $S \subseteq V(G)$ is called $r$-scattered if $N_{G}^{r}(u) \cap N_{G}^{r}(v)=\emptyset$ for all distinct $u, v \in S$.

Proposition 4. Let $G$ be a graph and let $S=\left\{s_{1}, s_{2}, \ldots, s_{k}\right\} \subseteq V(G)$ be a 2-scattered set of size $k$ in $G$. Then the closed neighborhoods of the vertices in $S$ form a sunflower with $k$ petals and an empty core.

Definition 5. A class $\mathcal{C}$ of graphs is uniformly quasi-wide with margin $s_{\mathcal{C}}: \mathbb{N} \rightarrow \mathbb{N}$ and $N_{\mathcal{C}}: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ if for all $r, k \in \mathbb{N}$, if $G \in \mathcal{C}$ and $W \subseteq V(G)$ with $|W|>N_{\mathcal{C}}(r, k)$, then there is a set $S \subseteq W$ with $|S|<s_{\mathcal{C}}(r)$, such that $W$ contains an r-scattered set of size at least $k$ in $G[V(G) \backslash S]$. $\mathcal{C}$ is effectively uniformly quasi-wide if $s_{\mathfrak{C}}(r)$ and $N_{\mathcal{C}}(r, k)$ are computable.

Examples of effectively uniformly quasi-wide classes include graphs of bounded degree with margin 1 and $H$-minor-free graphs with margin $|V(H)|-1$.

Theorem 3 ([14]). A class $\mathcal{C}$ of graphs is effectively nowhere-dense if and only if $\mathcal{C}$ is effectively uniformly quasi-wide.

Theorem 4 ([14]). Let $\mathcal{C}$ be an effectively nowhere-dense class of graphs and $h$ be the computable function such that $K_{h(r)} \npreceq m_{r}^{r} G$ for all $G \in \mathcal{C}$. Let $G$ be an n-vertex graph in $\mathcal{C}, r, k \in \mathbb{N}$, and $W \subseteq V(G)$ with $|W| \geq N(h(r), r, k)$, for some computable function $N$. Then in $\mathcal{O}\left(n^{2}\right)$ time, we can compute a set $B \subseteq V(G)$, $|B| \leq h(r)-2$, and a set $A \subseteq W$ such that $|A| \geq k$ and $A$ is an $r$-scattered set in $G[V(G) \backslash B]$.

Lemma 2. Let $\mathcal{C}$ be an effectively nowhere-dense class of graphs and $h$ be the computable function such that $K_{h(r)} \not_{m}^{r} G$ for all $G \in \mathcal{C}$. Let $\left(G, I_{s}, I_{t}, k\right)$ be an instance of ISR where $G \in \mathcal{C}$ and let $R$ be the set of vertices in $V(G) \backslash\left\{I_{s} \cup I_{t}\right\}$. Moreover, let $\mathcal{P}=\left\{P_{1}, P_{2}, \ldots\right\}$ be a family of sets which partitions $R$ such that for any two distinct vertices $u, v \in R, u, v \in P_{i}$ if and only if $N_{G}(u) \cap\left\{I_{s} \cup I_{t}\right\}=N_{G}(v) \cap\left\{I_{s} \cup I_{t}\right\}$. If there exists a set $P_{i} \in \mathcal{P}$ such that $\left|P_{i}\right|>N\left(h(2), 2,2^{h(2)+1} k\right)$, for some computable function $N$, then there exists an irrelevant vertex $v \in V(G) \backslash\left\{I_{s} \cup I_{t}\right\}$ such that $\left(G, I_{s}, I_{t}, k\right)$ is a yes-instance if and only if $\left(G^{\prime}, I_{s}, I_{t}, k\right)$ is a yes-instance, where $G^{\prime}$ is obtained from $G$ by deleting $v$ and all edges incident on $v$.

Proof. By construction, we known that the family $\mathcal{P}$ contains at most $4^{k}$ sets, as we partition $R$ based on their neighborhoods in $I_{s} \cup I_{t}$. Note that some vertices in $R$ have no neighbors in $I_{s} \cup I_{t}$ and will therefore belong to the same set in $\mathcal{P}$.

Assume that there exists a $P \in \mathcal{P}$ such that $|P|>N\left(h(2), 2,2^{h(2)+1} k\right)$. Consider the graph $G[R]$. By Theorem 4, we can, in $\mathcal{O}\left(|R|^{2}\right)$ time, compute a set $B \subseteq R,|B| \leq h(2)-2$, and a set $A \subseteq P$ such that $|A| \geq 2^{h(2)+1} k$ and $A$ is a 2 -scattered set in $G[R \backslash B]$. Now let $\mathcal{P}^{\prime}=\left\{P_{1}^{\prime}, P_{2}^{\prime}, \ldots\right\}$ be a family of sets which partitions $A$ such that for any two distinct vertices $u, v \in A, u, v \in P_{i}^{\prime}$ if and only if $N_{G}(u) \cap B=N_{G}(v) \cap B$. Since $|A| \geq 2^{h(2)+1} k$ and $\left|\mathcal{P}^{\prime}\right| \leq 2^{h(2)}$, we know that at least one set in $\mathcal{P}^{\prime}$ will contain at least $2 k$ vertices of $A$. Denote these $2 k$ vertices by $A^{\prime}$. All vertices in $A^{\prime}$ have the same neighborhood in $B$ and the same neighborhood in $I_{s} \cup I_{t}$ (as all vertices in $A^{\prime}$ belonged to the same set $P \in \mathcal{P}$ ). Moreover, $A^{\prime}$ is a 2-scattered set in $G[R \backslash B]$. Hence, the sets $\left\{N_{G}\left[a_{1}^{\prime}\right], N_{G}\left[a_{2}^{\prime}\right], \ldots, N_{G}\left[a_{2 k}^{\prime}\right]\right\}$, i.e. the closed neighborhoods of the vertices in $A^{\prime}$, form a sunflower with $2 k$ petals (Proposition 4); the core of this sunflower is contained in $B \cup I_{s} \cup I_{t}$. Using the same arguments as we did in the proof of Lemma 1, we can show that there exists at least one irrelevant vertex $v \in V(G) \backslash\left\{B \cup I_{s} \cup I_{t}\right\}$.

Theorem 5. ISR restricted to any effectively nowhere-dense class $\mathcal{C}$ of graphs is fixed-parameter tractable parameterized by $k$.

Proof. If after partitioning $V(G) \backslash\left\{I_{s} \cup I_{t}\right\}$ into at most $4^{k}$ sets the size of every set $P \in \mathcal{P}$ is bounded by $N\left(h(2), 2,2^{h(2)+1} k\right)$, then we can solve the problem by exhaustive enumeration, as $|V(G)| \leq$ $2 k+4^{k} N\left(h(2), 2,2^{h(2)+1} k\right)$. Otherwise, we can apply Lemma 2 and reduce the size of the graph in polynomial time.

## 4 Dominating set reconfiguration

### 4.1 W [1]-hardness

The W[1]-hardness of the DSR problem can be shown using only minor modifications to the standard parameterized reduction from IS to DS. That is, instead of reducing from IS to DS, we can instead give a reduction from ISR to DSR. We include a proof for completeness.

Theorem 6. DSR parameterized by $k$ is $\mathrm{W}[1]$-hard on general graphs.
Proof. We let $\left(G, I_{s}, I_{t}, k\right)$ be an instance of ISR, where $V(G)=\left\{v_{1}, \ldots, v_{n}\right\}, E(G)=\left\{e_{1}, \ldots, e_{m}\right\}, I_{s}=$ $\left\{v_{i_{1}}, \ldots, v_{i_{k}}\right\}$, and $I_{t}=\left\{v_{j_{1}}, \ldots, v_{j_{k}}\right\}$. We first construct a graph $G^{\prime}$ as follows. $G^{\prime}$ consists of the disjoint union of $k$ vertex-disjoint cliques $C_{1}, \ldots, C_{k}$, each of size $n, k$ vertex-disjoint independent sets $F_{1}, \ldots, F_{k}$, each of size at most $k+2$, and at most $n^{2} k^{2}$ vertex-disjoint independent sets $R_{1}, R_{2}, \ldots$, each of size $k+2$. Intuitively, each set $F_{i}$ will force any dominating set of $G^{\prime}$ of size $k$ (or $k+1$ ) to pick a vertex from each $C_{i}$ and the " $R$ sets" will guarantee that the selected vertices form an independent set in $G$. Formally, we have:
(1) For every vertex $v \in V(G)$ there is a corresponding vertex in each $C_{i}, 1 \leq i \leq k$ and we let $C_{i}=$ $\left\{c_{1}^{i}, \ldots, c_{n}^{i}\right\}$.
(2) For every $1 \leq i \leq k$, we make the set $C_{i}$ a clique in $G^{\prime}$.
(3) For each set $C_{i}, 1 \leq i \leq k$, we introduce a set $F_{i}$ of $k+2$ new independent vertices and add an edge between each vertex in $C_{i}$ and all vertices in $F_{i}$.
(4) For a vertex $c_{p}^{i} \in C_{i}$ and a vertex $c_{q}^{j} \in C_{j}, i \neq j, 1 \leq i, j \leq k$, and $1 \leq p, q \leq n$, if $p=q$ or $v_{p} v_{q} \in E(G)$ we introduce $k+2$ new independent vertices and make them adjacent to all vertices in $C_{i} \cup C_{j} \backslash\left\{c_{p}^{i}, c_{q}^{j}\right\}$. In other words, each new vertex dominates all but two vertices in $C_{i} \cup C_{j}$, namely $c_{p}^{i}$ and $c_{q}^{j}$.

We let $\left(G^{\prime}, D_{s}, D_{t}, k\right)$ denote the corresponding DSR instance, where $D_{s}=\left\{c_{i_{1}}^{1}, \ldots, c_{i_{k}}^{k}\right\}$ and $D_{t}=$ $\left\{c_{j_{1}}^{1}, \ldots, c_{j_{k}}^{k}\right\}$. Clearly, any dominating set $D$ of $G^{\prime}$ of size $k$ must pick exactly one vertex from each $C_{i}$, $1 \leq i \leq k$, and each such set corresponds to an independent set of size $k$ in $G$. Moreover, any reconfiguration sequence between $D_{s}$ and $D_{t}$ starts by adding a vertex (since $G^{\prime}$ has no dominating set of size $k-1$ ) and then removing another (since dominating sets larger than $k+1$ are not allowed). By swapping the order of consecutive vertex additions and removals we obtain a one-to-one correspondence between reconfiguration sequences of independent sets of (of size $k$ and $k-1$ ) and reconfiguration sequences (of the same length) between dominating sets of $G^{\prime}$ (of size $k$ and $k+1$ ). The instances are thus equivalent.

### 4.2 Graphs excluding $K_{d, d}$ as a subgraph

The parameterized complexity of the Dominating SET problem (parameterized by $k$ ) on various classes of graphs has been studied extensively in the literature; the main goal has been to push the tractability frontier as far as possible. The problem was shown fixed-parameter tractable on planar graphs by Alber et al. [1], on bounded genus graphs by Ellis et al. [19], on $H$-minor-free graphs by Demaine et al. [15], on bounded expansion graphs by Nesetril and Ossona de Mendez [36], on nowhere-dense graphs by Dawar and Kreutzer [14], on degenerate graphs by Alon and Gutner [2], and finally on $K_{d, d}$-free graphs by Philip et al. [40] and Telle and Villanger [42]. Figure 1 illustrates the inclusion relationship among these classes of graphs, which all fall under the category of sparse graphs. Our fixed-parameter tractable algorithm relies on many of these earlier results. Interestingly, and since the class of $K_{d, d}$-free graphs includes all those other graph classes, our algorithm (Theorem 9) implies that the diameter of the reconfiguration graph $R_{\mathrm{DS}}(G, k, k+1)$ (or of its connected components), for $G$ in any of the aforementioned classes, is bounded above by $f(k, c)$, where $f$ is a computable function and $c$ is constant which depends on the graph class at hand. We start with some definitions and known results.

Definition $6([18,41,40,42])$. Given a graph $G$, the domination core of $G$ is a set $C \subseteq V(G)$ such that any set $D \subseteq V(G)$ is a dominating set of $G$ if and only if $D$ dominates $C$. In other words, $D$ is a dominating set of $G$ if and only if $C \subseteq N_{G}[D]$.

Theorem $7([41,40,42])$. If $G$ is a graph which excludes $K_{d, d}$ as a subgraph and $G$ has a dominating set of size at most $k$ then the size of the domination core $C$ of $G$ is at most $d k^{d}$ and $C$ can be computed in $\mathcal{O}^{*}\left(d k^{d}\right)$ time.

Definition 7. A bipartite graph $G$ with bipartition $(A, B)$ is $B$-twinless if there are no vertices $u, v \in B$ such that $N(u)=N(v)$.

Theorem 8 ([41]). If $G$ is a bipartite graph with bipartition $(A, B)$ such that $G$ is $B$-twinless and excludes $K_{d, d}$ as a subgraph then

$$
|B| \leq 2(d-1)\left(\frac{|A| e}{d}\right)^{2 d}
$$

Since Theorem 7 implies a bound on the size of the domination core and allows us to compute it efficiently, our main concern is to deal with vertices outside of the core, i.e. vertices in $V(G) \backslash C$. The next lemma shows that we can in fact find strongly irrelevant vertices outside of the domination core of a graph.

Lemma 3. For $G$ an n-vertex graph, $C$ the domination core of $G$, and $D_{s}$ and $D_{t}$ two dominating sets of $G$, if there exist $u, v \in V(G) \backslash\left\{C \cup D_{s} \cup D_{t}\right\}$ such that $N_{G}(u) \cap C=N_{G}(v) \cap C$ then $u$ (or $v$ ) is strongly irrelevant.

Proof. Given a reconfiguration sequence $\sigma=\left\langle D_{0}=D_{s}, D_{1}, \ldots, D_{\ell}=D_{t}\right\rangle$ from $D_{s}$ to $D_{t}$ which touches $u$, we will show how to obtain a reconfiguration sequence $\sigma^{\prime}$ such that $\left|\sigma^{\prime}\right| \leq|\sigma|$ and $\sigma^{\prime}$ touches $v$ but not $u$.

We construct $\sigma^{\prime}$ in two stages. In the first stage, we construct the sequence $\alpha=\left\langle D_{0}^{\prime}, D_{1}^{\prime}, \ldots, D_{\ell}^{\prime}\right\rangle$ of dominating sets, where for all $0 \leq i \leq \ell$

$$
D_{i}^{\prime}=\left\{\begin{array}{l}
D_{i} \cup\{v\} \backslash\{u\} \text { if } u \in D_{i} \\
D_{i} \text { if } u \notin D_{i} .
\end{array}\right.
$$

Note that $\alpha$ is not necessarily a reconfiguration sequence from $D_{s}$ to $D_{t}$. In the second stage, we repeatedly delete from $\alpha$ any set $D_{i}^{\prime}$ such that $D_{i}^{\prime}=D_{i+1}^{\prime}, 0 \leq i<\ell$. We let $\sigma^{\prime}=\left\langle D_{0}^{\prime}, D_{1}^{\prime}, \ldots, D_{\ell^{\prime}}^{\prime}\right\rangle$ denote the resulting sequence, in which there are no two consecutive sets that are equal, and we claim that $\sigma^{\prime}$ is in fact a reconfiguration sequence from $D_{s}$ to $D_{t}$.

To prove the claim, we need to show that the following conditions hold:
(1) $D_{0}^{\prime}=D_{s}$ and $D_{\ell^{\prime}}^{\prime}=D_{t}$,
(2) $D_{i}^{\prime}$ is a dominating set of $G$ for all $0 \leq i \leq \ell^{\prime}$,
(3) $\left|D_{i}^{\prime} \Delta D_{i+1}^{\prime}\right|=1$ for all $0 \leq i<\ell^{\prime}$, and
(4) $k \leq\left|D_{i}^{\prime}\right| \leq k+1$ for all $0 \leq i \leq \ell^{\prime}$.

Since $u, v \notin D_{s} \cup D_{t}$, condition (1) clearly holds. Moreover, since replacing $u$ by $v$ in any set does not increase the size of the corresponding set, $k \leq\left|D_{i}^{\prime}\right| \leq k+1$ (condition (4) holds) and $\left|D_{i}^{\prime} \Delta D_{i+1}^{\prime}\right| \leq 1$. As there are no two consecutive sets in $\sigma^{\prime}$ that are equal, $\left|D_{i}^{\prime} \Delta D_{i+1}^{\prime}\right|>0$ and therefore $\left|D_{i}^{\prime} \Delta D_{i+1}^{\prime}\right|=1$ (condition (3) holds). The fact that $D_{i}^{\prime}$ is a dominating set of $G$ follows from the definition of a domination core. Since $D_{i}$ is a dominating set of $G, C \subseteq N_{G}\left[D_{i}\right]$. Moreover, since $N_{G}(u) \cap C=N_{G}(v) \cap C$ and $u, v \notin C$, we know that $C \subseteq N_{G}\left[D_{i}^{\prime}\right]$. By the definition of the domination core, it follows that $D_{i}^{\prime}$ (which still dominates $C$ ) is also a dominating set of $G$. Therefore, all four conditions hold, as needed.

Theorem 9. DSR parameterized by $k+d$ is fixed-parameter tractable on graphs that exclude $K_{d, d}$ as a subgraph.

Proof. Given a graph $G$, integer $k$, and two dominating sets $D_{s}$ and $D_{t}$ of $G$ of size at most $k$, we first compute the domination core $C$ of $G$, which by Theorem 7 can be accomplished in $\mathcal{O}^{*}\left(d k^{d}\right)$ time. Next, and due to Lemma 3, we can delete all strongly irrelevant vertices from $V(G) \backslash\left\{C \cup D_{s} \cup D_{t}\right\}$. We denote this new graph by $G^{\prime}$.

Now consider the bipartite graph $G^{\prime \prime}$ with bipartition $\left(A=C \backslash\left\{D_{s} \cup D_{t}\right\}, B=V\left(G^{\prime}\right) \backslash\left\{C \cup D_{s} \cup D_{t}\right\}\right)$. This graph is $B$-twinless, since for every pair of vertices $u, v \in V(G) \backslash\left\{C \cup D_{s} \cup D_{t}\right\}$ such that $N_{G}(u) \cap C=N_{G}(v) \cap C$ either $u$ or $v$ is strongly irrelevant and is therefore not in $V\left(G^{\prime}\right)$ nor $V\left(G^{\prime \prime}\right)$. Moreover, since every subgraph of a $K_{d, d}$-free graph is also $K_{d, d}$-free, $G^{\prime \prime}$ is $K_{d, d}$-free. Hence, by Theorems 7 and 8 , we have

$$
\begin{aligned}
|B| & \leq 2(d-1)\left(\frac{|A| e}{d}\right)^{2 d} \\
& \leq 2 d(3|A|)^{2 d} \leq 2 d\left(3 d k^{d}\right)^{2 d}
\end{aligned}
$$

Putting it all together, we know that after deleting all strongly irrelevant vertices, the number of vertices in the resulting graph $G^{\prime}$ is at most

$$
\begin{aligned}
\left|V\left(G^{\prime}\right)\right| & =|V(C)|+\left|D_{s} \cup D_{t}\right|+\left|V\left(G^{\prime}\right) \backslash\left\{C \cup D_{s} \cup D_{t}\right\}\right| \\
& \leq d k^{d}+2 k+2 d\left(3 d k^{d}\right)^{2 d}
\end{aligned}
$$

Hence, we can solve DSR by exhaustively enumerating all $2^{\left|V\left(G^{\prime}\right)\right|}$ subsets of $V\left(G^{\prime}\right)$ and building the reconfiguration graph $R_{\mathrm{DS}}\left(G^{\prime}, k, k+1\right)$.

## References

1. J. Alber, H. L. Bodlaender, H. Fernau, and R. Niedermeier. Fixed parameter algorithms for planar dominating set and related problems. In Proceedings of the $17^{\text {th }}$ Scandinavian Workshop on Algorithm Theory, pages 97-110. Springer Berlin Heidelberg, 2000.
2. N. Alon and S. Gutner. Linear time algorithms for finding a dominating set of fixed size in degenerated graphs. Algorithmica, 54(4):544-556, 2009.
3. M. Bonamy and N. Bousquet. Recoloring bounded treewidth graphs. Electronic Notes in Discrete Mathematics, 44:257-262, 2013.
4. P. Bonsma. The complexity of rerouting shortest paths. In Proceedings of the $37^{\text {th }}$ International Symposium on Mathematical Foundations of Computer Science, pages 222-233, 2012.
5. P. Bonsma. Rerouting shortest paths in planar graphs. In Proceedings of the $32^{\text {nd }}$ Annual Conference on Foundations of Software Technology and Theoretical Computer Science, pages 337-349, 2012.
6. P. Bonsma and L. Cereceda. Finding paths between graph colourings: PSPACE-completeness and superpolynomial distances. Theoretical Computer Science, 410(50):5215-5226, 2009.
7. P. Bonsma, A. E. Mouawad, N. Nishimura, and V. Raman. The complexity of bounded length graph recoloring and CSP reconfiguration. In Proceedings of the $9^{\text {th }}$ International Symposium on Parameterized and Exact Computation, pages 110-121, 2014.
8. A. Brandstädt, V. B. Le, and J. P. Spinrad. Graph Classes: A Survey. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1999.
9. L. Cereceda. Mixing graph colourings. PhD thesis, London School of Economics, 2007.
10. L. Cereceda, J. van den Heuvel, and M. Johnson. Mixing 3-colourings in bipartite graphs. European Journal of Combinatorics, 30(7):1593-1606, 2009.
11. L. Cereceda, J. van den Heuvel, and M. Johnson. Finding paths between 3-colorings. Journal of Graph Theory, 67(1):69-82, 2011.
12. A. Dawar. Finite model theory on tame classes of structures. In Proceedings of the $32^{\text {nd }}$ International Symposium on Mathematical Foundations of Computer Science, volume 4708 of Lecture Notes in Computer Science, pages 2-12. Springer Berlin Heidelberg, 2007.
13. A. Dawar. Homomorphism preservation on quasi-wide classes. Journal of Computer and System Sciences, 76(5):324-332, 2010. Workshop on Logic, Language, Information and Computation.
14. A. Dawar and S. Kreutzer. Domination problems in nowhere-dense classes of graphs. CoRR, 2009. arXiv:0907.42837.
15. E. D. Demaine, F. V. Fomin, M. Hajiaghayi, and D. M. Thilikos. Subexponential parameterized algorithms on bounded-genus graphs and H-minor-free graphs. Journal of the ACM, 52(6):866-893, 2005.
16. R. Diestel. Graph theory. Springer-Verlag, Electronic Edition, 2005.
17. R. G. Downey and M. R. Fellows. Parameterized complexity. Springer-Verlag, New York, 1997.
18. P. G. Drange, M. S. Dregi, F. V. Fomin, S. Kreutzer, D. Lokshtanov, M. Pilipczuk, M. Pilipczuk, F. Reidl, S. Saurabh, F. S. Villaamil, and S. Sikdar. Kernelization and sparseness: the case of dominating set. CoRR, 2014. arXiv:1411.4575.
19. J. Ellis, H. Fan, and M. Fellows. The dominating set problem is fixed parameter tractable for graphs of bounded genus. In Proceedings of the $19^{\text {th }}$ Scandinavian Workshop on Algorithm Theory, volume 2368 of Lecture Notes in Computer Science, pages 180-189. Springer Berlin Heidelberg, 2002.
20. P. Erdos and R. Rado. Intersection theorems for systems of sets. Journal of the London Mathematical Society, 35:85-90, 1960.
21. J. Flum and M. Grohe. Parameterized complexity theory. Springer-Verlag, Berlin, 2006.
22. P. Gopalan, P. G. Kolaitis, E. N. Maneva, and C. H. Papadimitriou. The connectivity of Boolean satisfiability: computational and structural dichotomies. SIAM Journal on Computing, 38(6):2330-2355, 2009.
23. M. Grohe, S. Kreutzer, and S. Siebertz. Characterisations of nowhere dense graphs. In Proceedings of the $33^{\text {rd }}$ Annual Conference on Foundations of Software Technology and Theoretical Computer Science, pages 21-40, 2013.
24. M. Grohe, S. Kreutzer, and S. Siebertz. Deciding first-order properties of nowhere dense graphs. In Proceedings of the $46^{\text {th }}$ Annual ACM Symposium on Theory of Computing, pages 89-98, 2014.
25. T. Ito, E. D. Demaine, N. J. A. Harvey, C. H. Papadimitriou, M. Sideri, R. Uehara, and Y. Uno. On the complexity of reconfiguration problems. Theoretical Computer Science, 412(12-14):1054-1065, 2011.
26. T. Ito, M. Kamiński, and E. D. Demaine. Reconfiguration of list edge-colorings in a graph. Discrete Applied Mathematics, 160(15):2199-2207, 2012.
27. T. Ito, M. Kamiski, and H. Ono. Fixed-parameter tractability of token jumping on planar graphs. In Algorithms and Computation, Lecture Notes in Computer Science, pages 208-219. Springer International Publishing, 2014.
28. T. Ito, M. Kamiski, H. Ono, A. Suzuki, R. Uehara, and K. Yamanaka. On the parameterized complexity for token jumping on graphs. In Proceedings of the $11^{\text {th }}$ Annual Conference on Theory and Applications of Models of Computation, volume 8402 of Lecture Notes in Computer Science, pages 341-351. Springer International Publishing, 2014.
29. M. Johnson, D. Kratsch, S. Kratsch, V. Patel, and D. Paulusma. Finding shortest paths between graph colourings. CoRR, 2014. arXiv:1403.6347.
30. M. Kamiński, P. Medvedev, and M. Milanič. Shortest paths between shortest paths. Theoretical Computer Science, 412(39):5205-5210, 2011.
31. M. Kamiński, P. Medvedev, and M. Milanič. Complexity of independent set reconfigurability problems. Theoretical Computer Science, 439:9-15, 2012.
32. D. R. Lick and A. T. White. k-degenerate graphs. Canadian Journal of Mathematics, 22:1082-1096, 1970.
33. A. E. Mouawad, N. Nishimura, and V. Raman. Vertex cover reconfiguration and beyond. CoRR, 2014. arXiv:1402.4926.
34. A. E. Mouawad, N. Nishimura, V. Raman, N. Simjour, and A. Suzuki. On the parameterized complexity of reconfiguration problems. In Proceedings of the $8^{\text {th }}$ International Symposium on Parameterized and Exact Computation, 2013.
35. A. E. Mouawad, N. Nishimura, V. Raman, and M. Wrochna. Reconfiguration over tree decompositions. CoRR, 2014. arXiv:1405.2447.
36. J. Nesetril and P. O. de Mendez. Structural properties of sparse graphs. In Building Bridges, volume 19 of Bolyai Society Mathematical Studies, pages 369-426. Springer Berlin Heidelberg, 2008.
37. J. Nesetril and P. O. de Mendez. First order properties on nowhere dense structures. Journal of Symbolic Logic, 75(3):868-887, 2010.
38. J. Nesetril and P. O. de Mendez. From sparse graphs to nowhere dense structures: Decompositions, independence, dualities and limits, 2010. European Congress of Mathematics.
39. R. Niedermeier. Invitation to fixed-parameter algorithms. Oxford University Press, Oxford, 2006.
40. G. Philip, V. Raman, and S. Sikdar. Solving dominating set in larger classes of graphs: FPT algorithms and polynomial kernels. In Proceedings of the $17^{\text {th }}$ Annual European Symposium on Algorithms, volume 5757 of Lecture Notes in Computer Science, pages 694-705. Springer Berlin Heidelberg, 2009.
41. S. Saurabh. Private communications, 2014.
42. J. A. Telle and Y. Villanger. FPT algorithms for domination in biclique-free graphs. In Proceedings of the $20^{\text {th }}$ Annual European Conference on Algorithms, pages 802-812, 2012.
43. J. van den Heuvel. The complexity of change. Surveys in Combinatorics 2013, 409:127-160, 2013.
44. M. Wrochna. Reconfiguration in bounded bandwidth and treedepth. CoRR, 2014. arXiv:1405.0847.
