

More Natural Models of Electoral Control by Partition*

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Abstract

“Control” studies attempts to set the outcome of elections through the addition, deletion, or partition of voters or candidates. The set of benchmark control types was largely set in the seminal 1992 paper by Bartholdi, Tovey, and Trick that introduced control, and there now is a large literature studying how many of the benchmark types various election systems are vulnerable to, i.e., have polynomial-time attack algorithms for.

However, although the longstanding benchmark models of addition and deletion model relatively well the real-world settings that inspire them, the longstanding benchmark models of partition model settings that are arguably quite distant from those they seek to capture.

In this paper, we introduce—and for some important cases analyze the complexity of—new partition models that seek to better capture many real-world partition settings. In particular, in many partition settings one wants the two parts of the partition to be of (almost) equal size, or is partitioning into more than two parts, or has groups of actors who must be placed in the same part of the partition. Our hope is that having these new partition types will allow studies of control attacks to include such models that more realistically capture many settings.

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1 Introduction, Motivation, and Discussion

Elections are an important framework for decision-making, both in human settings and in multiagent systems settings as varied as recommender systems [GMHS99], rank aggregation and web-spam filtering [DKNS01], similarity search and classification [FKS03], and planning [ER97].

Given elections’ importance, it is natural that people and other agents should attempt manipulative attacks on such systems, and that computational social choice researchers should investigate the computational complexity of conducting such attacks. Such studies have focused primarily on three broad streams of manipulative attacks, known as manipulation, bribery, and control.

Control, which is the focus of this paper because among the three attack streams its current model is by far the most troubled as to naturalness, studies whether by changing the structure of an election—adding or deleting or partitioning voters or candidates—an actor can ensure that a given candidate wins. A set of 11 benchmark attacks—“control by adding voters,” “control by deleting candidates,” etc.—are often studied to seek to understand which types of attacks a given election system computationally resists. The 11 benchmark attacks already were present in the seminal Bartholdi, Tovey, and Trick [BTT92] paper on control, at least as refined in subsequent papers that made the partition tie-breaking options explicit [HHR07] and addressed the one asymmetry in the seminal paper’s definitions [FHHR09].

With these benchmark types in hand, many election systems have been evaluated as to how many of these types of attacks they computationally resist, with the goal of identifying natural election systems that resist as many as possible of the 11 benchmark types—or the 22 (or 21, due to a collapse of types recently noticed by Hemaspaandra, Hemaspaandra, and Menton [HHM12]) such types, if one also looks at the “destructive” cases where the goal is instead to prevent a given candidate from winning.

For example, among the papers that do excellent, detailed analyses tallying how many resistances to control attacks are possessed by such broadly-control-resistant election systems as Bucklin, fallback, ranked pairs, Schulze, sincere-strategy preference-based approval, and normalized range voting are [PX12, MS13, Men13, ENR09, EFRS].

However, although the benchmark control models regarding addition/deletion of candidates/voters are relatively natural, the benchmark models of partitioning have always been far less so. For many natural, real-world settings, the longstanding benchmark partition models simply don’t come close to capturing the settings that are routinely used to motivate them. And so, despite the fact that an enormous amount of effort—in most of the above papers and very many others—has been focused on the standard partition models over the more than two decades since they were created, we feel that it is valuable to revisit the issue of how to best frame models of partition. Perhaps such a revisitation should have occurred ten or fifteen years ago, but since we don’t possess a time machine, the best we can do is to approach the issue now, and in this paper we do so.

In truth, the appropriateness of a model of partition will depend heavily on the setting

one is trying to model, and there indeed are some settings that are well-modeled by the benchmark partition types. Still, there are numerous settings that are not well-modeled by them, and so in this paper we present, and give some initial results regarding, new types of partitioning that we feel are worth studying as capturing naturally important notions of partitioning. In particular, we define three new general flavors of partitioning.

Before discussing the three new flavors, we should briefly describe control by partition. In the classic version of control by partition of voters, one is given the votes of each voter (most typically as a tie-free ordering, e.g., “Nader>Kerry>Bush”) and a distinguished candidate one is interested in, and one asks whether there is a way of partitioning the voters into V_1 and V_2 such that if one has subelections among the candidates by V_1 and V_2 , and then a final election by all the voters with the candidates being the winners of those subelections, the distinguished candidate is the winner. In the classic version of candidate partitioning (known as “runoff partition of candidates”), the input is the same, but the question is whether there is a partition of the candidates into C_1 and C_2 such that if all the voters have subelections regarding C_1 and regarding C_2 , and then a final election by all voters over just the winners of those subelections, the distinguished candidate is the winner.

The three variants we suggest and study, in order to create models that are often closer to the real-world settings that partition seeks to capture, are equipartition, multipartition, and partition by groups. In equipartition, the two parts of partitions must be of the same size (or within one if the things being partitioned are odd in cardinality). The motivation for this is that in real-world settings, such as apportioning people into districts, it is very common to want the districts to be of essentially equal size. Even when breaking alternatives into two groups, it is natural to keep the playing field somewhat level by expecting the groups to be of essentially equal sizes. Yet the classic models of partition have no constraint at all on the sizes of the parts of the partition; although it might in reality outrage people were this to happen, in the classic candidate-partition model it is completely legal to partition the candidates into c and $C - \{c\}$, so that candidate c gets a free pass into the final election, and it is completely legal in the classic voter-partition model to divide a population 100,000,000 state into one district having 3 voters and one district having all the rest of the voters. Can such a lopsided division ever be natural? Well, actually, yes, for example if our partitioning regards a party-based primary and there are 3 people in the Green party and the rest are in the Libertarian party. But in many settings, size-balanced partitioning is the natural and indeed compelling expectation. The tricky thing regarding discussing what is natural and what model is best to capture a given real-world setting is that these issues are highly contextual—the richness of the real world means that no single model will capture all cases. Our goal in this paper, thus, is not to give a new model that we claim will capture all cases, but rather to give new models that in many settings are natural and attractive, and whose addition as models significantly broadens the class of cases one has good models for.

If one is dividing a state or entity up into electoral districts, it may well be the case that having just two districts is simply not appropriate—perhaps the law sets the number of districts to a different number. Thus our second new model for the study of partition-control is *multipartition*, that is, partitioning not into two parts but into k parts.

Our third new model is partition by groups. In partition by groups, each actor has a color, and all actors having the same color must be put in the same part of the partition as each other. For example, regarding voter partitioning, if we are breaking an electorate into primary districts and it is forbidden to have voters who live in the same address/apartment building (or the same block) be in different districts, then we would make color groups for each address/apartment building (or block). Or regarding candidate partitioning, if in a department the chair is splitting hiring candidates up for a series of culling votes and it would be openly insane for candidates from the same subfield to be put into separate culling votes, each subarea would have a color. Control by groups is so natural that one might wish to study it not just for partition cases but also for addition/deletion of voters/candidates, and so we provide a result for adding and deleting voters by groups in this paper’s section on groups. We mention that group-based (in a somewhat different framing) control by addition of voters has been previously introduced by Chen et al. [CFNT14], and in this paper’s section on groups we discuss that interesting work and its relationship to the present work.

In this paper, we look at these three new models for partitioning, but due to space limitations and for simplicity, we don’t seek to prove results about simultaneous combinations of them. However, certainly some real-world setting will draw on combinations and so that might in the future be a potential area for study.

Although in this paper the new models themselves are very important, we also, for each of them, provide one or more results as to the complexity of control within that model, especially for the most important election system, plurality. In some cases, our versions of control leave the classic control problem’s complexity unchanged (although for P cases it often takes far more complicated algorithms to handle the new cases while remaining in P), in some cases our versions increase the complexity from P to NP-complete, and we have even built a system where our version lowers the complexity from NP-complete to P.

For example, regarding equipartition, we show that for plurality, approval, and Condorcet voting, every existing P result can be reestablished even for equipartition. However, we show that for weakCondorcet elections, equipartitioning turns the P classic case into NP-completeness. We prove that control by groups often jumps P classic cases up to NP-completeness. And regarding multipartition and plurality, we build a P algorithm for the most important case.

The remainder of the paper is organized as follows. The next section discusses whether raising, lowering, or maintaining complexity are good or bad things. A brief definitions section follows that. In the three sections after that, we cover each of our three new models, giving formal definitions of each as well as results on their behavior. The conclusions and open problems section ends the paper proper, followed by an appendix.

2 Meaning of Increasing or Lowering Complexity

Is it good if, for a given problem, our equipartition variant is harder than the classic partition version of the problem? Is it good if our version keeps the complexity the same? Is it good if our version lowers the complexity?

The simple answer is that there is no simple answer, and that the questions are themselves simplistic. If one views complexity as trying to “shield” elections from undesirable attacks, then having high levels of complexity for a control problem is a good thing. If one is an attacker such as a campaign strategist—or if one is using an election-attack problem to model a real-world problem that one wants to solve (as for example the election problem “bribery” can be used to model resource-allocation problems)—then having polynomial-time algorithms is a good thing.

So whether high levels of complexity of control are desirable or undesirable is highly perspective-based and highly setting-based. What computational social choice theorists can do, however, is make clear what the control complexity levels are for the most important control types and the most important election systems. Having that knowledge in hand will allow social choice theorists, election-system choosers, campaign strategists, and others to all openly see the lay of the land. Although we have had colleagues tell us they are troubled that efficient algorithms could facilitate what they consider unethical election manipulation, our view is that having it openly known what systems have what weaknesses will for example let election-system choosers adopt systems that are not vulnerable to whatever attacks they most fear. We feel that, just as in cryptography, the right approach is not a head-in-the-sand one, but one of open inquiry and discovery, so that not just “bad” people (or the government) know what is or isn’t vulnerable.

As a final comment here, we mention that this paper does contain NP-completeness results. NP-completeness is a worst-case theory, and so for our NP-hard cases seeking results for other notions of hardness would indeed be interesting. However, as always, despite the empirical evidence that heuristics often do well on SAT and even on many NP-hard election problems (see Rothe and Schend [RS13]), it is settled theoretical truth that (unless the polynomial hierarchy collapses) no heuristic algorithm for any NP-hard problem can asymptotically have a subexponential error rate (see Hemaspaandra and Williams [HW12] for the details on this result and a discussion of how to square it with the strong empirical performance of heuristics). Perhaps even more to the point, the majority of the results mentioned in this paper are not about NP-hardness but are about showing that, even for partition-control variants that might seem likely to increase control complexity, polynomial-time control algorithms do exist.

3 Definitions

An election system is a mapping that, given the candidates and the votes, outputs a subset of the candidates, who are said to be the winners under that election system. We will often use the symbol \mathcal{E} to denote an election system. For approval elections, voters give a 0 or 1 to each candidate, and the candidate(s) having the largest number of 1 votes is the winner(s). For the other election systems that we will study, votes are tie-free linear orderings, e.g., “Nader>Kerry>Bush.” In plurality elections, whichever candidate(s) is in the top spot on the most votes is the winner(s). In Condorcet (resp., weakCondorcet) elections, each candidate who is preferred to each of other candidate d in strictly more than half (resp.,

greater than or equal to half) the votes is a winner.

For conciseness, we sometimes use bracket notation (borrowed from linguistics) for *independent* choices, e.g., “The $\begin{bmatrix} \text{ball} \\ \text{book} \\ \text{car} \end{bmatrix}$ is $\begin{bmatrix} \text{red} \\ \text{heavy} \end{bmatrix}$ ” is shorthand for the natural six claims obtained by making each possible choice.

4 Equipartition

Let us now define our equipartition notion and the classic partition problems. (In all our problem definitions, we include “represented via preference lists over C .” But in fact the voters are always represented by whatever vote type is that of the election system; throughout this paper that is always preference lists, except for approval voting the votes are instead 0/1-vectors.) Recall that partitioning a set into two (or more) parts means that every element of the set must appear in exactly one of the parts.

\mathcal{E} -CCREPC (CONTROL BY RUNOFF EQUIPARTITION OF CANDIDATES)

Given: A set C of candidates, a collection V of voters represented via preference lists over C , and a distinguished candidate $p \in C$.

Quest.: Is there a partition of C into C_1 and C_2 such that $|\|C_1\| - \|C_2\|| \leq 1$ and p is the sole winner of the two-stage election where the winners of subelection (C_1, V) that survive the tie-handling rule compete against the winners of subelection (C_2, V) that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system \mathcal{E} .

\mathcal{E} -CCEPV (CONTROL BY EQUIPARTITION OF VOTERS)

Given: A set C of candidates, a collection V of voters represented via preference lists over C , and a distinguished candidate $p \in C$.

Quest.: Is there a partition of V into V_1 and V_2 such that $|\|V_1\| - \|V_2\|| \leq 1$ and p is the sole winner of the two-stage election where the winners of election (C, V_1) that survive the tie-handling rule compete against the winners of (C, V_2) that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system \mathcal{E} .

The classic cases, \mathcal{E} -CCRPC and \mathcal{E} -CCPV, are defined identically, except without the clause forcing the partition parts to be equal in size (or off-by-one if the set being partitioned is of odd cardinality).

For all of the above, there is the issue of whether if there are multiple winners of a subelection, all of them move on to the final election (called ties-promote, notated TP) or none of them move on to the final election (called ties-eliminate, notated TE; in this model, to move on to the final election one must be the unique winner of a subelection). Thus each problem will always appear with a TE or TP to specify the tie-handling approach, e.g., \mathcal{E} -CCPV-TE or \mathcal{E} -CCPV-TP.

The literature also contains a “bye” version of partitioning candidates, in which any number of candidates can be assigned to skip (“bye”) the first round and the rest compete to get into the second round. Since equipartition is not natural for that “bye” version (because the number of candidates skipping a first round is usually driven by such things as excesses relative to powers of two in the number of candidates), we have not defined here either the classic or an “equi” version of “bye” partition; a later version of this paper, however, will note that many of our results hold for the “equi” version of “bye” partition, and more importantly will give, and prove results about, a model in which one specifies as part of the input how many candidates get a “bye,” since that model provides the natural partition-size-sensitive variant of the “bye” candidate-partition problem.

As a final comment about model details, all of the partition problems discussed above are in what is called the unique-winner model, i.e., the goal is to make a given candidate the one and only overall winner. That is the model of the seminal Bartholdi, Tovey, and Trick [BTT92] control paper and all the immediately subsequent control papers, and probably even today is the most common model when studying control (although we ourselves prefer the alternate model known as the nonunique-winner model or the co-winner model, where one merely needs to make a given candidate become *a* winner). And so throughout all parts of this version, we focus solely on the unique-winner model, since doing so creates apples-to-apples contrasts for those cases where our models change the existing complexity from those found in that seminal paper and various other papers. (We have checked the vast majority of our results in both models, and in a later version of this paper will cover both models. This is not an issue one can safely take for granted, since there are examples in the literature where the complexity or behavior of the two models differs sharply, e.g., [FHS08, HHM12].)

Let us turn to our results, which give a sense of what holds when one partitions while required to have the parts be “equi.” We show that for many important systems, including approval, Condorcet, and plurality, partition problems remain in P even for equipartitioning, albeit typically with substantially more difficult algorithms and new tricks relative to what the non-“equi” cases required. However, in contrast to its sibling, the Condorcet case, we prove that for weakCondorcet an increase from P to NP-completeness occurs. We also construct an (admittedly artificial) election system having its winner problem in P, such that going from partition to equipartition lowers the control complexity from NP-completeness to P.

The following shows that for approval, Condorcet, and plurality, each P case of $CC \left[\begin{smallmatrix} \text{RPC} \\ \text{PV} \end{smallmatrix} \right] - \left[\begin{smallmatrix} \text{TE} \\ \text{TP} \end{smallmatrix} \right]$ —these can be found in Table 1 of Hemaspaandra, Hemaspaandra and Rothe [HHR07] and are variously due to that paper and Bartholdi, Tovey, and Trick [BTT92]—remains in P for equipartition.

Theorem 4.1 *Each of the problems Plurality-CCEPV-TE and $\left[\begin{smallmatrix} \text{Condorcet} \\ \text{approval} \end{smallmatrix} \right] - \text{CCREPC} - \left[\begin{smallmatrix} \text{TE} \\ \text{TP} \end{smallmatrix} \right]$ belongs to P.*

In the appendix, we include a proof of $\text{Plurality-CCEPV-PE} \in \text{P}$. The proof of this has two interesting issues that do not occur in the standard partition case.

First, even for those inputs where p is at the top of no more than $\lceil \|V\|/2 \rceil$ of the votes, one cannot assume within the proof that if candidate p can be made to win, it can be made to win with some partition that puts all votes with p at their top in the same partition part. Here is an example showing that that assumption, which works in the general case, fails here. If we have 5 votes for p , 6 votes for a , and 3 votes for b , we can make p a sole overall winner by letting V_1 consist of 4 votes for p and 3 votes for a . Then p is the unique winner in V_1 , and V_2 consists of 3 votes for a , 3 votes for b , and 1 vote for p , so no candidate from the second election makes it through to the runoff, and p is the overall unique winner. However, if we put all 5 votes for p in V_1 , then since there are 7 votes in V_1 , at least 4 votes for a are in V_2 and a is the unique winner of the second subelection, and so also of the overall election.

The second twist is that we need to find a “safe” way to legally distribute in an “equi” overall fashion certain “remaining” votes; and our proof does this by pushing them all to one side (violating “equi,” typically), and then correcting this in a way that is guaranteed to succeed if success is possible.

We have also shown that the NP-complete cases still hold for our systems of interest.

Theorem 4.2 *For approval, Condorcet, and plurality, each NP-complete case of $\text{CC} \left[\begin{smallmatrix} \text{RPC} \\ \text{PV} \end{smallmatrix} \right] - \left[\begin{smallmatrix} \text{TE} \\ \text{TP} \end{smallmatrix} \right]$ —these can be found in Table 1 of Hemaspaandra, Hemaspaandra and Rothe [HHR07] and are variously due to that paper and Bartholdi, Tovey, and Trick [BTT92]—remains NP-complete for the one among $\text{CC} \left[\begin{smallmatrix} \text{REPC} \\ \text{EPV} \end{smallmatrix} \right] - \left[\begin{smallmatrix} \text{TE} \\ \text{TP} \end{smallmatrix} \right]$ that is its equipartition analogue.*

Now, it might be natural to wonder: Is there a meta-theorem showing that NP-completeness always inherits from partition cases to equipartition cases? If so Theorem 4.2 would become a freebie consequence of the meta-theorem. However, although we do not have any example of a natural election system where the “equi” case drops the complexity from NP-completeness to P, we have constructed an election system displaying precisely that behavior. And so NP-completeness does not always inherit from the standard case to the “equi” case. Theorem 4.3’s proof is included in the appendix.

Theorem 4.3 *There exists an election system \mathcal{E} , whose winner problem is in P, such that $\mathcal{E}\text{-CCPV-TP}$ is NP-complete, yet $\mathcal{E}\text{-CCEPV-TP}$ belongs to P.*

In contrast to its close relative, Condorcet elections, weakCondorcet elections increase complexity from the partition case to the equipartition case for the RPC-TP case. (This complexity increase is not precluded by the general fact that subcases of problems cannot be harder than the original problem. Although each equipartition of a set is indeed a partition of that set, we are not dealing here with a subcase of a problem, but rather with a control problem whose allowed internal actions are a subset of those of a different control problem, and so there is no automatic prohibition on the complexity increasing.) We do not

yet have a complexity classification for weakCondorcet-CCREPC-TE, and consider that an interesting open problem.

Theorem 4.4 weakCondorcet-CCRPC- $\begin{bmatrix} \text{TE} \\ \text{TP} \end{bmatrix}$ are in P, but weakCondorcet-CCREPC-TP is NP-complete.

Briefly, what is behind the change in complexity here is that we can make one of the subelections represent a vertex cover and we use equipartition to limit the size of that vertex cover. The reason the same approach does not work also for Condorcet elections is that we crucially need that we can have multiple winners. We now prove the two TP parts of Theorem 4.4, as these are what bring out the contrasting behavior.

Proof. To show that weakCondorcet-CCRPC-TP is in P, it suffices to note that for all $p \in C$, if p can be made the unique weakCondorcet winner in (C, V) by RPC-TP, then this is established by candidate partition $(\{p\}, C - \{p\})$. To see this, suppose p is not the unique weakCondorcet winner in partition $(\{p\}, C - \{p\})$. Then there is a candidate $c \neq p$ such that c is a weakCondorcet winner in $(C - \{p\}, V)$ and c ties-or-defeats p in their head-to-head contest. But then c is a weakCondorcet winner in (C, V) and so in any (C', V) with $C' \subseteq C$ and $c \in C'$. It follows that c is always a winner by RPC-TP, and so p will never be the unique winner.

That was easy. But $(\{p\}, C - \{p\})$ is clearly not an equipartition. We will now show that for weakCondorcet-CCREPC-TP is NP-complete. We will prove this by a reduction from Cubic Vertex Cover: Given a graph $G = (V, E)$ that is cubic, i.e., where every vertex has degree three, and a positive integer $k \leq \|V\|$, we ask whether G has a vertex cover of size k , i.e., a set of vertices $V' \subseteq V$ of size k such that every edge in E is incident with at least one vertex in V' . Let $\|V\| = n$. Since G is cubic, $\|E\| = 2n/3$.

Using McGarvey's construction [McG53], we construct, in polynomial time, an election (C, \widehat{V}) with the following properties:

- $C = \{p\} \cup V \cup E \cup D$, where $D = \{d_1, \dots, d_{n/2+2k-1}\}$.
- The set of voters, \widehat{V} , is such that we have the following head-to-head contest results:
 - for every $e \in E$, e defeats p ,
 - for every $c \in V \cup D$, p defeats c ,
 - for every $e = \{v, v'\} \in E$, v defeats e and v' defeats e ,
 - all other head-to-head contests are ties.

Suppose V' is a vertex cover of size k of G . Then p can be made the unique weakCondorcet winner by REPC, using partition $(\{p\} \cup D \cup V - V', E \cup V')$. Note that $\|\{p\} \cup D \cup V - V'\| = 1 + n/2 + 2k - 1 + n - k = 3n/2 + k = \|E \cup V'\|$. p is the Condorcet winner in $(\{p\} \cup D \cup V - V', \widehat{V})$, and thus participates in the runoff. Since V' is a vertex cover, for every candidate $e \in E$, there is a candidate $v \in V'$ such that v defeats

e in their head-to-head contest. So no candidate in E makes it to the runoff. So p is the Condorcet winner in the runoff, and thus certainly the unique weakCondorcet winner.

For the converse, suppose p can be made the unique weakCondorcet winner by REPC-TP. Let (C_1, C_2) be an equipartition of C with $p \in C_1$ that witnesses this. Then p is a weakCondorcet winner in (C_1, \widehat{V}) . This implies that $E \subseteq C_2$. Since p is a weakCondorcet winner in the runoff, no candidate from E participates in the runoff. So for every $e \in E$, there is a $c \in C_2$ such that c defeats e in their head-to-head contest. The only candidates that defeat $e = \{v, v'\}$ are v and v' . It follows that $C_2 \cap V$ is a vertex cover of G . Since (C_1, C_2) is an equipartition, $\|C_2 \cap V\| \leq k$. \square

5 Multipartition

In many settings two simply is not the number of parts into which one's voter set must be divided. For example, the Dean may wish to have three study sections each passing forward a best choice on some issue. Multipartition, which we'll define here just for PV but it can just as well be defined for RPC, generalizes the 2-partition PV problem used in the seminal Bartholdi, Tovey, and Trick paper to each k -partitioning. For each integer $k \geq 2$, define the following problem.

\mathcal{E} -CCP $_k$ V (CONTROL BY k -PARTITION OF VOTERS)
<p>Given: A set C of candidates, a collection V of voters represented via preference lists over C, and a distinguished candidate $p \in C$.</p> <p>Quest.: Is there a partition of V into k parts, V_1, V_2, \dots, V_k, such that p is the sole winner of the two-stage election where the winners of each of the k elections (C, V_i) that survive the tie-handling rule compete against each other in a final election? Each subelection (in both stages) is conducted using election system \mathcal{E}.</p>

Plurality-CCP $_2$ V-TE is in P [HHR07]. As Theorem 5.1 states, we in fact have that P membership still holds for each k -partition version. The proof is by breaking things into a huge but polynomial number of cases by for each part of the partition either guessing a candidate who allegedly uniquely wins that part along with the score that candidate has within that part or guessing a pair of candidates who tie (perhaps along with others) as the winners of that part of the partition along with the score those two achieve in that part, and then for each such case appropriately checking in polynomial time whether it can be realized.

Theorem 5.1 *For each $k \geq 2$, Plurality-CCP $_k$ V-TE \in P.*

It would be interesting to study multipartition for other election systems, and also to study multipartition varied to allow the number of partitions to itself not be fixed but rather to be specified as part of the input.

6 Voter Control by Groups

In voter partition by groups, each vote has a color (i.e., a label), and all votes with the same label must be placed into the same partition part. We also define group voter-control problems for deleting voters (where all votes of a given color must be jointly deleted or kept) and for adding voters (where in the pool of potential additional voters each one has a color, and each color group must be added or not added as a block). As discussed in the introduction, these models capture cases where groups cannot be separated. One example might be due to living at the same address in a redistricting problem, and another might be a departmental study group process in which each of the department's area subfaculties must be placed within the same study group. We below define just the voter cases for control by groups, but one could completely analogously define candidate control by groups.

\mathcal{E} -CCPVG (CONTROL BY PARTITION OF VOTER GROUPS)

Given: A set C of candidates, a collection V of voters represented via preference lists over C , a partition of V into any number of groups G_1, \dots, G_k , and a distinguished candidate $p \in C$.

Quest.: Is there a partition of V into V_1 and V_2 such that for each i either $G_i \subseteq V_1$ or $G_i \subseteq V_2$ holds and p is the sole winner of the two-stage election where the winners of election (C, V_1) that survive the tie-handling rule compete against the winners of (C, V_2) that survive the tie-handling rule? Each subelection (in both stages) is conducted using election system \mathcal{E} .

\mathcal{E} -CCDVG (CONTROL BY DELETING VOTER GROUPS)

Given: A set C of candidates, a collection V of voters represented via preference lists over C , a partition of V into any number of groups G_1, \dots, G_k , a nonnegative integer ℓ , and a distinguished candidate $p \in C$.

Quest.: Is there a set $S \subseteq V$, $\|S\| \leq \ell$, such that p is the sole winner of the \mathcal{E} election over C with the vote collection set being V with S (multiset) removed, and for each i either $G_i \subseteq S$ or $G_i \cap S = \emptyset$?

\mathcal{E} -CCAVG (CONTROL BY ADDING VOTER GROUPS)

Given: A set C of candidates, a collection V of voters represented via preference lists over C , a collection W of potential additional voters represented via preference lists over C , a partition of W into any number of groups G_1, \dots, G_k , a nonnegative integer ℓ , and a distinguished candidate $p \in C$.

Quest.: Is there a collection $S \subseteq W$, $\|S\| \leq \ell$, such that p is the sole winner of the \mathcal{E} election over C with the vote set being V (multiset) unioned with S , and for each i either $G_i \subseteq S$ or $G_i \cap S = \emptyset$?

Before stating results for this model, let us quickly discuss whether these notions are in overlap with models in the literature. After all, votes are coming and going in blocks, and so one might wonder if this is related to for example the notion of weighted control

introduced by Faliszewski, Hemaspaandra, and Hemaspaandra [FHH13]. Briefly, the notions are different to their core in that a weighted vote puts a lot of weight *on that vote*, but in contrast, a vote group may consist of votes that have vastly different preferences from each other. It is true that if one took the Faliszewski, Hemaspaandra, and Hemaspaandra [FHH13] notion of weighted control, and restricted the weights to being input in unary, and for the adding/deleting voters cases shifted from that paper’s model of counting as one’s limit the number votes and instead adopted the model (mentioned but not adopted in that paper) of limiting by the total weight of votes added/deleted, then *those* weighted control problems would each indeed many-one polynomial-time reduce to our analogous voter control by groups problem; but that seems to be far as the connection goes between the two papers.

A closer connection is to the work of Chen et al. [CFNT14], who define and study a very general notion of “combinatorial voter control” for addition of voters. (Their paper is not concerned with partition problems, the main focus of the present paper.) Loosely put, for each voter they have a group of voters who in some sense follow that voter, so that if one takes an action on a voter, the group of the voter follows also. Note that this is a very flexible, general scheme, and for example does not require that the follower function breaks the voters into equivalence classes, as does our coloring scheme. Of course, when proving NP-hardness results, such flexibility *weakens* the results. So in their paper (which is in the nonunique-winner model) they define and study a number of restricted models of the follower function. However, even the most restrictive models of follower functions that they study are incomparable to our model (even when they add nice symmetry-like properties, they focus those on voters with the same preferences, and so are not focused on what we are focused on, which is, in effect, coloring voters, i.e., breaking voters into equivalence classes in whatever arbitrary way is specified by the coloring), and so our NP-hardness result for Plurality-CCAVG is incomparable with their NP-hardness results for their model of combinatorial control by adding voters. Nonetheless, it is important to mention their work appeared in conference two months before any version of our paper appeared, and so their paper certainly deserves the precedence and credit for the idea of grouping voters, and we mention that their paper establishes very interesting results—both hardness and easiness—for voter-addition control for a broad range of types of follower functions (although not for the case we study here).

Turning to our results for our model, unlike our earlier two models, the addition of groups very broadly increases P complexity levels to NP-completeness. For the three cases covered by the following theorem, the analogous result without groups is well known to be in P [BTT92, HHR07]. And the NP-completeness claims of Theorem 6.1 are each proved by building an appropriate reduction from an NP-complete problem, in particular Exact Cover by 3-Sets. Plurality-CCPVG-TP is also NP-complete, but we do not state that in the theorem since this follows immediately from the known result (see Hemaspaandra, Hemaspaandra, and Rothe [HHR07]) that Plurality-CCPV-TP is NP-complete.

Theorem 6.1 *Each of Plurality-CC $\begin{bmatrix} \text{PVG-TE} \\ \text{AVG} \\ \text{DVG} \end{bmatrix}$ is NP-complete.*

In the appendix, we include a proof of the PVG-TE case.

7 Conclusions and Open Directions

We introduced three models of partition control—equipartition, multipartition, and partition by groups—that seek to for many cases more closely model real-world situations than the twenty-year-old standard benchmark set. We obtained a number of results on the complexity of our new models with respect to important election systems, especially plurality, the most prevalent of election systems. We established many natural examples where the variants are of the same complexity as their analogous standard benchmark model, and also established many natural examples where the variants’ complexity increases relative to the analogous standard benchmark model.

The current version of the paper focused on the unique-winner model and so-called constructive control, but a later version will cover the nonunique-winner model (in which our results broadly still hold) and so-called destructive control.

Open directions include studying combinations of our new models, seeking additional models to better capture real-world settings, doing studies to determine how well various models do capture real-world settings, extending the present study to partial-information models, seeking typical-case hardness results, pursuing the additional multipartition studies mentioned at the end of the multipartition section, and resolving the complexity of weakCondorcet-CCREPC-TE to see whether it provides an additional natural example of an increasing complexity level (see Theorem 4.4 and the comments preceding it).

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A Appendix: Selected Additional Proofs

A.1 Proof of the Plurality Part of Theorem 4.1

We now prove the claim that Plurality-CCEPV-TE belongs to P.

Proof.

The example given in the main text immediately after the statement of Theorem 4.1 shows that we have to be really careful about what assumptions we make in our proof. However, we indeed can show that our control problem is in P. First of all, note that p can be made a unique winner by EPV-TE if and only if there exists an equipartition (V_1, V_2) such that p is the unique winner of (C, V_1) and

1. (C, V_2) has c as unique winner and p defeats c , or
2. (C, V_2) has p as a winner, or
3. (C, V_2) has more than one winner.

These three conditions can be checked as follows in polynomial time.

1. For every $c \in C - \{p\}$ such that p defeats c , for every $k_p \leq \text{score}_V(p)$ and for every $k_c \leq \text{score}_V(c)$, put k_p votes for p in V_1 and the remaining votes for p in V_2 and put k_c votes for c in V_2 and the remaining votes for c in V_1 .

We will now check whether (V_1, V_2) can be extended to a desired equipartition. If p is not the unique winner in V_1 or c is not the unique winner in V_2 , then this is not possible and we move on to the next loop iteration.

Otherwise, for each $d \in C - \{p, c\}$, put as many votes for d as possible into V_1 while keeping p the unique winner in V_1 (i.e., $\min(\text{score}_V(d), k_p - 1)$ votes).

If $\|V_1\| < \lfloor \|V\|/2 \rfloor$, there are not enough votes in V_1 and we move on to the next loop iteration. Otherwise, move votes for candidates in $C - \{p, c\}$ from V_1 to V_2 if this is possible while keeping c the unique winner in V_2 . Keep doing this until (V_1, V_2) becomes an equipartition, in which case we have found a successful equipartition, or until it is not possible to move votes for candidates in $C - \{p, c\}$ from V_1 to V_2 while keeping c the unique winner in V_2 , in which case we move on to the next loop iteration.

2. This is similar to the previous case. For every $0 < k_p \leq \text{score}_V(p)$ put k_p votes for p in V_1 and the remaining votes for p in V_2 .

We now will check whether (V_1, V_2) can be extended to a desired equipartition. For each $d \in C - \{p\}$, put as many votes for d as possible into V_1 while keeping p the unique winner in V_1 (i.e., $\min(\text{score}_V(d), k_p - 1)$ votes).

If $\|V_1\| < \lfloor \|V\|/2 \rfloor$, there are not enough votes in V_1 and we move on to the next loop iteration. Otherwise, move votes for candidates in $C - \{p\}$ from V_1 to V_2 if this is possible while keeping p a winner in V_2 . Keep doing this until (V_1, V_2) becomes an equipartition, in which case we have found a successful equipartition, or until it is

not possible to move votes for candidates in $C - \{p\}$ from V_1 to V_2 while keeping p a winner in V_2 , in which case we move on to the next loop iteration.

3. The case that p is one of the winners of (C, V_2) has been handled in the previous case, so it suffices to handle the case where (C, V_2) has at least two winners in $C - \{p\}$.

For every $c, c' \in C$ such that $\|\{p, c, c'\}\| = 3$, for every $k_p \leq \text{score}_V(p)$, and for every $k_c \leq \min(\text{score}_V(c), \text{score}_V(c'))$, put k_p votes for p in V_1 and the remaining votes for p in V_2 , put k_c votes for c in V_2 and the remaining votes for c in V_1 , and put k_c votes for c' in V_2 and the remaining votes for c' in V_1 .

We now will check whether (V_1, V_2) can be extended to a desired equipartition. If p is not the unique winner in V_1 or c and c' are not winners in V_2 , then this is not possible and we move on to the next loop iteration.

Otherwise, for each $d \in C - \{p, c, c'\}$, put as many votes for d as possible into V_1 while keeping p the unique winner in V_1 (i.e., $\min(\text{score}_V(d), k_p - 1)$ votes).

If $\|V_1\| < \lfloor \|V\|/2 \rfloor$, there are not enough votes in V_1 and we move on to the next loop iteration. Otherwise, move votes for candidates in $C - \{p, c, c'\}$ from V_1 to V_2 if this is possible while keeping c and c' winners in V_2 . Keep doing this until (V_1, V_2) becomes an equipartition, in which case we have found a successful equipartition, or until it is not possible to move votes for candidates in $C - \{p, c, c'\}$ from V_1 to V_2 while keeping c and c' winners in V_2 , in which case we move on to the next loop iteration.

□

A.2 Proof of Theorem 4.3

We prove Theorem 4.3, namely, that there exists an election system \mathcal{E} , whose winner problem is in P, such that \mathcal{E} -CCPV-TP is NP-complete, yet \mathcal{E} -CCEPV-TP belongs to P.

Proof. We define \mathcal{E} as follows. On input (C, V) :

- If $\|C\| \leq 4$ and $(C \cap \{0, 1, 2, 3\} = \{0, 2\} \text{ or } C \cap \{0, 1, 2, 3\} = \{1, 3\})$, then the winners are the approval winners of $(C - \{0, 1, 2, 3\}, V)$.
- If $\|C\| \leq 4$ and $C \cap \{0, 1, 2, 3\} \neq \{0, 2\}$ and $C \cap \{0, 1, 2, 3\} \neq \{1, 3\}$, there are no winners.
- If $\|C\| > 4$ and $\{0, 1, 2, 3\} \subseteq C$, then $\|V\| \bmod 4$ is a winner and if $(C - \{0, 1, 2, 3\}, V)$ has a unique approval winner, then that candidate is also a winner. There are no other winners.
- If $\|C\| > 4$ and $\{0, 1, 2, 3\} \not\subseteq C$, there are no winners.

We first show that \mathcal{E} -CCEPV-TP is in P. This is easy. If $\|C\| \leq 4$, there are no winners in the runoff, since 0, 1, 2, and 3 do not participate in the runoff. If $\|C\| > 4$

and $\{0, 1, 2, 3\} \not\subseteq C$, there are no candidates in the runoff. If $\|C\| > 4$ and $\{0, 1, 2, 3\} \subseteq C$, there are at most four candidates in the runoff. The candidates in $\{0, 1, 2, 3\}$ that participate in the runoff are exactly $\|V_1\| \bmod 4$ and $\|V_2\| \bmod 4$ for partition (V_1, V_2) . But if (V_1, V_2) is an equipartition, it is never the case that $\{\|V_1\| \bmod 4, \|V_2\| \bmod 4\} = \{0, 2\}$ or $\{\|V_1\| \bmod 4, \|V_2\| \bmod 4\} = \{1, 3\}$ and so there are no winners in the runoff.

To show that \mathcal{E} -CCPV-TP is NP-complete, we reduce from approval-CCPV-TE, which is NP-complete [HHR07]. Let (C, V) be an election and let p be the preferred candidate. Assume that $C \cap \{0, 1, 2, 3\} = \emptyset$. Let $\widehat{C} = C \cup \{0, 1, 2, 3\}$ and let \widehat{V} consist of the voters in V (extended to \widehat{C} by not approving of candidates in $\{0, 1, 2, 3\}$) plus two additional voters that don't approve of any candidate if $\|V\|$ is even and one additional voter that doesn't approve of any candidate if $\|V\|$ is odd. We claim that p can be made the unique approval winner in (C, V) by PV-TE if and only if p can be made the unique \mathcal{E} winner in $(\widehat{C}, \widehat{V})$ by PV-TP.

First suppose that (V_1, V_2) is a partition of V that makes p the unique approval winner by PV-TE. If $\|V\|$ is odd, add the one additional voter that doesn't approve of any candidate to V_1 or V_2 in such a way that $\|V_1\| \bmod 4 \neq \|V_2\| \bmod 4$. If $\|V\|$ is even and $\|V_1\| \bmod 4 = \|V_2\| \bmod 4$, add the two additional voters to V_1 . If $\|V\|$ is even and $\|V_1\| \bmod 4 \neq \|V_2\| \bmod 4$, add one additional voter to V_1 and one additional voter to V_2 . In all cases, we now have a partition $(\widehat{V}_1, \widehat{V}_2)$ of \widehat{V} with the same unique approval winners as before (when restricting the candidates to C) and such that $\{\|V_1\| \bmod 4, \|V_2\| \bmod 4\} = \{0, 2\}$ or $\{\|V_1\| \bmod 4, \|V_2\| \bmod 4\} = \{1, 3\}$. This immediately implies that this partition makes p the unique winner in \mathcal{E} -CCPV-TP.

For the converse, suppose that $(\widehat{V}_1, \widehat{V}_2)$ is a partition of \widehat{V} that makes p the unique \mathcal{E} winner by PV-TP. Then $(\widehat{V}_1, \widehat{V}_2)$ makes p the unique approval winner in (C, \widehat{V}) by PV-TE. Now simply delete the additional voters that don't approve of any candidate to obtain partition (V_1, V_2) of V that makes p the unique approval winner in (C, V) by PV-TE. \square

A.3 Proof of the PVG-TE Part of Theorem 6.1

We now prove the claim that Plurality-CCPVG-TE is NP-complete.

Proof. We reduce from X3C. Given a set $B = \{b_1, \dots, b_{3m}\}$, $m > 1$, and a collection $\mathcal{S} = \{S_1, \dots, S_n\}$ of subsets $S_i = \{b_{i,1}, b_{i,2}, b_{i,3}\} \subseteq B$ with $\|S_i\| = 3$ for each i , $1 \leq i \leq n$.

We assume $n > m + 1$. We may safely make this assumption, as X3C still remains NP-complete under this restriction. $n < m$ is automatically a no instance and the two cases $n = m$ and $n = m + 1$ are solvable in polynomial time. Thus the problem is still NP-complete under the restriction $n > m + 1$.

Define the election (C, V) , where $C = \{p, c, d, e\} \cup B$ is the set of candidates, p is the distinguished candidate, and V consists of the following $n + 3$ groups of voters. As a shorthand, when specifying votes we will sometimes include a set of candidates, when resolving those as any linear ordering among those voters (e.g., lexicographic) will be fine for the vote's role in the proof. For example, $p > S > b$, where $S = \{z, a, w\}$, may be taken as a shorthand for $p > a > w > z > b$.

- For each i , $1 \leq i \leq n$, there is a group, G_i , with six voters of the form:
 - $p > C - \{p\}$,
 - $p > C - \{p\}$,
 - $b_{i,1} > C - \{b_{i,1}, p\} > p$,
 - $b_{i,2} > C - \{b_{i,2}, p\} > p$,
 - $b_{i,3} > C - \{b_{i,3}, p\} > p$, and
 - $e > C - \{e, p\} > p$.
- There is a group G_B consisting of the following voters:
 - Let $\ell_j = \|\{S_i \in \mathcal{S} | b_j \in S_i\}\|$ for all j , $1 \leq j \leq 3m$. For each j , $1 \leq j \leq 3m$, there are $2n - \ell_j$ voters of the form $b_j > C - \{b_j, p\} > p$.
 - There are $2m$ voters of the form $p > C - \{p\}$.
 - There are $n + m - 1$ voters of the form $e > C - \{e, p\} > p$.
 - There is one voter v_c of the form $c > C - \{c, p\} > p$.
- There is a group G_c containing $2(n + m) + 1$ voters of the form $c > C - \{c, p\} > p$.
- There is a group G_d containing $2(n + m) + 1$ voters of the form $d > C - \{d, p\} > p$.

It is easy to see that in this election each $b_j \in B$ has a score of $2n$, candidate c has a score of $2(n + m) + 2$, candidate d has a score of $2(n + m) + 1$, candidate e has a score of $2n + m - 1$, and the distinguished candidate p has a score of $2(n + m)$.

We claim that B has an exact cover B' if and only if p can be made the unique winner of the election by control by partition of voters in the TE model.

Suppose B has an exact cover B' . Partition the set of voters as follows. Let V_2 contain the m groups corresponding to B' and the groups G_c and G_d . Let $V_1 = V - V_2$. Candidate p is the unique winner of subelection (C, V_1) , since p has a score of $2n$, each $b_j \in B$ has a score of $2n - 1$, candidate c has a score of 1, candidate d has no points at all, and candidate e has $2n - 1$ points. There is no unique winner in subelection (C, V_2) , since candidates c and d tie for first place, eliminating each other. Thus only candidate p moves to the final round of the election, and p is the unique winner of the final round.

For the converse direction, suppose p can be made a winner of the election by partition of voters in the TE model. Without loss of generality, assume that p is the unique winner of subelection (C, V_1) . Since both candidates c and d accumulate $2(n + m) + 1$ points in their respective groups, and all the other candidates p , e , and each $b_j \in B$, $1 \leq j \leq 3m$, have less than $2(n + m) + 1$ overall points, c and d have to eliminate each other in subelection (C, V_2) . Candidate c has an additional vote in group G_B , which can only be in subelection (C, V_1) , as otherwise c would be the unique winner of (C, V_2) . Since in group G_B candidate e beats p by $n - m - 1$ points, there have to be at least m additional groups from G_i in subelection (C, V_1) (this is the only way p can gain more points than e). On the other hand, there can

be at most m groups from G_i in subelection (C, V_1) , as otherwise there would exist at least one candidate in B who would beat p there. Thus there must be exactly m groups from G_i in subelection (C, V_1) , and these groups have to correspond to an exact cover of B , since otherwise p cannot beat all the b_j 's. \square