

Arithmetical Congruence Preservation: from Finite to Infinite

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Abstract

Various problems on integers lead to the class of congruence preserving functions on rings, i.e. functions verifying $a - b$ divides $f(a) - f(b)$ for all a, b . We characterized these classes of functions in terms of sums of rational polynomials (taking only integral values) and the function giving the least common multiple of $1, 2, \dots, k$. The tool used to obtain these characterizations is “lifting”: if $\pi: X \rightarrow Y$ is a surjective morphism, and f a function on Y a lifting of f is a function F on X such that $\pi \circ F = f \circ \pi$. In this paper we relate the finite and infinite notions by proving that the finite case can be lifted to the infinite one. For p -adic and profinite integers we get similar characterizations via lifting. We also prove that lattices of recognizable subsets of \mathbb{Z} are stable under inverse image by congruence preserving functions.

1 Introduction

A function f (on \mathbb{N} or \mathbb{Z}) is said to be congruence preserving if $a - b$ divides $f(a) - f(b)$. Polynomial functions are obvious examples of congruence preserving functions. In [3, 4] we characterized this notion (which we named “functions having the integral difference ratio property”) for functions $\mathbb{N} \rightarrow \mathbb{Z}$ and $\mathbb{Z} \rightarrow \mathbb{Z}$. In [5] we extended the characterization to functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ (for a suitable extension of the notion of congruence preservation).

In the present paper, we prove in §2 that every congruence preserving function $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ can be lifted to a congruence preserving function $\mathbb{N} \rightarrow \mathbb{N}$ (i.e. it is the projection of such a function). As a corollary (i) we show that such a lift also works replacing \mathbb{N} with $\mathbb{Z}/qn\mathbb{Z}$ and (ii) and we give an alternative

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proof of a representation (obtained in [5]) of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ as linear sums of “rational” polynomials.

In §3 we consider the rings of p -adic integers (resp. profinite integers) and prove that congruence preserving functions are inverse limits of congruence preserving functions on the $\mathbb{Z}/p^k\mathbb{Z}$ (resp. on the $\mathbb{Z}/n\mathbb{Z}$). Considering the Mahler representation of continuous functions by Newton series, we prove that congruence preserving functions correspond to those series for which the linear coefficient with rank k is divisible by the least common multiple of $1, \dots, k$.

We proved in [2] that lattices of regular subsets of \mathbb{N} are closed under inverse image by congruence preserving functions: in §4, we extend this result to functions $\mathbb{Z} \rightarrow \mathbb{Z}$.

2 Congruence preservation: exchanging finite and infinite

We characterize congruence preserving functions on $\mathbb{Z}/n\mathbb{Z}$ by first lifting each such function into a congruence preserving function $\mathbb{N} \rightarrow \mathbb{N}$. In a second step, we use our characterization of congruence preserving functions $\mathbb{N} \rightarrow \mathbb{Z}$ to characterize the congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$.

Definition 1. Let X be a subset of a commutative (semi-)ring $(R, +, \times)$. A function $f: X \rightarrow R$ is said to be congruence preserving if

$$\forall x, y \in X \quad \exists d \in R \quad f(x) - f(y) = d(x - y), \quad \text{i.e. } x - y \text{ divides } f(x) - f(y).$$

Definition 2 (Lifting). Let $\sigma: X \rightarrow N$ and $\rho: Y \rightarrow M$ be surjective maps. A function $F: X \rightarrow Y$ is said to be a (σ, ρ) -lifting of a function $f: N \rightarrow M$ (or simply lifting if σ, ρ are clear from the context) if the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{F} & Y \\ \sigma \downarrow & & \downarrow \rho \\ N & \xrightarrow{f} & M \end{array} \quad \text{i.e.} \quad \rho \circ F = f \circ \sigma.$$

We will consider elements of $\mathbb{Z}/k\mathbb{Z}$ as integers and vice versa via the following maps.

Notation 3. 1. Let $\pi_k: \mathbb{Z} \rightarrow \mathbb{Z}/k\mathbb{Z}$ be the canonical surjective homomorphism associating to an integer its class in $\mathbb{Z}/k\mathbb{Z}$.

2. Let $\iota_k: \mathbb{Z}/k\mathbb{Z} \rightarrow \mathbb{N}$ be the injective map associating to an element $x \in \mathbb{Z}/k\mathbb{Z}$ its representative in $\{0, \dots, k-1\}$.

3. Let $\pi_{n,m}: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ be the map $\pi_{n,m} = \pi_m \circ \iota_n$. In case m divides n , $\pi_{n,m}$ is a surjective homomorphism.

If $m \leq n$ let $\iota_{m,n}: \mathbb{Z}/m\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ be the injective map $\iota_{m,n} = \pi_n \circ \iota_m$.

Lemma 4. If m divides n , $\pi_m = \pi_{n,m} \circ \pi_n$.

The next theorem insures that congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ can be lifted to congruence preserving functions $\mathbb{N} \rightarrow \mathbb{Z}$.

Theorem 5 (Lifting functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ to $\mathbb{N} \rightarrow \mathbb{N}$). *Let $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ with $m \geq 2$. The following conditions are equivalent:*

- (1) f is congruence preserving.
- (2) f can be (π_n, π_n) -lifted to a congruence preserving function $F: \mathbb{N} \rightarrow \mathbb{N}$.

In view of applications in the context of p -adic and profinite integers, we state and prove a slightly more general version with an extended notion of congruence preservation defined below.

Definition 6. A function $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving if

$$\text{for all } x, y \in \mathbb{Z}/n\mathbb{Z}, \quad \pi_{n,m}(x - y) \text{ divides } f(x) - f(y) \text{ in } \mathbb{Z}/m\mathbb{Z}. \quad (1)$$

Theorem 7 (Lifting functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ to $\mathbb{N} \rightarrow \mathbb{N}$). *Let $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ with m divides n and $m \geq 2$. The following conditions are equivalent:*

- (1) f is congruence preserving.
- (2) f can be (π_n, π_m) -lifted to a congruence preserving function $F: \mathbb{N} \rightarrow \mathbb{N}$.
- (3) f can be (π_n, π_m) -lifted to a congruence preserving function $F: \mathbb{N} \rightarrow \mathbb{Z}$.

Proof. (2) \Rightarrow (3) is trivial.

(3) \Rightarrow (1). Assume f lifts to the congruence preserving function $F: \mathbb{N} \rightarrow \mathbb{Z}$. The following diagram commutes

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{F} & \mathbb{Z} \\ \pi_n \downarrow & & \downarrow \pi_m \\ \mathbb{Z}/n\mathbb{Z} & \xrightarrow{f} & \mathbb{Z}/m\mathbb{Z} \end{array} \quad \text{and thus} \quad \begin{cases} \pi_m \circ F &= f \circ \pi_n \\ f &= \pi_m \circ F \circ \iota_n \end{cases}$$

Let $x, y \in \mathbb{Z}/n\mathbb{Z}$. As F is congruence preserving, $\iota_n(x) - \iota_n(y)$ divides $F(\iota_n(x)) - F(\iota_n(y))$, hence $F(\iota_n(x)) - F(\iota_n(y)) = (\iota_n(x) - \iota_n(y))\delta$. Since π_m is a morphism and $\pi_m \circ \iota_n = \pi_{n,m}$, we get $\pi_m(F(\iota_n(x))) - \pi_m(F(\iota_n(y))) = \pi_{n,m}(x - y)\pi_{n,m}(\delta)$. As F lifts f we have $\pi_m(F(\iota_n(x))) - \pi_m(F(\iota_n(y))) = f(x) - f(y)$ whence (1).

(1) \Rightarrow (2). By induction on $t \in \mathbb{N}$ we define a sequence of functions $\varphi_t: \{0, \dots, t\} \rightarrow \mathbb{N}$ for $t \in \mathbb{N}$ such that φ_{t+1} extends φ_t and (*) and (**) below hold.

$$\begin{cases} (*) & \varphi_t \text{ is congruence preserving,} \\ (**) & \pi_m(\varphi_t(u)) = f(\pi_n(u)) \text{ for all } u \in \{0, \dots, t\}. \end{cases}$$

Basis. We choose $\varphi_0(0) \in \mathbb{N}$ such that $\pi_m(\varphi_0(0)) = f(\pi_n(0))$. Properties (*) and (**) clearly hold for φ_0 .

Induction: from φ_t to φ_{t+1} . Since the wanted φ_{t+1} has to extend φ_t to the

domain $\{0, \dots, t, t+1\}$, we only have to find a convenient value for $\varphi_{t+1}(t+1)$. By the induction hypothesis, (*) and (**) hold for φ_t ; in order for φ_{t+1} to satisfy (*) and (**), we have to find $\varphi_{t+1}(t+1)$ such that $t+1-i$ divides $\varphi_{t+1}(t+1) - \varphi_t(i)$, for $i = 0, \dots, t$, and $\pi_m(\varphi_{t+1}(t+1)) = f(\pi_n(t+1))$. Rewritten in terms of congruences, these conditions amount to say that $\varphi_{t+1}(t+1)$ is a solution of the following system of congruence equations:

$$\left. \begin{array}{l} \star(0) \\ \star(i) \\ \star(t-1) \\ \star\star \end{array} \right\} \begin{array}{l} \left| \begin{array}{lcl} \varphi_{t+1}(t+1) & \equiv & \varphi_t(0) \\ & \vdots & \\ \varphi_{t+1}(t+1) & \equiv & \varphi_t(i) \\ & \vdots & \\ \varphi_{t+1}(t+1) & \equiv & \varphi_t(t-1) \\ \varphi_{t+1}(t+1) & \equiv & \iota_m(f(\pi_n(t+1))) \end{array} \right. \\ \begin{array}{l} (\text{mod } t+1) \\ (\text{mod } t+1-i) \\ (\text{mod } 2) \\ (\text{mod } m) \end{array} \end{array} \right\} \quad (2)$$

Recall the Generalized Chinese Remainder Theorem (cf. §3.3, exercise 9 p. 114, in Rosen's textbook [12]): a system of congruence equations

$$\bigwedge_{i=0, \dots, t} x \equiv a_i \pmod{n_i}$$

has a solution if and only if $a_i \equiv a_j \pmod{\gcd(n_i, n_j)}$ for all $0 \leq i < j \leq t$.

Let us show that the conditions of application of the Generalized Chinese Remainder Theorem are satisfied for system (2).

- Lines $\star(i)$ and $\star(j)$ of system (2) (with $0 \leq i < j \leq t-1$).
Every common divisor to $t+1-i$ and $t+1-j$ divides their difference $j-i$ hence $\gcd(t+1-i, t+1-j)$ divides $j-i$. Since φ_t satisfies (*), $j-i$ divides $\varphi_t(j) - \varphi_t(i)$ and a fortiori $\gcd(t+1-i, t+1-j)$ divides $\varphi_t(j) - \varphi_t(i)$.
- Lines $\star(i)$ and $\star\star$ of system (2) (with $0 \leq i \leq t-1$).
Let $d = \gcd(t+1-i, m)$. We have to show that d divides $\iota_m(f(\pi_n(t+1))) - \varphi_t(i)$. Since f is congruence preserving, $\pi_{n,m}(\pi_n(t+1) - \pi_n(i))$ divides $f(\pi_n(t+1)) - f(\pi_n(i))$. As m divides n , by Lemma 4, $\pi_{n,m}(\pi_n(t+1) - \pi_n(i)) = \pi_m(t+1) - \pi_m(i) = \pi_m(t+1-i)$ and $f(\pi_n(t+1)) - f(\pi_n(i)) = k\pi_m(t+1-i)$ for some $k \in \mathbb{Z}/m\mathbb{Z}$. Applying ι_m , there exists $\lambda \in \mathbb{Z}$ such that

$$\iota_m(f(\pi_n(t+1))) - \iota_m(f(\pi_n(i))) = \iota_m(k)\iota_m(\pi_m(t+1-i)) + \lambda m$$

as $\iota_m(\pi_m(u)) \equiv u \pmod{m}$ for every $u \in \mathbb{Z}$, there exists $\mu \in \mathbb{Z}$ such that

$$\iota_m(f(\pi_n(t+1))) - \iota_m(f(\pi_n(i))) = \iota_m(k)(t+1-i) + \mu m + \lambda m. \quad (3)$$

Since φ_t satisfies (**), we have $\pi_m(\varphi_t(i)) = f(\pi_n(i))$ hence $\varphi_t(i) \equiv \iota_m(f(\pi_n(i))) \pmod{m}$. Thus equation (3) can be rewritten

$$\iota_m(f(\pi_n(t+1))) - \varphi_t(i) = (t+1-i)\iota_m(k) + \nu m \quad \text{for some } \nu. \quad (4)$$

As d divides m and $t+1-i$, (4) shows that d divides $\iota_n(f(\pi_n(t+1))) - \varphi_t(i)$ as wanted.

Thus, we can apply the Generalized Chinese Theorem and get the wanted value of $\varphi_{t+1}(t+1)$, concluding the induction step.

Finally, taking the union of the φ_t 's, $t \in \mathbb{N}$, we get a function $F : \mathbb{N} \rightarrow \mathbb{N}$ which is congruence preserving and lifts f . \square

Example 8 (counterexample to Theorem 7). *Lemma 4 and Theorem 7 do not hold if m does not divide n . Consider $f : \mathbb{Z}/6\mathbb{Z} \rightarrow \mathbb{Z}/8\mathbb{Z}$ defined by $f(0) = 0$, $f(1) = 3$, $f(2) = 4$, $f(3) = 1$, $f(4) = 4$, $f(5) = 7$. Note first that, in $\mathbb{Z}/8\mathbb{Z}$, 1, 3 and 5 are invertible, hence f is congruence preserving iff for $k \in \{2, 4\}$, for all $x \in \mathbb{Z}/6\mathbb{Z}$, k divides $f(x+k) - f(x)$ and this holds; nevertheless, f has no congruence preserving lift $F : \mathbb{Z} \rightarrow \mathbb{Z}$. If such a lift F existed, we should have*

(1) *because F lifts f , $\pi_8(F(0)) = f(\pi_6(0)) = 0$ and $\pi_8(F(8)) = f(\pi_6(8)) = f(2) = 4$;*

(2) *as F is congruence preserving, 8 must divide $F(8) - F(0)$; we already noted that 8 divides $F(0)$, hence 8 divides $F(8)$ and $\pi_8(F(8)) = 0$, contradicting $\pi_8(F(8)) = 4$.*

Note that $\pi_{6,8}$ is neither a homomorphism nor surjective and $0 = \pi_8(8) \neq \pi_{6,8} \circ \pi_6(8) = 2$.

As a first corollary of Theorem 7 we get a new proof of the representations of congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ as finite linear sums of polynomials with rational coefficients (cf. [5]). Let us recall the so-called binomial polynomials.

Definition 9. For $k \in \mathbb{N}$, let $P_k(x) = \binom{x}{k} = \frac{1}{k!} \prod_{\ell=0}^{k-1} (x - \ell)$.

Though P_k has rational coefficients, it maps \mathbb{N} into \mathbb{Z} . Also, observe that $P_k(x)$ takes value 0 for all $k > x$. This implies that for any sequence of integers $(a_k)_{k \in \mathbb{N}}$, the infinite sum $\sum_{k \in \mathbb{N}} a_k P_k(x)$ reduces to a finite sum for any $x \in \mathbb{N}$ hence defines a function $\mathbb{N} \rightarrow \mathbb{Z}$.

Definition 10. We denote by $\text{lcm}(k)$ the least common multiple of integers $1, \dots, k$ (with the convention $\text{lcm}(0) = 1$).

Definition 11. To each binomial polynomial P_k , $k \in \mathbb{N}$, we associate a function $P_k^{n,m} : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ which sends an element $x \in \mathbb{Z}/n\mathbb{Z}$ to $(\pi_m \circ P_k \circ \iota_n)(x) \in \mathbb{Z}/m\mathbb{Z}$.

In other words, consider the representative t of x lying in $\{0, \dots, n-1\}$, evaluate $P_k(t)$ in \mathbb{N} and then take the class of the results in $\mathbb{Z}/m\mathbb{Z}$.

Lemma 12. If $\text{lcm}(k)$ divides a_k in \mathbb{Z} , then the function $\pi_m(a_k) P_k^{n,m} : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ (represented by $a_k P_k$) is congruence preserving.

Proof. In [3] we proved that if $\text{lcm}(k)$ divides a_k then $a_k P_k$ is a congruence preserving function on \mathbb{N} . Let us now show that $\pi_m(a_k) P_k^{n,m}: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is also congruence preserving. Let $x, y \in \mathbb{Z}/n\mathbb{Z}$: as $a_k P_k$ is congruence preserving, $\iota_n(x) - \iota_n(y)$ divides $a_k P_k(\iota_n(x)) - a_k P_k(\iota_n(y))$. As π_m is a morphism, $\pi_m(\iota_n(x)) - \pi_m(\iota_n(y))$ divides $\pi_m(a_k) \pi_m(P_k(\iota_n(x))) - \pi_m(a_k) \pi_m(P_k(\iota_n(y))) = \pi_m(a_k) P_k^{n,m}(x) - \pi_m(a_k) P_k^{n,m}(y)$; as $\pi_m \circ \iota_n = \pi_{n,m}$ (Notation 3), we conclude that $\pi_m(a_k) P_k^{n,m}$ is congruence preserving. \square

Corollary 13 ([5]). *Let $1 \leq m = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$, p_i prime. Suppose m divides n and let $\nu(m) = \max_{i=1, \dots, \ell} p_i^{\alpha_i}$. A function $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving if and only if it is represented by a finite \mathbb{Z} -linear sum such that $\text{lcm}(k)$ divides a_k (in \mathbb{Z}) for all $k < \nu(m)$, i.e. $f = \sum_{k=0}^{\nu(m)-1} \pi_m(a_k) P_k^{n,m}$.*

Proof. Assume $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving. Applying Theorem 7, lift f to $F: \mathbb{N} \rightarrow \mathbb{N}$ which is congruence preserving.

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{F = \sum_{k=0}^{\nu(m)-1} a_k P_k} & \mathbb{Z} \\ \pi_n \downarrow & & \downarrow \pi_m \\ \mathbb{Z}/n\mathbb{Z} & \xrightarrow{f} & \mathbb{Z}/m\mathbb{Z} \end{array} \quad f \circ \pi_n = \pi_m \circ F$$

We proved in [5] that every congruence preserving function $F: \mathbb{N} \rightarrow \mathbb{N}$ is of the form $F = \sum_{k=0}^{\infty} a_k P_k$ where $\text{lcm}(k)$ divides a_k for all k . Since F lifts f , for $u \in \mathbb{Z}$, we have

$$\begin{aligned} f(\pi_n(u)) &= \pi_m(F(u)) = \pi_m\left(\sum_{k=0}^{\infty} a_k P_k(u)\right) \\ &= \sum_{k=0}^{\infty} \pi_m(a_k) \pi_m(P_k(u)) = \sum_{k=0}^{k=\nu(m)-1} \pi_m(a_k) \pi_m(P_k(u)) \quad (5) \end{aligned}$$

The last equality is obtained by noting that for $k \geq \nu(m)$, m divides $\text{lcm}(k)$ hence as a_k is a multiple of $\text{lcm}(k)$, $\pi_m(a_k) = 0$. From (5) we get $f(\pi_n(u)) = \sum_{k=0}^{k=\nu(m)-1} \pi_m(a_k) \pi_m(P_k(u)) = \pi_m\left(\sum_{k=0}^{k=\nu(m)-1} a_k P_k(u)\right)$. This proves that f is lifted to the rational polynomial function $\sum_{k=0}^{k=\nu(m)-1} a_k P_k$.

The converse follows from Lemma 12 and the fact that any finite sum of congruence preserving functions is congruence preserving. \square

As a second corollary of Theorem 7 we can lift congruence preserving functions $\mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$ to congruence preserving functions $\mathbb{Z}/qn\mathbb{Z} \rightarrow \mathbb{Z}/qn\mathbb{Z}$.

We state a slightly more general result.

Corollary 14. *Assume $m, n, q, r \geq 1$, m divides both n and s , and n, s both divide r . If $f: \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ is congruence preserving then it can be $(\pi_{r,n}, \pi_{s,m})$ -lifted to $g: \mathbb{Z}/r\mathbb{Z} \rightarrow \mathbb{Z}/s\mathbb{Z}$ which is also congruence preserving.*

Proof. Using Theorem 7, lift f to a congruence preserving $F : \mathbb{N} \rightarrow \mathbb{N}$ and set $g = \pi_s \circ F \circ \iota_r$. We show that the following diagram commutes:

$$\begin{array}{ccccc}
 \mathbb{N} & & & & \mathbb{N} \\
 & \swarrow \iota_r & & \searrow \pi_s & \\
 & \mathbb{Z}/r\mathbb{Z} & \xrightarrow{g} & \mathbb{Z}/s\mathbb{Z} & \\
 \pi_n \swarrow & \downarrow \pi_{r,n} & & \downarrow \pi_{s,m} & \searrow \pi_m \\
 & \mathbb{Z}/n\mathbb{Z} & \xrightarrow{f} & \mathbb{Z}/m\mathbb{Z} &
 \end{array}$$

$$\begin{aligned}
 \pi_{s,m} \circ g &= \pi_{s,m} \circ (\pi_s \circ F \circ \iota_r) \\
 &= (\pi_m \circ F) \circ \iota_r && \text{by Lemma 4 since } \pi_m = \pi_{s,m} \circ \pi_s \\
 &= (f \circ \pi_n) \circ \iota_r && \text{since } F \text{ lifts } f \\
 &= f \circ \pi_{r,n} && \text{since } \pi_n \circ \iota_r = \pi_{r,n}
 \end{aligned}$$

Thus, $\pi_{s,m} \circ g = f \circ \pi_{r,n}$, i.e. g lifts f .

Finally, if $x, y \in \mathbb{Z}/r\mathbb{Z}$ then $\iota_r(x) - \iota_r(y)$ divides $F(\iota_r(x)) - F(\iota_r(y))$ (by congruence preservation of F). Since π_s is a morphism and $\pi_s = \pi_{r,s} \circ \pi_r$, we deduce that $\pi_s(\iota_r(x)) - \pi_s(\iota_r(y)) = (\pi_{r,s} \circ \pi_r \circ \iota_r)(x) - (\pi_{r,s} \circ \pi_r \circ \iota_r)(y) = \pi_{r,s}(x - y)$ (recall $\pi_r \circ \iota_r$ is the identity on $\mathbb{Z}/r\mathbb{Z}$) divides $\pi_s(F(\iota_r(x))) - \pi_s(F(\iota_r(y))) = g(x) - g(y)$ (by definition of g). Thus, g is congruence preserving. \square

Remark 15. *The previous diagram is completely commutative: F lifts both f and g , and g lifts f : as r divides $x - \iota_r \circ \pi_r(x)$ for all x , and F is congruence preserving, r divides $F(x) - F \circ \iota_r \circ \pi_r(x)$, and because s divides r , $\pi_s \circ F(x) = \pi_s \circ F \circ \iota_r \circ \pi_r(x)$ hence $\pi_s \circ F = g \circ \pi_r = \pi_s \circ F \circ \iota_r \circ \pi_r$.*

3 Congruence preservation on p -adic/profinite integers

All along this section, p is a prime number; we study congruence preserving functions on the rings \mathbb{Z}_p of p -adic integers and $\widehat{\mathbb{Z}}$ of profinite integers. \mathbb{Z}_p is the projective limit $\varprojlim \mathbb{Z}/p^n\mathbb{Z}$ relative to the projections π_{p^n, p^m} . Usually, $\widehat{\mathbb{Z}}$ is defined as the projective limit $\varprojlim \mathbb{Z}/n\mathbb{Z}$ of the finite rings $\mathbb{Z}/n\mathbb{Z}$ relative to the projections $\pi_{n,m}$, for m dividing n . We here use the following equivalent definition which allows to get completely similar proofs for \mathbb{Z}_p and $\widehat{\mathbb{Z}}$.

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n!\mathbb{Z} = \{\hat{x} = (x_n)_{n=1}^\infty \in \prod_{n=1}^\infty \mathbb{Z}/n!\mathbb{Z} \mid \forall m < n, x_m \equiv x_n \pmod{m!}\}$$

Recall that \mathbb{Z}_p (resp. $\widehat{\mathbb{Z}}$) contains the ring \mathbb{Z} and is a compact topological ring for the topology given by the ultrametric d such that $d(x, y) = 2^{-n}$ where n is largest such that p^n (resp. $n!$) divides $x - y$, i.e. x and y have the same first n digits in their base p (resp. base factorial) representation. We refer to

the Appendix for some basic definitions, representations and facts that we use about the compact topological rings \mathbb{Z}_p and $\widehat{\mathbb{Z}}$.

We first prove that on \mathbb{Z}_p and $\widehat{\mathbb{Z}}$ every congruence preserving function is continuous (Proposition 17).

Definition 16. 1. Let $\mu : \mathbb{N} \rightarrow \mathbb{N}$ be increasing. A function $\Psi : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ admits μ as modulus of uniform continuity if and only if $d(x, y) \leq 2^{-\mu(n)}$ implies $d(\Psi(x), \Psi(y)) \leq 2^{-n}$.
2. Ψ is 1-Lipschitz if it admits the identity as modulus of uniform continuity.

Since the rings \mathbb{Z}_p and $\widehat{\mathbb{Z}}$ are compact, every continuous function admits a modulus of uniform continuity.

Proposition 17. Every congruence preserving function $\Psi : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ is 1-Lipschitz. Idem with $\widehat{\mathbb{Z}}$ in place of \mathbb{Z}_p .

Proof. If $d(x, y) \leq 2^{-n}$ then p^n divides $x - y$ hence (by congruence preservation) p^n also divides $\Psi(x) - \Psi(y)$ which yields $d(\Psi(x), \Psi(y)) \leq 2^{-n}$. \square

The converse of Proposition 17 is false: a continuous function is not necessarily congruence preserving as will be seen in Example 28. Note the following

Corollary 18. There are functions $\mathbb{Z}_p \rightarrow \mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}} \rightarrow \widehat{\mathbb{Z}}$) which are not continuous hence not congruence preserving.

Proof. As \mathbb{Z}_p has cardinality 2^{\aleph_0} there are $2^{2^{\aleph_0}}$ functions $\mathbb{Z}_p \rightarrow \mathbb{Z}_p$. Since \mathbb{N} is dense in \mathbb{Z}_p , \mathbb{Z}_p is a separable space, hence there are at most 2^{\aleph_0} continuous functions. \square

In general an arbitrary continuous function on \mathbb{Z}_p is not the inverse limit of a sequence of functions $\mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$'s. However, this is true for congruence preserving functions. We first recall how any continuous function $\Psi : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ is the inverse limit of a sequence of an inverse system of continuous functions $\psi_n : \mathbb{Z}/p^{\mu(n)}\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$, $n \in \mathbb{N}$, i.e. the diagrams of Figure 1 commute for any $m \leq n$. For legibility, we use notations adapted to \mathbb{Z}_p : we write π_n^p for $\pi_{p^n} : \mathbb{Z}_p \rightarrow \mathbb{Z}/p^n\mathbb{Z}$, $\pi_{n,m}^p$ (resp. $\iota_{n,m}^p$) for π_{p^n, p^m} (resp. ι_{p^n, p^m}), and ι_n^p for $\iota_{p^n} : \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}_p$.

Proposition 19. Consider $\Psi : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ and a strictly increasing $\mu : \mathbb{N} \rightarrow \mathbb{N}$. Define $\psi_n : \mathbb{Z}/p^{\mu(n)}\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ as $\psi_n = \pi_n^p \circ \Psi \circ \iota_{\mu(n)}^p$ for all $n \in \mathbb{N}$. Then the following conditions are equivalent :

- (1) Ψ is uniformly continuous and admits μ as a modulus of uniform continuity.
- (2) For all $1 \leq m \leq n$, the diagrams of Figure 1 commute hence Ψ is the inverse limit of the ψ_n 's, $n \in \mathbb{N}$.

Idem with $\widehat{\mathbb{Z}}$ in place of \mathbb{Z}_p .

$$\begin{array}{ccccc}
\mathbb{Z}/p^{\mu(n)}\mathbb{Z} & \xrightarrow{\iota_{\mu(n)}^p} & \mathbb{Z}_p & \xrightarrow{\Psi} & \mathbb{Z}_p \\
& \searrow Id & \downarrow \pi_{\mu(n)}^p & & \downarrow \pi_n^p \\
& & \mathbb{Z}/p^{\mu(n)}\mathbb{Z} & \xrightarrow{\psi_n} & \mathbb{Z}/p^n\mathbb{Z} \\
& \nearrow \iota_{\mu(m),\mu(n)}^p & \downarrow \pi_{\mu(n),\mu(m)}^p & & \downarrow \pi_{n,m}^p \\
\mathbb{Z}/p^{\mu(m)}\mathbb{Z} & \xrightarrow{Id} & \mathbb{Z}/p^{\mu(m)}\mathbb{Z} & \xrightarrow{\psi_m} & \mathbb{Z}/p^m\mathbb{Z}
\end{array}$$

Figure 1: Ψ as the inverse limit of the ψ_n 's, $n \in \mathbb{N}$.

Proof. (1) and (2) are also equivalent to (3) below.

(3) For all $1 \leq m \leq n$, the lower half of the diagram of Figure 1 commutes.

(1) \Rightarrow (2). • We first show $\pi_n^p \circ \Psi = \psi_n \circ \pi_{\mu(n)}^p$. Let $u \in \mathbb{Z}_p$. Since $\pi_{\mu(n)}^p \circ \iota_{\mu(n)}^p$ is the identity on $\mathbb{Z}/p^{\mu(n)}\mathbb{Z}$, we have $\pi_{\mu(n)}^p(u) = \pi_{\mu(n)}^p(\iota_{\mu(n)}^p(\pi_{\mu(n)}^p(u)))$ hence $p^{\mu(n)}$ (considered as an element of \mathbb{Z}_p) divides the difference $u - \iota_{\mu(n)}^p(\pi_{\mu(n)}^p(u))$, i.e. the distance between these two elements is at most $2^{-\mu(n)}$. As μ is a modulus of uniform continuity for Ψ , the distance between their images under Ψ is at most 2^{-n} , i.e. p^n divides their difference, hence $\pi_n^p(\Psi(u)) = \pi_n^p(\Psi(\iota_{\mu(n)}^p(\pi_{\mu(n)}^p(u))))$. By definition, $\psi_n = \pi_n^p \circ \Psi \circ \iota_{\mu(n)}^p$. Thus, $\pi_n^p(\Psi(u)) = \psi_n(\pi_{\mu(n)}^p(u))$, i.e. Ψ lifts ψ_n .

• We now show $\pi_{n,m}^p \circ \psi_n = \psi_m \circ \pi_{\mu(n),\mu(m)}^p$. Since Ψ lifts ψ_m , we have

$$\begin{aligned}
\pi_m^p \circ \Psi &= \psi_m \circ \pi_{\mu(m)}^p \\
\text{hence } \pi_m^p \circ \Psi \circ \iota_{\mu(n)}^p &= \psi_m \circ \pi_{\mu(m)}^p \circ \iota_{\mu(n)}^p \\
\pi_{n,m}^p \circ \pi_n^p \circ \Psi \circ \iota_{\mu(n)}^p &= \psi_m \circ \pi_{\mu(n),\mu(m)}^p \circ \pi_{\mu(n)}^p \circ \iota_{\mu(n)}^p \\
\pi_{n,m}^p \circ \psi_n &= \psi_m \circ \pi_{\mu(n),\mu(m)}^p \quad \text{since } \pi_{\mu(n)}^p \circ \iota_{\mu(n)}^p \text{ is the identity.}
\end{aligned}$$

This last equality means that ψ_n lifts ψ_m .

(2) \Rightarrow (3). Trivial

(3) \Rightarrow (1). The fact that Ψ lifts ψ_n shows that two elements of \mathbb{Z}_p with the same first $\mu(n)$ digits (in the p -adic representation) have images with the same first n digits. This proves that μ is a modulus of uniform continuity for Ψ . $\square \square$

For congruence preserving functions $\Phi : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, the representation of Proposition 19 as an inverse limit gets smoother since then $\mu(n) = n$.

Theorem 20. For a function $\Phi : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$, letting $\varphi_n : \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^n\mathbb{Z}$ be defined as $\varphi_n = \pi_n^p \circ \Phi \circ \iota_n^p$, the following conditions are equivalent.

(1) Φ is congruence preserving.

$$\begin{array}{ccccc}
\mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{\iota_n^p} & \mathbb{Z}_p & \xrightarrow{\Phi} & \mathbb{Z}_p \\
& \searrow Id & \downarrow \pi_n^p & & \downarrow \pi_n^p \\
& & \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{\varphi_n} & \mathbb{Z}/p^n\mathbb{Z} \\
& \nearrow \iota_{m,n}^p & \downarrow \pi_{n,m}^p & & \downarrow \pi_{n,m}^p \\
\mathbb{Z}/p^m\mathbb{Z} & \xrightarrow{Id} & \mathbb{Z}/p^m\mathbb{Z} & \xrightarrow{\varphi_m} & \mathbb{Z}/p^m\mathbb{Z}
\end{array}$$

Figure 2: Φ as the inverse limit of the φ_n 's, $n \in \mathbb{N}$.

(2) Φ is 1-Lipschitz, all φ_n 's are congruence preserving and Φ is the inverse limit of the φ_n 's, $n \in \mathbb{N}$.

A similar equivalence also holds for functions $\Phi : \widehat{\mathbb{Z}} \rightarrow \widehat{\mathbb{Z}}$.

Proof. (1) and (2) are also equivalent to (3) and (4) below.

(3) All φ_n 's are congruence preserving and, for all $1 \leq m \leq n$, the diagrams of Figure 2 commute.

(4) All φ_n 's are congruence preserving and, for all $1 \leq m \leq n$, the lower half (dealing with φ_n and φ_m) of the diagrams of Figure 2 commute.

- (2) \Leftrightarrow (3) \Leftrightarrow (4). Instantiate Proposition 19 with μ the identity on \mathbb{N} .
- (1) \Rightarrow (2). Proposition 17 insures that Φ is 1-Lipschitz. We show that φ_n is congruence preserving. Since Φ is congruence preserving, if $x, y \in \mathbb{Z}/p^n\mathbb{Z}$ then $\iota_n^p(x) - \iota_n^p(y)$ divides $\Phi(\iota_n^p(x)) - \Phi(\iota_n^p(y))$. Now, the canonical projection π_n^p is a morphism hence $\pi_n^p(\iota_n^p(x)) - \pi_n^p(\iota_n^p(y))$ divides $\pi_n^p(\Phi(\iota_n^p(x))) - \pi_n^p(\Phi(\iota_n^p(y)))$. Recall that $\pi_n^p \circ \iota_n^p$ is the identity on $\mathbb{Z}/p^n\mathbb{Z}$. Thus, $x - y$ divides $\pi_n^p(\Phi(\iota_n^p(x))) - \pi_n^p(\Phi(\iota_n^p(y))) = \varphi_n(x) - \varphi_n(y)$ as wanted.
- (4) \Rightarrow (1). The fact that Φ lifts φ_n shows that two elements of \mathbb{Z}_p with the same first n digits (in the p -adic representation) have images with the same first n digits. This proves that Φ is 1-Lipschitz.

It remains to prove that Φ is congruence preserving. Let $x, y \in \mathbb{Z}_p$. Since φ_n is congruence preserving $\pi_n^p(x) - \pi_n^p(y)$ divides $\varphi_n(\pi_n^p(x)) - \varphi_n(\pi_n^p(y))$. Let

$$U_n^{x,y} = \{u \in \mathbb{Z}/p^n\mathbb{Z} \mid \varphi_n(\pi_n^p(x)) - \varphi_n(\pi_n^p(y)) = (\pi_n^p(x) - \pi_n^p(y))u\}.$$

If $m \leq n$ and $u \in U_n^{x,y}$ then, applying $\pi_{n,m}^p$ to the equality defining $U_n^{x,y}$, and using the commutative diagrams of Figure 2, we get

$$\begin{aligned}
\varphi_n(\pi_n^p(x)) - \varphi_n(\pi_n^p(y)) &= (\pi_n^p(x) - \pi_n^p(y))u \\
\pi_{n,m}^p(\varphi_n(\pi_n^p(x))) - \pi_{n,m}^p(\varphi_n(\pi_n^p(y))) &= (\pi_{n,m}^p(\pi_n^p(x)) - \pi_{n,m}^p(\pi_n^p(y)))\pi_{n,m}^p(u) \\
\varphi_m(\pi_{n,m}^p(\pi_n^p(x))) - \varphi_m(\pi_{n,m}^p(\pi_n^p(y))) &= (\pi_{n,m}^p(\pi_n^p(x)) - \pi_{n,m}^p(\pi_n^p(y)))\pi_{n,m}^p(u) \\
\varphi_m(\pi_m^p(x)) - \varphi_m(\pi_m^p(y)) &= (\pi_m^p(x) - \pi_m^p(y))\pi_{n,m}^p(u)
\end{aligned}$$

Thus, if $u \in U_n^{x,y}$ then $\pi_{n,m}^p(u) \in U_m^{x,y}$.

Consider the tree \mathcal{T} of finite sequences (u_0, \dots, u_n) such that $u_i \in U_i^{x,y}$ and $u_i = \pi_{n,i}^p(u_n)$ for all $i = 0, \dots, n$. Since each $U_n^{x,y}$ is nonempty, the tree \mathcal{T} is infinite. Since it is at most p -branching, using König's Lemma, we can pick an infinite branch $(u_n)_{n \in \mathbb{N}}$ in \mathcal{T} . This branch defines an element $z \in \mathbb{Z}_p$. The commutative diagrams of Figure 2 show that the sequences $(\pi_n^p(x) - \pi_n^p(y))_{n \in \mathbb{N}}$ and $\varphi_n(\pi_n^p(x)) - \varphi_n(\pi_n^p(y))$ represent $x - y$ and $\Phi(x) - \Phi(y)$ in \mathbb{Z}_p . Equalities $\varphi_m(\pi_m^p(x)) - \varphi_m(\pi_m^p(y)) = (\pi_m^p(x) - \pi_m^p(y)) \pi_{n,m}(u)$ show that (going to the projective limits) $\Phi(x) - \Phi(y) = (x - y)z$. This proves that Φ is congruence preserving. \square

Congruence preserving functions $\widehat{\mathbb{Z}} \rightarrow \widehat{\mathbb{Z}}$ are determined by their restrictions to \mathbb{N} since \mathbb{N} is dense in $\widehat{\mathbb{Z}}$. Let us state a (partial) converse result.

Theorem 21. *Every congruence preserving function $F : \mathbb{N} \rightarrow \mathbb{Z}$ has a unique extension to a congruence preserving function $\Phi : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ (resp. $\widehat{\mathbb{Z}} \rightarrow \widehat{\mathbb{Z}}$).*

Proof. Observe that \mathbb{N} is dense in \mathbb{Z}_p (resp. $\widehat{\mathbb{Z}}$) and congruence preservation implies uniform continuity. Thus, F has a unique uniformly continuous extension Φ to \mathbb{Z}_p (resp. $\widehat{\mathbb{Z}}$). To show that this extension Φ is congruence preserving, observe that Φ is the inverse limit of the $\varphi_n = \rho_n \circ \Phi \circ \iota_n$'s. Now, since ι_n has range \mathbb{N} , we see that $\varphi_n = \rho_n \circ F \circ \iota_n$ hence is congruence preserving as is F . Finally, Theorem 20 insures that Φ is also congruence preserving. \square

Polynomials in $\mathbb{Z}_p[X]$ obviously define congruence preserving functions $\mathbb{Z}_p \rightarrow \mathbb{Z}_p$. But non polynomial functions can also be congruence preserving.

Consequence 22. *The extensions to \mathbb{Z}_p and $\widehat{\mathbb{Z}}$ of the $\mathbb{N} \rightarrow \mathbb{Z}$ functions [3, 4]*

$$x \mapsto \lfloor e^{1/a} a^x x! \rfloor \quad (\text{for } a \in \mathbb{Z} \setminus \{0, 1\}) \quad , \quad x \mapsto \text{if } x = 0 \text{ then } 1 \text{ else } \lfloor e x! \rfloor$$

and the Bessel like function $f(n) = \sqrt{\frac{e}{\pi}} \times \frac{\Gamma(1/2)}{2 \times 4^n \times n!} \int_1^\infty e^{-t/2} (t^2 - 1)^n dt$ are congruence preserving.

We now characterize congruence preserving functions via their representation as infinite linear sums of the P_k s; this representation is similar to Mahler's characterization for continuous functions (Theorem 25). First recall the notion of valuation.

Definition 23. *The p -valuation (resp. the factorial valuation) $\text{Val}(x)$ of $x \in \mathbb{Z}_p$, or $x \in \mathbb{Z}/p^n\mathbb{Z}$ (resp. $x \in \widehat{\mathbb{Z}}$) is the largest s such that p^s (resp. $s!$) divides x or is $+\infty$ in case $x = 0$. It is also the length of the initial block of zeros in the p -adic (resp. factorial) representation of x .*

Note that for any polynomial P_k (or more generally any polynomial), the below diagram commutes for any $m \leq n$ (recall that $P_k^{p^n, p^n} = \pi_{p^n} \circ P_k \circ \iota_{p^n}$):

$$\begin{array}{ccc} \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{P_k^{p^n, p^n}} & \mathbb{Z}/p^n\mathbb{Z} \\ \pi_{p^n, p^m} \downarrow & & \downarrow \pi_{p^n, p^m} \\ \mathbb{Z}/p^m\mathbb{Z} & \xrightarrow{P_k^{p^m, p^m}} & \mathbb{Z}/p^m\mathbb{Z} \end{array} \quad \text{i.e.} \quad \pi_{p^n, p^m} \circ P_k^{p^n, p^n} = P_k^{p^m, p^m} \circ \pi_{p^n, p^m}$$

We now can define the interpretation $\widehat{P}_k(x)$ of $P_k(x)$ in \mathbb{Z}_p (similar for $\widehat{\mathbb{Z}}$).

Definition 24. Define $\widehat{P}_k: \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ as $\widehat{P}_k = \varprojlim_{n \in \mathbb{N}} P_k^{p^n, p^n}$. For $x \in \mathbb{Z}_p$, $x = (\varprojlim_{n \in \mathbb{N}} x_n)$, we have $\widehat{P}_k(x) = \varprojlim_{n \in \mathbb{N}} \pi_{p^n}(P_k(\iota_{p^n}(x_n)))$.

Moreover, the below diagrams commute for all n

$$\begin{array}{ccc} \mathbb{Z}_p & \xrightarrow{\widehat{P}_k} & \mathbb{Z}_p \\ \pi_n^p \downarrow & & \downarrow \pi_n^p \\ \mathbb{Z}/p^n\mathbb{Z} & \xrightarrow{P_k^{p^n, p^n}} & \mathbb{Z}/p^n\mathbb{Z} \\ \iota_{p^n} \downarrow & & \downarrow \iota_{p^n} \\ \mathbb{N} & \xrightarrow{P_k} & \mathbb{N} \end{array}$$

Theorem 25 (Mahler, 1956 [9]). 1. A series $\sum_{k \in \mathbb{N}} a_k \widehat{P}_k(x)$, $a_k \in \mathbb{Z}_p$, is convergent in \mathbb{Z}_p if and only if $\lim_{k \rightarrow \infty} a_k = 0$, i.e. the corresponding sequence of valuations $(\text{Val}(a_k))_{k \in \mathbb{N}}$ tends to $+\infty$.

2. The above series represent all uniformly continuous functions $\mathbb{Z}_p \rightarrow \mathbb{Z}_p$.

Idem with $\widehat{\mathbb{Z}}$.

We can also characterize of congruence preserving functions via their representation as infinite linear sums of the P_k s.

Theorem 26. A function $\Phi: \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ represented by a series $\Phi = \sum_{k \in \mathbb{N}} a_k \widehat{P}_k$ is congruence preserving if and only if $\text{lcm}(k)$ divides a_k for all k , i.e. $a_k = p^i b_k$ for $k \geq p^i$.

Proof. Suppose Φ is congruence preserving. By Theorem 20, Φ is uniformly continuous and by Theorem 25, $\Phi = \sum_{k \in \mathbb{N}} a_k \widehat{P}_k$ with $a_k \in \mathbb{Z}_p$. Substituting in $\varphi_n = \pi_n^p \circ \Phi \circ \iota_n^p$, we get $\varphi_n = \pi_n^p \circ (\sum_{k \in \mathbb{N}} a_k \widehat{P}_k) \circ \iota_n^p = \sum_{k \in \mathbb{N}} \pi_n^p(a_k) \pi_n^p \circ \widehat{P}_k \circ \iota_n^p = \sum_{k \in \mathbb{N}} \pi_n^p(a_k) P_k^{p^n, p^n}$. Theorem 20 insures that $\Phi = \varprojlim_{n \in \mathbb{N}} \varphi_n$ and the φ_n are congruence preserving on $\mathbb{Z}/p^n\mathbb{Z}$; thus by Corollary 13 : $\varphi_n = \sum_{k=0}^{\nu(n)-1} b_k^n P_k^{p^n, p^n}$, with $\text{lcm}(k)$ divides b_k^n for all $k \leq \nu(n) - 1$. We proved in [5]

that the $P_k^{p^n \cdot p^n}$ form a basis of the functions on $\mathbb{Z}/p^n\mathbb{Z}$, hence $\pi_n^p(a_k) = b_k^n$ and $lcm(k)$ divides $\pi_n^p(a_k)$. Noting that $Val(a_k) = Val(\pi_n^p(a_k))$ and applying Lemma 27, we deduce that $lcm(k)$ divides a_k , i.e. $\nu_p(k) \leq Val(a_k)$, and $a_k = p^{\nu_p(k)} b_k$. In particular, this implies that $d(a_k, 0) \leq 2^{-\nu_p(k)}$ and thus $\lim_{k \rightarrow \infty} a_k = 0$.

Conversely, if $\Phi = \sum_{k \in \mathbb{N}} a_k \widehat{P}_k$ and $lcm(k)$ divides a_k for all k , then $lcm(k)$ divides $\pi_n^p(a_k)$ for all n, k ; hence the associated φ_n are congruence preserving which implies that so is Φ . \square

Lemma 27. *Let $\nu_p(k)$ be the largest i such that $p^i \leq k < p^{i+1}$. In $\mathbb{Z}/p^n\mathbb{Z}$, $lcm(k)$ divides a number x iff $\nu_p(k) \leq Val(x)$.*

Proof. In $\mathbb{Z}/p^n\mathbb{Z}$ all numbers are invertible except multiples of p . Hence $lcm(k)$ divides x iff $p^{\nu_p(k)}$ divides x . \square

Example 28. *Let $\Phi = \sum_{k \in \mathbb{N}} a_k P_k$ with $a_k = p^{\nu_p(k)-1}$, with $\nu_p(k)$ as in Lemma 27. Φ is uniformly continuous by Theorem 25. By Lemma 27 $lcm(k)$ does not divide a_k hence by Theorem 26 Φ is not congruence preserving.*

4 Congruence preserving functions and lattices

A *lattice* of subsets of a set X is a family of subsets of X such that $L \cap M$ and $L \cup M$ are in \mathcal{L} whenever $L, M \in \mathcal{L}$. Let $f : X \rightarrow X$. A lattice \mathcal{L} of subsets of X is *closed* under f^{-1} if $f^{-1}(L) \in \mathcal{L}$ whenever $L \in \mathcal{L}$. *Closure under decrement* means closure under Suc^{-1} , where Suc is the successor function. For $L \subseteq \mathbb{Z}$ and $t \in \mathbb{Z}$, let $L - t = \{x - t \mid x \in L\}$.

Proposition 29. *Let X be \mathbb{N} or \mathbb{Z} or $\mathbb{N}_\alpha = \{x \in \mathbb{Z} \mid x \geq \alpha\}$ with $\alpha \in \mathbb{Z}$. For L a subset of X let $\mathcal{L}_X(L)$ be the family of sets of the form $\bigcup_{j \in J} \bigcap_{i \in I_j} X \cap (L - i)$ where J and the I_j 's are finite non empty subsets of \mathbb{N} . Then $\mathcal{L}_X(L)$ is the smallest sublattice of $\mathcal{P}(X)$, the class of subsets of X , containing L and closed under decrement.*

Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a non decreasing function, the following conditions are equivalent [2]:

- (1) $_{\mathbb{N}}$ For every finite subset L of \mathbb{N} , the lattice $\mathcal{L}_{\mathbb{N}}(L)$ is closed under f^{-1} .
- (2) $_{\mathbb{N}}$ The function f is congruence preserving and $f(a) \geq a$ for all $a \in \mathbb{N}$.
- (3) $_{\mathbb{N}}$ For every regular subset L of \mathbb{N} the lattice $\mathcal{L}_{\mathbb{N}}(L)$ is closed under f^{-1} .

We now extend this result to functions $\mathbb{Z} \rightarrow \mathbb{Z}$. First recall the notions of *recognizable* and *rational* subsets: a subset L of a monoid X is rational if it can be generated from finite sets by unions, products and stars; L is recognizable if there exists a morphism $\varphi : X \rightarrow M$, with M a finite monoid, and F a subset of M such that $L = \varphi^{-1}(F)$. For \mathbb{N} , recognizable and rational subsets coincide and are called regular subsets of \mathbb{N} . For \mathbb{Z} , recognizable subsets are finite unions of arithmetic sequences, while rational subsets are unions of the form $F \cup P \cup -N$, with F finite, and P, N two regular subsets of \mathbb{N} ; i.e. a recognizable subset of \mathbb{Z} is also rational, but the converse is false. It is known that

1. A subset $L \subseteq \mathbb{N}$ is regular if it is the union of a finite set with finitely many arithmetic progressions, i.e. $L = F \cup (R + d\mathbb{N})$ with $d \geq 1$, $F, R \subseteq \{x \mid 0 \leq x < d\}$ (possibly empty).
2. A subset $L \subseteq \mathbb{Z}$ is rational if it is of the form $L = L^+ \cup (-L^-)$ where L^+, L^- are regular subsets of \mathbb{N} , i.e. $L = -(d + S + d\mathbb{N}) \cup F \cup (d + R + d\mathbb{N})$ with $d \geq 1$, $R, S \subseteq \{x \mid 0 \leq x < d\}$, $F \subseteq \{x \mid -d < x < d\}$ (possibly empty). See [1].
3. A subset $L \subseteq \mathbb{Z}$ is recognizable if it is of the form $L = (F + d\mathbb{Z})$ with $d \geq 1$, $F \subseteq \{x \mid 0 \leq x < d\}$

Theorem 30. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a non decreasing function. The following conditions are equivalent:

- (1) $_{\mathbb{Z}}$ For every finite subset L of \mathbb{Z} , the lattice $\mathcal{L}_{\mathbb{Z}}(L)$ is closed under f^{-1} .
- (2) $_{\mathbb{Z}}$ The function f is congruence preserving and $f(a) \geq a$ for all $a \in \mathbb{Z}$.
- (3) $_{\mathbb{Z}}$ For every recognizable subset L of \mathbb{Z} the lattice $\mathcal{L}_{\mathbb{Z}}(L)$ is closed under f^{-1} .

Proof. For this proof, we need the \mathbb{Z} -version of Lemma 3.1 in [2].

Lemma 31. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a nondecreasing congruence preserving function. Then, for any set $L \subseteq \mathbb{Z}$, we have $f^{-1}(L) = \bigcup_{a \in f^{-1}(L)} \bigcap_{t \in L-a} (L-t)$.

Proof. Let $a \in f^{-1}(L)$. As $t \in L-a \Leftrightarrow a \in L-t$, we have $a \in \bigcap_{t \in L-a} L-t$, proving inclusion \subseteq .

For the other inclusion, let $b \in \bigcap_{t \in L-a} L-t$ with $a \in f^{-1}(L)$. To prove that $f(b) \in L$, we argue by way of contradiction. Suppose $f(b) \notin L$. Since $f(a) \in L$ we have $a \neq b$. The condition on f insures the existence of $k \in \mathbb{Z}$ such that $f(b) - f(a) = k(b-a)$. In fact, $k \in \mathbb{N}$ since f is nondecreasing.

Suppose first that $a < b$. Since $k \in \mathbb{N}$ and $f(a) + k(b-a) = f(b) \notin L$ there exists a least $r \in \mathbb{N}$ such that $f(a) + r(b-a) \notin L$. Moreover, $r \geq 1$ since $f(a) \in L$. Let $t = f(a) - a + (r-1)(b-a)$. By minimality of r , we get $t+a = f(a) + (r-1)(a-b) \in L$. Now $t+a \in L$ implies $t \in L-a$; as $b \in \bigcap_{t \in L-a} L-t$ this implies $b \in L-t$ hence $t+b \in L$. But $t+b = f(a) + r(b-a) \notin L$, this contradicts the definition of r .

Suppose next that $a > b$. Since $k \in \mathbb{N}$ and $f(b) + k(a-b) = f(a) \in L$ there exists a least $r \in \mathbb{N}$ such that $f(b) + r(a-b) \in L$. Moreover, $r \geq 1$ since $f(b) \notin L$. Let $t = f(b) - b + (r-1)(a-b)$. By minimality of r , we get $t+b = f(b) + (r-1)(a-b) \notin L$. Now $t+a \in L$ implies $t+b \in L$, contradiction. \square

• (1) $_{\mathbb{Z}} \Rightarrow$ (2) $_{\mathbb{Z}}$. Assume (1) $_{\mathbb{Z}}$ holds. We first prove inequality $f(x) \geq x$ for all $x \in \mathbb{Z}$. Observe that (by Proposition 29) $\mathcal{L}_{\mathbb{Z}}(\{x\}) = \{X \in \mathcal{P}_{<\omega}(\mathbb{Z}) \mid X = \emptyset \text{ or } \max X \leq x\}$. In particular, letting $z = f(x)$ and applying (1) $_{\mathbb{Z}}$ with $\mathcal{L}(\{f(x)\})$, we get $f^{-1}(\{f(x)\}) \in \mathcal{L}_{\mathbb{Z}}(\{f(x)\})$ hence $x \leq \max(f^{-1}(\{f(x)\})) \leq f(x)$.

To show that f is congruence preserving, we reduce to the \mathbb{N} case.

For $\alpha \in \mathbb{Z}$, let $Suc_{\alpha} : \mathbb{N}_{\alpha} \rightarrow \mathbb{N}_{\alpha}$ be the successor function on $\mathbb{N}_{\alpha} = \{z \in \mathbb{Z} \mid z \geq \alpha\}$. The structures $\langle \mathbb{N}, Suc \rangle$ and $\langle \mathbb{N}_{\alpha}, Suc_{\alpha} \rangle$ are isomorphic. Since $f(x) \geq x$ for all $x \in \mathbb{Z}$, the restriction $f \upharpoonright \mathbb{N}_{\alpha}$ maps \mathbb{N}_{α} into \mathbb{N}_{α} . In particular, using our result in \mathbb{N} , conditions (1) $_{\mathbb{N}_{\alpha}}$ and (2) $_{\mathbb{N}_{\alpha}}$ (relative to $f \upharpoonright \mathbb{N}_{\alpha}$) are equivalent.

We show that condition $(2)_{\mathbb{N}_\alpha}$ holds. Let $L \subseteq \mathbb{N}_\alpha$ be finite. Condition $(1)_{\mathbb{Z}}$ insures that $\mathcal{L}_{\mathbb{Z}}(L)$ is closed under f^{-1} . In particular, $f^{-1}(L) \in \mathcal{L}_{\mathbb{Z}}(L)$. Using Proposition 29, we get $f^{-1}(L) = \bigcup_{j \in J} \bigcap_{i \in I_j} (L - i)$ for finite J , I_j 's included in \mathbb{N} hence $(f \upharpoonright \mathbb{N}_\alpha)^{-1}(L) = f^{-1}(L) \cap \mathbb{N}_\alpha = \bigcup_{j \in J} \bigcap_{i \in I_j} (\mathbb{N}_\alpha \cap (L - i)) \in \mathcal{L}_{\mathbb{N}_\alpha}(L)$. This proves condition $(1)_{\mathbb{N}_\alpha}$. Since $(1)_{\mathbb{N}_\alpha} \Rightarrow (2)_{\mathbb{N}_\alpha}$ we see that $f \upharpoonright \mathbb{N}_\alpha$ is congruence preserving. Now, α is arbitrary in \mathbb{Z} and the fact that $f \upharpoonright \mathbb{N}_\alpha$ is congruence preserving for all $\alpha \in \mathbb{Z}$ implies that f is congruence preserving. Thus, condition $(2)_{\mathbb{Z}}$ holds.

- $(2)_{\mathbb{Z}} \Rightarrow (3)_{\mathbb{Z}}$. Assume $(2)_{\mathbb{Z}}$. It is enough to prove that $f^{-1}(L) \in \mathcal{L}_{\mathbb{Z}}(L)$ whenever L is recognizable. Let $L = (F + d\mathbb{Z})$ with $d \geq 1$, $F = \{f_1, \dots, f_n\} \subseteq \{x \mid 0 \leq x < d\}$. Then f is not constant since $f(x) \geq x$ for all $x \in \mathbb{Z}$. Also, $f^{-1}(\alpha)$ is finite for all α : let b be such that $f(b) = \beta \neq \alpha$, by congruence preservation the nonzero integer $\alpha - \beta$ is divided by $a - b$ for all $a \in f^{-1}(\alpha)$ hence $f^{-1}(\alpha)$ is finite. $f^{-1}(F)$ is thus finite too. Moreover, $L - t = F - t + d\mathbb{Z} = L - t - d + d\mathbb{Z} = L - t - d = L - t + d + d\mathbb{Z} = L - t + d$, hence there are only finitely many $L - t$'s. By Lemma 31 we have $f^{-1}(L) = \bigcup_{a \in f^{-1}(F)} \bigcap_{t \in L - a} (L - t)$; as there are only a finite number of $L - t$'s, all union and intersections reduce to finite unions and intersections and $f^{-1}(L) \in \mathcal{L}_{\mathbb{Z}}(L)$.
- $(3)_{\mathbb{Z}} \Rightarrow (2)_{\mathbb{Z}}$. Similar to $(1)_{\mathbb{Z}} \Rightarrow (2)_{\mathbb{Z}}$.
- $(2)_{\mathbb{Z}} \Rightarrow (1)_{\mathbb{Z}}$. Similar to $(2)_{\mathbb{Z}} \Rightarrow (3)_{\mathbb{Z}}$. □

Example 32. *Theorem 30 does not hold if we substitute rational for recognizable in $(3)_{\mathbb{Z}}$. Consider $L = (6 + 10\mathbb{N})$ and $f(x) = x^2$; L is rational and f is congruence preserving. However $f^{-1}(L) = (\{4, 6\} + 10\mathbb{N}) \cup -(\{4, 6\} + 10\mathbb{N})$ does not belong to $\mathcal{L}_{\mathbb{Z}}(L)$: $f^{-1}(L)$ contains infinitely many negative numbers, while each $L - t$ for $t \in f^{-1}(L)$ contains only finitely many negative numbers; hence any finite union of finite intersections of $L - t$'s can contain only a finite number of negative numbers and cannot be equal to $f^{-1}(L)$. □*

Theorem 30 does not hold for \mathbb{Z}_p : $f^{-1}(L)$ no longer belongs to $\mathcal{L}_{\mathbb{Z}_p}(L)$, the lattice of subsets of \mathbb{Z}_p containing L and closed under decrement. Consider the congruence preserving function $f(x) = (\sum_{i \geq 2} p^i)x$, and let $L = \{\sum_{i \geq 2} p^i\} = \{f(1)\}$. Then $f^{-1}(L) = \{1\} \notin \mathcal{L}_{\mathbb{Z}_p}(L)$ because all elements of the $(L - i)$ end with an infinity of 1s.

Thus integer decrements are not sufficient; but even if we substitute translations for decrements, Theorem 30 can't be generalized.

A recognizable subset of \mathbb{Z}_p is of the form $F + p^n\mathbb{Z}_p$ with F finite, $F \subseteq \mathbb{Z}/p^n\mathbb{Z}$. For L a subset of \mathbb{Z}_p let $\mathcal{L}_{\mathbb{Z}_p}^c(L)$ (resp. $\mathcal{L}_{\mathbb{Z}_p}(L)$) be the family of sets of the form $\bigcup_{j \in J} \bigcap_{i \in I_j} (L + a_i)$, $a_i \in \mathbb{Z}_p$, where J and the I_j 's are (resp. finite) non empty subsets of \mathbb{N} . Then $\mathcal{L}_{\mathbb{Z}_p}^c(L)$ (resp. $\mathcal{L}_{\mathbb{Z}_p}(L)$) is the smallest complete sublattice (resp. sublattice) of $\mathcal{P}(\mathbb{Z}_p)$, the class of subsets of \mathbb{Z}_p , containing L and closed under translation. It is easy to see that, for $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ and $L \subseteq \mathbb{Z}/p^k\mathbb{Z}$, we have $f^{-1}(L) = \bigcup_{a \in f^{-1}(L)} \bigcap_{t \in L} (L + (a - t))$. Hence for any $f : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$

$(1)_{\mathbb{Z}_p}$ If $f^{-1}(a)$ is finite for every a , then for every finite subset L of \mathbb{Z}_p , the lattice $\mathcal{L}_{\mathbb{Z}_p}(L)$ is closed under f^{-1} .

(3) $_{\mathbb{Z}_p}$ For every recognizable subset L of \mathbb{Z}_p the lattice $\mathcal{L}_{\mathbb{Z}_p}^c(L)$ is closed under f^{-1} .

However conditions (1) $_{\mathbb{Z}_p}$ and (3) $_{\mathbb{Z}_p}$ do not imply that f is congruence preserving: let f be inductively defined on \mathbb{Z} by: for $0 \leq x < p$, $f(x) = x$, and for $x \geq p$, $f(x) = f(x-p) + 1$. For $np \leq k < (n+1)p$, $f(k) = n + (k - np)$; hence f is uniformly continuous and has a unique uniformly continuous extension \hat{f} to \mathbb{Z}_p ; then \hat{f} satisfies (1) $_{\mathbb{Z}_p}$ and (3) $_{\mathbb{Z}_p}$ but is not congruence preserving as p does not divide 1.

5 Conclusion

We here studied functions having congruence preserving properties; these functions appeared in two ways at least: (i) as the functions such that lattices of regular subsets of \mathbb{N} are closed under f^{-1} (see [2]), and (ii) as the functions uniformly continuous in a variety of finite groups (see [10]).

The contribution of the present paper is to *characterize congruence preserving functions* on various sets derived from \mathbb{Z} such as $\mathbb{Z}/n\mathbb{Z}$, (resp. $\mathbb{Z}_p, \widehat{\mathbb{Z}}$) via polynomials (resp. series) with *rational coefficients* which share the following common property: $\text{lcm}(k)$ divides the k -th coefficient. Examples of *non polynomial* (Bessel like) congruence preserving functions can be found in [4].

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6 Appendix

Recall some classical equivalent approaches to the topological rings of p -adic integers and profinite integers, cf. Lenstra [7, 8], Lang [6] and Robert [11].

Proposition 33. *Let p be prime. The three following approaches lead to isomorphic structures, called the topological ring \mathbb{Z}_p of p -adic integers.*

- *The ring \mathbb{Z}_p is the inverse limit of the following inverse system:*
 - *the family of rings $\mathbb{Z}/p^n\mathbb{Z}$ for $n \in \mathbb{N}$, endowed with the discrete topology,*
 - *the family of surjective morphisms $\pi_{p^n, p^m} : \mathbb{Z}/p^n\mathbb{Z} \rightarrow \mathbb{Z}/p^m\mathbb{Z}$ for $0 \leq n \geq m$.*
- *The ring \mathbb{Z}_p is the set of infinite sequences $\{0, \dots, p-1\}^{\mathbb{N}}$ endowed with the Cantor topology and addition and multiplication which extend the usual way to perform addition and multiplication on base p representations of natural integers.*
- *The ring \mathbb{Z}_p is the Cauchy completion of the metric topological ring $(\mathbb{N}, +, \times)$ relative to the following ultrametric: $d(x, x) = 0$ and for $x \neq y$, $d(x, y) = 2^{-n}$ where n is the p -valuation of $|x - y|$, i.e. the maximum k such that p^k divides $x - y$.*

Recall the factorial representation of integers.

Lemma 34. *Every positive integer n has a unique representation as*

$$n = c_k k! + c_{k-1} (k-1)! + \dots + c_2 2! + c_1 1!$$

where $c_k \neq 0$ and $0 \leq c_i \leq i$ for all $i = 1, \dots, k$.

Proposition 35. *The four following approaches lead to isomorphic structures, called the topological ring $\widehat{\mathbb{Z}}$ of profinite integers.*

- *The ring $\widehat{\mathbb{Z}}$ is the inverse limit of the following inverse system:*

- the family of rings $\mathbb{Z}/k\mathbb{Z}$ for $k \geq 1$, endowed with the discrete topology,
- the family of surjective morphisms $\pi_{n,m} : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ for $m \mid n$.
- The ring $\widehat{\mathbb{Z}}$ is the inverse limit of the following inverse system:
 - the family of rings $\mathbb{Z}/k!\mathbb{Z}$ for $k \geq 1$, endowed with the discrete topology,
 - the family of surjective morphisms $\pi_{(n+1)!,n!} : \mathbb{Z}/n!\mathbb{Z} \rightarrow \mathbb{Z}/m!\mathbb{Z}$ for $n \geq m$.
- The ring $\widehat{\mathbb{Z}}$ is the set of infinite sequences $\prod_{n \geq 1} \{0, \dots, n\}$ endowed with the product topology and addition and multiplication which extend the obvious way to perform addition and multiplication on factorial representations of natural integers.
- The ring $\widehat{\mathbb{Z}}$ is the Cauchy completion of the metric topological ring $(\mathbb{N}, +, \times)$ relative to the following ultrametric: for $x \neq y \in \mathbb{N}$, $d(x, y) = 0$ and $d(x, y) = 2^{-n}$ where n is the maximum k such that $k!$ divides $x - y$.
- The ring $\widehat{\mathbb{Z}}$ is the product ring $\prod_{p \text{ prime}} \mathbb{Z}_p$ endowed with the product topology.

Proposition 36. *The topological rings \mathbb{Z}_p and $\widehat{\mathbb{Z}}$ are compact and zero dimensional (i.e. they have a basis of closed open sets).*

Proposition 37. *Let $\lambda : \mathbb{N} \rightarrow \mathbb{Z}_p$ (resp. $\lambda : \mathbb{N} \rightarrow \widehat{\mathbb{Z}}$) be the function which maps $n \in \mathbb{N}$ to the element of \mathbb{Z}_p (resp. $\widehat{\mathbb{Z}}$) with base p (resp. factorial) representation obtained by sufficing an infinite tail of zeros to the base p (resp. factorial) representation of n .*

The function λ is an embedding of the semiring \mathbb{N} onto a topologically dense semiring in the ring \mathbb{Z}_p (resp. $\widehat{\mathbb{Z}}$).

Remark 38. *In the base p representation, the opposite of an element $f \in \mathbb{Z}_p$ is the element $-f$ such that, for all $m \in \mathbb{N}$,*

$$(-f)(i) = \begin{cases} 0 & \text{if } \forall s \leq i \ f(s) = 0, \\ p - f(i) & \text{if } i \text{ is least such that } f(i) \neq 0, \\ p - 1 - f(i) & \text{if } \exists s < i \ f(s) \neq 0. \end{cases}$$

In particular,

– *Integers in \mathbb{N} correspond in \mathbb{Z}_p to infinite base p representations with a tail of 0's.*

– *Integers in $\mathbb{Z} \setminus \mathbb{N}$ correspond in \mathbb{Z}_p to infinite base p representations with a tail of digits $p - 1$.*

Similar results hold for the infinite factorial representation of profinite integers.