Decidability of Verification of Safety Properties of Spatial Families of Linear Hybrid Automata

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We consider systems composed of an unbounded number of uniformly designed linear hybrid automata, whose dynamic behavior is determined by their relation to neighboring systems. We present a class of such systems and a class of safety properties whose verification can be reduced to the verification of (small) families of "neighboring" systems of bounded size, and identify situations in which such verification problems are decidable, resp. fixed parameter tractable. We illustrate the approach with an example from coordinated vehicle guidance, and describe an implementation which allows us to perform such verification tasks automatically.

1 Introduction

Verification of families of interacting systems is very important nowadays. Next generations cars will perform cooperative maneuvers for collision avoidance, lane changing, overtaking, and passing intersections. They will rely on an internal digital representation of the environment – capturing relative distance and speed of surrounding vehicles through on board sensors, sensor fusion, and vehicle2vehicle communication in determining which coalition of vehicles will follow what dynamics to achieve e.g. collision freedom. While prototype realizations of such highly automated driving functions have been demonstrated (cf. e.g. HAVEit project [Hoeger et al., 2008]), the challenge in deploying such solutions rests in proving their safety.

In this paper, we propose a general mathematical model capturing the essence of such interacting systems as *spatial families of hybrid automata* and provide efficient verification methods for proving safety when abstracting the dynamics to linear hybrid automata. It thus provides efficient verification methods for systems composed of an unbounded dynamically communicating parallel composition of uniformly defined linear hybrid automata.

The main contributions can be summarized as follows:

- We identify a class of systems composed of dynamically communicating uniformly defined linear hybrid automata and a class of safety properties (with exhaustive entry conditions) for which the verification of the whole system can be reduced to the verification of subsystems of bounded size of "neighboring" components.
- We identify situations when verification is decidable and fixed parameter tractable.

- We identify situations when checking whether the safety property has "exhaustive entry conditions" is decidable resp. fixed parameter tractable.
- We analyze the complexity of parametric verification resp. synthesis.
- We illustrate all concepts we introduce and all steps of our method on a running example from coordinated vehicle guidance.
- We implemented these ideas in the tool HAHA (Hierarchical Analysis of Hybrid Automata), which employs H-PILoT for the reasoning tests. We present several tests and comparisons.

1.1 Related work

A considerable amount of work has been dedicated to identifying classes of hybrid automata for which checking safety is decidable. Reachability and safety in linear hybrid automata are in general undecidable, while invariant checking and bounded reachability are decidable. There are various approaches to the parametric verification of individual hybrid automata [Alur et al., 1996], the development of a dynamic hybrid logic [Platzer, 2008], and of tools (cf. e.g. [Frehse et al., 2008, Fribourg and Kühne, 2013]). A survey of existing decidability and undecidability results for individual hybrid automata can be found in [Sofronie-Stokkermans, 2010, Damm et al., 2011], which gives an overview of papers in which classes of hybrid automata resp. classes of verification problems for which decidability results can be established.

In this paper we analyze *systems* of hybrid automata. In recent years, systems of systems have been studied in various papers.

Small model or cutoff properties for the verification of families of systems have been studied, but only for systems of discrete (or even finite state) systems. In [Emerson and Srinivasan, 1990] an indexed temporal logic is introduced that can be used to specify programs with arbitrarily many similar processes. It is shown that the problems of checking "almost always satisfiability" and "almost always unsatisfiability" are decidable, and a small model property is given. In [Abdulla et al., 2013], a framework for the automatic verification of systems with a parametric number of communicating processes (organized in various topologies such as words, multisets, rings, or trees) is proposed; a method for the verification of such systems is given which needs to inspect only a small number of processes in order to show correctness of the whole system (the method relies on an abstraction function that views the system from the perspective of a fixed number of processes). In [Kaiser et al., 2010], the class of finite-state programs executed by an unbounded number of replicated threads communicating via shared variables is studied. The thread-state reachability problem for this class is decidable via Petri net coverability analysis, but as techniques solely based on coverability are inefficient, [Kaiser et al., 2010] presents an alternative method based on a thread-state cutoff. Modularity results (and similar cutoff results) are presented for the special case of systems of trains on a complex track topology in [Sofronie-Stokkermans, 2009] and [Faber et al., 2010]. In [Jacobs and Bloem, 2014] a cutoff property is used for parameterized synthesis in token ring networks (the synthesis problem is reduced to distributed synthesis in a network consisting of a few copies of a single process). Our work generalizes previous results on verification of classes of systems such as [Emerson and Srinivasan, 1990, Abdulla et al., 2013, Kaiser et al., 2010, Faber et al., 2010, Damm et al., 2013, Jacobs and Bloem, 2014 in supporting the much richer system model of linear hybrid automata. The temporal logic we use for specifying the safety properties we consider is similar to that introduced in [Emerson and Srinivasan, 1990].

Among the existing work in which the safety of cooperative driver assistance systems (modeling autonomous cars on highways performing lane-change maneuvers) we mention the results in [Frese and Beyerer, 2010], [Hilscher et al., 2011] and [Damm et al., 2013].

[Damm et al., 2013] proposes a design and verification methodology for cooperative driver assistance systems (with focus on applications where drivers are supported in complex driving tasks by safe strategies involving the coordinated movements of multiple vehicles to complete the driving task successfully). A "divide and conquer" approach for formally verifying timed probabilistic requirements on successful completion of the driving task and collision freedom is proposed. Our method is different, mainly because it relies on locality properties of the logical theories used for modeling the problems. In [Hilscher et al., 2011], an alternative approach to prove safety (collision freedom) of multi-lane motorway traffic with lane-change maneuvers is proposed, based on a new spatial interval logic based on the view of each car. The compositional approach [Hilscher et al., 2011] addresses an application class that is related to our running example, but does not use hybrid automata to model the systems and does not provide decidability or complexity results. [Frese and Beyerer, 2010] searches for strategies controlling all vehicles, and employs heuristic methods to determine strategies for coordinated vehicle movements. An excellent survey of alternative methods for controlling all vehicles to perform collision-free driving tasks is given in [Frese, 2010]. Both methods share the restriction of the analysis to a small number of vehicles, whereas we consider an unbounded number of systems.

[Henzinger et al., 2001] analyzes the interplay of fixed combinations of hybrid systems using assume-guarantee reasoning. In [Johnson and Mitra, 2012a, Johnson and Mitra, 2012b] a small model theorem for *finite* families of automata with constant derivatives, with a parametric bound on the number of components, is established; the discrete transitions describe changes in exactly one system (thus no global updates of sensors can be modeled). Our approach allows us to consider families with an *unbounded* or *infinite* number of components which are *parametric linear hybrid automata*. We moreover allow for parallel mode switches and global topology updates. In [Mickelin et al., 2014], robust finite abstractions with bounded estimation errors are provided for reducing the synthesis of winning strategies for LTL objectives to finite state synthesis; the approach is used for an aerospace control application. [Platzer, 2010] proposes a quantified differential dynamic logic for specifying and verifying distributed hybrid systems but the focus is not on providing decidability results or small model property results.

Our current work stands in the tradition of [Sofronie-Stokkermans, 2010, Damm et al., 2011, Sofronie-Stokkermans, 2013], where we studied linear hybrid systems in which both mode changes and the dynamics can be parametrized. We presented first results on the verification of families of LHA in [Damm et al., 2015]. This paper considerably extends the results presented in [Damm et al., 2015]. In particular, compared to [Damm et al., 2015], the theoretical results are extended and the experimental results reported in Section 7 are an order of magnitude faster than the ones reported in [Damm et al., 2015]; we also explain how to use our system and our theory prover H-PILoT for generating (and visualizing) counterexamples to safety.

1.2 Paper Structure

In Section 2 we present our model of spatial families of hybrid automata with its semantics. In Section 3 we introduce the verification properties we consider. The notions are illustrated on a running example of cars on a highway. In Section 4 we present classes of decidable and tractable logical theories, which we use in Section 5 for solving the verification tasks and proving modularity and complexity results. In Section 6 we summarize the main results in the form of a small model property, as well as a discussion of the decidability and complexity of the verification problems we consider. We identify situations in which the problems are fixed parameter tractable; and give decidability and complexity results also for parametric verification and parameter synthesis. In Section 7 we discuss our tests with our systems H-PILoT and HAHA. In Section 8 we present a summary of the results we obtained, followed by plans for future work.

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Figure 1: Traffic situation on a highway

2 Spatial Families of Hybrid Automata

We study families $\{S(i) \mid i \in I\}$ consisting of an unbounded number of similar systems. To describe them, we have to specify the properties of the component systems and the way they obtain information about neighboring systems:

- We model the systems S(i) using hybrid automata.
- For describing the information about neighboring or other observed systems we use structures $(I, \{p : I \to I\}_{p \in P})$, where I is a countably infinite set and $P = P_S \cup P_N$ is a finite set of unary function symbols which model the way the systems perceive other systems using sensors in P_S , or by neighborhood connections (e.g. established by communication channels) in P_N .

We use highway control as a running example.

Example 1 Let I be a set of car identities, including the special constant nil.

- (1) A car can observe other cars through sensors; these are modeled by a finite application-dependent set P_S of functions p: I → I, where p(i) = j represents the fact that i's p-sensor observes car j. We choose P_S to include back, front, sidefront, sideback, which indicate the closest car in the respective directions: In Figure 1, we have sidefront(7) = 5, back(7) = 18, front(7) = 8. If sensor p ∈ P of car i sees no car then p(i) = nil. We will make these notions more precise in Examples 3 and 4.
- (2) Car platoons of length at most n can be modeled e.g. by choosing a set of neighborhood connections P_N including leader, follower₁,... follower_n, next, prev. Car i is leader if leader(i) = i; if leader(j) = i ≠ j, then j = follower_k(i) for some k ≤ n.

Definition 2 (Hybrid automata, linear hybrid automata [Alur et al., 1996]) A hybrid automaton (HA) is a tuple

 $S = (X, Q, \mathsf{Init}, \mathsf{flow}, \mathsf{Inv}, E, \mathsf{guard}, \mathsf{jump})$

consisting of:

- (1) finite sets $X = \{x_1, \ldots, x_n\}$ (real-valued variables) and Q (control modes); a finite multiset E with elements in $Q \times Q$ (control switches);
- (2) families $\operatorname{Init}_q \mid q \in Q$ and $\operatorname{Inv}_q \mid q \in Q$ of predicates over X, defining the initial states and invariant conditions for each control mode, and flow = {flow_q | $q \in Q$ } of predicates over $X \cup \dot{X}$ specifying the dynamics in each control mode, where $\dot{X} = {\dot{x}_1, \ldots, \dot{x}_n}$ (\dot{x}_i is the derivative of x_i);
- (3) families {guard_e | $e \in E$ } of predicates over X (guards) and {jump_e | $e \in E$ } of predicates over $X \cup X'$ (jump conditions) for the control switches, where $X' = \{x'_1, \ldots, x'_n\}$ is a copy of X.

A linear hybrid automaton (LHA) is a HA in which for every $q \in Q, e \in E$:

(i) Inv_q , $Init_q$, $jump_e$ and $guard_e$ are convex linear predicates¹ and

(ii) flow_q is a convex linear predicate (with only non-strict inequalities) over \dot{X} .

A state of S is a pair (q, a), where $q \in Q$ and $a=(a_1, \ldots, a_n)$, where $a_i \in \mathbb{R}$ is a value for $x_i \in X$. A state s = (q, a) is *admissible* (resp. *initial*) if Inv_q (resp. Init_q) is true when each x_i is replaced by a_i . A state can change by a jump (instantaneous transition that changes the control mode and the values of the variables according to the jump conditions), or by a flow (evolution in a mode q where the values of the variables change according to the flow_q).

2.1 The language.

To describe the families $\{S(i) \mid i \in I\}$, the topology $(I, \{p : I \to I\}_{p \in P})$ and its updates, and the safety properties we are interested in, we use a two-sorted first-order language $\mathcal{L}_{index,num}$ of a theory of pointers with two sorts, index and num. Sort index is used for representing the indices and sort num is used for numerical values. The signature of the theory contains a constant nil of sort index, unary function symbols in P (sort index \rightarrow index) for modeling pointer fields, and a set X (sort index \rightarrow num) for modeling the scalar (numeric) information associated with the indices (values of the continuous variables of the systems). A theory \mathcal{T}_{num} (sort num) is used for describing properties of the values of the continuous variables of the systems (e.g. the theory \mathbb{R} of real numbers, or linear real arithmetic $LI(\mathbb{R})$). We consider first-order formulae in the language $\mathcal{L}_{index,num}$. Variables of sort index are denoted with indexed versions of i, j, k; variables of sort num are denoted x_1, \ldots, x_n .

2.2 Component systems.

The component systems are similar² hybrid automata $\{S(i) \mid i \in I\}$, with:

- the same set of control modes Q and the same mode switches $E \subseteq Q \times Q$,
- real valued variables $X_{S(i)}$, partitioned into a set $X(i) = \{x(i) \mid x \in X\}$ of variables describing the states of the system S(i) and a set $X_P(i) = \{x_p(i) \mid x \in X, p \in P\}$ describing the state of the neighbors $\{p(i) \mid p \in P\}$ of i, where $X = \{x_1, \ldots, x_n\}$.

We consider two possibilities for $x_p(i)$:

- (a) Continuous sensors: $x_p(i)$ is at any moment the value of x(p(i)), the value of variable x for the system S(p(i)) and is controlled by suitable flow/jump conditions of S(p(i));
- (b) Intermittent sensors: $x_p(i)$ is the value of x(p(i)) which was sensed by the sensor in the last measurement, and does not change between measurements.

We assume that all sets $X(i), i \in I$ are disjoint. Every component system S(i) has the form:

 $S(i) = (X(i) \cup X_P(i), Q, \mathsf{flow}(i), \mathsf{Inv}(i), \mathsf{Init}(i), E, \mathsf{guard}(i), \mathsf{jump}(i))$

where – with the notations in Definition 2:

 $^{^1\}mathrm{A}$ convex linear predicate is a finite conjunction of linear inequalities over $\mathbb{R}.$

²The results can be adapted to the situation when a finite number of types of systems are given and the description of each S(i) is of one of these types.



Figure 2: Hybrid automaton modeling the behavior of a car on a two-lane highway

- for every $q \in Q \operatorname{Inv}_q(i)$, $\operatorname{Init}_q(i)$ is a conjunction of formulae of the form $\mathcal{E} \vee C$, where C is a predicate over $X_{S(i)}$ and \mathcal{E} is a disjunction of equalities of the form $i = \operatorname{nil}$ and $p(i) = \operatorname{nil}$ if $x_p(i)$ occurs in C. We will in general assume that Init_q includes Inv_q as a conjunct.
- for every $q \in Q$, $\mathsf{flow}_q(i)$ is a conjunction of formulae of the form $\mathcal{E} \vee C$, where C is a predicate over $X_{S(i)} \cup \dot{X}_{S(i)}$ and \mathcal{E} is a disjunction of equalities of the form $i = \mathsf{nil}$ and $p(i) = \mathsf{nil}$ if $x_p(i)$ occurs in C.
- for every $e \in E$, $\operatorname{guard}_e(i)$ is a conjunction of formulae of the form $\neg(\mathcal{E} \lor C)$, where C is a predicate over $X_{S(i)}$ and \mathcal{E} is a disjunction of equalities of the form $i = \operatorname{nil}$ and $p(i) = \operatorname{nil}$ if $x_p(i)$ occurs in C.
- for every $e \in E$, $\operatorname{jump}_e(i)$ is a conjunction of formulae of the form $\mathcal{E} \vee C$, where C is a predicate over $X_{S(i)} \cup X'(i)$ and \mathcal{E} is a disjunction of equalities of the form $i = \operatorname{nil}$ and $p(i) = \operatorname{nil}$ if $x_p(i)$ occurs in C.

All these formulae can also be regarded as $\mathcal{L}_{index,num}$ -formulae; for all $i \in I$ they differ only in the variable index.

The component S(i) is *linear* if

- (i) for every $q \in Q$, flow_q(i) contains only variables in $X_{S(i)}$ and
- (ii) for every $q \in Q$ and $e \in E$, $\mathsf{flow}_q(i)$, $\mathsf{Inv}_q(i)$, $\mathsf{Init}_q(i)$, $\mathsf{guard}_e(i)$, $\mathsf{jump}_e(i)$ are conjunctions of formulae $\mathcal{E} \lor C$, as above, where C is a linear inequality (non-strict for flows).

We also consider systems of *parametric* LHA, in which some coefficients in the linear inequalities (and also bounds for invariants, guards or jumps) are parameters in a set Par.

Example 3 Consider the following model of a system of cars, which is also depicted in Figure 2: The controlled variables are the position and the lane of the car, so $X = \{pos, lane\}$. The car can drive on either lane 1 or lane 2. Its sensors provide information about the car in front and back on the same lane (front, back) and about the closest cars on the other lane (sidefront, sideback). Thus the set of sensors is

 $P = \{ back, front, sideback, sidefront \}$.

Each car is modeled by a hybrid automaton with set of continuous variables

 $X = \{ \mathsf{pos}(i), \mathsf{lane}(i) \} \cup \{ \mathsf{pos}_p(i), \mathsf{lane}_p(i) \mid p \in P \}$

and modes

$$Q = \{\mathsf{Appr}, \mathsf{Rec}\}$$
.

We assume that $x_p(i) = x(p(i))$ (continuous sensors, variant (a) above) and use parameters $Par = \{d, d', D, D'\}$.

Initial states: As initial states, we allow all states where $pos_{front}(i) - pos(i) \ge d'$ if $front(i) \ne nil$, and where the respective mode invariant is satisfied:

• $\operatorname{Init}_{\operatorname{Appr}}$ and $\operatorname{Init}_{\operatorname{Rec}}$ are $(i = \operatorname{nil} \lor \operatorname{front}(i) = \operatorname{nil} \lor \operatorname{pos}_{\operatorname{front}}(i) - \operatorname{pos}(i) \ge d')$.

Invariants; flow conditions:

Mode Appr: car *i* keeps its velocity high enough to approach the car ahead.

- Inv_{Appr} is $(i = nil \lor 1 \le lane(i) \le 2) \land (i = nil \lor front(i) = nil \lor pos_{front}(i) pos(i) \ge d)$;
- flow_{Appr} is $(i = \mathsf{nil} \lor \mathsf{lane}(i) = 0) \land (i = \mathsf{nil} \lor \mathsf{front}(i) = \mathsf{nil} \lor \mathsf{pos}_{\mathsf{front}}(i) \le \mathsf{pos}(i)) \land (i = \mathsf{nil} \lor 0 \le \mathsf{pos}(i) \le 100).$

Mode Rec: car i maintains a lower velocity to fall back.

- Inv_{Rec} is $(i = nil \lor 1 \le lane(i) \le 2) \land (i = nil \lor front(i) = nil \lor pos_{front}(i) pos(i) \le D);$
- flow_{Rec} is $((i = nil \lor la\dot{n}e(i) = 0) \land (i = nil \lor 0 \le p\dot{o}s(i))$ $\land (i = nil \lor front(i) = nil \lor p\dot{o}s(i) \le p\dot{o}s_{front}(i))).$

Mode switches:

A mode switch (without resets) can happen if $i \neq \text{nil}$, front $(i) \neq \text{nil}$ (there is a car ahead) and the distance to that car leaves a predefined range, i.e.

- $pos_{front}(i) pos(i) \le D'$ (switch from Appr to Rec) or
- $\mathsf{pos}_{\mathsf{front}}(i) \mathsf{pos}(i) \ge d'$ (switch from Rec to Appr).

Another mode switch to mode Appr, which changes between lanes 1 and 2 with reset lane'(i)=3-lane(i), can happen when $i \neq nil$ and:

- the car in front is too close $(\text{front}(i) \neq \text{nil} \land \text{pos}_{\text{front}}(i) \text{pos}(i) \leq D')$ and
- there is space to start the maneuver: back(i)=nil ∨ pos(i)-pos_{back}(i)≥d'. Similarly for sideback(i) and sidefront(i).

2.3 Topology

We now present a possibility of modeling the topology of the family of systems using a one-state automaton, where the transitions are labeled with updates of the values of the pointers (Section 2.3.1), and a refinement of this model in which clocks are additionally used (Section 2.3.2).

2.3.1 Topology automata

We model the topology of the family of systems and its updates using an automaton Top with one mode, having as read-only-variables all variables in $\{x(i) \mid x \in X, i \in I\}$ and as write variables $\{x_p(i) \mid p \in P, i \in I\}$, where $P = P_S \cup P_N$. In addition, Top updates the functions $p: I \to I$, where $P = P_S \cup P_N$.

The initial states lnit are described using $\mathcal{L}_{index,num}$ -formulae. The jumps can represent updates of the sensor values $p(i), p \in P_S$, for a single system S(i), but also synchronized global updates of the sensors $p \in P_S$ or neighborhood connections $p \in P_N$ for subsets of systems with a certain property (described by a formula). This can be useful when modeling systems of systems with an external controller (e.g. systems of car platoons) and entails a simultaneous update of an unbounded set of variables.³ Therefore, the description of the mode switches (topology updates) in **Top** is of a global nature and is done using $\mathcal{L}_{index,num}$ -formulae.

The update rules for $p \in P$, which we denote as $\mathsf{Update}(p, p')$, are conjunctions of implications of the form

$$\forall i (i \neq \mathsf{nil} \land \phi_k^p(i) \to F_k^p(p'(i), i)), \qquad k \in \{1, \dots, m\},\tag{1}$$

which describe how the values of the pointer p change depending on a set of mutually exclusive conditions $\{\phi_1^p(i), \ldots, \phi_m^p(i)\}$ such that:

- $\phi_k^p(i)$ and $F_k^p(j,i)$ are formulae over the 2-sorted language $\mathcal{L}_{index,num}$ without any occurrence of unary functions in P';
- if $p \in P_S$ (*p* represents a sensor), the formulae $\phi_k^p(i)$ and $F_k^p(j,i)$ also do not contain functions in P;
- under the condition $\phi_k^p(i)$, the existence of a value for p'(i) such that $F_k^p(p'(i), i)$ holds must be guaranteed, i.e.

$$\models \phi_k^p(i) \to \exists j \, F_k^p(j,i);$$

- The variables $\{x(i) \mid x \in X, i \in I\}$ can be used in the guards of $\mathsf{Update}(p, p')$, but cannot be updated by Top.
- If $x_p(i)$ stores the value of x(p(i)) at the update of p (variant (b) on page 6), then the update rules also change $x_p(i)$, so $F_k^p(p'(i), i)$ must contain $x'_p(i) = x(p'(i))$ as a conjunct.

Example 4 We present possible update rules for the topology and initial states for the model of cars in Example 3. Consider the following formulae:

- ASL(j,i): j ≠ nil ∧ lane(j) = lane(i) ∧ pos(j) > pos(i), which expresses the fact that j is ahead of i on the same lane, and
- Closest_f(j, i): ASL(j, i) ∧ ∀k(ASL(k, i)→pos(k) ≥ pos(j)), which expresses the fact that j is ahead of i on the same lane and there is no car between them.

Update rules. The rule for updating the front sensor of all cars with a given property expressed by a formula Prop and of no other car is described by Update(front, front'):

$$\begin{aligned} \forall i (i \neq \mathsf{nil} \land \mathsf{Prop}(i) \land \neg \exists j (\mathsf{ASL}(j,i)) \to \mathsf{front}'(i) = \mathsf{nil}) \\ \forall i (i \neq \mathsf{nil} \land \mathsf{Prop}(i) \land \exists j (\mathsf{ASL}(j,i)) \to \mathsf{Closest}_{\mathsf{f}}(\mathsf{front}'(i),i)) \\ \forall i (i \neq \mathsf{nil} \land \neg \mathsf{Prop}(i) \to \mathsf{front}'(i) = \mathsf{front}(i)) \end{aligned}$$

Below are three examples of formulae which can describe a property Prop:

- (1) If $Prop(i) = (i = i_0)$, only the front sensor of car i_0 is updated.
- (2) For car platoons, Prop(i) can be $leader(i) = i_0$; we then obtain a coordinated update for all platoon members.

 $^{^3}$ Our choice allows us to uniformly represent various types of topology updates, from purely local ones to global updates, without loss of generality.

(3) If Prop(i) = true, Update(front, front') describes a global update.

Initial states. The initial states can e.g. be the states in which all sensor pointers have the correct value, as if they had just been updated. For front this can be expressed by the following set of formulae:

$$\begin{split} \forall i(i \neq \mathsf{nil} \land \mathsf{front}(i) = \mathsf{nil} & \to \forall k(k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \ge \mathsf{pos}(i) \to \mathsf{lane}(k) \neq \mathsf{lane}(i))) \\ \forall i(i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} & \to \mathsf{pos}_{\mathsf{front}}(i) > \mathsf{pos}(i) \land \mathsf{lane}_{\mathsf{front}}(i) = \mathsf{lane}(i) \land \\ & \forall k(k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \ge \mathsf{pos}(i) \land \mathsf{lane}(k) = \mathsf{lane}(i) \\ & \to \mathsf{pos}(k) \ge \mathsf{pos}_{\mathsf{front}}(i)) \land \\ & \mathsf{pos}(\mathsf{front}(i)) = \mathsf{pos}_{\mathsf{front}}(i) \land \mathsf{lane}(\mathsf{front}(i)) = \mathsf{lane}_{\mathsf{front}}(i)). \end{split}$$

Alternatively, we can express this using formulae similar to the update rules:

$$\forall i (i \neq \mathsf{nil} \land \mathsf{Prop}(i) \land \neg \exists j (\mathsf{ASL}(j,i)) \to \mathsf{front}(i) = \mathsf{nil}) \\ \forall i (i \neq \mathsf{nil} \land \mathsf{Prop}(i) \land \exists j (\mathsf{ASL}(j,i)) \to \mathsf{Closest}_{\mathsf{f}}(\mathsf{front}(i),i))$$

Example 5 Consider a car platoon as in Example 1 (2). The situation when a car i_0 (who is not a leader) leaves the platoon can e.g. be described by:

$$\begin{split} \mathsf{leader}'(i_0) &= i_0 \qquad \mathsf{next}'(i_0) = \mathsf{nil} \qquad \mathsf{prev}'(i_0) = \mathsf{nil} \\ \mathsf{prev}(i_0) &\neq \mathsf{nil} \to \mathsf{next}'(\mathsf{prev}(i_0)) = \mathsf{next}(i_0) \\ \mathsf{next}(i_0) &\neq \mathsf{nil} \to \mathsf{prev}'(\mathsf{next}(i_0)) = \mathsf{prev}(i_0) \\ \forall i(i \neq i_0 \land i \neq \mathsf{prev}(i_0) \to \mathsf{next}'(i) = \mathsf{next}(i)) \\ \forall i(i \neq i_0 \land i \neq \mathsf{next}(i_0) \to \mathsf{prev}'(i) = \mathsf{prev}(i)) \end{split}$$

2.3.2 Timed topology automata

If we want to ensure that the component systems update the information about their neighbors sufficiently often, we can use additional clock variables $\{c_p(i) \mid i \in I, p \in P\}$, satisfying flow conditions of the form $\dot{c}_p(i) = 1$. Every topology update involving a set of systems and pointer field p has the effect that the clocks $c_p(i)$ for all systems i in that set are set to 0 (added to the conclusion of the topology updates).

Example 6 In Example 4 the consequence of the update rules Update(front, front') for front would contain as a conjunct the formula $c'_{\text{front}}(i) = 0$.

In addition, we can require that for every system i the interval between two updates of $p \in P$ is at most $\Delta t(i)$. Then $\operatorname{Init}_{\mathsf{Top}}$ contains $\forall i c_p(i) = 0$ as a conjunct; the invariant of the mode of Top contains $\forall i \ 0 \leq c_p(i) \leq \Delta t(i)$; and if $c_p(i) = \Delta t(i)$ a topology update for system i must take place.

2.4 Spatial family of hybrid automata

Definition 7 (Spatial Family of Hybrid Automata) A spatial family of hybrid automata (SFHA) is a family of the form

$$S = (\mathsf{Top}, \{S(i) \mid i \in I\}),$$

where $\{S(i) \mid i \in I\}$ is a system of similar hybrid automata and Top is a topology automaton. If for every $i \in I$, S(i) is a linear hybrid automaton, we talk about a spatial family of linear hybrid automata (SFLHA). If the topology automaton is timed, we speak of a spatial family of timed (linear) hybrid automata (SFT(L)HA).



Figure 3: Visualization of a run of an SFLHA with components $S(1), \ldots, S(n)$. Time passes towards the right. Jumps in a system S(i) are marked by a short vertical line, and local or global updates by a dashed line. The curve corresponding to S(i) between such lines represents a flow in system S(i).

Definition 8 (Decoupling) An SFLHA S is decoupled if the real-valued variables in the guard of a mode switch of S(i) can only be reset in a jump by S(i) or by Top.

Remark: In the variant with continuous sensors (variant (a) on page 6), we have $x_p(i) = x(p(i))$ for every $i \in I$. If $x_p(i)$ is used in the guard of a mode switch of S(i), then in order to ensure that S is decoupled, no jump of S(p(i)) should reset x(p(i)).

In the variant with intermittent sensors (variant (b)), $x_p(i)$ is the value sensed by the sensor p in the last measurement and so S is always decoupled.

Example 9 In our running highway example (Example 3, 4) only the variables pos(i), $pos_{front}(i)$, $pos_{back}(i)$, $pos_{back}(i)$, $pos_{back}(i)$, and $pos_{back}(i)$ are used in jump guards. Since no jump of a car resets its position, the system is decoupled. Note that if $lane_{front}(i)$ were used in any jump guard, the system would not be decoupled in variant (a), because front(i) can reset its lane during a jump.

Definition 10 (States and Runs) Let $S = (\text{Top}, \{S(i) \mid i \in I\})$ be a spatial family of hybrid automata.

- A state s = (q, a) of S consists of a tuple $q = (q_i)_{i \in I} \in Q^I$ of modes of the component automata and a tuple a of values of the variables of all components. A state (q, a) is admissible if the values in a satisfy the invariants of Top and the restriction to the variables of S(i) satisfies $\ln v_{q_i}(i)$, for all $i \in I$.
- Initial states of S are the initial states of Top whose restriction to the variables of S(i) are initial states of S(i), for all i ∈ I.
- A state change (s, s') is a flow of length t if its restriction to the variables of S(i) is a flow of length t, for all $i \in I$.
- A state change (s, s') is a jump if its restriction to the variables of S(i) is a jump or else a flow of length 0, for all i ∈ I.
- A run of S is a sequence s_0, s_1, \ldots of admissible states where:
 - (i) s_0 is an initial state of S,
 - (ii) each pair (s_j, s_{j+1}) is a jump, a flow or a topology update, and
 - (iii) each flow is followed by a jump or a topology update.

A visualization of a run of an SFLHA is depicted in Figure 3. (Note that property (iii) of runs does not restrict the set of states that are reachable in a run.)

3 Verification Tasks

The properties of SFLHA we consider are specified in a logic which combines first-order logic over the language $\mathcal{L}_{index,num}$ and temporal logic: Formulae are constructed inductively from atoms using temporal operators and quantification over variables of sort index. Since runs of the system define valuations of variables for each point in time, the semantics of such formulae is defined canonically, see e.g. [Hungar et al., 1995]. We consider *safety properties* of the form:

$$\Phi_{entry} \rightarrow \Box \Phi_{safe}$$

which state that for every run of the composed system, if Φ_{entry} holds at the beginning of the run then Φ_{safe} always holds during the run.

Example 11 Collision freedom can be expressed using the formula

$$\Phi^{g}_{\mathsf{safe}}: \forall i, j(i \neq \mathsf{nil} \land j \neq \mathsf{nil} \land \mathsf{lane}(i) = \mathsf{lane}(j) \land \mathsf{pos}(i) > \mathsf{pos}(j) \to \mathsf{pos}(i) - \mathsf{pos}(j) \ge d_s)$$

for a suitably chosen constant $d_s > 0$ (global safety distance) or by referring only to the "neighbors", using $\Phi_{\mathsf{safe}}^l = \bigwedge_{\mathsf{index} \in P} \Phi_{\mathsf{safe}}^{\mathsf{index}}$, where e.g. $\Phi_{\mathsf{safe}}^{\mathsf{front}}$ is:

 $\forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \rightarrow \mathsf{pos}(\mathsf{front}(i)) - \mathsf{pos}(i) \geq d_s).$

In Section 3.1 we identify a class of general safety properties with what we call exhaustive entry conditions (Definition 12) which can be reduced to invariant checking for certain mode reachable states (Definition 15). In Section 3.2 we then show that for decoupled SFLHA we can reduce checking invariance for mode reachable states of Φ_{safe} to satisfiability checking in suitable logical theories, which are combinations of $LI(\mathbb{R})$ possibly extended with functions x_i satisfying additional properties (boundedness, continuity, boundedness conditions for the slope), and theories of pointers for modeling the information provided by the sensors.

Using decidability results presented in Section 4, in Section 5 we identify situations in which the analysis of safety properties $\Phi_{entry} \rightarrow \Box \Phi_{safe}$ can be precisely reduced to a neighborhood of bounded size of the systems for which Φ_{safe} could fail. This allows us to prove a small model property and to identify safety properties which are decidable resp. fixed parameter tractable.

Notation. In what follows, sequences i_1, \ldots, i_k of variables of sort index are denoted with \overline{i} , sequences x_1, \ldots, x_n (resp. $\dot{x}_1, \ldots, \dot{x}_n$) with \overline{x} (resp. \overline{x}). The sequence $x_1(i), \ldots, x_n(i)$ of all variables of S(i) is denoted with $\overline{x}(i)$, and $\dot{x}_1(i), \ldots, \dot{x}_n(i)$ with $\overline{x}(i)$. To refer to the value of x(i) at time t, we write x(i,t). The sequence $x_1(i,t), \ldots, x_n(i,t)$ of values of variables of system S_i at a time t is denoted $\overline{x}(i,t)$.

3.1 Safety properties

Safety of LHA is in general undecidable; classes of LHA and safety properties which are decidable have been identified in several papers. In [Damm et al., 2011] we discuss such approaches and propose weaker conditions guaranteeing decidability. The approach described here continues this line of research. The choice of the class of safety properties we consider is based on the observation that industrial style guides for designing hybrid automata make sure that modes are entered in an "inner envelope", chosen such that modes cannot be left before a fixed minimal dwelling time; this avoids immediate context switching. In [Damm et al., 2011] we showed that using inner envelopes for individual LHA allows us to reduce safety checking to invariant checking and the proof of bounded liveness properties to checking bounded unfoldings.

3.1.1 Safety properties with exhaustive entry conditions

In this paper we study possibilities of automatically verifying a certain class of safety properties, namely safety properties with *exhaustive entry conditions*.

Definition 12 (Exhaustive Entry Conditions) A safety property with exhaustive entry conditions has the form

$$\Phi_{\mathsf{entry}} \to \Box \Phi_{\mathsf{safe}}$$

where $\Phi_{entry} = \forall i_1, \ldots, i_m \phi_{entry}(\overline{x}(i_1), \ldots, \overline{x}(i_m))$ is a formula in the language $\mathcal{L}_{index,num}$ such that:

(i) If Φ_{entry} holds in a state s, s is an initial state of S;

(ii) For every jump or topology update (s, s'), Φ_{entry} holds in s'.

Condition (i) guarantees that we make minimal restrictions on initial states: runs can start in any state satisfying Φ_{entry} . The formula Φ_{entry} can be seen as a description of certain "inner envelopes" of the modes. Condition (ii) expresses the fact that a jump leads into a state satisfying Φ_{entry} (in the inner envelope of the target mode).

For instance, if $\mathsf{Init}_{\mathsf{top}}$ describes the fact that the information about all variables detected by sensors in P_S is precise, then condition (ii) imposes the restriction that sensors have to be globally updated after any jump or local topology update, which is clearly too restrictive. We can instead require that the initial states contain all states in which the positions indicated by sensors are within a given margin ε of error (the entry condition Φ_{entry} could describe such states).

Remark 13 Conditions (i) and (ii) ensure that if we start from a state in which Φ_{entry} holds for a given combination a of the values of the variables, then there exists at least one tuple $q=(q_i)_{i\in I}\in Q^I$ of modes of the component automata such that (q, a) is an admissible state (i.e. the combination a of the values satisfies the invariants in mode q), and that any jump or topology update starting in a state satisfying Φ_{entry} leads again to an admissible state.

Example 14 Assume that Φ_{entry} describes such a small margin of error between the information given by sensors and the real positions in the running example, e.g.

$$\Phi_{\mathsf{entry}} = \forall i \ (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \rightarrow |\mathsf{lane}_{\mathsf{front}}(i) - \mathsf{lane}(\mathsf{front}(i))| < \varepsilon).$$

Since lane can be modified by a mode change (from value 1 to 2 or vice versa), condition (ii) is not guaranteed to hold. For example, directly after a lane change, front may point to a car which is now on a different lane, thus violating Φ_{entry} .

In order to guarantee (ii), we need to ensure that

- Top is a timed topology automaton where the interval Δt between sensor updates is small enough and
- after lane changes the sensors of all systems affected by the change are simultaneously updated.

In what follows we show that checking safety properties with exhaustive entry conditions can be reduced to checking invariance of Φ_{safe} under all flows, and under jumps and topology updates in states which are reachable through a flow from a state satisfying Φ_{entry} (we call such state changes GMR jumps and topology updates, cf. Definition 15).



Figure 4: Global mode reachability

3.1.2 Reduction to GMR invariant checking

We prove that checking safety properties with exhaustive entry conditions for decoupled SFHA can be reduced to checking whether the safety property Φ_{safe} is invariant under certain jumps, flows, and topology updates.

Definition 15 (Globally Mode Reachable) Let S be an SFHA. A state s = (q, a) of S is globally mode reachable (GMR, for short) if there exists a state $s_0 = (q, a_0)$ of S such that a_0 satisfies Φ_{entry} and there is a flow in S from (q, a_0) to (q, a).

A state change (s, s') of S (which can be a flow, a jump, or a topology update) is globally mode reachable if s is globally mode reachable.

Figure 4 visualizes the concept of global mode reachability of a state.

Theorem 16 An SFHA $S = (\text{Top}, \{S(i) \mid i \in I\})$ satisfies a safety property with exhaustive entry conditions $\Phi_{entry} \rightarrow \Box \Phi_{safe}$ if and only if the following hold:

- (1) All states satisfying Φ_{entry} satisfy Φ_{safe} .
- (2) Φ_{safe} is preserved under all flows starting from a state satisfying Φ_{entry} .
- (3) Φ_{safe} is preserved under all GMR jumps.
- (4) Φ_{safe} is preserved under all GMR topology updates.

Proof: Assume S satisfies the safety property $\Phi_{entry} \rightarrow \Box \Phi_{safe}$. We prove that (1)–(4) hold.

(1) Consider a state s satisfying condition Φ_{entry} . By condition (i) from Definition 12, all states satisfying Φ_{entry} are initial. Since S satisfies the condition $\Phi_{entry} \rightarrow \Box \Phi_{safe}$, all runs consisting of only one state s (satisfying Φ_{entry}) have the property that Φ_{safe} holds during the run. Hence Φ_{safe} holds at state s.

(2) Consider now a flow (s, s') starting from a state satisfying condition Φ_{entry} . Then s is initial by condition (i) from Definition 12, i.e. s, s' is a run of S. The assumption that S satisfies the safety property implies that this flow is safe as well (so all states during this flow are safe).

(3) Consider a jump (s, s'), where s is globally mode reachable. Then s is reachable using a flow in S from a state s_0 satisfying condition Φ_{entry} (by condition (i) from Definition 12, s_0 is an initial state). Because s_0, s, s' is a run of S and S satisfies the safety property $\Phi_{entry} \to \Box \Phi_{safe}$, it follows that Φ_{safe} holds at s'.

(4) The proof for topology updates is similar to the one for jumps. The fact that every topology update leads to an admissible state is a consequence of condition (ii) from Definition 12.

Assume now that (1)–(4) hold. We prove that S satisfies the safety property $\Phi_{entry} \rightarrow \Box \phi_{safe}$. Let s_0, s_1, \ldots be a run in the composed system S, starting in an initial state satisfying condition Φ_{entry} . We prove by induction on n that for every state s_n in the run:

(a) all states in the run up to state s_n are GMR.

(b) Φ_{safe} holds during the run up to state s_n .

 Φ_{entry} holds in state s_0 , hence by (1), s_0 is both safe and GMR.

Assume that we have proved that for all $1 \le i \le n-1$, s_i has properties (a) and (b) above. If the change of state (s_{n-1}, s_n) is due to a flow, then s_{n-1} must be reached by a jump or topology update; so Φ_{entry} holds at s_{n-1} , hence (a) s_n is GMR and (b) by (2) all the states in which the system is during the flow from s_{n-1} to s_n are also safe.

Assume that the change of state (s_{n-1}, s_n) is due to a jump or a topology update. By the induction hypothesis, s_{n-1} is GMR and safe. Then (a) s_n satisfies Φ_{entry} by property (ii) of exhaustive entry conditions, hence is GMR and (b) the jump or topology update (s_{n-1}, s_n) is mode reachable, so s_n is safe by (3) if (s_{n-1}, s_n) is a jump, and by (4) if it is a topology update.

3.1.3 Safety properties with GMR-exhaustive entry conditions

Systems tend to be specified in such a way that their behavior is also defined for situations that cannot occur in practice. E.g. a car in our running example could – looking only at our specification – be in mode Rec while $pos_{front}(i) = pos(i)$. Jumps and updates in such a practically impossible situation may lead to more and more meaningless states and are nothing that we want to worry about when designing entry conditions. In this sense, condition (ii) in Definition 12 is too strong. One way of avoiding such situations is to adapt Definition 12 by requiring that condition (ii) is relative to GMR jumps or topology updates.

Definition 17 (GMR-Exhaustive Entry Conditions) Safety properties with GMR-exhaustive entry conditions have the form

$$\Phi_{\mathsf{entry}} \to \Box \Phi_{\mathsf{safe}}$$

where $\Phi_{\text{entry}} = \forall i_1, \ldots, i_m \phi_{\text{entry}}(\overline{x}(i_1), \ldots, \overline{x}(i_m))$ is a formula in the language $\mathcal{L}_{\text{index,num}}$ such that:

- (i) If Φ_{entry} holds in a state s, s is an initial state of S;
- (ii) For every GMR jump or GMR topology update (s, s'), Φ_{entry} holds in s'.

The proof of Theorem 16 can easily be adapted to the case of safety properties with GMR-exhaustive entry conditions.

Theorem 18 An SFHA $S = (\text{Top}, \{S(i) \mid i \in I\})$ satisfies a safety property with GMRexhaustive entry conditions $\Phi_{\text{entry}} \rightarrow \Box \Phi_{\text{safe}}$ if and only if the following hold:

- (1) All states satisfying Φ_{entry} satisfy Φ_{safe} .
- (2) Φ_{safe} is preserved under all flows starting from a state satisfying Φ_{entry} .
- (3) Φ_{safe} is preserved under all GMR jumps.
- (4) Φ_{safe} is preserved under all GMR topology updates.

Remark 19 In fact, often safety cannot be guaranteed for all runs but only for runs with a certain structure: In the running example, we might be interested only in runs in which lane changes are preceded and followed by local or global updates of the sensors. The definitions and results presented before can be adapted without problems such that they are relative to classes of runs. The tests in Section 7 show that in many cases it is not possible to guarantee safety for all runs, but safety can be guaranteed for runs in which jumps (corresponding e.g. to lane changes) are preceded by local or global updates of the sensors.

Example 20 Consider the running example and the safety property

 $\Phi^g_{\mathsf{safe}}: \forall i, j (i \neq \mathsf{nil} \land j \neq \mathsf{nil} \land \mathsf{lane}(i) = \mathsf{lane}(j) \land \mathsf{pos}(i) > \mathsf{pos}(j) \to \mathsf{pos}(i) - \mathsf{pos}(j) \geq d_s)$

We showed (using the method described in this paper) that this formula is invariant under globally mode reachable flows and topology updates, but not under globally mode reachable jumps (see also the remarks in Section 7.3); the problems with the jumps can occur because the information provided by sensors at the moment of a line change is outdated. In order to prevent this, it is necessary to ensure that a topology update takes place immediately before any lane change. We proved that for all runs in which topology updates take place before lane changes, formula Φ_{safe}^{g} is invariant under all jumps.

3.2 Reducing verification tasks to satisfiability checking

We consider safety properties $\Phi_{entry} \rightarrow \Box \Phi_{safe}$ with exhaustive entry conditions, where Φ_{entry} and Φ_{safe} are of the form

$$\Phi_{\mathsf{entry}} = \forall i_1 \dots i_m \phi_{\mathsf{entry}}(\overline{x}(i_1), \dots, \overline{x}(i_m))$$

$$\Phi_{\mathsf{safe}} = \forall i_1 \dots i_n \phi_{\mathsf{safe}}(\overline{x}(i_1), \dots, \overline{x}(i_n))$$

with quantifier-free ϕ_{entry} and ϕ_{safe} . We show that for decoupled SFLHA S we can reduce checking whether such a property holds, to checking whether certain formulae $F_q^{\text{init}}, F_q^{\text{flow}}, F_q^{\text{iump}}, F_q^{\text{top}}$ are unsatisfiable for all combinations of modes $q = (q_i)_{i \in I} \in Q^I$.

3.2.1 Sequentializing parallel jumps

We first show that for decoupled SFLHA we do not need to consider parallel jumps.

Lemma 21 Let $S = (\text{Top}, \{S(i) \mid i \in I\})$ be a decoupled SFHA.

- (1) Φ_{safe} is invariant under all (GMR) jumps in S iff it is invariant under all (GMR) jumps which reset the variables of a finite family of systems in S.
- (2) Φ_{safe} is invariant under all (GMR) jumps involving a finite family of systems in S iff it is invariant under all (GMR) jumps in any component of S.

Proof: (1) The direct implication is obviously true. Assume that Φ_{safe} is invariant under all (GMR) jumps which reset the variables of a finite family of systems in S. Consider a jump in S which resets the variables of an infinite family of systems in S. Assume that Φ_{safe} is not invariant under this jump, i.e. Φ_{safe} holds before the jump but there exist systems $S(i_1), \ldots, S(i_n)$ such that after the jump $\phi_{safe}(\overline{x}(i_1), \ldots, \overline{x}(i_n))$ is not true. Since S is decoupled, the value of the variables $\overline{x}(i_1), \ldots, \overline{x}(i_n)$ cannot be reset by systems not in $S(i_1), \ldots, S(i_n)$. This shows that already the combination of mode switches in the finite family $S(i_1), \ldots, S(i_n)$ would lead from a safe to an unsafe state. Contradiction.

(2) The direct implication is obviously true. We prove the converse implication. Let $C = \{c_1, \ldots, c_k\} \subseteq \{S(i) \mid i \in I\}$, let $guard_C$ and $jump_C$ be the formulae describing the guards resp. updates of a simultaneous (GMR) mode switch for all systems in C (the other variables do not change). Assume that Φ_{safe} is not invariant under this jump. Then the formula

$$\Phi_{\mathsf{safe}}(\overline{x}_0) \wedge \mathsf{guard}_C(\overline{x}_0) \wedge \mathsf{jump}_C(\overline{x}_0, \overline{x}_k) \wedge \neg \Phi_{\mathsf{safe}}(\overline{x}_k)$$

is satisfied by some variable assignment β . Because of the assumptions on resets in a decoupled SFHA, a jump in some S(i) cannot invalidate the guard of a simultaneous transition in another

S(j). In particular, none of c_1, \ldots, c_k can invalidate the guard of a later element of this sequence. In other words, if $\operatorname{guard}_C(\overline{x}_0)$ is true for a variable assignment, then – if we sequentialize C as the succession of jumps c_1, c_2, \ldots, c_k , sequentially changing the values of the variables from \overline{x}_0 to $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_k$, $\operatorname{guard}_{c_i}(\overline{x}_{i-1})$ is also true.⁴ Therefore,

$$\Phi_{\mathsf{safe}}(\overline{x}_0) \land \bigwedge_{i \in \{1, \dots, k\}} \left(\mathsf{guard}_{c_i}(\overline{x}_{i-1}) \land \mathsf{jump}_{\{c_i\}}(\overline{x}_{i-1}, \overline{x}_i) \right) \land \neg \Phi_{\mathsf{safe}}(\overline{x}_k)$$

is satisfiable for some extension β' of β to the fresh variables $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_{k-1}$. Since for each i obviously either $\Phi_{\mathsf{safe}}(\overline{x}_i)$ or $\neg \Phi_{\mathsf{safe}}(\overline{x}_i)$ is satisfied by β' , there must be at least one index $i_0 \in \{1, \ldots, k\}$ for which $\Phi_{\mathsf{safe}}(\overline{x}_{i_0-1})$ and $\neg \Phi_{\mathsf{safe}}(\overline{x}_{i_0})$, and thus all of

$$\Phi_{\mathsf{safe}}(\overline{x}_{i_0-1}) \wedge \mathsf{guard}_{\{c_{i_0}\}}(\overline{x}_{i_0-1}) \wedge \mathsf{jump}_{\{c_{i_0}\}}(\overline{x}_{i_0-1}, \overline{x}_{i_0}) \wedge \neg \phi_{\mathsf{safe}}(\overline{x}_{i_0})$$

is satisfied by β' . So Φ_{safe} is not invariant under jumps of a single component.

3.2.2 Verification of safety properties and satisfiability checking

We show that for decoupled SFLHA we can express the verification tasks (1)-(4) in Theorem 16 as satisfiability problems.

Theorem 22 Let S be a decoupled SFLHA. Let c_1, \ldots, c_n be the Skolem constants obtained from the negation of Φ_{safe} .

(1) The entry states of S satisfy Φ_{safe} iff the following formula F^{entry} is unsatisfiable:

$$F^{\text{entry}}: \Phi_{\text{entry}} \wedge \neg \phi_{\text{safe}}(\overline{x}(c_1), \dots, \overline{x}(c_n))$$

(2) Φ_{safe} is invariant under flows starting in a state satisfying Φ_{entry} iff for all $q = (q_i)_{i \in I} \in Q^I$ the following formula F_q^{flow} is unsatisfiable:

$$\begin{split} F_q^{\mathsf{flow}} &: t_0 < t_1 \land \Phi_{\mathsf{entry}}(\overline{x}(t_0)) \land \forall i_1, \dots, i_n \phi_{\mathsf{safe}}(\overline{x}(i_1, t_0), \dots, \overline{x}(i_n, t_0)) \\ & \land \forall i \, \mathsf{Flow}_{q_i}(\overline{x}(i, t_0), \overline{x}(i, t_1)) \land \neg \phi_{\mathsf{safe}}(\overline{x}(c_1, t_1), \dots, \overline{x}(c_n, t_1)) \end{split}$$

where if $flow_q(i) = \bigwedge \left(\mathcal{E}_f \vee \sum_{k=1}^n a_k^q(i) \dot{x}_k(i) \le a^q(i) \right)$ then

$$\begin{aligned} \mathsf{Flow}_{q_i}(\overline{x}(i,t_0),\overline{x}(i,t_1)) : & \bigwedge \left(\mathcal{E}_f \vee \sum_{k=1}^n a_k^{q_i}(i)(x_k(i,t_1) - x_k(i,t_0)) \le a^{q_i}(i)(t_1 - t_0) \right) \\ & \wedge \mathsf{Inv}_{q_i}(\overline{x}(i,t_0)) \wedge \mathsf{Inv}_{q_i}(\overline{x}(i,t_1)) \; . \end{aligned}$$

(3) Φ_{safe} is invariant under GMR jumps in S iff for all $q=(q_i)_{i\in I}\in Q^I$ the following formula $F^{\text{jump}_e^q}(i_0)$ is unsatisfiable for every $i_0 \in I$ and $e = (q_{i_0}, q'_{i_0}) \in E$, s.t. if $p(i_0)$ occurs in guard_e it is not nil:

$$\begin{split} F^{\mathsf{jump}_{e}^{q}}(i_{0}): \ \ \Phi_{\mathsf{entry}}(\overline{x}(t_{0})) \wedge \left(\left(t_{0} < t_{1} \wedge \forall i \mathsf{Flow}_{q_{i}}(\overline{x}(i,t_{0}),\overline{x}(i,t_{1})) \right) \vee t_{0} = t_{1} \right) \\ & \wedge \forall i_{1}, \dots, i_{n} \phi_{\mathsf{safe}}(\overline{x}(i_{1},t_{1}), \dots, \overline{x}(i_{n},t_{1})) \\ & \wedge \operatorname{guard}_{e}(\overline{x}(i_{0},t_{1})) \wedge \operatorname{jump}_{e}(\overline{x}(i_{0},t_{1}),\overline{x}'(i_{0})) \wedge \operatorname{Inv}_{q_{i_{0}}'}(\overline{x}'(i_{0})) \\ & \wedge \forall j (j \neq i_{0} \to \overline{x}'(j) = \overline{x}(j)) \wedge \neg \phi_{\mathsf{safe}}(\overline{x}'(c_{1}), \dots, \overline{x}'(c_{n})) \ . \end{split}$$

⁴In general, if c_i is a jump in a system S(j), $guard_{c_i}$ is expressed using only the variables of the system S(j), since the values of those variables are not changed by previous jumps, $guard_{c_i}(\overline{x}_{i-1})$ is in fact identical with $guard_{c_i}(\overline{x}_0)$.

(4) Φ_{safe} is invariant under GMR topology updates for pointers in a set P_1 iff for all $q = (q_i)_{i \in I} \in Q^I$ the following formula F_q^{top} is unsatisfiable:

$$\begin{split} F_q^{\mathsf{top}}: \ \Phi_{\mathsf{entry}}(\overline{x}(t_0)) \wedge \left(\left(t_0 < t_1 \wedge \forall i \mathsf{Flow}_{q_i}(\overline{x}(i,t_0),\overline{x}(i,t_1)) \right) \vee t_0 = t_1 \right) \\ & \wedge \forall i_1, \dots, i_n \phi_{\mathsf{safe}}(\overline{x}(i_1,t_1), \dots, \overline{x}(i_n,t_1)) \\ & \wedge \bigwedge_{p \in P_1} \mathsf{Update}(p,p') \wedge \neg \phi'_{\mathsf{safe}}(\overline{x}(c_1), \dots, \overline{x}(c_n)) \ , \end{split}$$

where ϕ'_{safe} is obtained from ϕ_{safe} by replacing every $p \in P_1$ with p'.

Proof: (1) is immediate.

(2) Assume that Φ_{safe} is not invariant under flows in some state q. Then there are functions $x(i) : \mathbb{R} \to \mathbb{R}$ satisfying all flow conditions and such that Φ_{safe} holds at the beginning of the flow and does not hold at the end of the flow. Then (using the mean value theorem) one can show that these functions can be used for constructing a model for the formula F_q^{flow} . See [Damm et al., 2011] for more details.

Conversely, assume that formula F_q^{flow} is satisfiable. We can define the functions x(i) by taking the linear interpolation of the functions defined at t_0 and t_1 . Then $\text{flow}_{q_i}(\overline{x}(i, t_0, t_1))$ holds; it follows that the functions $x(i) : \mathbb{R} \to \mathbb{R}$ satisfy the flow condition. So Φ_{safe} is not invariant under flows.

In particular, the results presented in [Damm et al., 2011] ensure that if the numerical constraints in the mode invariants are conjunctions of linear inequalities (and hence convex) we do not need to express explicitly that the invariant needs to hold at all points between t_0 and t_1 . (If we can construct a model of the formula in which the invariant holds at t_0 and t_1 we can construct a model in which the invariant holds at all points between t_0 and t_1 using linear interpolation of the functions x_i .)

- (3) is a consequence of Lemma 21 using arguments from (2).
- (4) is immediate (again, using arguments from (2)).

3.2.3 Checking exhaustive entry conditions

We now show that for decoupled SFLHA S we can reduce checking conditions (i) and (ii) in Definition 12 to satisfiability tests.

Theorem 23 Let S be a decoupled SFHA S, and $\Phi_{entry} \rightarrow \Box \Phi_{safe}$ be a safety condition as above. Then conditions (i) and (ii) in Definition 12 hold iff:

(i) Initial states:

$$\Phi_{\mathsf{entry}}(\overline{x}) \land \left(\neg(\bigvee_{q \in Q} \mathsf{Init}_q(\overline{x}(i_0))) \lor \neg\mathsf{Init}_{\mathsf{top}}(\overline{x})\right) \text{ is unsatisfiable}$$

(ii) For all $q = (q_i)_{i \in I} \in Q^I$:

(a) Topology updates:

 $(\forall i \operatorname{Inv}_{q_i}(\overline{x}_i)) \land \operatorname{Update}(p, p') \land \neg \Phi'_{\operatorname{entry}}(\overline{x}) \text{ is unsatisfiable,}$

where Φ'_{entry} arises from Φ_{entry} by replacing p with p', and

(b) Jumps: For all $e = (q_{i_0}, q'_{i_0}) \in E, i_0 \in I$:

$$\begin{array}{l} (\forall i \operatorname{Inv}_{q_i}(\overline{x}_i)) \land \operatorname{guard}_e(\overline{x}_{i_0}) \land \operatorname{jump}_e(\overline{x}_{i_0}, \overline{x}'_{i_0}) \land \\ \forall j (j \neq i_0 \to \overline{x}'(j) = \overline{x}(j)) \land \neg \Phi_{\operatorname{entry}}(\overline{x}') \quad is \ unsatisfiable \end{array}$$

Proof: (i) Condition (i) in Definition 12 states that if Φ_{entry} holds in a state *s* then *s* is initial. This is the case if and only if whenever Φ_{entry} holds for given values of the variables, then for these values:

- for all $i \in I$ there exists a mode $q \in Q$ such that the initial condition of mode q is satisfied in system S(i), and
- Init_{top} holds.

It can be easily checked that this is the case if and only if it cannot happen that Φ_{entry} holds for given values of the variables and for these values Init_{top} does not hold, or there exists a system i_0 such that for these values none of the initial conditions in $\{\mathsf{Inv}_q(i_0) \mid q \in Q\}$ holds, i.e. if and only if the following formula is unsatisfiable:

$$\Phi_{\mathsf{entry}}(\overline{x}) \land \big(\neg (\bigvee_{q \in Q} \mathsf{Init}_q(\overline{x}(i_0))) \lor \neg \mathsf{Init}_{\mathsf{top}}(\overline{x}) \big).$$

(ii) Condition (ii) in Definition 12 states that for every state change (s, s') due to (a) a topology update or (b) a jump, Φ_{entry} holds in s'. This happens if and only if the formulae in (a) and (b) are unsatisfiable (i.e. if and only if it cannot happen that S is in a mode $q = (q_i)_{i \in I}$ (i.e. the invariants of the systems S(i) in these modes hold), and (a) there is an update after which Φ_{entry} does not hold or (b) there is a jump after which Φ_{entry} does not hold).

For spatial families of *linear* hybrid automata, a similar result can be used for recognizing safety conditions with GMR-exhaustive entry conditions.

Theorem 24 For a decoupled SFLHA S, conditions (i) and (ii') in Definition 17 hold iff:

(i)

$$\Phi_{\mathsf{entry}}(\overline{x}) \land \left(\neg(\bigvee_{q \in Q} \mathsf{Init}_q(\overline{x}(i_0))) \lor \neg\mathsf{Init}_{\mathsf{top}}(\overline{x})\right) \text{ is unsatisfiable}$$

(ii') For all $(q_i)_{i \in I} \in Q^I$, $e \in E$, $i_0 \in I$:

- the following conjunction is unsatisfiable:

$$\begin{array}{l} t_0 < t_1 \ \land \ \Phi_{\mathsf{entry}}(\overline{x}(t_0)) \ \land \ \forall i \ \mathsf{Flow}_{q_i}(\overline{x}(i,t_0),\overline{x}(i,t_1)) \ \land \\ \mathsf{Update}(p,p') \ \land \ \neg \Phi_{\mathsf{entry}}'(\overline{x}(t_1)), \end{array}$$

where Φ'_{entry} arises from Φ_{entry} by replacing p with p'; and

- the following conjunction is unsatisfiable:

t

$$\begin{array}{rcl} _{0} < t_{1} & \wedge & \Phi_{\mathsf{entry}}(\overline{x}(t_{0})) & \wedge & \forall i\mathsf{Flow}(\overline{x}(i,t_{0}),\overline{x}(i,t_{1})) \wedge \\ & \mathsf{guard}_{e}(\overline{x}(i_{0},t_{1})) & \wedge & \mathsf{jump}_{e}(\overline{x}(i_{0},t_{1}),\overline{x}'_{i_{0}}) \wedge \\ & \forall j(j \neq i_{0} \rightarrow \overline{x}'(j) = \overline{x}(j,t_{1})) & \wedge & \neg \Phi_{\mathsf{entry}}(\overline{x}') \end{array}$$

where if $flow_q(i) = \bigwedge \left(\mathcal{E}_f \vee \sum_{k=1}^n a_k^q(i) \dot{x}_k(i) \le a^q(i) \right)$ then

$$\begin{aligned} \mathsf{Flow}_{q_i}(\overline{x}(i,t_0),\overline{x}(i,t_1)) : & \bigwedge \left(\mathcal{E}_f \vee \sum_{k=1}^n a_k^{q_i}(i)(x_k(i,t_1) - x_k(i,t_0)) \le a^{q_i}(i)(t_1 - t_0) \right) \\ & \wedge \forall i \operatorname{Inv}_{q_i}(\overline{x}(i,t_0)) \wedge \forall i \operatorname{Inv}_{q_i}(\overline{x}(i,t_1)) \end{aligned}$$

Proof: The proof of (ii') is similar to the proof of Theorem 23(ii), with the only difference that we need to additionally take flows into account. \Box

4 Automated Reasoning

We present classes of theories for which decidable fragments relevant for the verification tasks above exist. We use the following complexity results for fragments of linear arithmetic:

- The satisfiability over ℝ of conjunctions of linear inequalities can be checked in PTIME [Khachian, 1979].
- The problem of checking the satisfiability of sets of clauses in $LI(\mathbb{R})$ is in NP [Sontag, 1985].
- The satisfiability of any conjunction of Horn disjunctive linear (HDL) constraints⁵ over ℝ [Koubarakis, 2001] and the satisfiability of any conjunction of Ord-Horn constraints⁶ over ℝ [Nebel and Bürckert, 1995] can be decided in PTIME.

4.1 Local theory extensions

Let \mathcal{T}_0 be a base theory with signature Σ_0 . We consider extensions $\mathcal{T}_1 := \mathcal{T}_0 \cup \mathcal{K}$ of \mathcal{T}_0 with new function symbols in a set Σ_1 of extension functions whose properties are axiomatized with a set \mathcal{K} of augmented clauses, i.e. of axioms of the form $\forall x_1 \dots x_n (\Phi(x_1, \dots, x_n) \lor C(x_1, \dots, x_n))$, where $\Phi(x_1, \dots, x_n)$ is a first-order formula in signature Σ_0 and $C(x_1, \dots, x_n)$ is a clause containing extension functions. In this case we refer to the (theory) extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$. In [Sofronie-Stokkermans, 2005] we introduced and studied local theory extensions. In [Ihlemann and Sofronie-Stokkermans, 2010], various notions of locality of theory extensions were introduced and studied.

Definition 25 (Local theory extension) An extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is a local extension if for every set G of ground $\Sigma_0 \cup \Sigma_1 \cup \Sigma_c$ -clauses (where Σ_c is a set of additional constants), if G is unsatisfiable w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$ then unsatisfiability can be detected using the set $\mathcal{K}[G]$ consisting of those instances of \mathcal{K} in which the terms starting with extension functions are ground terms occurring in \mathcal{K} or G.

Stably local extensions are defined similarly, with the difference that $\mathcal{K}[G]$ is replaced with $\mathcal{K}^{[G]}$, the set of instances of \mathcal{K} in which the variables are instantiated with ground terms which occur in \mathcal{K} or G.

4.2 Hierarchical reasoning in local theory extensions

For local theory extensions (or stably local theory extensions) hierarchical reasoning is possible. If $\mathcal{T}_0 \cup \mathcal{K}$ is a (stably) local extension of \mathcal{T}_0 and G is a set of ground $\Sigma_0 \cup \Sigma_1 \cup \Sigma_c$ -clauses then, by Definition 25, $\mathcal{T}_0 \cup \mathcal{K} \cup G$ is unsatisfiable iff $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ (or resp. $\mathcal{T}_0 \cup \mathcal{K}^{[G]} \cup G$) is unsatisfiable. We can reduce this last satisfiability test to a satisfiability test w.r.t. \mathcal{T}_0 . The idea is to purify $\mathcal{K}[G] \cup G$ (resp. $\mathcal{K}^{[G]} \cup G$) by

- introducing (bottom-up) new constants c_t for subterms $t = f(g_1, \ldots, g_n)$ with $f \in \Sigma$, g_i ground $\Sigma_0 \cup \Sigma_c$ -terms,
- replacing the terms t with the constants c_t , and
- adding the definitions $c_t = t$ to a set D.

⁵A Horn-disjunctive linear constraint is a disjunction $d_1 \vee \cdots \vee d_n$ where each d_i is a linear inequality or disequation, and the number of inequalities does not exceed one.

⁶Ord-Horn constraints are implications $\bigwedge_{i=1}^{n} x_i \leq y_i \rightarrow x_0 \leq y_0$, $(x_i, y_i \text{ are variables})$.

We denote by $\mathcal{K}_0 \cup G_0 \cup D$ the set of formulae obtained this way. Then G is satisfiable w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$ iff $\mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0$ is satisfiable w.r.t. \mathcal{T}_0 , where

$$\mathsf{Con}_0 = \{ (\bigwedge_{i=1}^n c_i = d_i) \to c = d \mid f(c_1, \dots, c_n) = c, f(d_1, \dots, d_n) = d \in D \}.$$

Theorem 26 ([Sofronie-Stokkermans, 2005]) If $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ is a (stably) local extension and G is a set of (augmented) ground clauses then we can reduce the problem of checking whether G is satisfiable w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$ to checking the satisfiability w.r.t. \mathcal{T}_0 of the formula $\mathcal{K}_0 \cup \mathcal{G}_0 \cup \mathsf{Con}_0$ constructed as explained above.

If $\mathcal{K}_0 \cup G_0 \cup \mathsf{Con}_0$ belongs to a decidable fragment of \mathcal{T}_0 we can use the decision procedure for this fragment to decide whether $\mathcal{T}_0 \cup \mathcal{K} \cup G$ is unsatisfiable.

As the size of $\mathcal{K}_0 \cup \mathcal{G}_0 \cup \mathsf{Con}_0$ is polynomial in the size of G (for a given \mathcal{K}), locality allows us to express the complexity of the ground satisfiability problem w.r.t. \mathcal{T}_1 as a function of the complexity of the satisfiability of \mathcal{F} -formulae w.r.t. \mathcal{T}_0 .

4.3 Examples of local theories and theory extensions

In establishing the decidability results for the verification of safety properties of SFLHA we will use locality results for updates and for theories of pointers.

4.3.1 Update rules

We first consider update rules, in which some of the function symbols change the way they are defined, depending on a partition of their domain of definition. Many update rules define local theory extensions.

Theorem 27 ([Jacobs and Kuncak, 2011, Ihlemann et al., 2008]) Let \mathcal{T}_0 be a base theory with signature Σ_0 and $\Sigma \subseteq \Sigma_0$. Consider a family Update (Σ, Σ') of update axioms of the form:

$$\forall \overline{x}(\phi_i^f(\overline{x}) \to F_i^f(f'(\overline{x}), \overline{x})) \qquad i = 1, \dots, m, \quad f \in \Sigma$$
⁽²⁾

which describe how the values of the Σ -functions change, depending on a partition of the state space, described by a finite set $\{\phi_i^f \mid i \in I\}$ of Σ_0 -formulae and using Σ_0 -formulae F_i^f such that

(i)
$$\phi_i(\overline{x}) \wedge \phi_j(\overline{x}) \models_{\mathcal{T}_0} \bot$$
 for $i \neq j$ and

(*ii*) $\mathcal{T}_0 \models \forall \overline{x}(\phi_i(\overline{x}) \to \exists y(F_i(y, \overline{x}))) \text{ for all } i \in I.$

Then the extension of \mathcal{T}_0 with axioms $\mathsf{Update}(\Sigma, \Sigma')$ is local.

4.3.2 A theory of pointers

We present a fragment of the theory of pointers studied in [McPeak and Necula, 2005] and later analyzed in [Ihlemann et al., 2008]. Consider the language $\mathcal{L}_{index,num}$ with sorts index and num introduced before, with sets of unary pointer (numeric) fields P(X), and with a constant nil of sort index. The only predicate of sort index is equality; the signature Σ_{num} of sort num depends on the theory \mathcal{T}_{num} modeling the scalar domain. A guarded index-positive extended clause is a clause of the form:

$$C := \forall i_1 \dots i_n \ \mathcal{E}(i_1, \dots, i_n) \lor \mathcal{C}(\overline{x}_i(i_1), \dots, \overline{x}_i(i_n))$$
(3)

where C is a \mathcal{T}_{num} -formula over terms of sort num, $x_i \in X$, and \mathcal{E} is a disjunction of equalities between terms of sort index, containing all atoms of the form $i = \operatorname{nil}, f_n(i) = \operatorname{nil}, \ldots, f_2(\ldots f_n(i)) =$ nil for all terms $f_1(f_2(\ldots f_n(i)))$ occurring in $\mathcal{E} \vee C$, where $f_1 \in P \cup X, f_2, \ldots, f_n \in P$.

Theorem 28 ([Ihlemann et al., 2008]) Every set \mathcal{K} of guarded index-positive extended clauses defines a stably local extension of $\mathcal{T}_{num} \cup Eq_{index}$, where Eq_{index} is the pure theory of equality of sort index.

4.4 Chains of local theory extensions

The results we obtain in this paper will be justified by locality properties for certain theory extensions. In many cases we need to perform reasoning tasks in an extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}$ in which the set \mathcal{K} of axioms of the extension can be written as a union $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ such that both

- (1) $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1$ and
- (2) $\mathcal{T}_0 \cup \mathcal{K}_1 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$

are (stably) local theory extensions. In this case we say that we have a chain of (stably) local theory extensions; the reasoning task can be hierarchically reduced to reasoning in \mathcal{T}_0 in two steps:

Step 1: In a first step, we reduce checking whether $\mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \cup G$ is satisfiable to checking whether $\mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 * [G] \cup G$ is satisfiable (where $\mathcal{K}_2 * [G]$ is $\mathcal{K}_2[G]$ if the extension is local and $\mathcal{K}_2^{[G]}$ if the extension is stably local).

We can further reduce this task to checking the satisfiability of $\mathcal{T}_0 \cup \mathcal{K}_1 \cup (\mathcal{K}_2)_0 \cup \mathcal{G}_0 \cup \mathsf{Con}_0$ as explained in Theorem 26.

Step 2: if $G_1 = (\mathcal{K}_2)_0 \cup G_0 \cup \mathsf{Con}_0$ is a set of ground clauses, and the theory extension $\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1$ is (stably) local, we can use again Theorem 26 to reduce the problem of checking the satisfiability of $\mathcal{T}_0 \cup \mathcal{K}_1 \cup G_1$ to a satisfiability test w.r.t. \mathcal{T}_0 .

The idea can be used also for longer chains of (stably) local theory extensions:

$$\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \subseteq \cdots \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 \cup \cdots \cup \mathcal{T}_n.$$

A similar reduction can be used for chains of extensions

$$\mathcal{T}_0 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \subseteq \mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2$$

in which the second extension is (stably) local, if after using Step 1 above (i) the set of clauses obtained by instantiation $\mathcal{T}_0 \cup \mathcal{K}_1 \cup \mathcal{K}_2 * [G]$ or (ii) the set of clauses $\mathcal{T}_0 \cup \mathcal{K}_1 \cup (\mathcal{K}_2)_0$ obtained after the hierarchical reduction described in Theorem 26, define a (stably) local extension of \mathcal{T}_0 .

Example 29 We can for instance consider a set $\mathcal{K} = \mathsf{Update}(\Sigma, \Sigma')$ of update rules of the form in Theorem 27, which, by Theorem 27, defines a local extension of a base theory \mathcal{T}_0 .

Then, for every set G of ground clauses, $\mathcal{T}_0 \cup \mathcal{K} \cup G$ is satisfiable iff $\mathcal{T}_0 \cup \mathcal{K}[G] \cup G$ is satisfiable. It can happen that $\mathcal{K}[G]$ (hence also the purified set of clauses \mathcal{K}_0) is not ground, and that the purified set of clauses $\mathcal{K}_0 \cup G_0$ contains additional function symbols in a set $P \cup X$.

If, for instance, \mathcal{K}_0 is a set of guarded index-positive extended clauses then, by Theorem 28, \mathcal{K}_0 defines a stably local extension of $\mathcal{T}_{num} \cup \mathsf{Eq}_{index}$, where Eq_{index} is the pure theory of equality of sort index.

In order to check the satisfiability of G w.r.t. $\mathcal{T}_0 \cup \mathcal{K}$ we need to consider the following instances of \mathcal{K} : $(\mathcal{K}[G])^{[T_G]}$ where T_G is the set of ground terms occurring in $G \cup \mathcal{K}[G]$.

5 Verification: Decidability and Complexity

As mentioned in Section 3, we consider safety properties with exhaustive entry conditions $\Phi_{entry} \rightarrow \Box \Phi_{safe}$. We make the following assumptions:

Assumption 1: $S = (\text{Top}, \{S(i) \mid i \in I\})$ is a decoupled SFLHA.

- Assumption 2: Φ_{safe} is a set (conjunction) of guarded index-positive extended clauses of the form $\forall i_1, \ldots, i_n \mathcal{E} \lor \mathcal{C}$, such that \mathcal{C} is a conjunction of linear inequalities, and Φ_{entry} is a set (conjunction) containing either
 - (1) only guarded index-positive extended clauses of the form $\forall i_1, \ldots, i_n \mathcal{E} \lor \mathcal{C}$, such that \mathcal{C} is a conjunction of linear inequalities;
 - (2) or only $\mathcal{L}_{index,num}$ -formulae of the form $\forall i (i \neq nil \land \phi_k \rightarrow F(f(i), i)), k \in \{1, \ldots, m\}$ where $f \in \Sigma_1 \subseteq P \cup X$, the ϕ_k and F are formulae satisfying the conditions in Theorem 27 which do not contain any symbol in Σ_1 , such that all ϕ_k are quantifierfree;
 - (3) or only formulae of the form $\forall i (i \neq \mathsf{nil} \land \phi \to F_1(f'(i), i)) \land \forall i (i \neq \mathsf{nil} \land \neg \phi \to F_2(f'(i), i))$, where $f \in \Sigma_1 \subseteq P \cup X$, the ϕ and F_1, F_2 are formulae which do not contain any symbol in Σ_1 , and such that after the instantiation of the variable *i*, and computing the prenex normal form and Skolemization, the remaining formulae are either ground or guarded index-positive extended clauses of the form $\mathcal{E} \lor \mathcal{C}$, where \mathcal{C} is a conjunction of linear inequalities.

Assumption 3: The formulas Update(p, p') either

- (1) are of the form described in Theorem 27, with ϕ_k quantifier-free; or
- (2) contain only formulae of the form $\forall i (i \neq \operatorname{nil} \land \phi \to F_1(p'(i), i)) \land \forall i (i \neq \operatorname{nil} \land \neg \phi \to F_2(p'(i), i))$ where for every $p \in P \cup X$, p' is a new function symbol denoting the updated value of p, the formulae ϕ and F do not contain primed function symbols and:
 - (i) $\phi = \forall j_1, \dots, j_m \psi(i, j_1, \dots, j_m)$ with $m \ge 0$ and all free variables in F(p'(i), i) occur below p', or
 - (ii) $\phi = \exists \overline{j} \psi(i, \overline{j})$ and $i \neq \mathsf{nil} \land \psi(i, \overline{j}) \rightarrow F(i', i)$ is a guarded index-positive extended clause $\mathcal{E} \lor \mathcal{C}$, where \mathcal{C} is a conjunction of linear inequalities.
- Assumption 4: The numeric constraints in the description of the SFLHA S (including the conditions $\phi_k^p \rightarrow F_k^p(j,i)$ obtained from $\phi_k^p \rightarrow F_k^p(p'(i),i)$ in Update(p,p') by replacing all occurrences of p'(i) with j) and the numerical constraints in Φ_{safe} and Φ_{entry} are all HDL constraints or all Ord-Horn constraints.

Example 30 We illustrate the restrictions imposed by Assumptions 1-4 by examples:

- Assumption 1: The formulae used in the description of our running example (e.g. in Example 3) satisfy Assumption 1.
- Assumption 2: The safety conditions in Example 11, namely:
 - $\begin{array}{l} \ \Phi^g_{\mathsf{safe}} : \forall i, j (i \neq \mathsf{nil} \land j \neq \mathsf{nil} \land \mathsf{lane}(i) = \mathsf{lane}(j) \land \mathsf{pos}(i) > \mathsf{pos}(j) \to \mathsf{pos}(i) \mathsf{pos}(j) \geq d_s), \\ \ \Phi^l_{\mathsf{safe}} = \bigwedge_{\mathsf{index} \in P} \Phi^{\mathsf{index}}_{\mathsf{safe}}, \text{ where e.g. } \Phi^{\mathsf{front}}_{\mathsf{safe}} \text{ is:} \\ \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \rightarrow \mathsf{pos}(\mathsf{front}(i)) \mathsf{pos}(i) \geq d_s) \end{array}$

satisfy the conditions on Φ_{safe} in Assumption 2.

• Assumption 2(1): The entry condition in Example 14:

$$\Phi_{\mathsf{entry}} = \forall i \ (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \rightarrow |\mathsf{lane}_{\mathsf{front}}(i) - \mathsf{lane}(\mathsf{front}(i))| < \varepsilon)$$

satisfies the conditions in Assumption 2(1).

• Assumption 2(2): The entry condition Φ_{entry} :

$$\begin{split} \forall i(i \neq \mathsf{nil} \land \mathsf{front}(i) = \mathsf{nil} & \to \forall k(k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \geq \mathsf{pos}(i) \to \mathsf{lane}(k) \neq \mathsf{lane}(i))) \\ \forall i(i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} & \to \mathsf{pos}_{\mathsf{front}}(i) > \mathsf{pos}(i) + d' \land \mathsf{lane}_{\mathsf{front}}(i) = \mathsf{lane}(i) \land \\ & \forall k(k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \geq \mathsf{pos}(i) \land \mathsf{lane}(k) = \mathsf{lane}(i) \\ & \to \mathsf{pos}(k) \geq \mathsf{pos}_{\mathsf{front}}(i)) \land \\ & \mathsf{pos}(\mathsf{front}(i)) = \mathsf{pos}_{\mathsf{front}}(i) \land \mathsf{lane}(\mathsf{front}(i)) = \mathsf{lane}_{\mathsf{front}}(i)) \end{split}$$

satisfies the conditions in Assumption 2(2).

• Assumption 2(3): The entry condition Φ_{entry} :

$$\forall i (i \neq \mathsf{nil} \land \mathsf{Prop}(i) \land \neg \exists j (\mathsf{ASL}(j,i)) \to \mathsf{front}(i) = \mathsf{nil}) \\ \forall i (i \neq \mathsf{nil} \land \mathsf{Prop}(i) \land \exists j (\mathsf{ASL}(j,i)) \to \mathsf{Closest}_{\mathsf{f}}(\mathsf{front}(i),i))$$

with the notations in Example 4, namely:

- $ASL(j,i): j \neq nil \land lane(j) = lane(i) \land pos(j) > pos(i)$, which expresses the fact that j is ahead of i on the same lane, and
- $\mathsf{Closest}_{\mathsf{f}}(j,i)$: $\mathsf{ASL}(j,i) \land \forall k(\mathsf{ASL}(k,i) \rightarrow \mathsf{pos}(k) \ge \mathsf{pos}(j))$, which expresses the fact that j is ahead of i and there is no car between them

satisfies the conditions in Assumption 2(3).

• Assumption 3: The formula Update(front, front') used for the update rules in Example 4:

$$\begin{aligned} \forall i (i \neq \mathsf{nil} \land \mathsf{Prop}(i) \land \neg \exists j (\mathsf{ASL}(j,i)) \to \mathsf{front}'(i) = \mathsf{nil}) \\ \forall i (i \neq \mathsf{nil} \land \mathsf{Prop}(i) \land \exists j (\mathsf{ASL}(j,i)) \to \mathsf{Closest}_{\mathsf{f}}(\mathsf{front}'(i),i)) \\ \forall i (i \neq \mathsf{nil} \land \neg \mathsf{Prop}(i) \to \mathsf{front}'(i) = \mathsf{front}(i)) \end{aligned}$$

satisfies the conditions in Assumption 3.

• Assumption 4: The numeric constraints in the formulae describing the invariants, the initial states, the flows, guards and jumps in Example 3 are conjunctions of HDL constraints, hence satisfy Assumption 4.

In the condition Φ_{safe}^l above, the numeric constraint is $\mathsf{pos}(\mathsf{front}(i)) - \mathsf{pos}(i) \ge d_s$, hence is a HDL constraint.

We prove that under Assumptions 1–3 the verification problems of Theorem 16 are decidable, and analyze their complexity.

We analyze the complexity of verifying safety properties with exhaustive entry conditions, by analyzing the complexity of checking the satisfiability of the formulae F_q^{entry} , F_q^{jump} , F_q^{flow} , and F_q^{top} (cf. Theorem 22). Since the number of systems to be considered is unbounded, a naive approach to analyzing the satisfiability of these formulae for all tuples $q = (q_i)_{i \in I} \in Q^I$ can be problematic. We identify situations which allow us to limit the analysis to a "neighborhood" of the systems for which ϕ_{safe} fails. For this we use the specific form of the axioms we consider.

5.1 Verification tasks: Chains of local theory extensions

We show that under Assumptions 1–4 the theories used for specifying the various verification tasks in Theorem 16 and the corresponding satisfiability problems in Theorem 22 can be structured as chains of (stably) local theory extensions.

Theorem 31 For all $(q_i)_{i \in I} \in Q^I$ the following hold:

(1) Safety of entry conditions:

- (a) Under Assumption 2 (1): $\mathbb{R} \cup \mathsf{Eq}_{\mathsf{index}} \subseteq \mathbb{R} \cup \Phi_{\mathsf{entry}}$ is a stably local theory extension.
- (b) Under Assumption 2 (2) both theory extensions below: $\mathbb{R} \cup \mathsf{Eq}_{\mathsf{index}} \subseteq \mathbb{R} \cup \mathsf{UIF}_{(P \cup X) \setminus \Sigma_1} \subseteq \mathbb{R} \cup \Phi_{\mathsf{entry}}$ are local theory extensions.⁷
- (c) Under Assumption 2 (3) both extensions below: $\mathbb{R} \cup \mathsf{Eq}_{\mathsf{index}} \subseteq \mathbb{R} \cup \mathsf{UIF}_{(P \cup X) \setminus \Sigma_1} \subseteq \mathbb{R} \cup \Phi_{\mathsf{entry}}$ are local theory extensions. However, there exist sets G of ground clauses for which $\Phi_{\mathsf{entry}}[G]$ may not be a set of ground clauses. In this case, the requirements in Assumption 2 (3) ensure that $\mathbb{R} \cup \mathsf{UIF}_{(P \cup X) \setminus \Sigma_1} \subseteq \mathbb{R} \cup \Phi_{\mathsf{entry}}[G]$ is a stably local theory extension.
- (2) Invariance under flows:

Under Assumptions 1 and 2(1): $\mathbb{R} \cup \mathsf{Eq}_{\mathsf{index}} \subseteq \mathbb{R} \cup (\Phi_{\mathsf{entry}}(\overline{x}(t_0)) \cup \Phi_{\mathsf{safe}}(\overline{x}(t_0)) \cup \{\forall i (\mathsf{Flow}_{q_i}(\overline{x}(i,t_0), \overline{x}(i,t_1)))\}) \text{ is a stably local theory extension.}$

(3) Invariance under GMR jumps:

 $\begin{array}{l} \textit{Under Assumptions 1 and 2(1):} \\ \mathbb{R} \cup \mathsf{Eq}_{\mathsf{index}} \subseteq \mathbb{R} \cup (\Phi_{\mathsf{entry}}(\overline{x}(t_0)) \cup \{\forall i \, (\mathsf{Flow}_{q_i}(\overline{x}(i,t_0),\overline{x}(i,t_1)))\} \cup \Phi_{\mathsf{safe}}(\overline{x}(t_1)) \cup \\ \{ \mathsf{guard}_e(\overline{x}(i_0,t_1)), \ \mathsf{jump}_e(\overline{x}(i_0,t_1),\overline{x}'(i_0)), \ \mathsf{Inv}_{q_{i_0}'}(\overline{x}'(i_0,t_1))\} \} \\ \textit{is a stably local theory extension for every } i_0 \in I \textit{ and } e \in E \textit{ s.t. if } p(i_0) \textit{ occurs in guard}_e \textit{ it is not nil.} \end{array}$

(4) Invariance under topology updates:

Under Assumptions 1, 2(1), and 3, the first extension below is stably local: $\mathbb{R} \cup \mathsf{Eq}_{\mathsf{index}} \subseteq \mathbb{R} \cup (\Phi_{\mathsf{entry}}(\overline{x}(t_0)) \cup \Phi_{\mathsf{safe}}(\overline{x}(t_0)) \cup \{\forall i \, (\mathsf{Flow}_{q_i}(\overline{x}(i,t_0),\overline{x}(i,t_1)))\}) \\
\subseteq \mathbb{R} \cup (\Phi_{\mathsf{entry}}(\overline{x}(t_0)) \cup \Phi_{\mathsf{safe}}(\overline{x}(t_0)) \cup \{\forall i \, (\mathsf{Flow}_{q_i}(\overline{x}(i,t_0),\overline{x}(i,t_1)))\}) \\
\cup \mathsf{Update}(\mathsf{index},\mathsf{index'}).$

and the last extension is local.

Proof: This follows immediately from the form of the formulae and from the locality results in Theorem 27 and 28. $\hfill \Box$

Notation. In the following sections let $G = \neg \phi_{\mathsf{safe}}(\overline{x}(c_1), \ldots, \overline{x}(c_n))$. By Assumption 2, G consists of a conjunction of ground linear inequalities and a set of disequalities, consisting of unit clauses of the form $g \neq \mathsf{nil}$ for every ground term g of sort index occurring in G below a pointer or scalar field. We will denote by $\mathsf{st}(G)$ the set of all (ground) subterms of G. The results in the next subsections follow from Theorem 31.

⁷If Σ is a set of functions then UIF_{Σ} is the theory of uninterpreted function symbols in Σ axiomatized only by the congruence axioms for the functions in Σ . Any extension of a theory with uninterpreted function symbols is local [Sofronie-Stokkermans, 2005].

5.2 Verification of safety properties.

We now analyze the decidability and complexity of verifying safety properties with exhaustive entry conditions, by analyzing the complexity of checking the satisfiability of the formulae F_q^{entry} , F_q^{jump} , F_q^{flow} , and F_q^{top} (cf. Theorem 22).

5.2.1 Entry conditions

We first analyze the decidability and complexity of checking whether entry states are safe. By Theorem 22(1), this is the case iff $\Phi_{entry} \wedge G$ is unsatisfiable, where $G = \neg \phi_{safe}(\overline{x}(c_1), \ldots, \overline{x}(c_n))$. In what follows we identify conditions in which the problem of checking the satisfiability of this formula is decidable and study its complexity.

Lemma 32 Under Assumption 2 the following hold:

- (1) Under Assumption 2 (1), $\Phi_{entry} \wedge G$ is unsatisfiable iff $\Phi_{entry}^{[G]} \wedge G$ is unsatisfiable.
- (2) Under Assumptions 2 (2) or (3), $\Phi_{entry} \wedge G$ is unsatisfiable iff $(\Phi_{entry}[G])^{[T_G]} \wedge G$ is unsatisfiable, where T_G is the set of all ground terms of sort index occurring in $\Phi_{entry}[G]$.
- (3) The size of the set of terms of sort index in st(G) and hence also the number of instances in Φ_{entry}^[G] (in case (1)) resp. (Φ_{entry}[G])^[T_G] (in case (2)) is polynomial in the number of terms of sort index in Φ_{safe}. Therefore also the cardinality of the set I^G_{entry} of ground terms of sort index contained in these sets of instances is polynomial in the number of terms of sort index in Φ_{safe}.

Proof: (1) Under Assumption 2 (1), by Theorem 31(1)(a), Φ_{entry} defines a stably local theory extension of $\mathbb{R} \cup \mathsf{Eq}_{index}$, so in order to check whether $\Phi_{entry} \wedge G$ is satisfiable it is sufficient to check whether $\Phi_{entry} \cap G$ is satisfiable.

(2) Under Assumption 2 (2) or (3), by Theorem 31(1)(b) or (c), Φ_{entry} defines a local theory extension of $\mathbb{R} \cup \mathsf{UIF}_X$. Therefore, in order to check whether there exists a model of $\mathbb{R} \cup \Phi_{entry}[G]$ which is a model for G it is sufficient to check whether there is a model of $\mathbb{R} \cup \Phi_{entry}[G]$ which is a model for G. Note however that $\Phi_{entry}[G]$ is in general not a set of ground formulae. The conditions in Assumption 2 (2) and (3) ensure that this set of instances is a guarded index-positive extended clause. By Theorem 28, in order to check whether there is a model of $\mathbb{R} \cup \Phi_{entry}[G]$ which is a model for G it is sufficient to check whether there is a model of $\mathbb{R} \cup \Phi_{entry}[G]$ which is a model for G it is sufficient to check whether there is a model of $\mathbb{R} \cup \Phi_{entry}[G]^{T_G}$ which is a model for G, where T_G is the set of all ground terms of sort index occurring in $\Phi_{entry}[G] \wedge G$.

(3) We show that the number of instances (and size) of $\Phi_{entry}^{[G]}$ (resp. $(\Phi_{entry}[G])^{T_G})$ – hence also the size of I_{entry}^G – is polynomial in the number of terms of sort index in Φ_{safe} .

Let np_G be the number of terms of sort index occurring in G, and np_{entry} the number of terms of sort index occurring in Φ_{entry} , and let:

- nv_{entry} be the number of universally quantified variables in Φ_{entry} under Assumption 2(1) or 2(2),
- na_{entry} (ne_{entry}) be the maximal number of universally (existentially) quantified variables in a formula in Φ_{entry} under Assumption 2(3).

The number n_{entry} of instances in $\Phi_{\text{entry}}^{[G]}$ is at most $np_G^{nv_{\text{entry}}}$; the size s_{entry} (number of literals) in $\Phi_{\text{entry}}^{[G]}$ is at most $np_G^{nv_{\text{entry}}} \cdot \text{size}(\Phi_{\text{entry}})$.

 I_{entry}^{G} contains all terms of sort index in $\Phi_{\text{entry}}^{[G]} \wedge G$. Under Assumption 2(1) and (2), there can be at most $np_{\text{entry}} \cdot np_{G}$ such terms in $\Phi_{\text{entry}}^{[G]}$. Under Assumption 2(3) we have to additionally take into account the Skolem constants introduced for the existentially quantified variables

after instantiation. For each combination of values for the universally quantified variables, we introduce a tuple of Skolem functions for the existentially quantified variables. We have at most $np_G^{na_{entry}}$ possible such combinations of values, thus at most $np_G^{na_{entry}}$ tuples of Skolem functions. Since in Assumption 2(3), $na_{entry} = 1$, we have at most np_G tuples of Skolem functions for every formula in Φ_{entry} containing existential quantifiers. Thus, in this case the number of terms of sort index in $\Phi_{entry}^{[G]} \wedge G$ is at most $np_{entry} \cdot (np_G + np_G)$ (the terms which can be used as arguments are either the np_G subterms of G or the newly introduced Skolem constants).

In all cases, the cardinality $n_{i_{entry}}$ of I_{entry}^G is at most $2 \cdot np_{entry} \cdot np_G$, hence is linear in the number of terms of sort index in Φ_{safe} and in the number of variables occurring in Φ_{entry} . \Box .

Theorem 33 Under Assumption 2 the problem of checking the satisfiability of $\mathsf{F}^{\mathsf{entry}}$: $\Phi_{\mathsf{entry}} \wedge G$ is decidable (and in NP).

Proof: The hierarchical method for reasoning in stably local theory extensions allows us to reduce the task of checking the satisfiability of $\mathsf{F}^{\mathsf{entry}}$ to the problem of checking the satisfiability of a formula which is a conjunction of guarded index-positive extended clauses of the form $\mathcal{E} \vee \mathcal{C}$, where \mathcal{E} is a disjunction of equalities between terms of sort index and \mathcal{C} a constraint over real numbers w.r.t. the disjoint combination of the theory of real numbers \mathbb{R} and the theory of uninterpreted functions symbols in $P \cup C$. The reduction is done in one step if Assumption 2(1) holds, and in two steps if Assumption 2(2) or (3) holds. The problem of checking the satisfiability of such formulae is decidable.

In both cases the variant of Assumption 2 we use guarantees that all the clauses we obtain are ground or index-positive extended clauses of the form $\mathcal{E} \vee \mathcal{C}$, where \mathcal{C} is a conjunction of linear inequalities.⁸ After the hierarchical reduction we obtain a set of ground clauses in the combination of $LI(\mathbb{R})$ and $\mathsf{Eq}_{\mathsf{index}}$; the complexity of checking decidability of ground clauses in such a combination is in NP.

Corollary 34 Let $S = (\text{Top}, \{S(i) \mid i \in I\})$ be an SFHA. Under Assumption 2, the following are equivalent:

- (1) There exist indices c_1, \ldots, c_n for which the safety condition Φ_{safe} does not hold although Φ_{entry} holds.
- (2) There exists a finite set $I_{entry} \subseteq I$ of indices, of size polynomial in the number of terms of sort index in Φ_{safe} (assuming that the lengths of the formulae describing the SFHA S are considered constants) such that the entry conditions are not safe already in the systems $S_{entry} = (\mathsf{Top}_{|I_{entry}}, \{S(i) \mid i \in I_{entry}\}).$

 I_{entry} and the system S_{entry} describe a suitable neighborhood of c_1, \ldots, c_n which can effectively be described (the indices in I_{entry} correspond to the terms in I_{entry}^G in Theorem 33).

Proof: (1) ⇒ (2) Assume that (1) holds. Then $\Phi_{entry} \wedge G$ is satisfiable. Then $\Phi_{entry} {}^{[G]} \wedge G$ (or resp. $\Phi_{entry} {}^{[G]} \wedge G$) is satisfiable, i.e. there is a model \mathcal{A} for this formula. Let I_{entry}^{G} be as defined in Theorem 33, and let I_{entry} be the set of the values in \mathcal{A} of the terms in I_{entry}^{G} . The model \mathcal{A} can easily be transformed into a model of Φ_{entry} , describing a system referring to the neighborhood I_{entry} of the indices c_1, \ldots, c_n at which Φ_{entry} holds, but Φ_{safe} does not hold. But then the entry conditions are not safe already for the system $S_{entry} = (\mathsf{Top}_{|I_{entry}}, \{S(i) \mid i \in I_{entry}\}).$

By Lemma 32 (3), the size of I_{entry}^G (hence also the size of I_{entry}) is polynomial in the number of terms of sort index in Φ_{safe} .

⁸The latter can happen only under Assumption 3 (2); the remaining free variables occur only as arguments of the variables x; in this case we instantiate again, the size of the set of clauses grows polynomially.

 $(2) \Rightarrow (1)$ Conversely, assume that there exists a finite set $I_{entry} \subseteq I$ of indices, corresponding to terms in I_{entry}^G , such that in S_{entry} there are indices c_1, \ldots, c_n at which the safety property does not hold. Then $\Phi_{entry} \wedge G$ is satisfiable, if quantification is considered to be made on the finite set I_{entry} . The model for this formula is a model of $(\Phi_{entry})^{[G]} \wedge G$ (or resp. of $(\Phi_{entry}[G])^{T_G} \wedge G$). By Lemma 32 it follows that $\Phi_{entry} \wedge G$ is satisfiable, i.e. (1) holds.

Parametric Verification. We can consider parametric systems, in which we assume that some of the constants used in the specification of the entry conditions and safety properties are parameters. If we impose constraints on these parameters (in the form of constraints between real numbers) then the results in Theorem 33 can still be used to prove that the verification problems remain decidable. The complexity of the problems depends on the form of the constraints (for linear constraints we still can show that the problem is in NP).

Alternatively, we can use the method for hierarchical reasoning combined with quantifier elimination for the theory of real numbers for generating constraints on the parameters which guarantee that $\Phi_{\text{entry}} \wedge G$ is unsatisfiable, as explained in [Sofronie-Stokkermans, 2013] (the complexity is then exponential).

Example 35 Consider the running example, with entry states being states in which the information provided by the sensors is correct and every car is sufficiently far away from the following car on the same lane, described by the following formula Φ_{entry} :

$$\begin{split} \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) = \mathsf{nil} & \to \forall k (k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \ge \mathsf{pos}(i) \to \mathsf{lane}(k) \neq \mathsf{lane}(i))) \\ \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} & \to \mathsf{pos}_{\mathsf{front}}(i) > \mathsf{pos}(i) + d' \land \mathsf{lane}_{\mathsf{front}}(i) = \mathsf{lane}(i) \land \\ & \forall k (k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \ge \mathsf{pos}(i) \land \mathsf{lane}(k) = \mathsf{lane}(i) \\ & \to \mathsf{pos}(k) \ge \mathsf{pos}_{\mathsf{front}}(i)) \land \\ & \mathsf{pos}(\mathsf{front}(i)) = \mathsf{pos}_{\mathsf{front}}(i) \land \mathsf{lane}(\mathsf{front}(i)) = \mathsf{lane}_{\mathsf{front}}(i)). \end{split}$$

This formula clearly satisfies Assumption 2(2), as an extension of the theory of front, lane_{front} and pos_{front} with the functions pos and lane, satisfying the formulae above. Consider the following safety property:

$$\Phi^g_{\mathsf{safe}} = \forall i, j (i \neq \mathsf{nil} \land j \neq \mathsf{nil} \land i \neq j \land \mathsf{pos}(i) > \mathsf{pos}(j) \land \mathsf{lane}(i) = \mathsf{lane}(j) \to \mathsf{pos}(i) - \mathsf{pos}(j) \ge d_s) \land \mathsf{pos}(i) = \mathsf{lane}(j) \to \mathsf{pos}(i) = \mathsf{lane}(j) \to \mathsf{pos}(j) \ge d_s) \land \mathsf{pos}(j) \ge d_s \land$$

We check the satisfiability of $\Phi_{entry} \wedge G$, where $G = \neg \Phi_{safe}^g$ is:

$$\begin{split} G: i_0 \neq \mathsf{nil} \land j_0 \neq \mathsf{nil} \land i_0 \neq j_0 \land \mathsf{lane}(i_0) = \mathsf{lane}(j_0) \\ \land \mathsf{pos}(i_0) > \mathsf{pos}(j_0) \land \mathsf{pos}(i_0) - \mathsf{pos}(j_0) < d_s \end{split}$$

as follows: We compute $\Phi_{entry}[G]$. For instance, by instantiating *i* with j_0 and *k* with i_0 in both formulae, we obtain:

 $\begin{array}{l} (j_0 \neq \mathsf{nil} \land \mathsf{front}(j_0) = \mathsf{nil} \land i_0 \neq \mathsf{nil} \land i_0 \neq j_0 \land \mathsf{pos}(i_0) \geq \mathsf{pos}(j_0) \rightarrow \mathsf{lane}(i_0) \neq \mathsf{lane}(j_0)) \\ (j_0 \neq \mathsf{nil} \land \mathsf{front}(j_0) \neq \mathsf{nil} \rightarrow \mathsf{pos}_{\mathsf{front}}(j_0) > \mathsf{pos}(j_0) + d' \land \mathsf{lane}_{\mathsf{front}}(j_0) = \mathsf{lane}(j_0)) \\ (j_0 \neq \mathsf{nil} \land \mathsf{front}(j_0) \neq \mathsf{nil} \land i_0 \neq \mathsf{nil} \land i_0 \neq j_0 \land \mathsf{pos}(i_0) \geq \mathsf{pos}(j_0) \land \mathsf{lane}(i_0) = \mathsf{lane}(j_0) \\ \rightarrow \mathsf{pos}(i_0) \geq \mathsf{pos}_{\mathsf{front}}(j_0)). \end{array}$

After the hierarchical reduction, we obtain a set of clauses which is clearly unsatisfiable if $d' \ge d_s$. Below a short intuitive justification: From the literals in G and the first formula above we derive that front $(j_0) \ne nil$. Together with the second formula we then obtain:

$$\mathsf{pos}_{\mathsf{front}}(j_0) > \mathsf{pos}(j_0) + d' \wedge \mathsf{lane}_{\mathsf{front}}(j_0) = \mathsf{lane}(j_0),$$

and together with the third formula we obtain:

 $pos(i_0) \ge pos_{front}(j_0)$, hence $pos(i_0) > pos(j_0) + d'$.

If d_s and d' are numerical values such that $d' \ge d_s$, this is unsatisfiable.

Parametric verification. In this problem d_s and d' can also be considered to be parameters. If we assume that $d' \ge d_s$, we can easily see that $\mathsf{pos}(i_0) > \mathsf{pos}(j_0) + d' \land \mathsf{pos}(i_0) - \mathsf{pos}(j_0) < d_s$ is unsatisfiable. Alternatively, we can use quantifier elimination after the hierarchical reduction to prove that $\mathsf{pos}(i_0) > \mathsf{pos}(j_0) + d' \land \mathsf{pos}(i_0) - \mathsf{pos}(j_0) < d_s$ is unsatisfiable iff $d' \ge d_s$.

Small model property The instantiation we used justifies a small model property as explained in Corollary 34: In order to check whether the states satisfying the entry condition Φ_{entry} also satisfy the safety property expressed by Φ_{safe}^g , we first choose two different cars for which the safety condition may not hold, corresponding to the indices i_0 and j_0 in G. The instances of $\Phi_{entry}[G]$ contain two additional terms of sort index, namely front (i_0) and front (j_0) . We know that i_0 and j_0 are not nil and that they are different. We do not know however whether front (i_0) or front (j_0) are nil (neither whether they are equal to each other, or whether front $(i_0) = j_0$ or front $(j_0) = i_0$). We need to consider all such combinations, i.e. check whether Φ_{entry} entails Φ_{safe} in all systems $S_{I_0} = (\text{Top}_{|I_0}, \{S(i) \mid i \in I_0\})$, where I_0 are indices corresponding to the set of terms $I_{entry}^G = \{\text{nil}, i_0, j_0, \text{front}(i_0), \text{front}(j_0)\}$ (taking into account that one or more of the elements of I_0 might be equal).

We now analyze the complexity of checking whether in $S_{I_G} \Phi_{entry}[G] \cup G$ is satisfiable for a given I_G . Such systems describe models of $\Phi_{entry} \cup G$ obtained by using the usual completion – which sets all undefined functions of sort index to nil – from models of $\Phi_{entry}[G] \cup G$. Given one such system, we know precisely the equality relationships between the terms in I_G . Depending on this, we have the one of the following situations:

- some of the premises of the formulae in $\Phi_{entry}[G]$ may be false: then the corresponding instance is true in this model
- all premises of the formulae in $\Phi_{entry}[G]$ are true: We then only need to check the satisfiability of the conjunctions of linear constraints on the left-hand side, which can be done in polynomial time.

Note that if the guards of sort index in the formulae in Φ_{safe} and Φ_{entry} are terms of the form $t = \mathsf{nil}$ then we do not need to take into account all possible equality relationships between the terms in I_{entry}^G , but only possible equality of such terms with nil . The number of all possible systems which need to be tested is then $2^{|I_{\mathsf{entry}}^G \setminus \mathsf{st}[G]|}$, in our example $2^{|\{\mathsf{front}(i_0),\mathsf{front}(j_0)\}|} = 2^2$.

5.2.2 Flows

We now analyze the decidability and complexity of checking whether Φ_{safe} is preserved under all flows starting from a state satisfying Φ_{entry} . According to Theorem 22(2), this can be expressed as the problem of checking, for all $q = (q_i)_{i \in I} \in Q^I$, the satisfiability of the formula:

$$\begin{split} F_q^{\mathsf{flow}} &: t_0 < t_1 \land \Phi_{\mathsf{entry}}(\overline{x}(t_0)) \land \forall i_1, \dots, i_n \phi_{\mathsf{safe}}(\overline{x}(i_1, t_0), \dots, \overline{x}(i_n, t_0)) \\ & \land \forall i \, \mathsf{Flow}_{q_i}(\overline{x}(i, t_0), \overline{x}(i, t_1)) \land G \end{split}$$

where if $\mathsf{flow}_q(i) = \bigwedge \left(\mathcal{E}_f \vee \sum_{k=1}^n a_k^q(i) \dot{x}_k(i) \leq a^q(i) \right)$ then

$$\begin{aligned} \mathsf{Flow}_{q_i}(\overline{x}(i,t_0),\overline{x}(i,t_1)) : & \bigwedge \left(\mathcal{E}_f \vee \sum_{k=1}^n a_k^{q_i}(i)(x_k(i,t_1) - x_k(i,t_0)) \le a^{q_i}(i)(t_1 - t_0) \right) \\ & \wedge \mathsf{Inv}_{q_i}(\overline{x}(i,t_0)) \wedge \mathsf{Inv}_{q_i}(\overline{x}(i,t_1)) \end{aligned}$$

and $G = \neg \phi_{\mathsf{safe}}(\overline{x}(c_1, t_1), \dots, \overline{x}(c_n, t_1)).$

Lemma 36 (Flows) Under Assumptions 1 and 2(1) the following hold:

- (1) For every $q = (q_i)_{i \in I} \in Q^I$, $\mathsf{F}_q^{\mathsf{flow}}$ is unsatisfiable iff $F_q^{\mathsf{flow}}[G]$ is unsatisfiable.
- (2) The size of the set of terms of sort index in st(G) and hence also the size of $F_q^{flow}{}^{[G]}$ is polynomial in the number of terms of sort index in Φ_{safe} . Therefore also the size of the set I_{flow}^G of ground terms of sort index in $F_q^{flow}{}^{[G]}$ is polynomial in the number of terms of sort index in $F_q^{flow}{}^{[G]}$.

The set of instances $F_q^{\text{flow}[G]}$ contain formulae Inv_{q_i} and Flow_{q_i} for indices *i* corresponding to terms in I_{flow}^G .

Proof: (1) If Φ_{entry} satisfies Assumption 2(1) then, by Theorem 31(2), for every $q = (q_i)_{i \in I} \in Q^I$ the set of axioms:

 $\mathcal{K}_{\mathsf{flow}} = \Phi_{\mathsf{entry}}(\overline{x}_0) \land \Phi_{\mathsf{safe}}(\overline{x}_0) \land \forall i \, \mathsf{Flow}_{q_i}(\overline{x}_0(i), \overline{x}_1(i))$

defines a stably local theory extension of $\mathbb{R} \cup \mathsf{Eq}_{\mathsf{index}}$, so in order to check whether $\mathsf{F}_q^{\mathsf{flow}}$ is satisfiable it is sufficient to check whether $\mathcal{K}_{\mathsf{flow}}{}^{[G]} \wedge G$ is satisfiable. (2) Clearly, the size of $\mathcal{K}_{\mathsf{flow}}{}^{[G]}$ (hence also the size of I_{flow}^G) is polynomial in the number of

(2) Clearly, the size of $\mathcal{K}_{\mathsf{flow}}^{[G]}$ (hence also the size of I_{flow}^{G}) is polynomial in the number of terms of sort index in Φ_{safe} . Because of Assumption 2, this set of instances contains only the instances of $\forall i \mathsf{lnv}_{q_i}$ in which *i* is replaced by a term in I_{flow}^G . But this means that only the states q_i , where $i \in I_{\mathsf{flow}}^G$ need to be considered. (This also means that in order to check invariance of the safety condition under all flows, we only need to consider combinations of states of systems corresponding to the indices in I_{flow}^G).

With the notation used in the proof of Lemma 32 (3) we have the following upper bounds for the size of $\mathcal{K}_{\text{flow}}^{[G]}$ and of I_{flow}^{G} :

• the number n_{flow} of clauses in $\mathcal{K}_{\text{flow}}^{[G]}$ is $n_{\text{flow}} = n_{\text{entry}} + n_{\text{safe}} + n_{\text{Flow}} \leq n p_G^{nv_{\text{entry}}} + n p_G^{nv_{\text{safe}}} + c \cdot n p_G$,

where n_{entry} is the number of instances in $\Phi_{\text{entry}}^{[G]}$ (thus at most $np_G^{nv_{\text{entry}}}$); n_{safe} is the number of instances in $\Phi_{\text{safe}}^{[G]}$ (thus at most $np_G^{nv_{\text{safe}}}$, proof analogous to the proof of Lemma 32(3)), and n_{Flow} is the number of instances of $\forall i \text{Flow}_{q_i}(\overline{x}_0(i), \overline{x}_1(i))$. Since Flow is a conjunction of c formulae, each having only one universally quantified variable, the number of instances is at most $c \cdot np_G$.

• the number ni_{flow} of elements in I_{flow}^G is $ni_{\text{flow}} = ni_{\text{entry}} + ni_{\text{safe}} + ni_{\text{Flow}} \leq (np_{\text{entry}} + np_{\text{safe}} + np_{\text{Flow}}) \cdot np_G$ (the justification is the same as that used in the proof of Lemma 32(3)). \Box

Theorem 37 For every $q \in Q^I$, the satisfiability of the formulae F_q^{flow} is decidable (and in NP).

Proof: The hierarchical method for reasoning in stably local theory extensions allows us to reduce the task of checking the satisfiability of $\mathsf{F}_q^{\mathsf{flow}}$ to the problem of checking the satisfiability of a formula which is a conjunction of guarded index-positive extended clauses of the form $\mathcal{E} \vee \mathcal{C}$, where \mathcal{E} is a disjunction of equalities of sort index and \mathcal{C} a constraint over real numbers w.r.t. the disjoint combination of the theory of real numbers \mathbb{R} and the theory of uninterpreted function symbols in $P \cup X$.

Due to Assumption 1, all the clauses in $\mathsf{F}_q^{\mathsf{flow}}$ are ground or index-positive extended clauses of the form $\mathcal{E} \lor \mathcal{C}$, where \mathcal{C} is a conjunction of linear inequalities. We obtain a set of ground clauses in the combination of $LI(\mathbb{R})$ and $\mathsf{Eq}_{\mathsf{index}}$.

The locality result mentioned above shows that in order to check invariance of the safety condition under all flows, we only need to consider combinations of states of systems corresponding to the indices in I_{flow}^G . Therefore checking invariance under all flows is decidable.

Corollary 38 Under Assumptions 1 and 2(1), there exists a finite set $I_{\text{flow}} \subseteq I$ of indices, such that the following are equivalent:

- (1) F_q^{flow} is satisfiable for some $q \in Q^I$
- (2) $F_{q_0}^{\text{flow}}$ is satisfiable for some $q_0 \in Q^{I_{\text{flow}}}$.

Therefore checking invariance under all flows is decidable (and in NP).

Proof: (1) \Rightarrow (2) Assume that for some $q \in Q^I$, $\mathsf{F}_q^{\mathsf{flow}}$ is satisfiable. By Theorem 33, $F_q^{\mathsf{flow}}{}^{[G]}$ is satisfiable. Then there is a model \mathcal{A} for this formula. Let I_{entry}^G be the set of ground terms of sort index in $F_q^{\mathsf{flow}}{}^{[G]}$, and let I_{flow} be the set of the values of the terms in the model \mathcal{A} . The model \mathcal{A} can easily be transformed into a model of $\mathsf{F}_{q_0}^{\mathsf{flow}}$, where q_0 is the restriction of q to I_{flow} . (2) \Rightarrow (1) Conversely, assume that there exists a finite set $I_{\mathsf{flow}} \subseteq I$ of indices, corresponding

 $(2) \Rightarrow (1)$ Conversely, assume that there exists a finite set $I_{\text{flow}} \subseteq I$ of indices, corresponding to terms in I_{flow}^G (and thus to a neighborhood of the indices of cars that may violate the safety condition) and a tuple of modes $q_0 \in Q^{I_{\text{flow}}}$ such that $\mathsf{F}_{q_0}^{\text{flow}}$ is satisfiable. This model is a model of $\mathsf{F}_q^{\text{flow}[G]}$. By Theorem 33 it follows that $\mathsf{F}_q^{\text{flow}}$ is satisfiable. \Box

The results in Lemma 36, Theorem 37 and Corollary 38 immediately imply the following small model property.

Corollary 39 Let $S = (\text{Top}, \{S(i) \mid i \in I\})$ be an SFHA. Under Assumption 1 and 2(1), the following are equivalent:

- (1) There exist indices c_1, \ldots, c_n for which the safety condition Φ_{safe} is not preserved under flows starting in a state in which Φ_{entry} holds.
- (2) There exists a finite set $I_{\text{flow}} \subseteq I$ of indices, of size polynomial in the size of n (assuming that the lengths of the formulae describing the SFHA S are considered constants) describing a suitable neighborhood of c_1, \ldots, c_n which can effectively be described (they correspond to the terms in I_{flow}^G in Theorem 33) such that already in the systems $S_{\text{flow}} = (\text{Top}_{|I_{\text{flow}}}, \{S(i) \mid i \in I_{\text{flow}}\})$ the safety condition Φ_{safe} is not preserved under flows starting in a state in which Φ_{entry} holds.

Proof: (1) \Rightarrow (2) Assume that (1) holds. Then for some $q = (q_i)_{i \in I} \in Q^I$, $\mathcal{K}_{\mathsf{flow}} \wedge G$ is satisfiable (with the notation in the proof of Lemma 36). By Theorem 33, $\mathcal{K}_{\mathsf{flow}}{}^{[G]} \wedge G$ is satisfiable. Then there is a model \mathcal{A} for this formula. Let I_{flow}^G be as defined in Theorem 37, and let I_{flow} be the set of the values in \mathcal{A} of the terms in I_{flow}^G . The model \mathcal{A} can easily be transformed into a model of $\mathcal{K}_{\mathsf{flow}} \wedge G$, describing a system referring to the neighborhood I_{flow} of the indices c_1, \ldots, c_n at which Φ_{safe} does not hold, although Φ_{safe} and Φ_{entry} hold at the beginning of the flow. But then for $q = (q_i)_{i \in I} \in Q^{I_{\mathsf{flow}}}, \Phi_{\mathsf{safe}}$ is not invariant under flows starting in a state in which Φ_{entry} holds already for the system $S_{\mathsf{entry}} = (\mathsf{Top}_{|I_{\mathsf{flow}}}, \{S(i) \mid i \in I_{\mathsf{flow}}\}).$

 $(2) \Rightarrow (1)$ Conversely, assume that there exists a finite set $I_{\text{flow}} \subseteq I$ of indices, corresponding to terms in I_{flow}^G , a tuple $q = (q_i)_{i \in I} \in Q^{I_{\text{flow}}}$, and that in S_{flow} there are indices c_1, \ldots, c_n at which the safety property does not hold at the end of a flow starting in a state in which Φ_{safe} and Φ_{entry} hold. Then $\mathcal{K}_{\text{flow}} \wedge G$ (with instantiation over I_{flow} is satisfiable, i.e. it has a model. As I_{flow} corresponds to I_{flow}^G , we can obtain a model of $\mathcal{K}_{\text{flow}}^{[G]} \wedge G$. By Theorem 33 it follows that $\mathcal{K}_{\text{flow}}^{[G]} \wedge G$ is satisfiable, i.e. (1) holds. **Parametric Verification.** If we consider parametric systems, in which some of the constants used in the specification of the entry conditions, flows, and safety properties are parameters, we have again the following options: If we impose constraints on these parameters (in the form of constraints between real numbers) then the results in Theorem 37 and Corollary 38 can still be used to prove that the verification problems remain decidable. The complexity of the problems depends on the form of the constraints (for linear constraints, in particular when Assumptions 1-3 hold and parameters are not allowed as coefficients and do not appear as bounds in the flow conditions we still can show that the problem is in NP). For systems in which parameters are allowed as coefficients or appear in the flow conditions, the complexity is exponential.

We can use the method for hierarchical reasoning combined with quantifier elimination for the theory of real numbers for generating constraints on the parameters which guarantee that $F_{q_0}^{\text{flow}}$ is unsatisfiable for all $q_0 \in Q^{I_{\text{flow}}}$ (the complexity is exponential).

Example 40 We consider the following safety property:

$$\Phi_{\mathsf{safe}}^l: \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \rightarrow \mathsf{pos}(\mathsf{front}(i)) - \mathsf{pos}(i) \geq d_s).$$

Consider the tuple $(q_i)_{i \in I}$ consisting of the acceleration modes for all systems

$$\operatorname{Inv}_{q_i}(i) := i \neq \operatorname{nil} \land \operatorname{front}(i) \neq \operatorname{nil} \rightarrow \operatorname{pos}(\operatorname{front}(i), t_0) - \operatorname{pos}(i, t_0) \geq d.$$

 Φ_{safe}^l is invariant under flows in mode $(q_i)_{i \in I}$ if and only if the following formula is unsatisfiable:

$$\begin{array}{ll} 0 \leq t_0 < t_1 \leq \Delta t & \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \rightarrow \mathsf{pos}(\mathsf{front}(i), t_0) - \mathsf{pos}(i, t_0) > d_s) & \Phi_{\mathsf{safe}}^l(t_0) \\ & \land & \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \rightarrow \mathsf{pos}(\mathsf{front}(i), t_0) - \mathsf{pos}(i, t_0) \geq d) & \forall i \, \mathsf{Inv}_{q_i}(t_0) \\ & \land & \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \rightarrow \mathsf{pos}(\mathsf{front}(i), t_1) - \mathsf{pos}(i, t_1) \geq d) & \forall i \, \mathsf{Inv}_{q_i}(t_1) \\ & \land & \forall i (i \neq \mathsf{nil} \rightarrow \mathsf{pos}(i, t_1) - \mathsf{pos}(i, t_0) \leq v_{\mathsf{max}}(t_1 - t_0)) & \mathsf{Flow}(t_0, t_1) \\ & \land & i_0 \neq \mathsf{nil} \land \mathsf{front}(i_0) \neq \mathsf{nil} \land \mathsf{pos}(\mathsf{front}(i_0), t_1) - \mathsf{pos}(i, t_1) \leq d_s & G \end{array}$$

The universally quantified conjuncts in the formula are guarded index-positive clauses. After instantiation and purification, we obtain:

| D | | $\mathcal{K}^{[G]}_{flow0} \wedge G_0$ |
|--------------------------|---------------------------|--|
| $f = front(i_0)$ | | $0 \le t_0 < t_1 \le \Delta t$ |
| f' = front(f) | $\Phi^{l [G]}_{safe 0}$ | $i_0 \neq nil \land f \neq nil ightarrow p_{10} - p_{00} > d_s$ |
| $p_{00} = pos(i_0, t_0)$ | 6 | $f eq nil \wedge f' eq nil 	o p_{20} - p_{10} > d_s$ |
| $p_{01} = pos(i_0, t_1)$ | $Inv_a(t_0)^{[G]}_{0}$ | $i_0 eq nil \land f eq nil ightarrow p_{10} - p_{00} > d$ |
| $p_{10} = pos(f, t_0)$ | | $f \neq nil \land f' \neq nil \rightarrow p_{20} - p_{10} > d$ |
| $p_{11} = pos(f, t_1)$ | $Inv_a(t_1)^{[G]}{}_0$ | $i_0 \neq nil \land f \neq nil ightarrow p_{11} - p_{01} > d$ |
| $p_{20} = pos(f', t_0)$ | | $f \neq nil \land f' \neq nil \rightarrow p_{21} - p_{11} > d$ |
| $p_{21} = pos(f', t_1)$ | $Flow_{q=0}^{[G]}$ | $i_0 \neq nil \land f \neq nil \rightarrow p_{01} - p_{00} \le v_{max}(t_1 - t_0)$ |
| | | $f \neq nil \land f' \neq nil \rightarrow p_{11} - p_{10} \leq v_{max}(t_1 - t_0)$ |
| | G_0 | $i_0 eq nil \land f eq nil \land p_{11} - p_{01} \le d_s$ |
| | N_0 (ins | tances of the congruence axioms) |

It is easy to check unsatisfiability if $d > d_s$. This proves that if $d > d_s$ then Φ^l_{Safe} is invariant under flows.

The modularity/small model property result in Corollary 39 can be used as follows: From the safety property, we can determine the index set I_{flow} which we need to consider (which describes

the instances of the universally quantified formulae which we need to take into account). For the example described above, $I_{\text{flow}}^G = \{i_0, \text{front}(i_0), \text{front}(\text{front}(i_0))\}$. Since we know that $i_0 \neq 0$ and $\text{front}(i_0) \neq 0$, we have two situations to consider: one in which $\text{front}(\text{front}(i_0)) = \text{nil}$ and one in which $\text{front}(\text{front}(i_0)) \neq \text{nil}$ (equalities between $i_0, \text{front}(i_0)$ and $\text{front}(\text{front}(i_0))$ are ruled out by the conditions on pos).

By Corollary 39, in order to check whether all initial states are safe, it is sufficient to restrict to families of systems $(\text{Top}_{I_{\text{flow}}}, \{S(i) \mid i \in I_{\text{flow}}\})$ for the two situations:

- $I_{\text{flow}} = \{c_0, c_2\}$ where $\text{front}(c_0) = c_1$ and $\text{front}(c_1) = \text{nil}$, and
- $I_{\text{flow}} = \{c_0, c_2, c_3\}$, where front $(c_0) = c_1$ and front $(c_2) = c_3$, front $(c_3) = \text{nil}$.

We will need to consider combinations of modes (Appr/Rec) only for the systems in this family, thus we need to try only $2^2 + 2^3$ possible combinations of modes.

The global safety condition: $\forall i, j (i \neq \text{nil} \land j \neq \text{nil} \land \text{lane}(i) = \text{lane}(j) \land \text{pos}(i) > \text{pos}(j) \rightarrow \text{pos}(i) - \text{pos}(j) \ge d$) can be checked only together with properties which guarantee that the imprecise information of the sensors does not impact on safety. For proving such properties, we use timed topologies and timed topology updates.

5.2.3 Jumps

We now analyze the decidability and complexity of checking whether Φ_{safe} is preserved under all jumps starting from a state reachable by a flow from a state satisfying Φ_{entry} . According to Theorem 22(3), this can be expressed as the problem of checking whether for all $q=(q_i)_{i\in I}\in Q^I$ the following formula $F^{\mathsf{jump}^q}_{e}(i_0)$ is unsatisfiable for every $i_0 \in I$ and $e = (q_{i_0}, q'_{i_0}) \in E$, s.t. if $p(i_0)$ occurs in guard_e it is not nil:

$$\begin{split} F^{\mathsf{jump}_{e}^{q}}(i_{0}): \ \Phi_{\mathsf{entry}}(\overline{x}(t_{0})) \wedge \left(\left(t_{0} < t_{1} \wedge \forall i \mathsf{Flow}_{q_{i}}(\overline{x}(i,t_{0}),\overline{x}(i,t_{1})) \right) \vee t_{0} = t_{1} \right) \\ & \wedge \forall i_{1}, \dots, i_{n} \phi_{\mathsf{safe}}(\overline{x}(j_{1},t_{1}), \dots, \overline{x}(i_{n},t_{1})) \\ & \wedge \operatorname{guard}_{e}(\overline{x}(i_{0},t_{1})) \wedge \operatorname{jump}_{e}(\overline{x}(i_{0},t_{1}),\overline{x}'(i_{0})) \wedge \operatorname{Inv}_{q_{i_{0}}'}(\overline{x}'(i_{0})) \\ & \wedge \forall j (j \neq i_{0} \to \overline{x}'(j) = \overline{x}(j)) \wedge G, \end{split}$$

where $G = \neg \phi_{\mathsf{safe}}(\overline{x}(c_1, t_1), \dots, \overline{x}(c_n, t_1)).$

Lemma 41 (Jumps) Under Assumptions 1 and 2(1) the following hold:

- (1) For every $q \in Q^I$, $\mathsf{F}^{\mathsf{jump}}_a$ is unsatisfiable iff $F^{\mathsf{jump}}_a^{[G]}$ is unsatisfiable.
- (2) The size of the set of terms of sort index in st(G) and hence also the size of F^{jump[G]}_q is polynomial in the number of terms of sort index in Φ_{safe}. Therefore also the size of the set I^G_{jump} of ground terms of sort index in F^{jump[G]}_q is polynomial in the number of terms of sort index in Φ_{safe}.

The set of instances $F_q^{\text{jump}[G]}$ contain formulae Inv_{q_i} and Flow_{q_i} corresponding to terms $i \in I_{\text{jump}}^G$.

Proof: The proof is similar to the one of Lemma 32 and Lemma 36 using Theorem 31(3). The set of terms I_{jump}^G corresponding to i_0 is the set of all ground terms of sort index in $\mathcal{K}_{jump}^{[G]}$.

The estimation of the number n_{jump} of instances in $\mathcal{K}_{jump}^{[G]}$ and on the number of terms n_{ijump} in I_{jump}^{G} is similar to that made in the proofs of Lemma 32(3) and Lemma 36(2). With the notations used in the proofs of these Lemmata we have:

- $n_{\text{jump}} = n_{\text{entry}} + n_{\text{safe}} + n_{\text{Flow}} + np_G \le np_G^{nv_{\text{entry}}} + np_G^{nv_{\text{safe}}} + (c+1) \cdot np_G;$
- $ni_{jump} = ni_{entry} + ni_{safe} + ni_{Flow} + ni_{Jump} \le (np_{sfentry} + np_{safe} + np_{Flow}) \cdot np_G + ni_{Jump}$, where ni_{Jump} is the number of terms of sort index occurring in

$$\mathsf{guard}_e(\overline{x}(i_0,t_1)) \wedge \mathsf{jump}_e(\overline{x}(i_0,t_1),\overline{x}'(i_0)) \wedge \mathsf{Inv}_{q'_{i_0}}(\overline{x}'(i_0)).$$

Theorem 42 (Jumps) For every $q \in Q^I$, the satisfiability of F_a^{jump} is decidable (and in NP).

Proof: Follows from Lemma 41 and the fact that for every $q_0 \in Q^{I_{jump}}$, the satisfiability of $F_q^{jump}[G]$ is decidable (and it is in NP).

The following two results can be proved as in the case of flows.

Corollary 43 Let $S = (\text{Top}, \{S(i) \mid i \in I\})$ be an SFHA. Under Assumptions 1 and 2(1), there exists a finite set $I_{\text{jump}} \subseteq I$ of indices, such that the following are equivalent:

- (1) F_a^{jump} is satisfiable for some $q \in Q^I$
- (2) $F_{q_0}^{\text{jump}}$ is satisfiable for some $q_0 \in Q^{I_{\text{jump}}}$.

Therefore checking invariance under all GMR jumps is decidable (and in NP).

Corollary 44 Under Assumptions 1 and 2(1), the following are equivalent:

- (1) There exist indices c_1, \ldots, c_n for which the safety condition Φ_{safe} does not hold after a jump following a flow starting in a state satisfying Φ_{entry} .
- (2) There exists a finite set $I_{jump} \subseteq I$ of indices, of size polynomial in the size of n (assuming that the length of the formulae describing the SFHA S are considered constants) such that already in the system $S_{jump} = (\text{Top}_{|I_{jump}}, \{S(i) \mid i \in I_{jump}\})$ the safety condition Φ_{safe} does not hold after a jump following a flow starting in a state satisfying Φ_{entry} .

The set of indices I_{jump} and the system $S_{jump} = (Top_{|I_{jump}}, \{S(i) \mid i \in I_{jump}\})$ describe a suitable neighborhood of the systems c_1, \ldots, c_n at which the safety property is not preserved under jumps, which can effectively be described (they correspond to the terms in I_{jump}^G in Lemma 41).

Parametric Verification. If we impose constraints on these parameters (in the form of constraints between real numbers) then the results in Theorem 42 and Corollary 43 can be used to prove that the verification problems remain decidable. For linear constraints, in particular when Assumptions 1-3 hold and parameters are not allowed as coefficients and do not appear as bounds in the flow conditions, the problem is in NP. For systems in which parameters are allowed as coefficients or appear in the flow conditions, the complexity is exponential.

We can use the method for hierarchical reasoning combined with quantifier elimination for the theory of real numbers for generating constraints on the parameters which guarantee Φ_{safe} is preserved under GMR jumps (the complexity is exponential). **Example 45** We consider the following safety property Φ_{safe} :

 $\Phi_{\mathsf{safe}} \colon \forall i, j (i \neq \mathsf{nil} \land j \neq \mathsf{nil} \land \mathsf{lane}(i) = \mathsf{lane}(j) \land \mathsf{pos}(i) = \mathsf{pos}(j) \to i = j).$

Because jumps are instantaneous and **pos** is a continuous variable, Φ_{safe} is obviously invariant under jumps where the lane is not changed, i.e. where no variables are updated. To verify a jump where an update of the lane occurs, we look at a transition from the first to the second lane. We assume that car i_0 is in mode Appr; the modes of other cars will not affect the verification.

Verifying the safety condition in general for such a jump will require the afore-mentioned interplay with other components of a global safety condition, because front(i) may not actually be the car in front of i if another car cut in in front of i after the last topology update. To keep the presentation simple, we instead assume for this example that the lane change follows directly on an update, so that the sensors show correct information (i.e. the state of Top is an initial state). This is a special case of global mode reachability that is much easier to follow by hand than the general case. In particular, we use that there is no car between sidefront and sideback. Invariance under lane-changing jumps can then be reduced to checking whether the following set is unsatisfiable:

$$\begin{split} & \Phi_{\mathsf{safe}} \\ & \mathsf{guard:} \ k_0 \neq \mathsf{nil} \land \mathsf{front}(k_0) \neq \mathsf{nil} \land \mathsf{lane}(k_0) = 1 \land \mathsf{pos}(\mathsf{front}(k_0)) - \mathsf{pos}(k_0) \leq D' \\ & \mathsf{back}(k_0) \neq \mathsf{nil} \to \mathsf{pos}(k_0) - \mathsf{pos}(\mathsf{back}(k_0)) \geq d' \\ & \mathsf{sideback}(k_0) \neq \mathsf{nil} \to \mathsf{pos}(k_0) - \mathsf{pos}(\mathsf{sideback}(k_0)) \geq d' \\ & \mathsf{sidefront}(k_0) \neq \mathsf{nil} \to \mathsf{pos}(\mathsf{sidefront}(k_0)) - \mathsf{pos}(k_0) \geq d' \\ & \mathsf{Inv}_{\mathsf{before}} \colon \forall i((\mathsf{lane}(i) = 1 \lor \mathsf{lane}(i) = 2) \land \\ & i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \to \mathsf{pos}(\mathsf{front}(i)) - \mathsf{pos}(i) \geq d \\ & \mathsf{Inv}_{\mathsf{after}} \colon \forall i((\mathsf{lane}'(i) = 1 \lor \mathsf{lane}'(i) = 2) \land \\ & \mathsf{front}(i) \neq \mathsf{nil} \to \mathsf{pos}'(\mathsf{front}(i)) - \mathsf{pos}'(i) \geq d \\ & \mathsf{jump:} \ \mathsf{lane}'(k_0) = 2 \land \forall i(i \neq k_0 \to \mathsf{lane}'(i) = \mathsf{lane}(i)) \\ & \forall i(\mathsf{pos}'(i) = \mathsf{pos}(i)) \\ & \mathsf{Init}_{\mathsf{Top}} \colon \forall i, j(\mathsf{sideback}(i), \mathsf{sidefront}(i), j \neq \mathsf{nil} \land \mathsf{lane}(j) = 2 \to \mathsf{pos}(j) \leq \mathsf{pos}(\mathsf{sideback}(i)) \lor \mathsf{pos}(j) \geq \mathsf{pos}(\mathsf{sidefront}(i)) \\ & \mathsf{sidefront}(i), j \neq \mathsf{nil} \land \mathsf{sidefront}(i) = \mathsf{nil} \land \mathsf{lane}(j) = 2 \to \mathsf{pos}(j) \leq \mathsf{pos}(\mathsf{sideback}(i)) \\ & \mathsf{sidefront}(i), j \neq \mathsf{nil} \land \mathsf{sideback}(i) = \mathsf{nil} \land \mathsf{lane}(j) = 2 \to \mathsf{pos}(j) \geq \mathsf{pos}(\mathsf{sideback}(i)) \\ & \mathsf{pos}(\mathsf{sideback}(i), \mathsf{pos}(\mathsf{nil} \land \mathsf{nic}(i) = \mathsf{nil} \land \mathsf{lane}(j) = 2 \to \mathsf{pos}(j) \leq \mathsf{pos}(\mathsf{sideback}(i)) \\ & \mathsf{pos}(\mathsf{sideback}(i), \mathsf{pos}(\mathsf{nil} \land \mathsf{nic}(i) = \mathsf{nil} \land \mathsf{lane}(j) = 2 \to \mathsf{pos}(j) \leq \mathsf{pos}(\mathsf{sideback}(i)) \\ & \mathsf{pos}(\mathsf{sideback}(i), \mathsf{pos}(\mathsf{nil} \land \mathsf{nic}(i) = \mathsf{nil} \land \mathsf{lane}(j) = 2 \to \mathsf{pos}(j) \geq \mathsf{pos}(\mathsf{sideback}(i)) \\ & \mathsf{pos}(\mathsf{sidefront}(i), \mathsf{pos}(\mathsf{nil} \land \mathsf{nic}(i) = \mathsf{nil} \land \mathsf{lane}(j) = 2 \to \mathsf{pos}(j) \geq \mathsf{pos}(\mathsf{sidefront}(i))) \\ & \neg \Phi_{\mathsf{rafe}}' \mathsf{i}_0 \neq \mathsf{nil} \land \mathsf{j}_0 \neq \mathsf{nil} \land \mathsf{j}_0 \neq \mathsf{lane}'(\mathsf{i}_0) = \mathsf{lane}'(\mathsf{j}_0 \land \mathsf{pos}'(\mathsf{i}_0) = \mathsf{pos}'(\mathsf{j}_0) \end{aligned}$$

 $\neg \Psi_{safe}$. $i_0 \neq \min \land j_0 \neq \min \land i_0 \neq j_0 \land \text{rane}(i_0) = \text{rane}(j_0) \land \text{pos}(i_0) = \text{pos}(j_0)$

These axioms define a chain of local theory extensions:

 $\mathbb{R} \cup \mathsf{Eq}_{\mathsf{index}} \subseteq \mathbb{R} \cup \mathsf{Inv}_{\mathsf{before}} \cup \mathsf{Init}_{\mathsf{top}} \subseteq \mathbb{R} \cup \mathsf{Inv}_{\mathsf{before}} \cup \mathsf{Init}_{\mathsf{top}} \cup \mathsf{jump} \cup \mathsf{Inv}_{\mathsf{after}}$

After instantiation and purification the problem is reduced to a satisfiability test in the combination of linear arithmetic with pure equality (for the index sort). Below, we explain intuitively why the set of clauses above is unsatisfiable.

Due to the implication in the jump condition, the verification will be a case distinction on whether or not i_0 or j_0 equals k_0 . Since the case $k_0 \notin \{i_0, j_0\}$ is trivial, we concentrate the manual analysis on $k_0 = i_0 \neq j_0$. From the jump condition, we obtain:

 $lane'(j_0) = lane(j_0) \quad pos'(k_0) = pos(k_0) \quad pos'(i_0) = pos(i_0) \quad pos'(j_0) = pos(j_0)$

From the information from **Top**, we obtain:

 $\begin{aligned} \mathsf{sideback}(k_0), \mathsf{sidefront}(k_0), j_0 \neq \mathsf{nil} \land \mathsf{lane}(j_0) &= 2 \to \mathsf{pos}(j_0) \leq \mathsf{pos}(\mathsf{sideback}(k_0)) \lor \mathsf{pos}(j_0) \geq \mathsf{pos}(\mathsf{sidefront}(k_0)) \\ \mathsf{sideback}(k_0), j_0 \neq \mathsf{nil} \land \mathsf{sidefront}(k_0) &= \mathsf{nil} \land \mathsf{lane}(j_0) = 2 \to \mathsf{pos}(j_0) \leq \mathsf{pos}(\mathsf{sideback}(k_0)) \\ \mathsf{sidefront}(k_0), j_0 \neq \mathsf{nil} \land \mathsf{sideback}(k_0) &= \mathsf{nil} \land \mathsf{lane}(j_0) = 2 \to \mathsf{pos}(j_0) \geq \mathsf{pos}(\mathsf{sidefront}(k_0)) \end{aligned}$

We know that $j_0 \neq \text{nil}$ and $\text{lane}(j_0) = 2$ (because $\text{lane}(j_0) = \text{lane}'(j_0) = \text{lane}'(i_0) = \text{lane}'(k_0)$). If either of sideback (k_0) or sidefront (k_0) is defined, then the guard condition states that they are at least d' away from k_0 , and the instances that we just derived state that then the same must hold for j_0 . In particular, $j_0 \neq k_0$ if d' > 0. This means that the derived set of ground instances is unsatisfiable if d' > 0.

5.2.4 Topology updates

We now analyze the decidability and complexity of checking whether Φ_{safe} is preserved under all GMR topology updates. By Theorem 22, this can be reduced to checking whether for all $q = (q_i)_{i \in I} \in Q^I$ the following formula F_q^{top} is unsatisfiable:

$$\begin{split} F_q^{\mathsf{top}}: \ \Phi_{\mathsf{entry}}(\overline{x}(t_0)) \wedge \left(\left(t_0 < t_1 \wedge \forall i \mathsf{Flow}_{q_i}(\overline{x}(i,t_0),\overline{x}(i,t_1)) \right) \vee t_0 = t_1 \right) \\ \wedge \forall i_1, \dots, i_n \phi_{\mathsf{safe}}(\overline{x}(j_1,t_1), \dots, \overline{x}(i_n,t_1)) \wedge \bigwedge_{p \in P_1} \mathsf{Update}(p,p') \wedge G, \end{split}$$

where $G = \neg \phi'_{\mathsf{safe}}(\overline{x}(c_1), \dots, \overline{x}(c_n))$ and ϕ'_{safe} is obtained from ϕ_{safe} by replacing every $p \in P_1$ with p'.

Lemma 46 (Topology updates) Under Assumptions 1, 2(1) and 3 the following hold:

- (1) For every $q = (q_i)_{i \in I} \in Q^I$, $\mathsf{F}_q^{\mathsf{top}}$ is unsatisfiable iff $F_q^{\mathsf{top}}[T_G]$ is unsatisfiable for a suitable set of ground terms T_G .
- (2) The size of the set of terms of sort index in st(G) and hence also the size of $F_q^{top}[T_G]$ is polynomial in the number of terms of sort index in Φ_{safe} . Therefore also the size of the set I_{top}^G of ground terms of sort index in $F_q^{top}[T_G]$ is polynomial in the number of terms of sort index in Φ_{safe} .

The set of instances $F_q^{top[T_G]}$ contains only formulae corresponding states q_i where *i* are indices corresponding to terms in I_{top}^G .

Proof: The proof is similar to the one of Lemma 36, using Theorem 31(4) and is only sketched here. Let $\mathcal{K}_{top} = \mathcal{K}_1 \cup \bigwedge_{p \in P_1} \mathsf{Update}(p, p')$, where \mathcal{K}_1 is the following formula:

$$\mathcal{K}_1 = \Phi_{\mathsf{entry}}(\overline{x}(t_0)) \land \left(\left(t_0 < t_1 \land \forall i \mathsf{Flow}_{q_i}(\overline{x}(i, t_0), \overline{x}(i, t_1)) \right) \lor t_0 = t_1 \right) \land \Phi_{\mathsf{safe}}(\overline{x}(t_1))$$

By Theorem 31(4), the extension of the theory $\mathbb{R} \cup \mathcal{K}_1$ with the additional function symbols $\{p' \mid p \in P_1\}$ axiomatzed by $\bigwedge_{p \in P_1} \mathsf{Update}(p, p')$ is local. Thus, $\mathbb{R} \cup \mathcal{K}_1 \cup \bigwedge_{p \in P_1} \mathsf{Update}(p, p') \cup G$ is satisfiable iff $\mathbb{R} \cup \mathcal{K}_1 \cup \bigwedge_{p \in P_1} \mathsf{Update}(p, p')[G] \cup G$ is satisfiable.

We can distinguish two cases:

Case 1: $\bigwedge_{p \in P_1} \text{Update}(p, p')[G] \cup G$ is a ground formula G'. Then we can proceed as in the proof of Lemma 36, with the difference that G is replaced by G'. $\mathbb{R} \cup \mathcal{K}_1 \cup G'$ is satisfiable iff

proof of Lemma 36, with the difference that G is replaced by G'. $\mathbb{R} \cup \mathcal{K}_1 \cup G'$ is satisfiable iff $\mathbb{R} \cup \mathcal{K}_1^{[G']}$ is satisfiable. The set T_G consists of all the ground terms of sort index in $\mathsf{st}(G')$, and depends not only of G but also on the form of the update rules.

Case 2: $\bigwedge_{p \in P_1} \text{Update}(p, p')[G] \cup G$ contains free variables. Then the proof proceeds as the

proof of Lemma 32. The conditions in Assumption 2(1) and 3 ensure also in this case that after

at most two instantiation steps we can reduce the satisfiability test to testing the satisfiability of ground clauses. Under Assumption 3(1), the set T_G contains the ground terms of sort index in $\bigwedge_{p \in P_1} \mathsf{Update}(p, p')[G] \cup G$. Under Assumption 3(2) it contains additional Skolem constants which

need to be introduced because of the existential quantifiers in some of the updates.

 I_{top}^G consists of the set of all ground terms of sort index in $\mathcal{K}_{top}^{[T_G^{top}]}$ together with all terms obtained by replacing the variables with Skolem constants $c_p, p \in P$ which occur from Skolemization in the instances of Update(p, p').

The estimation of the number n_{update} of instances in $F_q^{top}[T_G^{top}]$ and on the number of terms ni_{update} in I_{top}^G is similar to that made in the proofs of Lemma 32(3), Lemma 36(2) and Lemma 41(2). With the notations used in the proofs of these Lemmata we have:

- $n_{update} = n_{entry} + n_{safe} + n_{Flow} + n_{Update} \le n p_G^{nv_{entry}} + n p_G^{nv_{safe}} + n p_G^{nv_{update}} + c \cdot n p_G;$
- $ni_{update} = ni_{entry} + ni_{safe} + ni_{Flow} + ni_{update} \leq (np_{entry} + np_{safe} + np_{Flow} + 2np_{update}) \cdot np_G$, where ni_{update} is the number of terms of sort index occurring in $\bigwedge_{p \in P_1} Update(p, p')$. \Box

Theorem 47 For every $q \in Q^I$, the satisfiability of the formulae F_q^{top} is decidable (and in NP).

Corollary 48 Under Assumptions 1, 2(1) and 3 there exists a finite set $I_{top} \subseteq I$ of indices, such that the following are equivalent:

- (1) F_q^{top} is satisfiable for some $q \in Q^I$
- (2) $F_{q_0}^{\text{top}}$ is satisfiable for some $q_0 \in Q^{I_{\text{flow}}}$.

Therefore checking invariance under all topology updates is decidable (and in NP).

Corollary 49 Let $S = (\text{Top}, \{S(i) \mid i \in I\})$ be an SFHA. Under Assumption 1, 2(1) and 3, the following are equivalent:

- (1) There exist indices c_1, \ldots, c_n for which the safety condition Φ_{safe} is not preserved under updates reachable from a state in which Φ_{entry} holds.
- (2) There exists a finite set $I_{update} \subseteq I$ of indices, of size polynomial in the size of n (assuming that the lengths of the formulae describing the SFHA S are considered constants) describing a suitable neighborhood of c_1, \ldots, c_n which can effectively be described (they correspond to the terms in I_{update}^G in Theorem 33) such that already in the systems $S_{update} = (\text{Top}_{|I_{update}}, \{S(i) \mid i \in I_{update}\})$ the safety condition Φ_{safe} is not preserved under updates in states reachable from a state in which Φ_{entry} holds.

The proofs are in all cases analogous to the proofs for the case of flows and jumps (Corollaries 44 and 44).

Parametric Verification. Also in this case, if we impose constraints on these parameters (in the form of constraints between real numbers) then the results in Lemma 46 and Corollary 48 can be used to prove that the verification problems remain decidable. The complexity of the problems is similar to that for jumps. We can also use hierarchical reasoning combined with quantifier elimination for the theory of real numbers for generating constraints on the parameters which guarantee Φ_{safe} is preserved under GMR updates, as in [Sofronie-Stokkermans, 2013] (the complexity is exponential).

Example 50 Consider the topology updates in Example 4. Invariance of Φ_{safe}^g under these updates can be proved (cf. Section 7). Φ_{safe}^l is not invariant. We now consider a variant $\overline{\Phi}_{\mathsf{safe}}^l$ of Φ_{safe}^l where:

$$\overline{\Phi}_{\mathsf{safe}}^{\mathsf{front}} \colon \forall i \big(i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \land \mathsf{lane}(i) = \mathsf{lane}(\mathsf{front}(i)) \to \mathsf{pos}(\mathsf{front}(i)) - \mathsf{pos}(i) > d_s \big)$$

In order to prove that $\overline{\Phi}_{safe}^{front}$ is preserved by topology updates, we prove that the formula

$$\overline{\Phi}_{\mathsf{safe}}^{\mathsf{front}} \wedge \mathsf{Update}(\mathsf{front},\mathsf{front}') \wedge G$$

is unsatisfiable, where $G=\neg\overline{\Phi}_{\mathsf{safe}}^{\mathsf{front}'}$ is the ground clause

$$i_0 \neq \mathsf{nil} \land \mathsf{front}'(i_0) \neq \mathsf{nil} \land \mathsf{lane}(i_0) = \mathsf{lane}(\mathsf{front}'(i_0)) \land \mathsf{pos}(\mathsf{front}'(i_0)) - \mathsf{pos}(i_0) \leq d_s.$$

The extension: $\mathbb{R} \cup \overline{\Phi}_{\mathsf{safe}}^{\mathsf{front}} \subseteq \mathbb{R} \cup \overline{\Phi}_{\mathsf{safe}}^{\mathsf{front}} \cup \mathsf{Update}(\mathsf{front}, \mathsf{front}')$ is local. We determine the conjuncts of $\mathsf{Update}(\mathsf{front}, \mathsf{front}')[G]$, where $\mathsf{st}(K, G) = \{\mathsf{front}'(i_0)\}$. After instantiation and purification (replacing $\mathsf{front}'(i_0)$ with f') we obtain:

$$\begin{array}{l} i_0 \neq \mathsf{nil} \land \neg \exists j(\mathsf{ASL}(j, i_0)) \to f' = \mathsf{nil} \\ i_0 \neq \mathsf{nil} \land \quad \exists j(\mathsf{ASL}(j, i_0)) \to \mathsf{Closest}_f(f', i_0) \end{array}$$

with the notations in Example 4. Transforming these formulae into prenex form and skolemizing the existential quantifier, we obtain (with Skolem constant c_0):

$$C_1: i_0 \neq \mathsf{nil} \land \neg \mathsf{ASL}(c_0, i_0) \to f' = \mathsf{nil}$$

$$C_2: i_0 \neq \mathsf{nil} \land \mathsf{ASL}(j, i_0) \to \mathsf{Closest}(f', i_0).$$

The formula C_1 is ground. To check the satisfiability of $\Phi_{\mathsf{safe}} \cup C_2 \cup G_1$ where $G_1 = C_1 \wedge G_0$ (where G_0 is $i_0 \neq \mathsf{nil} \wedge f' \neq \mathsf{nil} \wedge \mathsf{lane}(i_0) = \mathsf{lane}(f') \wedge \mathsf{pos}(f') - \mathsf{pos}(i_0) \leq d_s$), it is sufficient to check the satisfiability of $\Phi_{\mathsf{safe}}[G_1] \cup C_2[G_1] \cup G_1$.

5.3 Checking exhaustive entry conditions

In Theorem 23 we showed that for decoupled SFLHA S we can reduce checking conditions (i) and (ii) in Definition 12 (exhaustive entry conditions) to checking the satisfiability of the following formulae:

- (i) $\Phi_{entry}(\overline{x}) \wedge \left(\neg (\bigvee_{q \in Q} \operatorname{Init}_q(\overline{x}(i_0))) \vee \neg \operatorname{Init}_{top}(\overline{x})\right)$ is unsatisfiable.
- (ii) for all $(q_i)_{i \in I} \in Q^I$:
 - (a) Topology updates:

 $(\forall i \operatorname{Inv}_{q_i}(\overline{x}_i)) \land \operatorname{Update}(p, p') \land \neg \Phi'_{\operatorname{entry}}(\overline{x})$ is unsatisfiable,

where Φ'_{entry} arises from Φ_{entry} by replacing p with p', and

(b) Jumps: For all $e \in E, i_0 \in I$:

$$\begin{array}{l} (\forall i \operatorname{Inv}_{q_i}(\overline{x}_i)) \land \operatorname{guard}_e(\overline{x}_{i_0}) \land \operatorname{jump}_e(\overline{x}_{i_0}, \overline{x}'_{i_0}) \land \\ \forall j (j \neq i_0 \to \overline{x}'(j) = \overline{x}(j)) \land \neg \Phi_{\operatorname{entry}}(\overline{x}') \quad \text{ is unsatisfiable.} \end{array}$$

We now identify conditions under which these tasks are decidable and analyze their complexity.

Theorem 51 Under Assumption 1, and if both Φ_{entry} and $Init_{top}$ satisfy the conditions on Φ_{entry} in Assumption 2(1), then the following hold:

- (i) The following are equivalent:
 - (1) $\Phi_{entry}(\overline{x}) \land \left(\neg(\bigvee_{q \in Q} \operatorname{Init}_{q}(\overline{x}(i_{0}))) \lor \neg\operatorname{Init}_{top}(\overline{x})\right)$ is unsatisfiable.
 - (2) $\Phi_{\text{entry}}(\overline{x}) \wedge G_1$ is unsatisfiable, where $G_1 = \bigwedge_{q \in Q} \neg \text{lnit}_q(\overline{x}(i_0))$ and $\Phi_{\text{entry}}(\overline{x}) \wedge G_2$ is unsatisfiable, where $G_2 = \neg \text{lnit}_{\text{top}}(\overline{x})$.
 - (3) $\Phi_{\text{entry}}(\overline{x})^{[G_1]} \wedge G_1$ is unsatisfiable, where $G_1 = \bigwedge_{q \in Q} \neg \text{Init}_q(\overline{x}(i_0))$ and

 $\Phi_{\text{entry}}(\overline{x})^{[G_2]} \wedge G_2$ is unsatisfiable, where $G_2 = \neg \text{Init}_{\text{top}}(\overline{x})$.

The size of the set of terms of sort index in $st(G_1)$, $st(G_2)$ and hence also the size of the sets of instances in (3) is polynomial in the number of terms of sort index in G_1, G_2 .

- (ii) (a) For every $q = (q_i)_{i \in I} \in Q^I$ the following are equivalent:
 - (a1) $(\forall i \operatorname{Inv}_{q_i}(\overline{x}(i))) \land \operatorname{Update}(p, p') \land G_3 \text{ is unsatisfiable, where } G_3 = \neg \Phi'_{\operatorname{entry}}(\overline{x}).$
 - (a2) $(\forall i \operatorname{Inv}_{q_i}(\overline{x}(i))) \land \operatorname{Update}(p, p')[G_3] \land G_3 \text{ is unsatisfiable.}$
 - (a3) $[(\forall i \operatorname{Inv}_{q_i}(\overline{x}(i))) \land \operatorname{Update}(p, p')[G_3]]^{[T_{G_3}]} \land G_3$ is unsatisfiable, where T_{G_3} is the set of all ground terms of sort index in the formula in (2).
 - (b) For every $q = (q_i)_{i \in I} \in Q^I$ the following are equivalent:
 - (b1) $(\forall i \operatorname{Inv}_{q_i}(\overline{x}_i)) \land \operatorname{guard}_e(\overline{x}_{i_0}) \land \operatorname{jump}_e(\overline{x}_{i_0}, \overline{x}'_{i_0}) \land \forall j (j \neq i_0 \rightarrow \overline{x}'(j) = \overline{x}(j)) \land G_4, is unsatisfiable, where <math>G_4 = \neg \Phi_{\operatorname{entry}}(\overline{x}').$
 - $\begin{array}{l} (b2) \ ((\forall i \ \mathsf{Inv}_{q_i}(\overline{x}_i)) \land \mathsf{guard}_e(\overline{x}_{i_0}) \land \mathsf{jump}_e(\overline{x}_{i_0}, \overline{x}'_{i_0}) \land \forall j (j \neq i_0 \rightarrow \overline{x}'(j) = \overline{x}(j)))^{[G_4]} \land G_4 \ is unsatisfiable. \end{array}$

Theorem 52 (Decidability and complexity) The problem of checking the satisfiability of the formula in (i)(3) is decidable (and in NP). For every $q = (q_i)_{i \in I} \in Q^I$, the problem of checking the satisfiability of the formulae in (ii)(a3) and (ii)(b2) is decidable (and in NP).

Corollary 53 Under Assumption 1, and if Φ_{entry} and $Init_{top}$ satisfy the conditions in Assumption 2(1), there exists a finite set $I_0 \subseteq I$ of indices, such that the following are equivalent:

- (1) The formula in (i)(a) is satisfiable for some $q \in Q^I$
- (2) The formula in (i)(a) is satisfiable for some $q \in Q^{I_0}$.

Therefore checking invariance under all GMR jumps is decidable (and in NP).

Parametric Verification. These results can be used also for parametric systems, either for *checking* whether a safety property has exhaustive entry conditions (assuming that certain constraints on the parameters are known) or for generating constraints on parameters used in the specification of the system, and of Φ_{entry} under which Definition 12 holds.

Example 54 Consider the running example. Assume that the initial conditions for the topology automaton are expressed by the formulae $Init_{Top}$, stating that all sensor pointers have the correct value, as if they had just been updated. For front this can be expressed by the following set of formulae:

$$\begin{split} \forall i(i \neq \mathsf{nil} \land \mathsf{front}(i) = \mathsf{nil} & \to \forall k(k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \geq \mathsf{pos}(i) \to \mathsf{lane}(k) \neq \mathsf{lane}(i))) \\ \forall i(i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} & \to \mathsf{pos}_{\mathsf{front}}(i) > \mathsf{pos}(i) \land \mathsf{lane}_{\mathsf{front}}(i) = \mathsf{lane}(i) \land \\ & \forall k(k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \geq \mathsf{pos}(i) \land \mathsf{lane}(k) = \mathsf{lane}(i) \\ & \to \mathsf{pos}(k) \geq \mathsf{pos}_{\mathsf{front}}(i)) \land \\ & \mathsf{pos}(\mathsf{front}(i)) = \mathsf{pos}_{\mathsf{front}}(i) \land \mathsf{lane}(\mathsf{front}(i)) = \mathsf{lane}_{\mathsf{front}}(i)). \end{split}$$

In Example 3, the initial conditions of the two modes Appr and Rec are:

 $\mathsf{Init}_{\mathsf{Appr}} = \mathsf{Init}_{\mathsf{Rec}} = \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \to \mathsf{pos}_{\mathsf{front}}(i) - \mathsf{pos}(i) \ge d').$

Consider a safety property $\Phi_{entry} \rightarrow \Box \Phi_{safe}$, with entry states being states in which the information provided by the sensors is correct and every car is sufficiently far away from the following car on the same lane, described by the following formula Φ_{entry} (again stated only for front):

$$\begin{array}{ll} \forall i(i \neq \mathsf{nil} \land \mathsf{front}(i) = \mathsf{nil} & \rightarrow \forall k(k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \geq \mathsf{pos}(i) \rightarrow \mathsf{lane}(k) \neq \mathsf{lane}(i))) \\ \forall i(i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} & \rightarrow \mathsf{pos}_{\mathsf{front}}(i) > \mathsf{pos}(i) + d' \land \mathsf{lane}_{\mathsf{front}}(i) = \mathsf{lane}(i) \land \\ & \forall k(k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \geq \mathsf{pos}(i) \land \mathsf{lane}(k) = \mathsf{lane}(i) \\ & \rightarrow \mathsf{pos}(k) \geq \mathsf{pos}_{\mathsf{front}}(i)) \land \\ & \mathsf{pos}(\mathsf{front}(i)) = \mathsf{pos}_{\mathsf{front}}(i) \land \mathsf{lane}(\mathsf{front}(i)) = \mathsf{lane}_{\mathsf{front}}(i)). \end{array}$$

It can be easily checked that $\Phi_{entry} \wedge \neg \mathsf{Init}_{\mathsf{top}}$ is unsatisfiable and that $\Phi_{\mathsf{entry}} \wedge G_1$, where

$$G_1 = \neg \mathsf{Init}_a \land \neg \mathsf{Init}_r = (c \neq \mathsf{nil} \land \mathsf{front}(c) \neq \mathsf{nil} \land \mathsf{pos}_{\mathsf{front}}(c) - \mathsf{pos}(c) < d')$$

is unsatisfiable.

In general, we can only guarantee that $\forall i \operatorname{Inv}_{q_i}(\overline{x}(i)) \wedge \operatorname{Update}(p, p') \wedge \neg \Phi'_{entry}$ is unsatisfiable if the invariants and the update rules are designed such that after an update each car is sufficiently far away from the following car on the same lane.

Similarly, we can only guarantee that $\forall i \mathsf{Inv}_{q_i}(\overline{x}(i)) \land \mathsf{guard}_e(\overline{x}) \land \mathsf{jump}_e(\overline{x}, \overline{x}') \land \neg \Phi_{\mathsf{entry}}(\overline{x}')$ is unsatisfiable if the jump rules are designed such that after a jump that resets some of the variables (e.g. after a lane change) each car is sufficiently far away from the following car on the same lane.

6 Consequences of Locality

In what follows we present two applications of the previous results: a small model property and a complexity result which refines the NP-complexity results established in Section 5.

6.1 A small model property

From Corollaries 34, 39, 44 and 49 we obtain the following small model property for the verification of safety properties with exhaustive entry conditions.

Theorem 55 (Small model property) Under Assumptions 1, 2(1) and 3, a decoupled SFLHA S satisfies a safety property with exhaustive entry conditions iff the property holds in all systems of the form $S_0 = (\text{Top}, \{S(i) \mid i \in I_0\})$, where I_0 is a set of indices corresponding to ground terms in $G = \neg \Phi_{\text{safe}}$ occurring in the instances of the formulae $F^{\text{entry}[G]}, F_q^{\text{flow}[G]}, F_q^{\text{flow}[G]}, or F_q^{\text{top}[G]}$. The size $|I_0|$ of I_0 is molemomial in the number of terms of corresponding to $\mathcal{F}_q^{\text{top}[G]}$.

The size $|I_0|$ of I_0 is polynomial in the number of terms of sort index occurring in Φ_{safe} , and can be precisely determined from the form of the formulae Φ_{safe} , F_q^{intry} , F_q^{jump} , or F_q^{jop} .

Proof: Direct consequence of Corollaries 34, 39, 44 and 49. From the proofs of Lemma 32, 36, 41 and 46, we know that for checking the safety of entry conditions and invariance under flows and GMR jumps and topology update we only need to analyze systems with set of indices of cardinality at most $(np_{entry} + np_{safe} + np_{Flow} + 2np_{Update}) \cdot np_G + np_{jump}$, where $np_{entry}, np_{sfsafe}, np_{Flow}, np_{Update}$ is the number of all terms of sort index occurring in the corresponding formulae ($\Phi_{entry}, \Phi_{safe}, Flow, Update(p, p')$) and np_G is the set of ground terms of sort index occurring in G.

6.2 Decidability, Complexity

From Theorems 33, 37, 42 and 47 and from Theorem 52 and Corollaries 34, 39, 44 and 49 we obtain the following decidability and complexity results:

Theorem 56 Under Assumptions 1, 2(1) and 3, the problem of checking invariance of a safety condition in an SFLHA S is decidable (and in NP).

 \Box .

 \Box .

Proof: Direct consequence of Theorems 33, 37, 42 and 47.

Theorem 57 Under Assumptions 1, 2(1), and 3, and if $Init_{top}$ consists of guarded index-positive extended clauses where the scalar constraint is a conjunction of linear inequalities, the problem of checking whether a safety property $\Phi_{entry} \rightarrow \Box \Phi_{safe}$ has extended entry condition in an SFLHA S is decidable (and in NP).

Proof: Direct consequence of Theorem 52.

Under Assumption 4, some of the verification problems can be solved in PTIME:

Theorem 58 With the notation introduced in Theorem 22 and used in Sections 5.2.1–5.2.4, and under Assumptions 1, 2(1), 3 and 4, the following hold for every conjunction $\text{Def} : \bigwedge_{p(t)\in T_1} p(t)=\text{nil} \land \bigwedge_{p(t)\in T_2} p(t) \neq \text{nil}$, where $T_1 \cup T_2 = \{p(t) \mid t \text{ subterm of sort index of } G, p \in P, p(t) \text{ not in } G\}$ and every $q \in Q^{I_{\text{entry}}}$ (resp. $Q^{I_{\text{flow}}}$ or $Q^{I_{\text{update}}}$):

- (1) The satisfiability of $F_a^{entry} \wedge Def$ can be checked in PTIME.
- (2) The satisfiability of $F_a^{\text{flow}} \wedge \text{Def}$ can be checked in PTIME.
- (2) The satisfiability of $F_a^{jump} \wedge Def$ can be checked in PTIME.
- (4) Assuming that either (a) P_S is empty, or else (b) Update(p, p') has the form in Theorem 27, the satisfiability of $\mathsf{F}_{a}^{\mathsf{update}} \wedge \mathsf{Def}$ can be checked in PTIME.

If we consider |Q|, |E| and |P| to be constant and the number of terms of sort index in Φ_{safe} , and the maximal number of variables in the update axioms as a parameter, these problems can be considered to be fixed parameter tractable.

Proof: All transformations in the hierarchical reduction increase the size of the ground formulae to be checked polynomially. If the constraints over \mathbb{R} we obtain after this reduction lie in a tractable fragment of linear arithmetic, and if the ground constraints involving terms of sort index are unit and contain definedness or undefinedness conditions⁹ for all ground terms of sort index, then checking satisfiability can be done in PTIME. The number of possible choices for Def is $2^{(T_1 \cup T_2) \setminus \text{st}(G)}$. Since each of the verification tasks for a fixed Def can be solved in PTIME, this yields the fixed parameter tractability result.

⁹A definedness condition for a term t of sort index is a literal $t \neq \text{nil}$; an undefinedness condition for t is a literal of the form t = nil.

Theorem 59 (Parametric systems) The complexity results in Theorems 33–47 and 58, as well as the small model property, also hold for parametric SFLHA in which only the bounds in $\Phi_{entry}, \Phi_{safe}, Inv_q, Init_q, guard_e, jump_e, and Update are parameters. For systems in which parameters are allowed as coefficients or appear in the flow conditions, the complexity is exponential.$

Proof: This follows from the fact that all verification problems can be reduced to checking satisfiability for quantifier-free formulae (i.e. validity of existentially quantified formulae). If the parameters occur only in the bounds in Φ_{entry} , Φ_{safe} , Inv_q , Init_q , guard_e , jump_e , and Update then the numerical constraints are still linear hence the complexity is as in the non-parametric case, and the satisfiability of quantifier-free formulae over the theory of real-closed fields (\mathbb{R}) can be checked in EXPTIME [Ben-Or et al., 1986].

Theorem 60 (Parametric synthesis) Under Assumptions 1, 2(1) and 3, the complexity of synthesizing constraints on parameters which guarantee that a parametric SFLHA satisfies a safety condition with exhaustive entries (using quantifier elimination) is exponential.

Proof: The proof is similar to the proof of Theorem 59, taking into account that the complexity of quantifier elimination for formulae without alternation quantifiers (hence also for existential formulae) is EXPTIME [Collins, 1975, Ben-Or et al., 1986].

Similar methods can be used for showing that under Assumptions 1, 2(1) and 3 the problem of checking conditions (i) and (ii) in the definition of exhaustive entry conditions is in NP. We can also express Φ_{entry} and S parametrically and infer constraints on parameters under which conditions (i) and (ii) hold.

Remark 61 Similar results can also be obtained under Assumption 2(2) or 2(3), but because in those cases we need to instantiate in two steps the description of the instances needed is a bit more complicated (the number of instances and the size of I_0 is still polynomial in these situations.

In fact, all decidability results directly translate to situations where the involved formulas do not satisfy Assumptions 2 or 3 but belong to other fragments for which the theory extensions in Theorem 31 are local or stably local; the complexity depends on the complexity of checking satisfiability for formulae obtained after instantiation.

7 Tool Support

In order to perform the verification tasks automatically, we implemented our approach in the tool HAHA (Hierarchic Analysis of Hybrid Automata)¹⁰. HAHA employs H-PILoT¹¹, a program for hierarchical reasoning in extensions of logical theories [Ihlemann and Sofronie-Stokkermans, 2009], to perform reductions of the verification proof tasks to satisfiability problems in a combination of linear arithmetic over \mathbb{R} and pure equality. These are then solved using the theorem prover Z3 [de Moura and Bjørner, 2008].

7.1 Input syntax

We specify spatial families of linear hybrid automata in XML files, whose structure directly mirrors the constituent structure of such a family. For example, the specifications of the approach mode and the lane-changing jump for our running example are presented in Figure 5. Note that we do not explicitly specify the definedness guards \mathcal{E} . Instead, they are added automatically by H-PILoT.

¹⁰http://userp.uni-koblenz.de/~sofronie/horbach/haha.html

¹¹http://userp.uni-koblenz.de/~sofronie/hpilot/

```
<mode id="approach">
    <invariant>OR(lane(i) = 1,lane(i) = 2)</invariant>
    <invariant>pos(front(i))-pos(i) >= mindistance</invariant>
    <flow derivatives=".lane(i)" value="0"/>
    <flow derivatives=".pos(i)" upperbound="100"/>
    <flow derivatives=".pos(i)" lowerbound="0"/>
    <flow derivatives=".pos(i)-.pos(front(i))" lowerbound="0"/>
    </mode>
<jump source="__any__" target="approach">
    <guard>mindistance >= pos(front(i))-pos(i)</guard>
    <guard>pos(sidefront(i))-pos(i) >= mindistance</guard>
    <guard>pos(sidefront(i)) >= mindistance</guard>
    <guard>pos(i)-pos(sideback(i)) >= mindistance</guard>
    <guard>pos(i)-pos(back(i)) >= mindistance</guard>
    </guard>pos(i)-pos(back(i)) >= mindistance</guard>
    </guard>
    </guard>pos(i)-pos(back(i)) >= mindistance</guard>
    </guard>
    </guar
```





Figure 6: Implementation Data Flow Overview.

7.2 System architecture

An overview of our implementation is depicted in Figure 6. In a first step, HAHA parses the problem from the XML specification and creates internal representations of the four verification tasks explained in Theorem 22.

Each of them is then translated into H-PILoT syntax, and H-PILoT performs the reduction to quantifier-free problems as in the proofs of Theorems 33–47. H-PILoT's output consists of problems in linear real and integer arithmetic, whose satisfiability is checked by Z3.

If Z3 detects unsatisfiability, the proof task was successful. For satisfiable formulae, H-PILoT returns a model which can be used to visualize the counterexample to the invariance properties [Krawez, 2012]. Finally, HAHA collects statistics on run times, satisfiability, and model sizes for the individual verification problems.

The check whether a given entry condition satisfies the properties in Definition 12 or 17 works similarly.

The use of GMR constraints is not always necessary to prove safety, because some safety properties are maintained by *all* jumps and updates, not just by globally mode reachable ones. Because the inclusion of GMR constraints affects the performance of the approach, HAHA can also run in a mode that does not create them (c.f. our experimental results below).



Figure 7: The property $\Phi_{\mathsf{safe}}^{\mathsf{front}}$ is violated by the depicted update if the distance between cars 7 and 5 is below the minimal safe distance d_s . Restriction to globally mode reachable jumps avoids this situation.

7.3 Experiments

We evaluated HAHA on variations of our running problem and on examples from the Passel benchmark suite [Johnson and Mitra, 2012b]. In the following sections, we describe the results of the verification of some of the safety conditions presented throughout the paper. The list is not exhaustive, but includes safety properties that demonstrates a variety of features of our approach. On the HAHA homepage, we provide all source data for these examples, including an xml description of the automaton, the verification problems that are handed over to H-PILoT, and finally the SMT problems handled by Z3. We also provide formalizations of several of the examples from the Passel benchmark suite.

7.3.1 Decision Problems

We considered our running example with the entry condition Φ_{entry} from Example 54:

$$\begin{split} \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) = \mathsf{nil} \to \forall k (k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \ge \mathsf{pos}(i) \to \mathsf{lane}(k) \neq \mathsf{lane}(i))) \\ \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \to \mathsf{pos}_{\mathsf{front}}(i) > \mathsf{pos}(i) + d' \land \mathsf{lane}_{\mathsf{front}}(i) = \mathsf{lane}(i) \land \\ \forall k (k \neq \mathsf{nil} \land k \neq i \land \mathsf{pos}(k) \ge \mathsf{pos}(i) \land \mathsf{lane}(k) = \mathsf{lane}(i) \\ \to \mathsf{pos}(k) \ge \mathsf{pos}_{\mathsf{front}}(i)) \land \\ \mathsf{pos}(\mathsf{front}(i)) = \mathsf{pos}_{\mathsf{front}}(i) \land \mathsf{lane}(\mathsf{front}(i)) = \mathsf{lane}_{\mathsf{front}}(i)) \end{split}$$

As safety conditions, we chose the following:

$$\begin{split} \Phi^{\mathsf{top}}_{\mathsf{safe}} &: \forall i (i \neq \mathsf{nil} \to \mathsf{front}(i) \neq i) \\ \Phi^{g}_{\mathsf{safe}} &: \forall i, j (i \neq \mathsf{nil} \land j \neq \mathsf{nil} \land \mathsf{lane}(i) = \mathsf{lane}(j) \land \mathsf{pos}(i) > \mathsf{pos}(j) \to \mathsf{pos}(i) - \mathsf{pos}(j) \geq d_s) \\ \Phi^{\mathsf{front}}_{\mathsf{safe}} &: \forall i (i \neq \mathsf{nil} \land \mathsf{front}(i) \neq \mathsf{nil} \to \mathsf{pos}(\mathsf{front}(i)) - \mathsf{pos}(i) \geq d_s) \end{split}$$

The first condition states a basic consistency property of the sensor information; the next two are the ones first introduced in Example 11. We provided constraints for all parameters, stating e.g. that the minimal distance between cars in mode Appr does not exceed the maximal distance between cars in mode Rec $(d \le D)$, and both are nonnegative $(d \ge 0, D \ge 0)$.

Results of experiments with our running example are summarized in Figure 9. The left half of the diagram shows the results and run times as well as the maximal model sizes (cf. Theorem 55) of verification attempts that ignore the entry condition and global mode reachability. A result of unsat means that HAHA could prove the respective verification task, sat means that it found a counter example. As can be seen, the analysis without regard to global mode reachability is faster but not always powerful enough. For example, Φ_{safe}^{front} is not invariant under all updates; Figure 7 shows an example of such an update that violates Φ_{safe}^{front} .

The right half of the diagram shows the results of verification including global mode reachability. In this mode, we could prove that Φ_{safe}^{front} holds in all runs.

From the tests presented in Figure 9, we observe the following facts:



Figure 8: The property Φ_{safe}^g is violated by by a lane change if there is another car between sidefront and sideback. This can happen even for globally mode reachable jumps.

- The formula Φ_{safe}^{top} is an invariant of the system, and is also invariant under globally mode reachable flows, jumps and topology updates.
- The formula Φ_{safe}^{front} is true in the initial states and is invariant under jumps and flows, but not under all topology updates. It is however invariant under all globally mode reachable topology updates.
- The formula Φ_{safe}^g is true in the initial states and is invariant under topology updates. However, the formula is not invariant under jumps and flows. We could show that it is invariant under globally mode reachable flows and topology updates, but not under globally mode reachable jumps.

7.3.2 Model generation

The fact that we could show that Φ_{safe}^g is not invariant under globally mode reachable jumps contradicted our intuition, because a lane change (and no other jump could be the culprit) can only take place if the adjacent cars front, back, sidefront and sideback are sufficiently far away. In order to understand the problem, we used the model returned by H-PILoT to construct a counterexample to safety. After simplifying this model, we obtained a model describing the situation presented in Figure 8: Because we do not specify in Φ_{entry} that sensors have to be set correctly, there may be another car between sidefront and sideback which will cause a lane change to lead to a collision.

A jump in the situation described in Figure 8 can only occur because the information provided by sensors at the moment of a line change is outdated. One way to avoid this is to ensure that a topology update takes place immediately before any lane change. This is exactly what a human driver would do: to recheck the surroundings immediately before a lane change. We proved that for all runs in which topology updates take place before lane changes, formula Φ^g_{safe} is invariant under all jumps. The detailed results are presented in the bottom rows of Figure 9.

7.3.3 Complexity

From the detailed run times in Figure 9, one can see that the locality-based reduction of the problem usually dominates the overall run time. The final satisfiability check with Z3 is much faster, especially when the problem size increases. We could partially reduce the gap by adding several optimizations to H-PILoT. The results reported in the table are thus an order of magnitude faster than the ones we reported in [Damm et al., 2015].

Comparing runs with and without consideration of entry states, we can see that the analysis of entry conditions and flows starting in an entry state is only marginally slower than the analysis of initial conditions and general flows. For jumps and topology updates, on the other hand, the additional flow formulae lead to larger ground problems, corresponding to larger potential counter models (cf. Theorem 55). Of course, a similar effect also occurs when every jump is preceded by an update.

| | without mode reachability | | | with mode reachability | | | | | |
|---|---------------------------|----------------|----------------|------------------------|-------|-------|----------------|--------|--------------|
| | init | flow | jump | update | entry | flow | jump | update | |
| Φ_{safe}^{top} | unsat | unsat | unsat | unsat | unsat | unsat | unsat | unsat | verified |
| constants | 11 | 43 | 141 | 19 | 11 | 43 | 206 | 51 | |
| reduction | 0.028 | 0.072 | 0.460 | 0.024 | 0.028 | 0.060 | 2.224 | 0.108 | |
| SMT | 0.008 | 0.020 | 0.050 | 0.012 | 0.004 | 0.000 | 0.072 | 0.020 | |
| total time | 0.036 | 0.092 | 0.510 | 0.036 | 0.032 | 0.060 | 2.296 | 0.128 | |
| Φ_{safe}^{front} | unsat | unsat | unsat | sat | unsat | unsat | unsat | unsat | verified |
| constants | 11 | 43 | 141 | 19 | 11 | 43 | 206 | 51 | |
| reduction | 0.020 | 0.048 | 0.420 | 0.024 | 0.020 | 0.060 | 2.260 | 0.140 | |
| SMT | 0.000 | 0.008 | 0.060 | 0.008 | 0.000 | 0.000 | 0.080 | 0.020 | |
| total time | 0.020 | 0.056 | 0.480 | 0.032 | 0.020 | 0.060 | 2.340 | 0.160 | |
| Φ^g_{safe} | unsat | \mathbf{sat} | \mathbf{sat} | unsat | unsat | unsat | \mathbf{sat} | unsat | not verified |
| constants | 9 | 33 | 131 | 15 | 9 | 33 | 191 | 39 | |
| reduction | 0.012 | 0.028 | 0.692 | 0.020 | 0.020 | 0.044 | 2.372 | 0.100 | |
| SMT | 0.012 | 0.004 | 0.048 | 0.000 | 0.000 | 0.012 | 0.292 | 0.000 | |
| total time | 0.024 | 0.032 | 0.740 | 0.020 | 0.020 | 0.056 | 2.664 | 0.100 | |
| including forced topology updates before every jump | | | | | | | | | |
| Φ^g_{safe} | unsat | sat | unsat | unsat | unsat | unsat | unsat | unsat | verified |
| constants | 9 | 33 | 155 | 15 | 9 | 33 | 215 | 39 | |
| reduction | 0.012 | 0.032 | 2.240 | 0.016 | 0.016 | 0.040 | 4.784 | 0.072 | |
| SMT | 0.008 | 0.012 | 0.070 | 0.012 | 0.012 | 0.010 | 0.144 | 0.020 | |
| total time | 0.020 | 0.044 | 2.310 | 0.028 | 0.028 | 0.050 | 4.928 | 0.092 | |
| | | | | | | | | | |

Figure 9: Verification times (in seconds) for the given safety properties and number of constants of index type in the reduced satisfiability problem

8 Conclusions

8.1 Summary of results

We proved that safety properties with exhaustive entry conditions for spatial families of similar linear hybrid automata can be verified efficiently: We reduced the proof task to invariant checking for certain mode reachable states and analyzed the complexity of such problems. As a by-product, we obtained a modularity result for checking safety properties. The results can also be used for invariant checking (for this the information about mode reachability in the formulae is ignored). The results we obtained are summarized in Figure 10.

The decidability and complexity results and the small model property were established under Assumptions 1, 2(1), 3 (and possibly 4 for tractability). Similar results can also be obtained under Assumption 2(2) or (3) (we did not present these situations explicitly in this paper because the instances obtained due to the locality results are more complicated to describe (the instantiation takes place in several steps); however it can be proved that the number of instances and the size of T_0 is still polynomial.

All decidability results directly translate to situations where the involved formulas do not satisfy Assumptions 2 or 3 but belong to other fragments for which the theory extensions in Theorem 31 are local or stably local; the complexity depends on the complexity of checking satisfiability for formulae obtained after instantiation.

| | Safety of Φ_{E} | $Harrow \Box \Phi_{Safe}$ | Exh. entry conds Φ_{Entry} | | | |
|---|----------------------|------------------------------|---------------------------------|------------------------------|--|--|
| | Assumptions $1-3$ | Assumptions 1–4 | Assumptions $1–3$ | Assumptions 1–4 | | |
| Verification | decidable | decidable | decidable | decidable | | |
| (Thm. 33–47,56,57,58) | NP | fixed parameter tractable | NP | fixed parameter tractable | | |
| Small model property | yes | yes | yes | yes | | |
| (Thm. 55) | | | | | | |
| Parametric verification | decidable | decidable | decidable | decidable | | |
| (Thm. 59) | | | | | | |
| non-param. coefficients/ bounds flows: | NP | fixed parameter tractable | NP | fixed parameter tractable | | |
| parametric coefficients | EXPTIME | EXPTIME | EXPTIME | EXPTIME | | |
| parametric bounds flows | EXPTIME | EXPTIME | EXPTIME | EXPTIME | | |
| Parameter Synthesis (Thm. 60) | EXPTIME | EXPTIME | EXPTIME | EXPTIME | | |

Figure 10: Summary of Results

We would like to point out that although in this paper we refer to a countable set I of car identities, due to the verification method we use the concrete identities of the cars are not important. If we prove safety, then we prove it for any model (and thus for any possible index set); if we cannot prove it then a counterexample gives us a possible index set for which the safety propery fails (thus a set of possible identities of the cars for which we can construct a counterexample to safety). On the other hand, fixing a set of car identities is not a restriction. In all the models that can be obtained in case the formulae we consider are satisfiable, the index sets are quotients (finite or countably infinite) of a countable set (which can for instance be chosen to be I or the set of natural numbers); all countable models are isomorphic to this set (Ior the set of natural numbers). In the paper this is handled by introducing Skolem constants for the indexes of the cars at which the safety condition might not hold. A model gives values for these constants (in I or in \mathbb{N}).

8.2 Plans for further work

Another important class of properties, related to timely completion of maneuvers, are bounded reachability properties. They state that for every run starting in a suitable initial configuration Φ_{entry} , a maneuver completion condition $\Phi_{complete}$ becomes true in a given bounded time frame. Similar methods can be used for efficiently checking also this type of properties if we guarantee that the number of jumps and topology updates in any fixed interval is bounded. We did not include such considerations here in order to keep the presentation and the required logics simpler.

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