# Learning with a Drifting Target Concept 

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#### Abstract

We study the problem of learning in the presence of a drifting target concept. Specifically, we provide bounds on the error rate at a given time, given a learner with access to a history of independent samples labeled according to a target concept that can change on each round. One of our main contributions is a refinement of the best previous results for polynomial-time algorithms for the space of linear separators under a uniform distribution. We also provide general results for an algorithm capable of adapting to a variable rate of drift of the target concept. Some of the results also describe an active learning variant of this setting, and provide bounds on the number of queries for the labels of points in the sequence sufficient to obtain the stated bounds on the error rates.


## 1 Introduction

Much of the work on statistical learning has focused on learning settings in which the concept to be learned is static over time. However, there are many application areas where this is not the case. For instance, in the problem of face recognition, the concept to be learned actually changes over time as each individual's facial features evolve over time. In this work, we study the problem of learning with a drifting target concept. Specifically, we consider a statistical learning setting, in which data arrive i.i.d. in a stream, and for each data point, the learner is required to predict a label for the data point at that time. We are then interested in obtaining low error rates for these predictions. The target labels are generated from a function known to reside in a given concept space, and at each time $t$ the target function is allowed to change by at most some distance $\Delta_{t}$ : that is, the probability the new target function disagrees with the previous target function on a random sample is at most $\Delta_{t}$.

This framework has previously been studied in a number of articles. The classic works of [HL91|HL94|BH96 Lon99 BBDK00] and BL97] together provide a general analysis of a very-much related setting. Though the objectives in these works are specified slightly differently, the results established there are easily translated into our present framework, and we summarize many of the relevant results from this literature in Section 3,

While the results in these classic works are general, the best guarantees on the error rates are only known for methods having no guarantees of computational efficiency. In a more recent effort, the work of [CMEDV10] studies this problem in the specific context of learning a homogeneous linear separator, when all the $\Delta_{t}$ values are identical. They propose a polynomial-time algorithm (based on the modified Perceptron algorithm of (DKM09), and prove a bound on the number of mistakes it makes as a function of the number of samples, when the data distribution satisfies a certain condition called " $\lambda$-good" (which generalizes a useful property of the uniform distribution on the origin-centered unit sphere). However, their result is again worse than that obtainable by the known computationally-inefficient methods.

Thus, the natural question is whether there exists a polynomial-time algorithm achieving roughly the same guarantees on the error rates known for the inefficient methods. In the present work, we resolve this question in the case of learning homogeneous linear separators under the uniform distribution, by proposing a polynomial-time algorithm that indeed achieves roughly the same bounds on the error rates known for the inefficient methods in the literature. This represents the main technical contribution of this work.

We also study the interesting problem of adaptivity of an algorithm to the sequence of $\Delta_{t}$ values, in the setting where $\Delta_{t}$ may itself vary over time. Since the values $\Delta_{t}$ might typically not be accessible in practice, it seems important to have learning methods having no explicit dependence on the sequence $\Delta_{t}$. We propose such a method below, and prove that it achieves roughly the same bounds on the error rates known for methods in the literature which require direct access to the $\Delta_{t}$ values. Also in the context of variable $\Delta_{t}$ sequences, we discuss conditions on the sequence $\Delta_{t}$ necessary and sufficient for there to exist a learning method guaranteeing a sublinear rate of growth of the number of mistakes.

We additionally study an active learning extension to this framework, in which, at each time, after making its prediction, the algorithm may decide whether or not to request access to the label assigned to the data point at that time. In addition to guarantees on the error rates (for all times, including those for which the label was not observed), we are also interested in bounding the number of labels we expect the algorithm to request, as a function of the number of samples encountered thus far.

## 2 Definitions and Notation

Formally, in this setting, there is a fixed distribution $\mathcal{P}$ over the instance space $\mathcal{X}$, and there is a sequence of independent $\mathcal{P}$-distributed unlabeled data $X_{1}, X_{2}, \ldots$. There is also a concept space $\mathbb{C}$, and a sequence of target functions $\mathbf{h}^{*}=$ $\left\{h_{1}^{*}, h_{2}^{*}, \ldots\right\}$ in $\mathbb{C}$. Each $t$ has an associated target label $Y_{t}=h_{t}^{*}\left(X_{t}\right)$. In this context, a (passive) learning algorithm is required, on each round $t$, to produce a classifier $\hat{h}_{t}$ based on the observations $\left(X_{1}, Y_{1}\right), \ldots,\left(X_{t-1}, Y_{t-1}\right)$, and we denote by $\hat{Y}_{t}=\hat{h}_{t}\left(X_{t}\right)$ the corresponding prediction by the algorithm for
the label of $X_{t}$. For any classifier $h$, we define $\operatorname{er}_{t}(h)=\mathcal{P}\left(x: h(x) \neq h_{t}^{*}(x)\right)$. We also say the algorithm makes a "mistake" on instance $X_{t}$ if $\hat{Y}_{t} \neq Y_{t}$; thus, $\operatorname{er}_{t}\left(\hat{h}_{t}\right)=\mathbb{P}\left(\hat{Y}_{t} \neq Y_{t} \mid\left(X_{1}, Y_{1}\right), \ldots,\left(X_{t-1}, Y_{t-1}\right)\right)$.

For notational convenience, we will suppose the $h_{t}^{*}$ sequence is chosen independently from the $X_{t}$ sequence (i.e., $h_{t}^{*}$ is chosen prior to the "draw" of $\left.X_{1}, X_{2}, \ldots \sim \mathcal{P}\right)$, and is not random.

In each of our results, we will suppose $\mathbf{h}^{*}$ is chosen from some set $S$ of sequences in $\mathbb{C}$. In particular, we are interested in describing the sequence $\mathbf{h}^{*}$ in terms of the magnitudes of changes in $h_{t}^{*}$ from one time to the next. Specifically, for any sequence $\boldsymbol{\Delta}=\left\{\Delta_{t}\right\}_{t=2}^{\infty}$ in $[0,1]$, we denote by $S_{\boldsymbol{\Delta}}$ the set of all sequences $\mathbf{h}^{*}$ in $\mathbb{C}$ such that, $\forall t \in \mathbb{N}, \mathcal{P}\left(x: h_{t}(x) \neq h_{t+1}(x)\right) \leq \Delta_{t+1}$.

Throughout this article, we denote by $d$ the VC dimension of $\mathbb{C}$ VC71, and we suppose $\mathbb{C}$ is such that $1 \leq d<\infty$. Also, for any $x \in \mathbb{R}$, define $\log (x)=$ $\ln (\max \{x, e\})$.

## 3 Background: $(\epsilon, S)$-Tracking Algorithms

As mentioned, the classic literature on learning with a drifting target concept is expressed in terms of a slightly different model. In order to relate those results to our present setting, we first introduce the classic setting. Specifically, we consider a model introduced by [HL94, presented here in a more-general form inspired by BBDK00. For a set $S$ of sequences $\left\{h_{t}\right\}_{t=1}^{\infty}$ in $\mathbb{C}$, and a value $\epsilon>0$, an algorithm $\mathcal{A}$ is said to be $(\epsilon, S)$-tracking if $\exists t_{\epsilon} \in \mathbb{N}$ such that, for any choice of $\mathbf{h}^{*} \in S, \forall T \geq t_{\epsilon}$, the prediction $\hat{Y}_{T}$ produced by $\mathcal{A}$ at time $T$ satisfies

$$
\mathbb{P}\left(\hat{Y}_{T} \neq Y_{T}\right) \leq \epsilon
$$

Note that the value of the probability in the above expression may be influenced by $\left\{X_{t}\right\}_{t=1}^{T},\left\{h_{t}^{*}\right\}_{t=1}^{T}$, and any internal randomness of the algorithm $\mathcal{A}$.

The focus of the results expressed in this classical model is determining sufficient conditions on the set $S$ for there to exist an $(\epsilon, S)$-tracking algorithm, along with bounds on the sufficient size of $t_{\epsilon}$. These conditions on $S$ typically take the form of an assumption on the drift rate, expressed in terms of $\epsilon$. Below, we summarize several of the strongest known results for this setting.

### 3.1 Bounded Drift Rate

The simplest, and perhaps most elegant, results for $(\epsilon, S)$-tracking algorithms is for the set $S$ of sequences with a bounded drift rate. Specifically, for any $\Delta \in[0,1]$, define $S_{\Delta}=S_{\Delta}$, where $\boldsymbol{\Delta}$ is such that $\Delta_{t+1}=\Delta$ for every $t \in \mathbb{N}$. The study of this problem was initiated in the original work of [HL94. The best known general results are due to Lon99: namely, that for some $\Delta_{\epsilon}=$ $\Theta\left(\epsilon^{2} / d\right)$, for every $\epsilon \in(0,1]$, there exists an $\left(\epsilon, S_{\Delta}\right)$-tracking algorithm for all values of $\Delta \leq \Delta_{\epsilon} 4_{4}^{4}$ This refined an earlier result of HL94 by a logarithmic

[^0]factor. Lon99] further argued that this result can be achieved with $t_{\epsilon}=\Theta(d / \epsilon)$. The algorithm itself involves a beautiful modification of the one-inclusion graph prediction strategy of HLW94; since its specification is somewhat involved, we refer the interested reader to the original work of Lon99] for the details.

### 3.2 Varying Drift Rate: Nonadaptive Algorithm

In addition to the concrete bounds for the case $\mathbf{h}^{*} \in S_{\Delta}$, HL94 additionally present an elegant general result. Specifically, they argue that, for any $\epsilon>0$, and any $m=\Omega\left(\frac{d}{\epsilon} \log \frac{1}{\epsilon}\right)$, if $\sum_{i=1}^{m} \mathcal{P}\left(x: h_{i}^{*}(x) \neq h_{m+1}^{*}(x)\right) \leq m \epsilon / 24$, then for $\hat{h}=\operatorname{argmin}_{h \in \mathbb{C}} \sum_{i=1}^{m} \mathbb{1}\left[h\left(X_{i}\right) \neq Y_{i}\right], \mathbb{P}\left(\hat{h}\left(X_{m+1}\right) \neq h_{m+1}^{*}\left(X_{m+1}\right)\right) \leq \epsilon 5$ This result immediately inspires an algorithm $\mathcal{A}$ which, at every time $t$, chooses a value $m_{t} \leq t-1$, and predicts $\hat{Y}_{t}=\hat{h}_{t}\left(X_{t}\right)$, for $\hat{h}_{t}=\operatorname{argmin}_{h \in \mathbb{C}} \sum_{i=t-m_{t}}^{t-1} \mathbb{1}\left[h\left(X_{i}\right) \neq Y_{i}\right]$. We are then interested in choosing $m_{t}$ to minimize the value of $\epsilon$ obtainable via the result of HL94]. However, that method is based on the values $\mathcal{P}\left(x: h_{i}^{*}(x) \neq\right.$ $h_{t}^{*}(x)$ ), which would typically not be accessible to the algorithm. However, suppose instead we have access to a sequence $\boldsymbol{\Delta}$ such that $\mathbf{h}^{*} \in S_{\boldsymbol{\Delta}}$. In this case, we could approximate $\mathcal{P}\left(x: h_{i}^{*}(x) \neq h_{t}^{*}(x)\right)$ by its upper bound $\sum_{j=i+1}^{t} \Delta_{j}$. In this case, we are interested choosing $m_{t}$ to minimize the smallest value of $\epsilon$ such that $\sum_{i=t-m_{t}}^{t-1} \sum_{j=i+1}^{t} \Delta_{j} \leq m_{t} \epsilon / 24$ and $m_{t}=\Omega\left(\frac{d}{\epsilon} \log \frac{1}{\epsilon}\right)$. One can easily verify that this minimum is obtained at a value

$$
m_{t}=\Theta\left(\underset{m \leq t-1}{\operatorname{argmin}} \frac{1}{m} \sum_{i=t-m}^{t-1} \sum_{j=i+1}^{t} \Delta_{j}+\frac{d \log (m / d)}{m}\right)
$$

and via the result of HL94 (applied to the sequence $X_{t-m_{t}}, \ldots, X_{t}$ ) the resulting algorithm has

$$
\begin{equation*}
\mathbb{P}\left(\hat{Y}_{t} \neq Y_{t}\right) \leq O\left(\min _{1 \leq m \leq t-1} \frac{1}{m} \sum_{i=t-m}^{t-1} \sum_{j=i+1}^{t} \Delta_{j}+\frac{d \log (m / d)}{m}\right) \tag{1}
\end{equation*}
$$

As a special case, if every $t$ has $\Delta_{t}=\Delta$ for a fixed value $\Delta \in[0,1]$, this result recovers the bound $\sqrt{d \Delta \log (1 / \Delta)}$, which is only slightly larger than that obtainable from the best bound of [Lon99]. It also applies to far more general and more intersting sequences $\boldsymbol{\Delta}$, including some that allow periodic large jumps (i.e., $\Delta_{t}=1$ for some indices $t$ ), others where the sequence $\Delta_{t}$ converges to 0 , and so on. Note, however, that the algorithm obtaining this bound directly depends on the sequence $\boldsymbol{\Delta}$. One of the contributions of the present work is to remove this requirement, while maintaining essentially the same bound, though in a slightly different form.

[^1]
### 3.3 Computational Efficiency

HL94 also proposed a reduction-based approach, which sometimes yields computationally efficient methods, though the tolerable $\Delta$ value is smaller. Specifically, given any (randomized) polynomial-time algorithm $\mathcal{A}$ that produces a classifier $h \in \mathbb{C}$ with $\sum_{t=1}^{m} \mathbb{1}\left[h\left(x_{t}\right) \neq y_{t}\right]=0$ for any sequence $\left(x_{1}, y_{1}\right), \ldots,\left(x_{m}, y_{m}\right)$ for which such a classifier $h$ exists (called the consistency problem), they propose a polynomial-time algorithm that is $\left(\epsilon, S_{\Delta}\right)$-tracking for all values of $\Delta \leq \Delta_{\epsilon}^{\prime}$, where $\Delta_{\epsilon}^{\prime}=\Theta\left(\frac{\epsilon^{2}}{d^{2} \log (1 / \epsilon)}\right)$. This is slightly worse (by a factor of $d \log (1 / \epsilon)$ ) than the drift rate tolerable by the (typically inefficient) algorithm mentioned above. However, it does sometimes yield computationally-efficient methods. For instance, there are known polynomial-time algorithms for the consistency problem for the classes of linear separators, conjunctions, and axis-aligned rectangles.

### 3.4 Lower Bounds

HL94 additionally prove lower bounds for specific concept spaces: namely, linear separators and axis-aligned rectangles. They specifically argue that, for $\mathbb{C}$ a concept space

$$
\operatorname{BASIC}_{n}=\left\{\cup_{i=1}^{n}\left[i / n,\left(i+a_{i}\right) / n\right): \mathbf{a} \in[0,1]^{n}\right\}
$$

on $[0,1]$, under $\mathcal{P}$ the uniform distribution on $[0,1]$, for any $\epsilon \in\left[0,1 / e^{2}\right]$ and $\Delta_{\epsilon} \geq e^{4} \epsilon^{2} / n$, for any algorithm $\mathcal{A}$, and any $T \in \mathbb{N}$, there exists a choice of $\mathbf{h}^{*} \in S_{\Delta_{\epsilon}}$ such that the prediction $\hat{Y}_{T}$ produced by $\mathcal{A}$ at time $T$ satisfies $\mathbb{P}\left(\hat{Y}_{T} \neq Y_{T}\right)>\epsilon$. Based on this, they conclude that no $\left(\epsilon, S_{\Delta_{\epsilon}}\right)$-tracking algorithm exists. Furthermore, they observe that the space $\mathrm{BASIC}_{n}$ is embeddable in many commonly-studied concept spaces, including halfspaces and axis-aligned rectangles in $\mathbb{R}^{n}$, so that for $\mathbb{C}$ equal to either of these spaces, there also is no $\left(\epsilon, S_{\Delta_{\epsilon}}\right)$-tracking algorithm.

## 4 Adapting to Arbitrarily Varying Drift Rates

This section presents a general bound on the error rate at each time, expressed as a function of the rates of drift, which are allowed to be arbitrary. Mostimportantly, in contrast to the methods from the literature discussed above, the method achieving this general result is adaptive to the drift rates, so that it requires no information about the drift rates in advance. This is an appealing property, as it essentially allows the algorithm to learn under an arbitrary sequence $\mathbf{h}^{*}$ of target concepts; the difficulty of the task is then simply reflected in the resulting bounds on the error rates: that is, faster-changing sequences of target functions result in larger bounds on the error rates, but do not require a change in the algorithm itself.

### 4.1 Adapting to a Changing Drift Rate

Recall that the method yielding (11) (based on the work of (HL94]) required access to the sequence $\boldsymbol{\Delta}$ of changes to achieve the stated guarantee on the expected number of mistakes. That method is based on choosing a classifier to predict $\hat{Y}_{t}$ by minimizing the number of mistakes among the previous $m_{t}$ samples, where $m_{t}$ is a value chosen based on the $\boldsymbol{\Delta}$ sequence. Thus, the key to modifying this algorithm to make it adaptive to the $\boldsymbol{\Delta}$ sequence is to determine a suitable choice of $m_{t}$ without reference to the $\boldsymbol{\Delta}$ sequence. The strategy we adopt here is to use the data to determine an appropriate value $\hat{m}_{t}$ to use. Roughly (ignoring logarithmic factors for now), the insight that enables us to achieve this feat is that, for the $m_{t}$ used in the above strategy, one can show that $\sum_{i=t-m_{t}}^{t-1} \mathbb{1}\left[h_{t}^{*}\left(X_{i}\right) \neq Y_{i}\right]$ is roughly $\tilde{O}(d)$, and that making the prediction $\hat{Y}_{t}$ with any $h \in \mathbb{C}$ with roughly $\tilde{O}(d)$ mistakes on these samples will suffice to obtain the stated bound on the error rate (up to logarithmic factors). Thus, if we replace $m_{t}$ with the largest value $m$ for which $\min _{h \in \mathbb{C}} \sum_{i=t-m}^{t-1} \mathbb{1}\left[h\left(X_{i}\right) \neq Y_{i}\right]$ is roughly $\tilde{O}(d)$, then the above observation implies $m \geq m_{t}$. This then implies that, for $\hat{h}=\operatorname{argmin}_{h \in \mathbb{C}} \sum_{i=t-m}^{t-1} \mathbb{1}\left[h\left(X_{i}\right) \neq Y_{i}\right]$, we have that $\sum_{i=t-m_{t}}^{t-1} \mathbb{1}\left[\hat{h}\left(X_{i}\right) \neq Y_{i}\right]$ is also roughly $\tilde{O}(d)$, so that the stated bound on the error rate will be achieved (aside from logarithmic factors) by choosing $\hat{h}_{t}$ as this classifier $\hat{h}$. There are a few technical modifications to this argument needed to get the logarithmic factors to work out properly, and for this reason the actual algorithm and proof below are somewhat more involved. Specifically, consider the following algorithm (the value of the universal constant $K \geq 1$ will be specified below).

```
0 . For \(T=1,2, \ldots\)
1. Let \(\hat{m}_{T}=\max \left\{m \in\{1, \ldots, T-1\}: \min _{h \in \mathbb{C}} \max _{m^{\prime} \leq m} \frac{\sum_{t=T-m^{\prime}}^{T-1} 1\left[h\left(X_{t}\right) \neq Y_{t}\right]}{d \log \left(m^{\prime} / d\right)+\log (1 / \delta)}<K\right\}\)
2. Let \(\hat{h}_{T}=\underset{h \in \mathbb{C}}{\operatorname{argmin}} \max _{m^{\prime} \leq \tilde{m}_{T}} \frac{\sum_{t=T-m^{\prime}}^{T-1} \mathbb{1}\left[h\left(X_{t}\right) \neq Y_{t}\right]}{d \log \left(m^{\prime} / d\right)+\log (1 / \delta)}\)
```

Note that the classifiers $\hat{h}_{t}$ chosen by this algorithm have no dependence on $\boldsymbol{\Delta}$, or indeed anything other than the data $\left\{\left(X_{i}, Y_{i}\right): i<t\right\}$, and the concept space $\mathbb{C}$.

Theorem 1. Fix any $\delta \in(0,1)$, and let $\mathcal{A}$ be the above algorithm. For any sequence $\boldsymbol{\Delta}$ in $[0,1]$, for any $\mathcal{P}$ and any choice of $\mathbf{h}^{*} \in S_{\boldsymbol{\Delta}}$, for every $T \in \mathbb{N} \backslash\{1\}$, with probability at least $1-\delta$,

$$
\operatorname{er}_{T}\left(\hat{h}_{T}\right) \leq O\left(\min _{1 \leq m \leq T-1} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log (1 / \delta)}{m}\right) .
$$

Before presenting the proof of this result, we first state a crucial lemma, which follows immediately from a classic result of (Vap82|Vap98, combined with the fact (from Vid03, Theorem 4.5) that the VC dimension of the collection of sets $\{\{x: h(x) \neq g(x)\}: h, g \in \mathbb{C}\}$ is at most 10d.

Lemma 1. There exists a universal constant $c \in[1, \infty)$ such that, for any class $\mathbb{C}$ of $V C$ dimension $d, \forall m \in \mathbb{N}, \forall \delta \in(0,1)$, with probability at least $1-\delta$, every $h, g \in \mathbb{C}$ have

$$
\begin{array}{r}
\left|\mathcal{P}(x: h(x) \neq g(x))-\frac{1}{m} \sum_{t=1}^{m} \mathbb{1}\left[h\left(X_{t}\right) \neq g\left(X_{t}\right)\right]\right| \\
\leq c \sqrt{\left(\frac{1}{m} \sum_{t=1}^{m} \mathbb{1}\left[h\left(X_{t}\right) \neq g\left(X_{t}\right)\right]\right) \frac{d \log (m / d)+\log (1 / \delta)}{m}} \\
+c \frac{d \log (m / d)+\log (1 / \delta)}{m}
\end{array}
$$

We are now ready for the proof of Theorem (1) For the constant $K$ in the algorithm, we will choose $K=145 c^{2}$, for $c$ as in Lemma 1 .

Proof (Proof of Theorem 1). Fix any $T \in \mathbb{N}$ with $T \geq 2$, and define

$$
\begin{aligned}
& m_{T}^{*}=\max \left\{m \in\{1, \ldots, T-1\}: \forall m^{\prime} \leq m,\right. \\
& \left.\sum_{t=T-m^{\prime}}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{t}\right) \neq Y_{t}\right]<K\left(d \log \left(m^{\prime} / d\right)+\log (1 / \delta)\right)\right\} .
\end{aligned}
$$

Note that

$$
\begin{equation*}
\sum_{t=T-m_{T}^{*}}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{t}\right) \neq Y_{t}\right] \leq K\left(d \log \left(m_{T}^{*} / d\right)+\log (1 / \delta)\right) \tag{2}
\end{equation*}
$$

and also note that (since $\left.h_{T}^{*} \in \mathbb{C}\right) \hat{m}_{T} \geq m_{T}^{*}$, so that (by definition of $\hat{m}_{T}$ and $\hat{h}_{T}$ )

$$
\sum_{t=T-m_{T}^{*}}^{T-1} \mathbb{1}\left[\hat{h}_{T}\left(X_{t}\right) \neq Y_{t}\right] \leq K\left(d \log \left(m_{T}^{*} / d\right)+\log (1 / \delta)\right)
$$

as well. Therefore,

$$
\begin{aligned}
\sum_{t=T-m_{T}^{*}}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{t}\right) \neq \hat{h}_{T}\left(X_{t}\right)\right] & \leq \sum_{t=T-m_{T}^{*}}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{t}\right) \neq Y_{t}\right]+\sum_{t=T-m_{T}^{*}}^{T-1} \mathbb{1}\left[Y_{t} \neq \hat{h}_{T}\left(X_{t}\right)\right] \\
& \leq 2 K\left(d \log \left(m_{T}^{*} / d\right)+\log (1 / \delta)\right) .
\end{aligned}
$$

Thus, by Lemma $\mathbb{1}$, for each $m \in \mathbb{N}$, with probability at least $1-\delta /\left(6 m^{2}\right)$, if $m_{T}^{*}=m$, then

$$
\mathcal{P}\left(x: \hat{h}_{T}(x) \neq h_{T}^{*}(x)\right) \leq(2 K+c \sqrt{2 K}+c) \frac{d \log \left(m_{T}^{*} / d\right)+\log \left(6\left(m_{T}^{*}\right)^{2} / \delta\right)}{m_{T}^{*}}
$$

Furthermore, since $\log \left(6\left(m_{T}^{*}\right)^{2}\right) \leq \sqrt{2 K} d \log \left(m_{T}^{*} / d\right)$, this is at most

$$
2(K+c \sqrt{2 K}) \frac{d \log \left(m_{T}^{*} / d\right)+\log (1 / \delta)}{m_{T}^{*}}
$$

By a union bound (over values $m \in \mathbb{N}$ ), we have that with probability at least $1-\sum_{m=1}^{\infty} \delta /\left(6 m^{2}\right) \geq 1-\delta / 3$,

$$
\mathcal{P}\left(x: \hat{h}_{T}(x) \neq h_{T}^{*}(x)\right) \leq 2(K+c \sqrt{2 K}) \frac{d \log \left(m_{T}^{*} / d\right)+\log (1 / \delta)}{m_{T}^{*}}
$$

Let us denote

$$
\tilde{m}_{T}=\underset{m \in\{1, \ldots, T-1\}}{\operatorname{argmin}} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log (1 / \delta)}{m}
$$

Note that, for any $m^{\prime} \in\{1, \ldots, T-1\}$ and $\delta \in(0,1)$, if $\tilde{m}_{T} \geq m^{\prime}$, then

$$
\begin{aligned}
& \min _{m \in\{1, \ldots, T-1\}} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log (1 / \delta)}{m} \\
& \geq \min _{m \in\left\{m^{\prime}, \ldots, T-1\right\}} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}=\frac{1}{m^{\prime}} \sum_{i=T-m^{\prime}}^{T-1} \sum_{j=i+1}^{T} \Delta_{j},
\end{aligned}
$$

while if $\tilde{m}_{T} \leq m^{\prime}$, then

$$
\begin{aligned}
& \min _{m \in\{1, \ldots, T-1\}} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log (1 / \delta)}{m} \\
& \geq \min _{m \in\left\{1, \ldots, m^{\prime}\right\}} \frac{d \log (m / d)+\log (1 / \delta)}{m}=\frac{d \log \left(m^{\prime} / d\right)+\log (1 / \delta)}{m^{\prime}} .
\end{aligned}
$$

Either way, we have that

$$
\begin{align*}
& \min _{m \in\{1, \ldots, T-1\}} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log (1 / \delta)}{m} \\
& \geq \min \left\{\frac{d \log \left(m^{\prime} / d\right)+\log (1 / \delta)}{m^{\prime}}, \frac{1}{m^{\prime}} \sum_{i=T-m^{\prime}}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}\right\} . \tag{3}
\end{align*}
$$

For any $m \in\{1, \ldots, T-1\}$, applying Bernstein's inequality (see BLM13, equation 2.10) to the random variables $\mathbb{1}\left[h_{T}^{*}\left(X_{i}\right) \neq Y_{i}\right] / d, i \in\{T-m, \ldots, T-1\}$, and again to the random variables $-\mathbb{1}\left[h_{T}^{*}\left(X_{i}\right) \neq Y_{i}\right] / d, i \in\{T-m, \ldots, T-1\}$, together with a union bound, we obtain that, for any $\delta \in(0,1)$, with probability
at least $1-\delta /\left(3 m^{2}\right)$,

$$
\begin{align*}
& \frac{1}{m} \sum_{i=T-m}^{T-1} \mathcal{P}\left(x: h_{T}^{*}(x) \neq h_{i}^{*}(x)\right) \\
& \quad-\sqrt{\left(\frac{1}{m} \sum_{i=T-m}^{T-1} \mathcal{P}\left(x: h_{T}^{*}(x) \neq h_{i}^{*}(x)\right)\right) \frac{2 \ln \left(3 m^{2} / \delta\right)}{m}} \\
& <\frac{1}{m} \sum_{i=T-m}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{i}\right) \neq Y_{i}\right] \\
& <\frac{1}{m} \sum_{i=T-m}^{T-1} \mathcal{P}\left(x: h_{T}^{*}(x) \neq h_{i}^{*}(x)\right) \\
& \quad+\max \left\{\begin{array}{l}
\sqrt{\left(\frac{1}{m} \sum_{i=T-m}^{T-1}\right.}\left(\frac{4 / 3) \ln \left(3 m^{2} / \delta\right)}{m}\right.
\end{array}\right. \tag{4}
\end{align*}
$$

The left inequality implies that

$$
\frac{1}{m} \sum_{i=T-m}^{T-1} \mathcal{P}\left(x: h_{T}^{*}(x) \neq h_{i}^{*}(x)\right) \leq \max \left\{\frac{2}{m} \sum_{i=T-m}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{i}\right) \neq Y_{i}\right], \frac{8 \ln \left(3 m^{2} / \delta\right)}{m}\right\} .
$$

Plugging this into the right inequality in (4), we obtain that

$$
\begin{aligned}
& \frac{1}{m} \sum_{i=T-m}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{i}\right) \neq Y_{i}\right]<\frac{1}{m} \sum_{i=T-m}^{T-1} \mathcal{P}\left(x: h_{T}^{*}(x) \neq h_{i}^{*}(x)\right) \\
& \quad+\max \left\{\sqrt{\left(\frac{1}{m} \sum_{i=T-m}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{i}\right) \neq Y_{i}\right]\right) \frac{8 \ln \left(3 m^{2} / \delta\right)}{m}}, \frac{\sqrt{32} \ln \left(3 m^{2} / \delta\right)}{m}\right\} .
\end{aligned}
$$

By a union bound, this holds simultaneously for all $m \in\{1, \ldots, T-1\}$ with probability at least $1-\sum_{m=1}^{T-1} \delta /\left(3 m^{2}\right)>1-(2 / 3) \delta$. Note that, on this event, we obtain

$$
\begin{aligned}
& \frac{1}{m} \sum_{i=T-m}^{T-1} \mathcal{P}\left(x: h_{T}^{*}(x) \neq h_{i}^{*}(x)\right)>\frac{1}{m} \sum_{i=T-m}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{i}\right) \neq Y_{i}\right] \\
& \quad-\max \left\{\sqrt{\left(\frac{1}{m} \sum_{i=T-m}^{T-1} \mathbb{1}\left[h_{T}^{*}\left(X_{i}\right) \neq Y_{i}\right]\right) \frac{8 \ln \left(3 m^{2} / \delta\right)}{m}}, \frac{\sqrt{32} \ln \left(3 m^{2} / \delta\right)}{m}\right\} .
\end{aligned}
$$

In particular, taking $m=m_{T}^{*}$, and invoking maximality of $m_{T}^{*}$, if $m_{T}^{*}<T-1$, the right hand side is at least

$$
(K-6 c \sqrt{K}) \frac{d \log \left(m_{T}^{*} / d\right)+\log (1 / \delta)}{m_{T}^{*}}
$$

Since $\frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j} \geq \frac{1}{m} \sum_{i=T-m}^{T-1} \mathcal{P}\left(x: h_{T}^{*}(x) \neq h_{i}^{*}(x)\right)$, taking $K=145 c^{2}$, we have that with probability at least $1-\delta$, if $m_{T}^{*}<T-1$, then

$$
\begin{aligned}
& 10(K+c \sqrt{2 K}) \min _{m \in\{1, \ldots, T-1\}} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log (1 / \delta)}{m} \\
& \geq 10(K+c \sqrt{2 K}) \min \left\{\frac{d \log \left(m_{T}^{*} / d\right)+\log (1 / \delta)}{m_{T}^{*}}, \frac{1}{m_{T}^{*}} \sum_{i=T-m_{T}^{*}}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}\right\} \\
& \geq 10(K+c \sqrt{2 K}) \frac{d \log \left(m_{T}^{*} / d\right)+\log (1 / \delta)}{m_{T}^{*}} \\
& \geq \mathcal{P}\left(x: \hat{h}_{T}(x) \neq h_{T}^{*}(x)\right)
\end{aligned}
$$

Furthermore, if $m_{T}^{*}=T-1$, then we trivially have (on the same $1-\delta$ probability event as above)

$$
\begin{aligned}
& 10(K+c \sqrt{2 K}) \min _{m \in\{1, \ldots, T-1\}} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log (1 / \delta)}{m} \\
& \geq 10(K+c \sqrt{2 K}) \min _{m \in\{1, \ldots, T-1\}} \frac{d \log (m / d)+\log (1 / \delta)}{m} \\
& =10(K+c \sqrt{2 K}) \frac{d \log ((T-1) / d)+\log (1 / \delta)}{T-1} \\
& =10(K+c \sqrt{2 K}) \frac{d \log \left(m_{T}^{*} / d\right)+\log (1 / \delta)}{m_{T}^{*}} \geq \mathcal{P}\left(x: \hat{h}_{T}(x) \neq h_{T}^{*}(x)\right) .
\end{aligned}
$$

### 4.2 Conditions Guaranteeing a Sublinear Number of Mistakes

One immediate implication of Theorem 1 is that, if the sum of $\Delta_{t}$ values grows sublinearly, then there exists an algorithm achieving an expected number of mistakes growing sublinearly in the number of predictions. Formally, we have the following corollary.

Corollary 1. If $\sum_{t=1}^{T} \Delta_{t}=o(T)$, then there exists an algorithm $\mathcal{A}$ such that, for every $\mathcal{P}$ and every choice of $\mathbf{h}^{*} \in S_{\boldsymbol{\Delta}}$,

$$
\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right]\right]=o(T)
$$

Proof. For every $T \in \mathbb{N}$ with $T \geq 2$, let

$$
\tilde{m}_{T}=\underset{1 \leq m \leq T-1}{\operatorname{argmin}} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log \left(1 / \delta_{T}\right)}{m}
$$

and define $\delta_{T}=\frac{1}{\bar{m}_{T}}$. Then consider running the algorithm $\mathcal{A}$ from Theorem 1 , except that in choosing $\hat{m}_{T}$ and $\hat{h}_{T}$ for each $T$, we use the above value $\delta_{T}$ in place of $\delta$. Then Theorem $\mathbb{1}$ implies that, for each $T$, with probability at least $1-\delta_{T}$,

$$
\operatorname{er}_{T}\left(\hat{h}_{T}\right) \leq O\left(\min _{1 \leq m \leq T-1} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log \left(1 / \delta_{T}\right)}{m}\right) .
$$

Since $\operatorname{er}_{T}\left(\hat{h}_{T}\right) \leq 1$, this implies that

$$
\begin{aligned}
& \mathbb{P}\left(\hat{Y}_{T} \neq Y_{T}\right)=\mathbb{E}\left[\operatorname{er}_{T}\left(\hat{h}_{T}\right)\right] \\
& \leq O\left(\min _{1 \leq m \leq T-1} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log \left(1 / \delta_{T}\right)}{m}\right)+\delta_{T} \\
& =O\left(\min _{1 \leq m \leq T-1} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)+\log (m)}{m}\right),
\end{aligned}
$$

and since $x \mapsto x \log (m / x)$ is nondecreasing for $x \geq 1, \log (m) \leq d \log (m / d)$, so that this last expression is

$$
O\left(\min _{1 \leq m \leq T-1} \frac{1}{m} \sum_{i=T-m}^{T-1} \sum_{j=i+1}^{T} \Delta_{j}+\frac{d \log (m / d)}{m}\right) .
$$

Now note that, for any $t \in \mathbb{N}$ and $m \in\{1, \ldots, t-1\}$,

$$
\begin{equation*}
\frac{1}{m} \sum_{s=t-m}^{t-1} \sum_{r=s+1}^{t} \Delta_{r} \leq \frac{1}{m} \sum_{s=t-m}^{t-1} \sum_{r=t-m+1}^{t} \Delta_{r}=\sum_{r=t-m+1}^{t} \Delta_{s} . \tag{5}
\end{equation*}
$$

Let $\beta_{t}(m)=\max \left\{\sum_{r=t-m+1}^{t} \Delta_{r}, \frac{d \log (m / d)}{m}\right\}$, and note that $\sum_{r=t-m+1}^{t} \Delta_{r}+$ $\frac{d \log (m / d)}{m} \leq 2 \beta_{t}(m)$. Thus, combining the above with (5), linearity of expectations, and the fact that the probability of a mistake on a given round is at most 1 , we obtain

$$
\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right]\right]=O\left(\sum_{t=1}^{T} \min _{m \in\{1, \ldots, t-1\}} \beta_{t}(m) \wedge 1\right) .
$$

Fixing any $M \in \mathbb{N}$, we have that for any $T>M$,

$$
\begin{aligned}
& \sum_{t=1}^{T} \min _{m \in\{1, \ldots, t-1\}} \beta_{t}(m) \wedge 1 \leq M+\sum_{t=M+1}^{T} \beta_{t}(M) \wedge 1 \\
& \leq M+\sum_{t=M+1}^{T} \mathbb{1}\left[\frac{d \log (M / d)}{M} \geq \sum_{r=t-M+1}^{t} \Delta_{r}\right] \frac{d \log (M / d)}{M} \\
& \quad+\sum_{t=M+1}^{T} \mathbb{1}\left[\sum_{r=t-M+1}^{t} \Delta_{r}>\frac{d \log (M / d)}{M}\right] \\
& \leq M+\frac{d \log (M / d)}{M} T+\sum_{t=M+1}^{T} \frac{M}{d \log (M / d)} \sum_{r=t-M+1}^{t} \Delta_{r} \\
& =\frac{d \log (M / d)}{M} T+g_{M}(T)
\end{aligned}
$$

where $g_{M}$ is a function satisfying $g_{M}(T)=o(T)$ (holding $M$ fixed). Since this is true of any $M \in \mathbb{N}$, we have that

$$
\begin{aligned}
\lim _{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^{T} \min _{m \in\{1, \ldots, t-1\}} \beta_{t}(m) \wedge 1 & \leq \lim _{M \rightarrow \infty} \lim _{T \rightarrow \infty} \frac{d \log (M / d)}{M}+\frac{g_{M}(T)}{T} \\
& =\lim _{M \rightarrow \infty} \frac{d \log (M / d)}{M}=0
\end{aligned}
$$

so that $\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right]\right]=o(T)$, as claimed.
For many concept spaces of interest, the condition $\sum_{t=1}^{T} \Delta_{t}=o(T)$ in Corollary 1 is also a necessary condition for any algorithm to guarantee a sublinear number of mistakes. For simplicity, we will establish this for the class of homogeneous linear separators on $\mathbb{R}^{2}$, with $\mathcal{P}$ the uniform distribution on the unit circle, in the following theorem. This can easily be extended to many other spaces, including higher-dimensional linear separators or axis-aligned rectangles in $\mathbb{R}^{k}$, by embedding an analogous setup into those spaces.

Theorem 2. If $\mathcal{X}=\left\{x \in \mathbb{R}^{2}:\|x\|=1\right\}, \mathcal{P}$ is $\operatorname{Uniform}(\mathcal{X})$, and $\mathbb{C}=\{x \mapsto$ $\left.2 \mathbb{1}[w \cdot x \geq 0]-1: w \in \mathbb{R}^{2},\|w\|=1\right\}$ is the class of homogeneous linear separators, then for any sequence $\boldsymbol{\Delta}$ in $[0,1]$, there exists an algorithm $\mathcal{A}$ such that $\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right]\right]=o(T)$ for every choice of $\mathbf{h}^{*} \in S_{\boldsymbol{\Delta}}$ if and only if $\sum_{t=1}^{T} \Delta_{t}=o(T)$.

Proof. The "if" part follows immediately from Corollary 1. For the "only if" part, suppose $\boldsymbol{\Delta}$ is such that $\sum_{t=1}^{T} \Delta_{t} \neq o(T)$. It suffices to argue that for any algorithm $\mathcal{A}$, there exists a choice of $\mathbf{h}^{*} \in S_{\boldsymbol{\Delta}}$ for which $\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right]\right] \neq$ $o(T)$. Toward this end, fix any algorithm $\mathcal{A}$. We proceed by the probabilistic
method, constructing a random sequence $\mathbf{h}^{*} \in S_{\boldsymbol{\Delta}}$. Let $B_{1}, B_{2}, \ldots$ be independent Bernoulli $(1 / 2)$ random variables (also independent from the unlabeled data $\left.X_{1}, X_{2}, \ldots\right)$. We define the sequence $\mathbf{h}^{*}$ inductively. For simplicity, we will represent each classifier in polar coordinates, writing $h_{\phi}($ for $\phi \in \mathbb{R})$ to denote the classifier that, for $x=\left(x_{1}, x_{2}\right)$, classifies $x$ as $h_{\phi}(x)=2 \mathbb{1}\left[x_{1} \cos (\phi)+x_{2} \sin (\phi) \geq\right.$ $0]-1$; note that $h_{\phi}=h_{\phi+2 \pi}$ for every $\phi \in \mathbb{R}$. As a base case, start by defining a function $h_{0}^{*}=h_{0}$, and letting $\phi_{0}=0$. Now for any $t \in \mathbb{N}$, supposing $h_{t-1}^{*}$ is already defined to be $h_{\phi_{t-1}}$, we define $\phi_{t}=\phi_{t-1}+\min \left\{\Delta_{t}, 1 / 2\right\} \pi B_{t}$, and $h_{t}^{*}=h_{\phi_{t}}$. Note that $\mathcal{P}\left(x: h_{t}^{*}(x) \neq h_{t-1}^{*}(x)\right)=\min \left\{\Delta_{t}, 1 / 2\right\}$ for every $t \in \mathbb{N}$, so that this inductively defines a (random) choice of $\mathbf{h}^{*} \in S_{\boldsymbol{\Delta}}$.

For each $t \in \mathbb{N}$, let $Y_{t}=h_{t}^{*}\left(X_{t}\right)$. Now fix any algorithm $\mathcal{A}$, and consider the sequence $\hat{Y}_{t}$ of predictions the algorithm makes for points $X_{t}$, when the target sequence $\mathbf{h}^{*}$ is chosen as above. Then note that, for any $t \in \mathbb{N}$, since $\hat{Y}_{t}$ and $B_{t}$ are independent,

$$
\begin{aligned}
\mathbb{P}\left(\hat{Y}_{t} \neq Y_{t}\right) & \geq \mathbb{E}\left[\mathbb{P}\left(\hat{Y}_{t} \neq Y_{t} \mid \hat{Y}_{t}, \phi_{t-1}\right)\right] \\
& \geq \mathbb{E}\left[\frac{1}{2} \mathbb{P}\left(h_{\phi_{t-1}+\min \left\{\Delta_{t}, 1 / 2\right\} \pi}\left(X_{t}\right) \neq h_{\phi_{t-1}-\min \left\{\Delta_{t}, 1 / 2\right\} \pi}\left(X_{t}\right) \mid \phi_{t-1}\right)\right] .
\end{aligned}
$$

Furthermore, since $\min \left\{\Delta_{t}, 1 / 2\right\} \pi \leq \pi / 2$, the regions $\left\{x: h_{\phi_{t-1}+\min \left\{\Delta_{t}, 1 / 2\right\} \pi}(x) \neq\right.$ $\left.h_{\phi_{t-1}}(x)\right\}$ and $\left\{x: h_{\phi_{t-1}-\min \left\{\Delta_{t}, 1 / 2\right\} \pi}(x) \neq h_{\phi_{t-1}}(x)\right\}$ have zero-probability overlap (indeed, are disjoint if $\Delta_{t}<1 / 2$ ), the above equals $\min \left\{\Delta_{t}, 1 / 2\right\}$.

By Fatou's lemma, linearity of expectations, and the law of total expectation, we have that

$$
\begin{aligned}
\mathbb{E}\left[\limsup _{T \rightarrow \infty} \frac{1}{T} \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right] \mid \mathbf{h}^{*}\right]\right] & \geq \limsup _{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^{T} \mathbb{P}\left(\hat{Y}_{t} \neq Y_{t}\right) \\
& \geq \limsup _{T \rightarrow \infty} \frac{1}{T} \sum_{t=1}^{T} \min \left\{\Delta_{t}, 1 / 2\right\}
\end{aligned}
$$

Since $\sum_{t=1}^{T} \Delta_{t} \neq o(T)$, the rightmost expression is strictly greater than zero. Thus, it must be that, with probility strictly greater than 0 ,

$$
\limsup _{T \rightarrow \infty} \frac{1}{T} \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right] \mid \mathbf{h}^{*}\right]>0
$$

In particular, this implies that there exists a (nonrandom) choice of the sequence $\mathbf{h}^{*} \in S_{\Delta}$ for which $\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right]\right] \neq o(T)$. Since this holds for any choice of the algorithm $\mathcal{A}$, this completes the proof.

## 5 Polynomial-Time Algorithms for Linear Separators

In this section, we suppose $\Delta_{t}=\Delta$ for every $t \in \mathbb{N}$, for a fixed constant $\Delta>0$, and we consider the special case of learning homogeneous linear separators in
$\mathbb{R}^{k}$ under a uniform distribution on the origin-centered unit sphere. In this case, the analysis of HL94 mentioned in Section 3.3 implies that it is possible to achieve a bound on the error rate that is $\tilde{O}(d \sqrt{\Delta})$, using an algorithm that runs in time poly $(d, 1 / \Delta, \log (1 / \delta)$ ) (and independent of $t$ ) for each prediction. This also implies that it is possible to achieve expected number of mistakes among $T$ predictions that is $\tilde{O}(d \sqrt{\Delta}) \times T$. CMEDV10] ${ }^{6}$ have since proven that a variant of the Perceptron algorithm is capable of achieving an expected number of mistakes $\tilde{O}\left((d \Delta)^{1 / 4}\right) \times T$.

Below, we improve on this result by showing that there exists an efficient algorithm that achieves a bound on the error rate that is $\tilde{O}(\sqrt{d \Delta})$, as was possible with the inefficient algorithm of HL94 Lon99 mentioned in Section 3.1. This leads to a bound on the expected number of mistakes that is $\tilde{O}(\sqrt{d \Delta}) \times T$. Furthermore, our approach also allows us to present the method as an active learning algorithm, and to bound the expected number of queries, as a function of the number of samples $T$, by $\tilde{O}(\sqrt{d \Delta}) \times T$. The technique is based on a modification of the algorithm of [HL94, replacing an empirical risk minimization step with (a modification of) the computationally-efficient algorithm of ABL13].

Formally, define the class of homogeneous linear separators as the set of classifiers $h_{w}: \mathbb{R}^{d} \rightarrow\{-1,+1\}$, for $w \in \mathbb{R}^{d}$ with $\|w\|=1$, such that $h_{w}(x)=$ $\operatorname{sign}(w \cdot x)$ for every $x \in \mathbb{R}^{d}$.

### 5.1 An Improved Guarantee for a Polynomial-Time Algorithm

We have the following result.
Theorem 3. When $\mathbb{C}$ is the space of homogeneous linear separators (with $d \geq 4$ ) and $\mathcal{P}$ is the uniform distribution on the surface of the origin-centered unit sphere in $\mathbb{R}^{d}$, for any fixed $\Delta>0$, for any $\delta \in(0,1 / e)$, there is an algorithm that runs in time $\operatorname{poly}(d, 1 / \Delta, \log (1 / \delta))$ for each time $t$, such that for any $\mathbf{h}^{*} \in S_{\Delta}$, for every sufficiently large $t \in \mathbb{N}$, with probability at least $1-\delta$,

$$
\operatorname{er}_{t}\left(\hat{h}_{t}\right)=O\left(\sqrt{\Delta d \log \left(\frac{1}{\delta}\right)}\right)
$$

Also, running this algorithm with $\delta=\sqrt{\Delta d} \wedge 1 / e$, the expected number of mistakes among the first $T$ instances is $O\left(\sqrt{\Delta d \log \left(\frac{1}{\Delta d}\right)} T\right)$. Furthermore, the algorithm can be run as an active learning algorithm, in which case, for this choice of $\delta$, the expected number of labels requested by the algorithm among the first $T$ instances is $O\left(\sqrt{\Delta d} \log ^{3 / 2}\left(\frac{1}{\Delta d}\right) T\right)$.

[^2]We first state the algorithm used to obtain this result. It is primarily based on a margin-based learning strategy of ABL13, combined with an initialization step based on a modified Perceptron rule from DKM09|CMEDV10. For $\tau>0$ and $x \in \mathbb{R}$, define $\ell_{\tau}(x)=\max \left\{0,1-\frac{x}{\tau}\right\}$. Consider the following algorithm and subroutine; parameters $\delta_{k}, m_{k}, \tau_{k}, r_{k}, b_{k}, \alpha$, and $\kappa$ will all be specified in the context of the proof; we suppose $M=\sum_{k=0}^{\left\lceil\log _{2}(1 / \alpha)\right\rceil} m_{k}$.

```
Algorithm: DriftingHalfspaces
0 . Let \(\tilde{h}_{0}\) be an arbitrary classifier in \(\mathbb{C}\)
1. For \(i=1,2, \ldots\)
2. \(\quad \tilde{h}_{i} \leftarrow \operatorname{ABL}\left(M(i-1), \tilde{h}_{i-1}\right)\)
```

Subroutine: ModPerceptron $(t, \tilde{h})$
0 . Let $w_{t}$ be any element of $\mathbb{R}^{d}$ with $\left\|w_{t}\right\|=1$

1. For $m=t+1, t+2, \ldots, t+m_{0}$
2. Choose $\hat{h}_{m}=\tilde{h}$ (i.e., predict $\hat{Y}_{m}=\tilde{h}\left(X_{m}\right)$ as the prediction for $\left.Y_{m}\right)$
3. Request the label $Y_{m}$
4. If $h_{w_{m-1}}\left(X_{m}\right) \neq Y_{m}$

$$
w_{m} \leftarrow w_{m-1}-2\left(w_{m-1} \cdot X_{m}\right) X_{m}
$$

6. Else $w_{m} \leftarrow w_{m-1}$
7. Return $w_{t+m_{0}}$

Subroutine: $\operatorname{ABL}(t, \tilde{h})$
0 . Let $w_{0}$ be the return value of $\operatorname{ModPerceptron}(t, \tilde{h})$

1. For $k=1,2, \ldots,\left\lceil\log _{2}(1 / \alpha)\right\rceil$
2. $W_{k} \leftarrow\{ \}$
3. For $s=t+\sum_{j=0}^{k-1} m_{j}+1, \ldots, t+\sum_{j=0}^{k} m_{j}$
4. Choose $\hat{h}_{s}=\tilde{h}$ (i.e., predict $\hat{Y}_{s}=\tilde{h}\left(X_{s}\right)$ as the prediction for $\left.Y_{s}\right)$
5. If $\left|w_{k-1} \cdot X_{s}\right| \leq b_{k-1}$, Request label $Y_{s}$ and let $W_{k} \leftarrow W_{k} \cup\left\{\left(X_{s}, Y_{s}\right)\right\}$
6. Find $v_{k} \in \mathbb{R}^{d}$ with $\left\|v_{k}-w_{k-1}\right\| \leq r_{k}, 0<\left\|v_{k}\right\| \leq 1$, and

$$
\sum_{(x, y) \in W_{k}} \ell_{\tau_{k}}\left(y\left(v_{k} \cdot x\right)\right) \leq \inf _{v:\left\|v-w_{k-1}\right\| \leq r_{k}} \sum_{(x, y) \in W_{k}} \ell_{\tau_{k}}(y(v \cdot x))+\kappa\left|W_{k}\right|
$$

7. Let $w_{k}=\frac{1}{\left\|v_{k}\right\|} v_{k}$
8. Return $h_{w_{\left\lceil\log _{2}(1 / \alpha)\right\rceil-1}}$

Before stating the proof, we have a few additional lemmas that will be needed. The following result for ModPerceptron was proven by CMEDV10.

Lemma 2. Suppose $\Delta<\frac{1}{512}$. Consider the values $w_{m}$ obtained during the execution of $\operatorname{ModPerceptron}(t, \tilde{h}) . \forall m \in\left\{t+1, \ldots, t+m_{0}\right\}, \mathcal{P}\left(x: h_{w_{m}}(x) \neq\right.$ $\left.h_{m}^{*}(x)\right) \leq \mathcal{P}\left(x: h_{w_{m-1}}(x) \neq h_{m}^{*}(x)\right)$. Furthermore, letting $c_{1}=\frac{\pi^{2}}{d \cdot 400 \cdot 2^{15}}$, if $\mathcal{P}\left(x: h_{w_{m-1}}(x) \neq h_{m}^{*}(x)\right) \geq 1 / 32$, then with probability at least $1 / 64, \mathcal{P}(x:$ $\left.h_{w_{m}}(x) \neq h_{m}^{*}(x)\right) \leq\left(1-c_{1}\right) \mathcal{P}\left(x: h_{w_{m-1}}(x) \neq h_{m}^{*}(x)\right)$.

This implies the following.

Lemma 3. Suppose $\Delta \leq \frac{\pi^{2}}{400 \cdot 2^{27}(d+\ln (4 / \delta))}$. For $m_{0}=\max \left\{\left\lceil 128\left(1 / c_{1}\right) \ln (32)\right\rceil\right.$, $\left.\left\lceil 512 \ln \left(\frac{4}{\delta}\right)\right\rceil\right\}$, with probability at least $1-\delta / 4$, ModPerceptron $(t, \tilde{h})$ returns a vector $w$ with $\mathcal{P}\left(x: h_{w}(x) \neq h_{t+m_{0}+1}^{*}(x)\right) \leq 1 / 16$.

Proof. By Lemma 2 and a union bound, in general we have

$$
\begin{equation*}
\mathcal{P}\left(x: h_{w_{m}}(x) \neq h_{m+1}^{*}(x)\right) \leq \mathcal{P}\left(x: h_{w_{m-1}}(x) \neq h_{m}^{*}(x)\right)+\Delta . \tag{6}
\end{equation*}
$$

Furthermore, if $\mathcal{P}\left(x: h_{w_{m-1}}(x) \neq h_{m}^{*}(x)\right) \geq 1 / 32$, then wth probability at least 1/64,

$$
\begin{equation*}
\mathcal{P}\left(x: h_{w_{m}}(x) \neq h_{m+1}^{*}(x)\right) \leq\left(1-c_{1}\right) \mathcal{P}\left(x: h_{w_{m-1}}(x) \neq h_{m}^{*}(x)\right)+\Delta \tag{7}
\end{equation*}
$$

In particular, this implies that the number $N$ of values $m \in\left\{t+1, \ldots, t+m_{0}\right\}$ with either $\mathcal{P}\left(x: h_{w_{m-1}}(x) \neq h_{m}^{*}(x)\right)<1 / 32$ or $\mathcal{P}\left(x: h_{w_{m}}(x) \neq h_{m+1}^{*}(x)\right) \leq$ $\left(1-c_{1}\right) \mathcal{P}\left(x: h_{w_{m-1}}(x) \neq h_{m}^{*}(x)\right)+\Delta$ is lower-bounded by a $\operatorname{Binomial}(m, 1 / 64)$ random variable. Thus, a Chernoff bound implies that with probability at least $1-\exp \left\{-m_{0} / 512\right\} \geq 1-\delta / 4$, we have $N \geq m_{0} / 128$. Suppose this happens.

Since $\Delta m_{0} \leq 1 / 32$, if any $m \in\left\{t+1, \ldots, t+m_{0}\right\}$ has $\mathcal{P}\left(x: h_{w_{m-1}}(x) \neq\right.$ $\left.h_{m}^{*}(x)\right)<1 / 32$, then inductively applying (6) implies that $\mathcal{P}\left(x: h_{w_{t+m_{0}}}(x) \neq\right.$ $\left.h_{t+m_{0}+1}^{*}(x)\right) \leq 1 / 32+\Delta m_{0} \leq 1 / 16$. On the other hand, if all $m \in\{t+1, \ldots, t+$ $\left.m_{0}\right\}$ have $\mathcal{P}\left(x: h_{w_{m-1}}(x) \neq h_{m}^{*}(x)\right) \geq 1 / 32$, then in particular we have $N$ values of $m \in\left\{t+1, \ldots, t+m_{0}\right\}$ satisfying (7). Combining this fact with (6) inductively, we have that

$$
\begin{aligned}
& \mathcal{P}\left(x: h_{w_{t+m_{0}}}(x) \neq h_{t+m_{0}+1}^{*}(x)\right) \leq\left(1-c_{1}\right)^{N} \mathcal{P}\left(x: h_{w_{t}}(x) \neq h_{t+1}^{*}(x)\right)+\Delta m_{0} \\
& \quad \leq\left(1-c_{1}\right)^{\left(1 / c_{1}\right) \ln (32)} \mathcal{P}\left(x: h_{w_{t}}(x) \neq h_{t+1}^{*}(x)\right)+\Delta m_{0} \leq \frac{1}{32}+\Delta m_{0} \leq \frac{1}{16}
\end{aligned}
$$

Next, we consider the execution of $\operatorname{ABL}(t, \tilde{h})$, and let the sets $W_{k}$ be as in that execution. We will denote by $w^{*}$ the weight vector with $\left\|w^{*}\right\|=1$ such that $h_{t+m_{0}+1}^{*}=h_{w^{*}}$. Also denote by $M_{1}=M-m_{0}$.

The proof relies on a few results proven in the work of ABL13], which we summarize in the following lemmas. Although the results were proven in a slightly different setting in that work (namely, agnostic learning under a fixed joint distribution), one can easily verify that their proofs remain valid in our present context as well.

Lemma 4. ABL13] Fix any $k \in\left\{1, \ldots,\left\lceil\log _{2}(1 / \alpha)\right\rceil\right\}$. For a universal constant $c_{7}>0$, suppose $b_{k-1}=c_{7} 2^{1-k} / \sqrt{d}$, and let $z_{k}=\sqrt{r_{k}^{2} /(d-1)+b_{k-1}^{2}}$. For a universal constant $c_{1}>0$, if $\left\|w^{*}-w_{k-1}\right\| \leq r_{k}$,

$$
\begin{array}{r}
\left|\mathbb { E } \left[\sum_{(x, y) \in W_{k}} \ell_{\tau_{k}}\left(\left|w^{*} \cdot x\right|\right)\left|w_{k-1},\left|W_{k}\right|\right]-\mathbb{E}\left[\sum_{(x, y) \in W_{k}} \ell_{\tau_{k}}\left(y\left(w^{*} \cdot x\right)\right)\left|w_{k-1},\left|W_{k}\right|\right] \mid\right.\right.\right. \\
\leq c_{1}\left|W_{k}\right| \sqrt{2^{k} \Delta M_{1}} \frac{z_{k}}{\tau_{k}}
\end{array}
$$

Lemma 5. [BL13] For any $c>0$, there is a constant $c^{\prime}>0$ depending only on $c$ (i.e., not depending on $d$ ) such that, for any $u, v \in \mathbb{R}^{d}$ with $\|u\|=\|v\|=1$, letting $\sigma=\mathcal{P}\left(x: h_{u}(x) \neq h_{v}(x)\right)$, if $\sigma<1 / 2$, then

$$
\mathcal{P}\left(x: h_{u}(x) \neq h_{v}(x) \text { and }|v \cdot x| \geq c^{\prime} \frac{\sigma}{\sqrt{d}}\right) \leq c \sigma .
$$

The following is a well-known lemma concerning concentration around the equator for the uniform distribution (see e.g., DKM09|BBZ07|ABL13]); for instance, it easily follows from the formulas for the area in a spherical cap derived by Li11.

Lemma 6. For any constant $C>0$, there are constants $c_{2}, c_{3}>0$ depending only on $C$ (i.e., independent of $d$ ) such that, for any $w \in \mathbb{R}^{d}$ with $\|w\|=1$, $\forall \gamma \in[0, C / \sqrt{d}]$,

$$
c_{2} \gamma \sqrt{d} \leq \mathcal{P}(x:|w \cdot x| \leq \gamma) \leq c_{3} \gamma \sqrt{d}
$$

Based on this lemma, ABL13 prove the following.
Lemma 7. ABL13] For $X \sim \mathcal{P}$, for any $w \in \mathbb{R}^{d}$ with $\|w\|=1$, for any $C>0$ and $\tau, b \in[0, C / \sqrt{d}]$, for $c_{2}, c_{3}$ as in Lemma (6).

$$
\mathbb{E}\left[\ell_{\tau}\left(\left|w^{*} \cdot X\right|\right)| | w \cdot X \mid \leq b\right] \leq \frac{c_{3} \tau}{c_{2} b}
$$

The following is a slightly stronger version of a result of ABL13 (specifically, the size of $m_{k}$, and consequently the bound on $\left|W_{k}\right|$, are both improved by a factor of $d$ compared to the original result).

Lemma 8. Fix any $\delta \in(0,1 / e)$. For universal constants $c_{4}, c_{5}, c_{6}, c_{7}, c_{8}, c_{9}, c_{10} \in$ $(0, \infty)$, for an appropriate choice of $\kappa \in(0,1)$ (a universal constant), if $\alpha=$ $c_{9} \sqrt{\Delta d \log \left(\frac{1}{\kappa \delta}\right)}$, for every $k \in\left\{1, \ldots,\left\lceil\log _{2}(1 / \alpha)\right\rceil\right\}$, if $b_{k-1}=c_{7} 2^{1-k} / \sqrt{d}$, $\tau_{k}=$ $c_{8} 2^{-k} / \sqrt{d}, r_{k}=c_{10} 2^{-k}, \delta_{k}=\delta /\left(\left\lceil\log _{2}(4 / \alpha)\right\rceil-k\right)^{2}$, and $m_{k}=\left\lceil c_{5} \frac{2^{k}}{\kappa^{2}} d \log \left(\frac{1}{\kappa \delta_{k}}\right)\right\rceil$, and if $\mathcal{P}\left(x: h_{w_{k-1}}(x) \neq h_{w^{*}}(x)\right) \leq 2^{-k-3}$, then with probability at least $1-$ $(4 / 3) \delta_{k},\left|W_{k}\right| \leq c_{6} \frac{1}{\kappa^{2}} d \log \left(\frac{1}{\kappa \delta_{k}}\right)$ and $\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x)\right) \leq 2^{-k-4}$.

Proof. By Lemma6, and a Chernoff and union bound, for an appropriately large choice of $c_{5}$ and any $c_{7}>0$, letting $c_{2}, c_{3}$ be as in Lemma (with $C=c_{7} \vee\left(c_{8} / 2\right)$ ), with probability at least $1-\delta_{k} / 3$,

$$
\begin{equation*}
c_{2} c_{7} 2^{-k} m_{k} \leq\left|W_{k}\right| \leq 4 c_{3} c_{7} 2^{-k} m_{k} \tag{8}
\end{equation*}
$$

The claimed upper bound on $\left|W_{k}\right|$ follows from this second inequality.
Next note that, if $\mathcal{P}\left(x: h_{w_{k-1}}(x) \neq h_{w^{*}}(x)\right) \leq 2^{-k-3}$, then

$$
\max \left\{\ell_{\tau_{k}}\left(y\left(w^{*} \cdot x\right)\right): x \in \mathbb{R}^{d},\left|w_{k-1} \cdot x\right| \leq b_{k-1}, y \in\{-1,+1\}\right\} \leq c_{11} \sqrt{d}
$$

for some universal constant $c_{11}>0$. Furthermore, since $\mathcal{P}\left(x: h_{w_{k-1}}(x) \neq\right.$ $\left.h_{w^{*}}(x)\right) \leq 2^{-k-3}$, we know that the angle between $w_{k-1}$ and $w^{*}$ is at most $2^{-k-3} \pi$, so that

$$
\begin{aligned}
&\left\|w_{k-1}-w^{*}\right\|=\sqrt{2-2 w_{k-1} \cdot w^{*}} \leq \sqrt{2-2 \cos \left(2^{-k-3} \pi\right)} \\
& \leq \sqrt{2-2 \cos ^{2}\left(2^{-k-3} \pi\right)}=\sqrt{2} \sin \left(2^{-k-3} \pi\right) \leq 2^{-k-3} \pi \sqrt{2}
\end{aligned}
$$

For $c_{10}=\pi \sqrt{2} 2^{-3}$, this is $r_{k}$. By Hoeffding's inequality (under the conditional distribution given $\left|W_{k}\right|$ ), the law of total probability, Lemma 4 and linearity of conditional expectations, with probability at least $1-\delta_{k} / 3$, for $X \sim \mathcal{P}$,

$$
\begin{align*}
\sum_{(x, y) \in W_{k}} \ell_{\tau_{k}}\left(y\left(w^{*} \cdot x\right)\right) & \leq\left|W_{k}\right| \mathbb{E}\left[\ell_{\tau_{k}}\left(\left|w^{*} \cdot X\right|\right)\left|w_{k-1},\left|w_{k-1} \cdot X\right| \leq b_{k-1}\right]\right. \\
& +c_{1}\left|W_{k}\right| \sqrt{2^{k} \Delta M_{1}} \frac{z_{k}}{\tau_{k}}+\sqrt{\left|W_{k}\right|(1 / 2) c_{11}^{2} d \ln \left(3 / \delta_{k}\right)} \tag{9}
\end{align*}
$$

We bound each term on the right hand side separately. By Lemma 7 the first term is at most $\left|W_{k}\right| \frac{c_{3} \tau_{k}}{c_{2} b_{k-1}}=\left|W_{k}\right| \frac{c_{3} c_{8}}{2 c_{2} c_{7}}$. Next,

$$
\frac{z_{k}}{\tau_{k}}=\frac{\sqrt{c_{10}^{2} 2^{-2 k} /(d-1)+4 c_{7}^{2} 2^{-2 k} / d}}{c_{8} 2^{-k} / \sqrt{d}} \leq \frac{\sqrt{2 c_{10}^{2}+4 c_{7}^{2}}}{c_{8}}
$$

while $2^{k} \leq 2 / \alpha$ so that the second term is at most

$$
\sqrt{2} c_{1} \frac{\sqrt{2 c_{10}^{2}+4 c_{7}^{2}}}{c_{8}}\left|W_{k}\right| \sqrt{\frac{\Delta m}{\alpha}}
$$

Noting that

$$
\begin{equation*}
M_{1}=\sum_{k^{\prime}=1}^{\left\lceil\log _{2}(1 / \alpha)\right\rceil} m_{k^{\prime}} \leq \frac{32 c_{5}}{\kappa^{2}} \frac{1}{\alpha} d \log \left(\frac{1}{\kappa \delta}\right) \tag{10}
\end{equation*}
$$

we find that the second term on the right hand side of (9) is at most

$$
\sqrt{\frac{c_{5}}{c_{9}}} \frac{8 c_{1}}{\kappa} \frac{\sqrt{2 c_{10}^{2}+4 c_{7}^{2}}}{c_{8}}\left|W_{k}\right| \sqrt{\frac{\Delta d \log \left(\frac{1}{\kappa \delta}\right)}{\alpha^{2}}}=\frac{8 c_{1} \sqrt{c_{5}}}{\kappa} \frac{\sqrt{2 c_{10}^{2}+4 c_{7}^{2}}}{c_{8} c_{9}}\left|W_{k}\right|
$$

Finally, since $d \ln \left(3 / \delta_{k}\right) \leq 2 d \ln \left(1 / \delta_{k}\right) \leq \frac{2 \kappa^{2}}{c_{5}} 2^{-k} m_{k}$, and (8) implies $2^{-k} m_{k} \leq$ $\frac{1}{c_{2} c_{7}}\left|W_{k}\right|$, the third term on the right hand side of (9) is at most

$$
\left|W_{k}\right| \frac{c_{11} \kappa}{\sqrt{c_{2} c_{5} c_{7}}}
$$

Altogether, we have

$$
\sum_{(x, y) \in W_{k}} \ell_{\tau_{k}}\left(y\left(w^{*} \cdot x\right)\right) \leq\left|W_{k}\right|\left(\frac{c_{3} c_{8}}{2 c_{2} c_{7}}+\frac{8 c_{1} \sqrt{c_{5}}}{\kappa} \frac{\sqrt{2 c_{10}^{2}+4 c_{7}^{2}}}{c_{8} c_{9}}+\frac{c_{11} \kappa}{\sqrt{c_{2} c_{5} c_{7}}}\right)
$$

Taking $c_{9}=1 / \kappa^{3}$ and $c_{8}=\kappa$, this is at most

$$
\kappa\left|W_{k}\right|\left(\frac{c_{3}}{2 c_{2} c_{7}}+8 c_{1} \sqrt{c_{5}} \sqrt{2 c_{10}^{2}+4 c_{7}^{2}}+\frac{c_{11}}{\sqrt{c_{2} c_{5} c_{7}}}\right)
$$

Next, note that because $h_{w_{k}}(x) \neq y \Rightarrow \ell_{\tau_{k}}\left(y\left(v_{k} \cdot x\right)\right) \geq 1$, and because (as proven above) $\left\|w^{*}-w_{k-1}\right\| \leq r_{k}$,

$$
\left|W_{k}\right| \operatorname{er}_{W_{k}}\left(h_{w_{k}}\right) \leq \sum_{(x, y) \in W_{k}} \ell_{\tau_{k}}\left(y\left(v_{k} \cdot x\right)\right) \leq \sum_{(x, y) \in W_{k}} \ell_{\tau_{k}}\left(y\left(w^{*} \cdot x\right)\right)+\kappa\left|W_{k}\right|
$$

Combined with the above, we have

$$
\left|W_{k}\right| \operatorname{er}_{W_{k}}\left(h_{w_{k}}\right) \leq \kappa\left|W_{k}\right|\left(1+\frac{c_{3}}{2 c_{2} c_{7}}+8 c_{1} \sqrt{c_{5}} \sqrt{2 c_{10}^{2}+4 c_{7}^{2}}+\frac{c_{11}}{\sqrt{c_{2} c_{5} c_{7}}}\right)
$$

Let $c_{12}=1+\frac{c_{3}}{2 c_{2} c_{7}}+8 c_{1} \sqrt{c_{5}} \sqrt{2 c_{10}^{2}+4 c_{7}^{2}}+\frac{c_{11}}{\sqrt{c_{2} c_{5} c_{7}}}$. Furthermore,

$$
\begin{aligned}
\left|W_{k}\right| \operatorname{er}_{W_{k}}\left(h_{w_{k}}\right)= & \sum_{(x, y) \in W_{k}} \mathbb{1}\left[h_{w_{k}}(x) \neq y\right] \\
& \geq \sum_{(x, y) \in W_{k}} \mathbb{1}\left[h_{w_{k}}(x) \neq h_{w^{*}}(x)\right]-\sum_{(x, y) \in W_{k}} \mathbb{1}\left[h_{w^{*}}(x) \neq y\right] .
\end{aligned}
$$

For an appropriately large value of $c_{5}$, by a Chernoff bound, with probability at least $1-\delta_{k} / 3$,

In particular, this implies

$$
\sum_{(x, y) \in W_{k}} \mathbb{1}\left[h_{w^{*}}(x) \neq y\right] \leq 2 e \Delta M_{1} m_{k}+\log _{2}\left(3 / \delta_{k}\right)
$$

so that

$$
\sum_{(x, y) \in W_{k}} \mathbb{1}\left[h_{w_{k}}(x) \neq h_{w^{*}}(x)\right] \leq\left|W_{k}\right| \operatorname{er}_{W_{k}}\left(h_{w_{k}}\right)+2 e \Delta M_{1} m_{k}+\log _{2}\left(3 / \delta_{k}\right)
$$

Noting that (10) and (8) imply

$$
\begin{aligned}
\Delta M_{1} m_{k} & \leq \Delta \frac{32 c_{5}}{\kappa^{2}} \frac{d \log \left(\frac{1}{\kappa \delta}\right)}{c_{9} \sqrt{\Delta d \log \left(\frac{1}{\kappa \delta}\right)}} \frac{2^{k}}{c_{2} c_{7}}\left|W_{k}\right| \leq \frac{32 c_{5}}{c_{2} c_{7} c_{9} \kappa^{2}} \sqrt{\Delta d \log \left(\frac{1}{\kappa \delta}\right)} 2^{k}\left|W_{k}\right| \\
& =\frac{32 c_{5}}{c_{2} c_{7} c_{9}^{2} \kappa^{2}} \alpha 2^{k}\left|W_{k}\right|=\frac{32 c_{5} \kappa^{4}}{c_{2} c_{7}} \alpha 2^{k}\left|W_{k}\right| \leq \frac{32 c_{5} \kappa^{4}}{c_{2} c_{7}}\left|W_{k}\right|
\end{aligned}
$$

and (8) implies $\log _{2}\left(3 / \delta_{k}\right) \leq \frac{2 \kappa^{2}}{c_{2} c_{5} c_{7}}\left|W_{k}\right|$, altogether we have

$$
\begin{aligned}
\sum_{(x, y) \in W_{k}} \mathbb{1}\left[h_{w_{k}}(x) \neq h_{w^{*}}(x)\right] & \leq\left|W_{k}\right| \operatorname{er}_{W_{k}}\left(h_{w_{k}}\right)+\frac{64 e c_{5} \kappa^{4}}{c_{2} c_{7}}\left|W_{k}\right|+\frac{2 \kappa^{2}}{c_{2} c_{5} c_{7}}\left|W_{k}\right| \\
& \leq \kappa\left|W_{k}\right|\left(c_{12}+\frac{64 e c_{5} \kappa^{3}}{c_{2} c_{7}}+\frac{2 \kappa}{c_{2} c_{5} c_{7}}\right) .
\end{aligned}
$$

Letting $c_{13}=c_{12}+\frac{64 e c_{5}}{c_{2} c_{7}}+\frac{2}{c_{2} c_{5} c_{7}}$, and noting $\kappa \leq 1$, we have $\sum_{(x, y) \in W_{k}} \mathbb{1}\left[h_{w_{k}}(x) \neq\right.$ $\left.h_{w^{*}}(x)\right] \leq c_{13} \kappa\left|W_{k}\right|$.

Lemma 1 (applied under the conditional distribution given $\left|W_{k}\right|$ ) and the law of total probability imply that with probability at least $1-\delta_{k} / 3$,

$$
\begin{aligned}
& \left|W_{k}\right| \mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x)| | w_{k-1} \cdot x \mid \leq b_{k-1}\right) \\
& \quad \leq \sum_{(x, y) \in W_{k}} \mathbb{1}\left[h_{w_{k}}(x) \neq h_{w^{*}}(x)\right]+c_{14} \sqrt{\left|W_{k}\right|\left(d \log \left(\left|W_{k}\right| / d\right)+\log \left(1 / \delta_{k}\right)\right)}
\end{aligned}
$$

for a universal constant $c_{14}>0$. Combined with the above, and the fact that (8) implies $\log \left(1 / \delta_{k}\right) \leq \frac{\kappa^{2}}{c_{2} c_{5} c_{7}}\left|W_{k}\right|$ and

$$
\begin{aligned}
d \log \left(\left|W_{k}\right| / d\right) & \leq d \log \left(\frac{8 c_{3} c_{5} c_{7} \log \left(\frac{1}{\kappa \delta_{k}}\right)}{\kappa^{2}}\right) \\
& \leq d \log \left(\frac{8 c_{3} c_{5} c_{7}}{\kappa^{3} \delta_{k}}\right) \leq 3 \log \left(8 \max \left\{c_{3}, 1\right\} c_{5}\right) c_{5} d \log \left(\frac{1}{\kappa \delta_{k}}\right) \\
& \leq 3 \log \left(8 \max \left\{c_{3}, 1\right\}\right) \kappa^{2} 2^{-k} m_{k} \leq \frac{3 \log \left(8 \max \left\{c_{3}, 1\right\}\right)}{c_{2} c_{7}} \kappa^{2}\left|W_{k}\right|
\end{aligned}
$$

we have

$$
\begin{aligned}
\left|W_{k}\right| \mathcal{P} & \left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x)| | w_{k-1} \cdot x \mid \leq b_{k-1}\right) \\
& \leq c_{13} \kappa\left|W_{k}\right|+c_{14} \sqrt{\left|W_{k}\right|\left(\frac{3 \log \left(8 \max \left\{c_{3}, 1\right\}\right)}{c_{2} c_{7}} \kappa^{2}\left|W_{k}\right|+\frac{\kappa^{2}}{c_{2} c_{5} c_{7}}\left|W_{k}\right|\right)} \\
& =\kappa\left|W_{k}\right|\left(c_{13}+c_{14} \sqrt{\frac{3 \log \left(8 \max \left\{c_{3}, 1\right\}\right)}{c_{2} c_{7}}+\frac{1}{c_{2} c_{5} c_{7}}}\right)
\end{aligned}
$$

Thus, letting $c_{15}=\left(c_{13}+c_{14} \sqrt{\frac{3 \log \left(8 \max \left\{c_{3}, 1\right\}\right)}{c_{2} c_{7}}+\frac{1}{c_{2} c_{5} c_{7}}}\right)$, we have

$$
\begin{equation*}
\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x)| | w_{k-1} \cdot x \mid \leq b_{k-1}\right) \leq c_{15} \kappa \tag{11}
\end{equation*}
$$

Next, note that $\left\|v_{k}-w_{k-1}\right\|^{2}=\left\|v_{k}\right\|^{2}+1-2\left\|v_{k}\right\| \cos \left(\pi \mathcal{P}\left(x: h_{w_{k}}(x) \neq\right.\right.$ $\left.\left.h_{w_{k-1}}(x)\right)\right)$. Thus, one implication of the fact that $\left\|v_{k}-w_{k-1}\right\| \leq r_{k}$ is that $\frac{\left\|v_{k}\right\|}{2}+$
$\frac{1-r_{k}^{2}}{2\left\|v_{k}\right\|} \leq \cos \left(\pi \mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w_{k-1}}(x)\right)\right)$; since the left hand side is positive, we have $\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w_{k-1}}(x)\right)<1 / 2$. Additionally, by differentiating, one can easily verify that for $\phi \in[0, \pi], x \mapsto \sqrt{x^{2}+1-2 x \cos (\phi)}$ is minimized at $x=\cos (\phi)$, in which case $\sqrt{x^{2}+1-2 x \cos (\phi)}=\sin (\phi)$. Thus, $\left\|v_{k}-w_{k-1}\right\| \geq$ $\sin \left(\pi \mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w_{k-1}}(x)\right)\right)$. Since $\left\|v_{k}-w_{k-1}\right\| \leq r_{k}$, we have $\sin (\pi \mathcal{P}(x:$ $\left.\left.h_{w_{k}}(x) \neq h_{w_{k-1}}(x)\right)\right) \leq r_{k}$. Since $\sin (\pi x) \geq x$ for all $x \in[0,1 / 2]$, combining this with the fact (proven above) that $\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w_{k-1}}(x)\right)<1 / 2$ implies $\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w_{k-1}}(x)\right) \leq r_{k}$.

In particular, we have that both $\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w_{k-1}}(x)\right) \leq r_{k}$ and $\mathcal{P}(x$ : $\left.h_{w^{*}}(x) \neq h_{w_{k-1}}(x)\right) \leq 2^{-k-3} \leq r_{k}$. Now Lemma 5 implies that, for any universal constant $c>0$, there exists a corresponding universal constant $c^{\prime}>0$ such that

$$
\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w_{k-1}}(x) \text { and }\left|w_{k-1} \cdot x\right| \geq c^{\prime} \frac{r_{k}}{\sqrt{d}}\right) \leq c r_{k}
$$

and

$$
\mathcal{P}\left(x: h_{w^{*}}(x) \neq h_{w_{k-1}}(x) \text { and }\left|w_{k-1} \cdot x\right| \geq c^{\prime} \frac{r_{k}}{\sqrt{d}}\right) \leq c r_{k}
$$

so that (by a union bound)

$$
\begin{aligned}
& \mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x) \text { and }\left|w_{k-1} \cdot x\right| \geq c^{\prime} \frac{r_{k}}{\sqrt{d}}\right) \\
& \leq \mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w_{k-1}}(x) \text { and }\left|w_{k-1} \cdot x\right| \geq c^{\prime} \frac{r_{k}}{\sqrt{d}}\right) \\
& +\mathcal{P}\left(x: h_{w^{*}}(x) \neq h_{w_{k-1}}(x) \text { and }\left|w_{k-1} \cdot x\right| \geq c^{\prime} \frac{r_{k}}{\sqrt{d}}\right) \leq 2 c r_{k}
\end{aligned}
$$

In particular, letting $c_{7}=c^{\prime} c_{10} / 2$, we have $c^{\prime} \frac{r_{k}}{\sqrt{d}}=b_{k-1}$. Combining this with (11), Lemma 6, and a union bound, we have that

$$
\begin{aligned}
& \mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x)\right) \\
& \leq \mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x) \text { and }\left|w_{k-1} \cdot x\right| \geq b_{k-1}\right) \\
& \quad+\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x) \text { and }\left|w_{k-1} \cdot x\right| \leq b_{k-1}\right) \\
& \leq 2 c r_{k}+\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x)| | w_{k-1} \cdot x \mid \leq b_{k-1}\right) \mathcal{P}\left(x:\left|w_{k-1} \cdot x\right| \leq b_{k-1}\right) \\
& \leq 2 c r_{k}+c_{15} \kappa c_{3} b_{k-1} \sqrt{d}=\left(2^{5} c c_{10}+c_{15} \kappa c_{3} c_{7} 2^{5}\right) 2^{-k-4} .
\end{aligned}
$$

Taking $c=\frac{1}{2^{6} c_{10}}$ and $\kappa=\frac{1}{2^{6} c_{3} c_{7} c_{15}}$, we have $\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{w^{*}}(x)\right) \leq 2^{-k-4}$, as required.

By a union bound, this occurs with probability at least $1-(4 / 3) \delta_{k}$.
Proof (Proof of Theorem [3). We begin with the bound on the error rate. If $\Delta>$ $\frac{\pi^{2}}{400 \cdot 2^{27}(d+\ln (4 / \delta))}$, the result trivially holds, since then $1 \leq \frac{400 \cdot 2^{27}}{\pi^{2}} \sqrt{\Delta(d+\ln (4 / \delta))}$. Otherwise, suppose $\Delta \leq \frac{\pi^{2}}{400 \cdot 2^{27}(d+\ln (4 / \delta))}$.

Fix any $i \in \mathbb{N}$. Lemma 3 implies that, with probability at least $1-\delta / 4$, the $w_{0}$ returned in Step 0 of $\operatorname{ABL}\left(M(i-1), \tilde{h}_{i-1}\right)$ satisfies $\mathcal{P}\left(x: h_{w_{0}}(x) \neq\right.$ $\left.h_{M(i-1)+m_{0}+1}^{*}(x)\right) \leq 1 / 16$. Taking this as a base case, Lemma 8 then inductively implies that, with probability at least

$$
1-\frac{\delta}{4}-\sum_{k=1}^{\left\lceil\log _{2}(1 / \alpha)\right\rceil}(4 / 3) \frac{\delta}{2\left(\left\lceil\log _{2}(4 / \alpha)\right\rceil-k\right)^{2}} \geq 1-\frac{\delta}{2}\left(1+(4 / 3) \sum_{\ell=2}^{\infty} \frac{1}{\ell^{2}}\right) \geq 1-\delta
$$

every $k \in\left\{0,1, \ldots,\left\lceil\log _{2}(1 / \alpha)\right\rceil\right\}$ has

$$
\begin{equation*}
\mathcal{P}\left(x: h_{w_{k}}(x) \neq h_{M(i-1)+m_{0}+1}^{*}(x)\right) \leq 2^{-k-4} \tag{12}
\end{equation*}
$$

and furthermore the number of labels requested during $\operatorname{ABL}\left(M(i-1), \tilde{h}_{i-1}\right)$ total to at most (for appropriate universal constants $\hat{c}_{1}, \hat{c}_{2}$ )

$$
\begin{aligned}
m_{0}+\sum_{k=1}^{\left\lceil\log _{2}(1 / \alpha)\right\rceil}\left|W_{k}\right| & \leq \hat{c}_{1}\left(d+\ln \left(\frac{1}{\delta}\right)+\sum_{k=1}^{\left\lceil\log _{2}(1 / \alpha)\right\rceil} d \log \left(\frac{\left(\left\lceil\log _{2}(4 / \alpha)\right\rceil-k\right)^{2}}{\delta}\right)\right) \\
& \leq \hat{c}_{2} d \log \left(\frac{1}{\Delta d}\right) \log \left(\frac{1}{\delta}\right)
\end{aligned}
$$

In particular, by a union bound, (12) implies that for every $k \in\left\{1, \ldots,\left\lceil\log _{2}(1 / \alpha)\right\rceil\right\}$, every

$$
m \in\left\{M(i-1)+\sum_{j=0}^{k-1} m_{j}+1, \ldots, M(i-1)+\sum_{j=0}^{k} m_{j}\right\}
$$

has

$$
\begin{aligned}
& \mathcal{P}\left(x: h_{w_{k-1}}(x) \neq h_{m}^{*}(x)\right) \\
& \leq \mathcal{P}\left(x: h_{w_{k-1}}(x) \neq h_{M(i-1)+m_{0}+1}^{*}(x)\right)+\mathcal{P}\left(x: h_{M(i-1)+m_{0}+1}^{*}(x) \neq h_{m}^{*}(x)\right) \\
& \leq 2^{-k-3}+\Delta M
\end{aligned}
$$

Thus, noting that

$$
\begin{aligned}
M & =\sum_{k=0}^{\left\lceil\log _{2}(1 / \alpha)\right\rceil} m_{k}=\Theta\left(d+\log \left(\frac{1}{\delta}\right)+\sum_{k=1}^{\left\lceil\log _{2}(1 / \alpha)\right\rceil} 2^{k} d \log \left(\frac{\left\lceil\log _{2}(1 / \alpha)\right\rceil-k}{\delta}\right)\right) \\
& =\Theta\left(\frac{1}{\alpha} d \log \left(\frac{1}{\delta}\right)\right)=\Theta\left(\sqrt{\frac{d}{\Delta} \log \left(\frac{1}{\delta}\right)}\right)
\end{aligned}
$$

with probability at least $1-\delta$,

$$
\mathcal{P}\left(x: h_{w_{\left\lceil\log _{2}(1 / \alpha)\right\rceil-1}}(x) \neq h_{M i}^{*}(x)\right) \leq O(\alpha+\Delta M)=O\left(\sqrt{\Delta d \log \left(\frac{1}{\delta}\right)}\right)
$$

In particular, this implies that, with probability at least $1-\delta$, every $t \in\{M i+$ $1, \ldots, M(i+1)-1\}$ has

$$
\begin{aligned}
\operatorname{er}_{t}\left(\hat{h}_{t}\right) & \leq \mathcal{P}\left(x: h_{w_{\left\lceil\log _{2}(1 / \alpha)\right\rceil-1}}(x) \neq h_{M i}^{*}(x)\right)+\mathcal{P}\left(x: h_{M i}^{*}(x) \neq h_{t}^{*}(x)\right) \\
& \leq O\left(\sqrt{\Delta d \log \left(\frac{1}{\delta}\right)}\right)+\Delta M=O\left(\sqrt{\Delta d \log \left(\frac{1}{\delta}\right)}\right)
\end{aligned}
$$

which completes the proof of the bound on the error rate.
Setting $\delta=\sqrt{\Delta d}$, and noting that $\mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right] \leq 1$, we have that for any $t>M$,

$$
\mathbb{P}\left(\hat{Y}_{t} \neq Y_{t}\right)=\mathbb{E}\left[\operatorname{er}_{t}\left(\hat{h}_{t}\right)\right] \leq O\left(\sqrt{\Delta d \log \left(\frac{1}{\delta}\right)}\right)+\delta=O\left(\sqrt{\Delta d \log \left(\frac{1}{\Delta d}\right)}\right)
$$

Thus, by linearity of the expectation,

$$
\mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}\left[\hat{Y}_{t} \neq Y_{t}\right]\right] \leq M+O\left(\sqrt{\Delta d \log \left(\frac{1}{\Delta d}\right)} T\right)=O\left(\sqrt{\Delta d \log \left(\frac{1}{\Delta d}\right)} T\right) .
$$

Furthermore, as mentioned, with probability at least $1-\delta$, the number of labels requested during the execution of $\operatorname{ABL}\left(M(i-1), \tilde{h}_{i-1}\right)$ is at most

$$
O\left(d \log \left(\frac{1}{\Delta d}\right) \log \left(\frac{1}{\delta}\right)\right)
$$

Thus, since the number of labels requested during the execution of $\operatorname{ABL}(M(i-$ 1), $\tilde{h}_{i-1}$ ) cannot exceed $M$, letting $\delta=\sqrt{\Delta d}$, the expected number of requested labels during this execution is at most

$$
\begin{aligned}
O\left(d \log ^{2}\left(\frac{1}{\Delta d}\right)\right)+\sqrt{\Delta d} M & \leq O\left(d \log ^{2}\left(\frac{1}{\Delta d}\right)\right)+O\left(d \sqrt{\log \left(\frac{1}{\Delta d}\right)}\right) \\
& =O\left(d \log ^{2}\left(\frac{1}{\Delta d}\right)\right)
\end{aligned}
$$

Thus, by linearity of the expectation, the expected number of labels requested among the first $T$ samples is at most

$$
O\left(d \log ^{2}\left(\frac{1}{\Delta d}\right)\left\lceil\frac{T}{M}\right\rceil\right)=O\left(\sqrt{\Delta d} \log ^{3 / 2}\left(\frac{1}{\Delta d}\right) T\right)
$$

which completes the proof.
Remark: The original work of CMEDV10 additionally allowed for some number $K$ of "jumps": times $t$ at which $\Delta_{t}=1$. Note that, in the above algorithm, since the influence of each sample is localized to the predictors trained within that
"batch" of $M$ instances, the effect of allowing such jumps would only change the bound on the number of mistakes to $\tilde{O}\left(\sqrt{d \Delta} T+\sqrt{\frac{d}{\Delta}} K\right)$. This compares favorably to the result of CMEDV10, which is roughly $O\left((d \Delta)^{1 / 4} T+\frac{d^{1 / 4}}{\Delta^{3 / 4}} K\right)$. However, the result of CMEDV10 was proven for a more general setting, allowing distributions $\mathcal{P}$ that are not uniform (though they do require a relation between the angle between any two separators and the probability mass they disagree on, similar to that holding for the uniform distribution, which seems to require that the distributions approximately retain some properties of the uniform distribution). It is not clear whether Theorem 3 can be generalized to this larger family of distributions.

## 6 General Results for Active Learning

As mentioned, the above results on linear separators also provide results for the number of queries in active learning. One can also state quite general results on the expected number of queries and mistakes achievable by an active learning algorithm. This section provides such results, for an algorithm based on the the well-known strategy of disagreement-based active learning. Throughout this section, we suppose $\mathbf{h}^{*} \in S_{\Delta}$, for a given $\Delta \in(0,1]$ : that is, $\mathcal{P}\left(x: h_{t+1}^{*}(x) \neq\right.$ $\left.h_{t}^{*}(x)\right) \leq \Delta$ for all $t \in \mathbb{N}$.

First, we introduce a few definitions. For any set $\mathcal{H} \subseteq \mathbb{C}$, define the region of disagreement

$$
\operatorname{DIS}(\mathcal{H})=\{x \in \mathcal{X}: \exists h, g \in \mathcal{H} \text { s.t. } h(x) \neq g(x)\}
$$

The analysis in this section is centered around the following algorithm. The Active subroutine is from the work of Han12 (slightly modified here), and is a variant of the $A^{2}$ (Agnostic Acive) algorithm of BBL06; the appropriate values of $M$ and $\hat{T}_{k}(\cdot)$ will be discussed below.

```
Algorithm: DriftingActive
0 . For \(i=1,2, \ldots\)
        Active \((M(i-1))\)
```

Subroutine: Active $(t)$
0 . Let $\hat{h}_{0}$ be an arbitrary element of $\mathbb{C}$, and let $V_{0} \leftarrow \mathbb{C}$

1. Predict $\hat{Y}_{t+1}=\hat{h}_{0}\left(X_{t+1}\right)$ as the prediction for the value of $Y_{t+1}$
2. For $k=0,1, \ldots, \log _{2}(M / 2)$
3. $Q_{k} \leftarrow\{ \}$
4. For $s=2^{k}+1, \ldots, 2^{k+1}$
5. Predict $\hat{Y}_{s}=\hat{h}_{k}\left(X_{s}\right)$ as the prediction for the value of $Y_{s}$
6. If $X_{s} \in \operatorname{DIS}\left(V_{k}\right)$

Request the label $Y_{s}$ and let $Q_{k} \leftarrow Q_{k} \cup\left\{\left(X_{s}, Y_{s}\right)\right\}$
8. Let $\hat{h}_{k+1}=\operatorname{argmin}_{h \in V_{k}} \sum_{(x, y) \in Q_{k}} \mathbb{1}[h(x) \neq y]$
9. Let $V_{k+1} \leftarrow\left\{h \in V_{k}: \sum_{(x, y) \in Q_{k}} \mathbb{1}[h(x) \neq y]-\mathbb{1}\left[\hat{h}_{k+1}(x) \neq y\right] \leq \hat{T}_{k}\right\}$

As in the DriftingHalfspaces algorithm above, this DriftingActive algorithm proceeds in batches, and in each batch runs an active learning algorithm designed to be robust to classification noise. This robustness to classification noise translates into our setting as tolerance for the fact that there is no classifier in $\mathbb{C}$ that perfectly classifies all of the data. The specific algorithm employed here maintains a set $V_{k} \subseteq \mathbb{C}$ of candidate classifiers, and requests the labels of samples $X_{s}$ for which there is some disagreement on the classification among classifiers in $V_{k}$. We maintain the invariant that there is a low-error classifier contained in $V_{k}$ at all times, and thus the points we query provide some information to help us determine which among these remaining candidates has low error rate. Based on these queries, we periodically (in Step 9) remove from $V_{k}$ those classifiers making a relatively excessive number of mistakes on the queried samples, relative to the minimum among classifiers in $V_{k}$. All predictions are made with an element of $V_{k}{ }^{7}$

We prove an abstract bound on the number of labels requested by this algorithm, expressed in terms of the disagreement coefficient Han07, defined as follows. For any $r \geq 0$ and any classifier $h$, define $\mathrm{B}(h, r)=\{g \in \mathbb{C}: \mathcal{P}(x:$ $g(x) \neq h(x)) \leq r\}$. Then for $r_{0} \geq 0$ and any classifier $h$, define the disagreement coefficient of $h$ with respect to $\mathbb{C}$ under $\mathcal{P}$ :

$$
\theta_{h}\left(r_{0}\right)=\sup _{r>r_{0}} \frac{\mathcal{P}(\mathrm{DIS}(\mathrm{~B}(h, r)))}{r}
$$

Usually, the disagreement coefficient would be used with $h$ equal the target concept; however, since the target concept is not fixed in our setting, we will make use of the worst-case value of the disagreement coefficient: $\theta_{\mathbb{C}}\left(r_{0}\right)=\sup _{h \in \mathbb{C}} \theta_{h}\left(r_{0}\right)$. This quantity has been bounded for a variety of spaces $\mathbb{C}$ and distributions $\mathcal{P}$ (see e.g., Han07EYW12 BL13). It is useful in bounding how quickly the region $\operatorname{DIS}\left(V_{k}\right)$ collapses in the algorithm. Thus, since the probability the algorithm requests the label of the next instance is $\mathcal{P}\left(\operatorname{DIS}\left(V_{k}\right)\right)$, the quantity $\theta_{\mathbb{C}}\left(r_{0}\right)$ naturally arises in characterizing the number of labels we expect this algorithm to request. Specifically, we have the following result $\sqrt[8]{ }$
Theorem 4. For an appropriate universal constant $c_{1} \in[1, \infty)$, if $\mathbf{h}^{*} \in S_{\Delta}$ for some $\Delta \in(0,1]$, then taking $M=\left\lceil c_{1} \sqrt{\frac{d}{\Delta}}\right\rceil_{2}$, and $\hat{T}_{k}=\log _{2}(1 / \sqrt{d \Delta})+2^{2 k+2} e \Delta$, and defining $\epsilon_{\Delta}=\sqrt{d \Delta} \log (1 /(d \Delta))$, the above DriftingActive algorithm makes an expected number of mistakes among the first $T$ instances that is

$$
O\left(\epsilon_{\Delta} \log (d / \Delta) T\right)=\tilde{O}(\sqrt{d \Delta}) T
$$

and requests an expected number of labels among the first $T$ instances that is

$$
O\left(\theta_{\mathbb{C}}\left(\epsilon_{\Delta}\right) \epsilon_{\Delta} \log (d / \Delta) T\right)=\tilde{O}\left(\theta_{\mathbb{C}}(\sqrt{d \Delta}) \sqrt{d \Delta}\right) T
$$

[^3]The proof of Theorem 4 relies on an analysis of the behavior of the Active subroutine, characterized in the following lemma.

Lemma 9. Fix any $t \in \mathbb{N}$, and consider the values obtained in the execution of Active $(t)$. Under the conditions of Theorem 4, there is a universal constant $c_{2} \in[1, \infty)$ such that, for any $k \in\left\{0,1, \ldots, \log _{2}(M / 2)\right\}$, with probability at least $1-2 \sqrt{d \Delta}$, if $h_{t+1}^{*} \in V_{k}$, then $h_{t+1}^{*} \in V_{k+1}$ and $\sup _{h \in V_{k+1}} \mathcal{P}(x: h(x) \neq$ $\left.h_{t+1}^{*}(x)\right) \leq c_{2} 2^{-k} d \log \left(c_{1} / \sqrt{d \Delta}\right)$.

Proof. By a Chernoff bound, with probability at least $1-\sqrt{d \Delta}$,

$$
\sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{1}\left[h_{t+1}^{*}\left(X_{s}\right) \neq Y_{s}\right] \leq \log _{2}(1 / \sqrt{d \Delta})+2^{2 k+2} e \Delta=\hat{T}_{k}
$$

Therefore, if $h_{t+1}^{*} \in V_{k}$, then since every $g \in V_{k}$ agrees with $h_{t+1}^{*}$ on those points $X_{s} \notin \operatorname{DIS}\left(V_{k}\right)$, in the update in Step 9 defining $V_{k+1}$, we have

$$
\begin{aligned}
& \sum_{(x, y) \in Q_{k}} \mathbb{1}\left[h_{t+1}^{*}(x) \neq y\right]-\mathbb{1}\left[\hat{h}_{k+1}(x) \neq y\right] \\
& =\sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{1}\left[h_{t+1}^{*}\left(X_{s}\right) \neq Y_{s}\right]-\min _{g \in V_{k}} \sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{1}\left[g\left(X_{s}\right) \neq Y_{s}\right] \\
& \leq \sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{1}\left[h_{t+1}^{*}\left(X_{s}\right) \neq Y_{s}\right] \leq \hat{T}_{k}
\end{aligned}
$$

so that $h_{t+1}^{*} \in V_{k+1}$ as well.
Furthermore, if $h_{t+1}^{*} \in V_{k}$, then by the definition of $V_{k+1}$, we know every $h \in V_{k+1}$ has

$$
\sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{1}\left[h\left(X_{s}\right) \neq Y_{s}\right] \leq \hat{T}_{k}+\sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{1}\left[h_{t+1}^{*}\left(X_{s}\right) \neq Y_{s}\right]
$$

so that a triangle inequality implies

$$
\begin{aligned}
\sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{1}\left[h\left(X_{s}\right) \neq h_{t+1}^{*}\left(X_{s}\right)\right] & \leq \sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{1}\left[h\left(X_{s}\right) \neq Y_{s}\right]+\mathbb{1}\left[h_{t+1}^{*}\left(X_{s}\right) \neq Y_{s}\right] \\
& \leq \hat{T}_{k}+2 \sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{1}\left[h_{t+1}^{*}\left(X_{s}\right) \neq Y_{s}\right] \leq 3 \hat{T}_{k}
\end{aligned}
$$

Lemma then implies that, on an additional event of probability at least 1 $\sqrt{d \Delta}$, every $h \in V_{k+1}$ has

$$
\begin{aligned}
& \mathcal{P}\left(x: h(x) \neq h_{t+1}^{*}(x)\right) \\
& \left.\leq 2^{-k} 3 \hat{T}_{k}+c 2^{-k} \sqrt{3 \hat{T}_{k}\left(d \log \left(2^{k} / d\right)+\log (1 / \sqrt{d \Delta})\right.}\right) \\
& \quad+c 2^{-k}\left(d \log \left(2^{k} / d\right)+\log (1 / \sqrt{d \Delta})\right) \\
& \leq 2^{-k} 3 \log _{2}(1 / \sqrt{d \Delta})+2^{k} 12 e \Delta+c 2^{-k} \sqrt{6 \log _{2}(1 / \sqrt{d \Delta}) d \log \left(c_{1} / \sqrt{d \Delta}\right)} \\
& \left.\quad+c 2^{-k} \sqrt{2^{2 k} 24 e \Delta d \log \left(c_{1} / \sqrt{d \Delta}\right.}\right)+2 c 2^{-k} d \log \left(c_{1} / \sqrt{d \Delta}\right) \\
& \leq 2^{-k} 3 \log _{2}(1 / \sqrt{d \Delta})+12 e c_{1} \sqrt{d \Delta}+3 c 2^{-k} \sqrt{d} \log \left(c_{1} / \sqrt{d \Delta}\right) \\
& \quad+\sqrt{24 e} c \sqrt{d \Delta \log \left(c_{1} / \sqrt{d \Delta}\right)}+2 c 2^{-k} d \log \left(c_{1} / \sqrt{d \Delta}\right),
\end{aligned}
$$

where $c$ is as in Lemma Since $\sqrt{d \Delta} \leq 2 c_{1} d / M \leq c_{1} d 2^{-k}$, this is at most

$$
\left(5+12 e c_{1}^{2}+3 c+\sqrt{24 e} c c_{1}+2 c\right) 2^{-k} d \log \left(c_{1} / \sqrt{d \Delta}\right) .
$$

Letting $c_{2}=5+12 e c_{1}^{2}+3 c+\sqrt{24 e} c c_{1}+2 c$, we have the result by a union bound.

We are now ready for the proof of Theorem 4.

Proof (Proof of Theorem [4). Fix any $i \in \mathbb{N}$, and consider running Active( $M(i-$ 1)). Since $h_{M(i-1)+1}^{*} \in \mathbb{C}$, by Lemma 9 a union bound, and induction, with probability at least $1-2 \sqrt{d \Delta} \log _{2}(M / 2) \geq 1-2 \sqrt{d \Delta} \log _{2}\left(c_{1} \sqrt{d / \Delta}\right)$, every $k \in$ $\left\{0,1, \ldots, \log _{2}(M / 2)\right\}$ has

$$
\begin{equation*}
\sup _{h \in V_{k}} \mathcal{P}\left(x: h(x) \neq h_{M(i-1)+1}^{*}(x)\right) \leq c_{2} 2^{1-k} d \log \left(c_{1} / \sqrt{d \Delta}\right) . \tag{13}
\end{equation*}
$$

Thus, since $\hat{h}_{k} \in V_{k}$ for each $k$, the expected number of mistakes among the predictions $\hat{Y}_{M(i-1)+1}, \ldots, \hat{Y}_{M i}$ is

$$
\begin{aligned}
& 1+\sum_{k=0}^{\log _{2}(M / 2)} \sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{P}\left(\hat{h}_{k}\left(X_{M(i-1)+s}\right) \neq Y_{M(i-1)+s}\right) \\
& \leq 1+\sum_{k=0}^{\log _{2}(M / 2)} \sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{P}\left(h_{M(i-1)+1}^{*}\left(X_{M(i-1)+s}\right) \neq Y_{M(i-1)+s}\right) \\
& \quad+\sum_{k=0}^{\log _{2}(M / 2)} \sum_{s=2^{k}+1}^{2^{k+1}} \mathbb{P}\left(\hat{h}_{k}\left(X_{M(i-1)+s}\right) \neq h_{M(i-1)+1}^{*}\left(X_{M(i-1)+s}\right)\right) \\
& \leq 1+\Delta M^{2}+\sum_{k=0}^{\log _{2}(M / 2)} 2^{k}\left(c_{2} 2^{1-k} d \log \left(c_{1} / \sqrt{d \Delta}\right)+2 \sqrt{d \Delta} \log _{2}(M / 2)\right) \\
& \leq 1+4 c_{1}^{2} d+2 c_{2} d \log \left(c_{1} / \sqrt{d \Delta}\right) \log _{2}\left(2 c_{1} \sqrt{d / \Delta}\right)+4 c_{1} d \log _{2}\left(c_{1} \sqrt{d / \Delta}\right) \\
& \\
& =O(d \log (d / \Delta) \log (1 /(d \Delta))) .
\end{aligned}
$$

Furthermore, (13) implies the algorithm only requests the label $Y_{M(i-1)+s}$ for $s \in\left\{2^{k}+1, \ldots, 2^{k+1}\right\}$ if $X_{M(i-1)+s} \in \operatorname{DIS}\left(\mathrm{~B}\left(h_{M(i-1)+1}^{*}, c_{2} 2^{1-k} d \log \left(c_{1} / \sqrt{d \Delta}\right)\right)\right)$, so that the expected number of labels requested among $Y_{M(i-1)+1}, \ldots, Y_{M i}$ is at most

$$
\begin{array}{r}
1+\sum_{k=0}^{\log _{2}(M / 2)} 2^{k}\left(\mathbb{E}\left[\mathcal{P}\left(\operatorname{DIS}\left(\mathrm{~B}\left(h_{M(i-1)+1}^{*}, c_{2} 2^{1-k} d \log \left(c_{1} / \sqrt{d \Delta}\right)\right)\right)\right)\right]\right. \\
\left.+2 \sqrt{d \Delta} \log _{2}\left(c_{1} \sqrt{d / \Delta}\right)\right) \\
\leq 1+\theta_{\mathbb{C}}\left(4 c_{2} d \log \left(c_{1} / \sqrt{d \Delta}\right) / M\right) 2 c_{2} d \log \left(c_{2} / \sqrt{d \Delta}\right) \log _{2}\left(2 c_{1} \sqrt{d / \Delta}\right) \\
+4 c_{1} d \log _{2}\left(c_{1} \sqrt{d / \Delta}\right) \\
=O\left(\theta_{\mathbb{C}}(\sqrt{d \Delta} \log (1 /(d \Delta))) d \log (d / \Delta) \log (1 /(d \Delta))\right)
\end{array}
$$

Thus, the expected number of mistakes among indices $1, \ldots, T$ is at most

$$
O\left(d \log (d / \Delta) \log (1 /(d \Delta))\left\lceil\frac{T}{M}\right\rceil\right)=O(\sqrt{d \Delta} \log (d / \Delta) \log (1 /(d \Delta)) T)
$$

and the expected number of labels requested among indices $1, \ldots, T$ is at most

$$
\begin{aligned}
& O\left(\theta_{\mathbb{C}}(\sqrt{d \Delta} \log (1 /(d \Delta))) d \log (d / \Delta) \log (1 /(d \Delta))\left\lceil\frac{T}{M}\right\rceil\right) \\
& \quad=O\left(\theta_{\mathbb{C}}(\sqrt{d \Delta} \log (1 /(d \Delta))) \sqrt{d \Delta} \log (d / \Delta) \log (1 /(d \Delta)) T\right)
\end{aligned}
$$

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[^0]:    ${ }^{4}$ In fact, [Lon99] also allowed the distribution $\mathcal{P}$ to vary gradually over time. For simplicity, we will only discuss the case of fixed $\mathcal{P}$.

[^1]:    ${ }^{5}$ They in fact prove a more general result, which also applies to methods approximately minimizing the number of mistakes, but for simplicity we will only discuss this basic version of the result.

[^2]:    ${ }^{6}$ This work in fact studies a much broader model of drift, which in fact allows the distribution $\mathcal{P}$ to vary with time as well. However, this $\tilde{O}\left((d \Delta)^{1 / 4}\right) \times T$ result can be obtained from their more-general theorem by calculating the various parameters for this particular setting.

[^3]:    ${ }^{7}$ One could alternatively proceed as in DriftingHalfspaces, using the final classifier from the previous batch, which would also add a guarantee on the error rate achieved at all sufficiently large $t$.
    ${ }^{8}$ Here, we define $\lceil x\rceil_{2}=2^{\left\lceil\log _{2}(x)\right\rceil}$, for $x \geq 1$.

