New Metric and Connections in Statistical Manifolds*

Rui F. Vigelis
† David C. de Souza † Charles C. Cavalcante § July 13, 2021

Abstract

We define a metric and a family of α -connections in statistical manifolds, based on φ -divergence, which emerges in the framework of φ -families of probability distributions. This metric and α -connections generalize the Fisher information metric and Amari's α -connections. We also investigate the parallel transport associated with the α -connection for $\alpha = 1$.

1 Introduction

In the framework of φ -families of probability distributions [11], the authors introduced a divergence $\mathcal{D}_{\varphi}(\cdot \| \cdot)$ between probabilities distributions, called φ -divergence, that generalizes the Kullback–Leibler divergence. Based on $\mathcal{D}_{\varphi}(\cdot \| \cdot)$ we can define a new metric and connections in statistical manifolds. The definition of metrics or connections in statistical manifolds is a common subject in the literature [2, 3, 7]. In our approach, the metric and α -connections are intrinsically related to φ -families. Moreover, they can be recognized as a generalization of the Fisher information metric and Amari's α -connections [1, 4].

Statistical manifolds are equipped with the Fisher information metric, which is given in terms of the derivative of $l(t;\theta) = \log p(t;\theta)$. Another metric can be defined if the logarithm $\log(\cdot)$ is replaced by the inverse of a φ -function $\varphi(\cdot)$ [11]. Instead of $l(t;\theta) = \log p(t;\theta)$, we can consider $f(t;\theta) = \varphi^{-1}(p(t;\theta))$. The manifold equipped with

^{*}This work was partially funded by CNPq (Proc. 309055/2014-8).

 $^{^\}dagger \text{Computer Engineering, Campus Sobral, Federal University of Ceará, Sobral-CE, Brazil, rfvigelis@ufc.br.}$

[‡]Federal Institute of Ceará, Fortaleza-CE, Brazil, davidcs@ifce.edu.br.

[§]Wireless Telecommunication Research Group, Department of Teleinformatics Engineering, Federal University of Ceará, Fortaleza-CE, Brazil, charles@ufc.br.

this metric, which coincides with the metric derived from $\mathcal{D}_{\varphi}(\cdot \| \cdot)$, is called a *generalized* statistical manifold.

Using the φ -divergence $\mathcal{D}_{\varphi}(\cdot \| \cdot)$, we can define a pair of mutually dual connections $D^{(1)}$ and $D^{(-1)}$, and then a family of α -connections $D^{(\alpha)}$. The connections $D^{(1)}$ and $D^{(-1)}$ corresponds to the exponential and mixture connections in classical information geometry. For example, in parametric φ -families, whose definition is found in Section 2.1, the connection $D^{(1)}$ is flat (i.e, its torsion tensor T and curvature tensor R vanish identically). As a consequence, a parametric φ -family admits a parametrization in which the Christoffel symbols $\Gamma_{ijk}^{(-1)}$ associated with $D^{(-1)}$ vanish identically. In addition, parametric φ -families are examples of Hessian manifolds [8].

The rest of the paper is organized as follows. In Section 2, we define the generalized statistical manifolds. Section 2.1 deals with parametric φ -families of probability distribution. In Section 3, α -connections are introduced. The parallel transport associated with $D^{(1)}$ is investigated in Section 3.1.

2 Generalized Statistical Manifolds

In this section, we provide a definition of generalized statistical manifolds. We begin with the definition of φ -functions. Let (T, Σ, μ) be a measure space. In the case $T = \mathbb{R}$ (or T is a discrete set), the measure μ is considered to be the Lebesgue measure (or the counting measure). A function $\varphi \colon \mathbb{R} \to (0, \infty)$ is said to be a φ -function if the following conditions are satisfied:

(a1) $\varphi(\cdot)$ is convex,

(a2)
$$\lim_{u\to-\infty}\varphi(u)=0$$
 and $\lim_{u\to\infty}\varphi(u)=\infty$.

Moreover, we assume that a measurable function $u_0: T \to (0, \infty)$ can be found such that, for each measurable function $c: T \to \mathbb{R}$ such that $\varphi(c(t)) > 0$ and $\int_T \varphi(c(t)) d\mu = 1$, we have

(a3)
$$\int_T \varphi(c(t) + \lambda u_0(t)) d\mu < \infty$$
, for all $\lambda > 0$.

The exponential function and the Kaniadakis' κ -exponential function [6] satisfy conditions (a1)–(a3) [11]. For $q \neq 1$, the q-exponential function $\exp_q(\cdot)$ [9] is not a φ -function, since its image is $[0,\infty)$. Notice that if the set T is finite, condition (a3) is always satisfied. Condition (a3) is indispensable in the definition of non-parametric families of probability distributions [11].

A generalized statistical manifold is a family of probability distributions $\mathcal{P} = \{p(t; \theta) : \theta \in \Theta\}$, which is defined to be contained in

$$\mathcal{P}_{\mu} = \left\{ p \in L^0 : p > 0 \text{ and } \int_T p d\mu = 1 \right\},$$

where L^0 denotes the set of all real-valued, measurable functions on T, with equality μ -a.e. Each $p_{\theta}(t) := p(t; \theta)$ is given in terms of parameters $\theta = (\theta^1, \dots, \theta^n) \in \Theta \subseteq \mathbb{R}^n$ by a one-to-one mapping. The family \mathcal{P} is called a *generalized statistical manifold* if the following conditions are satisfied:

- (P1) Θ is a domain (an open and connected set) in \mathbb{R}^n .
- (P2) $p(t;\theta)$ is a differentiable function with respect to θ .
- (P3) The operations of integration with respect to μ and differentiation with respect to θ^i commute.
- (P4) The matrix $g = (g_{ij})$, which is defined by

$$g_{ij} = -E_{\theta}' \left[\frac{\partial^2 f_{\theta}}{\partial \theta^i \partial \theta^j} \right], \tag{1}$$

is positive definite at each $\theta \in \Theta$, where $f_{\theta}(t) = f(t;\theta) = \varphi^{-1}(p(t;\theta))$ and

$$E'_{\theta}[\cdot] = \frac{\int_{T} (\cdot) \varphi'(f_{\theta}) d\mu}{\int_{T} u_{0} \varphi'(f_{\theta}) d\mu}.$$

Notice that expression (1) reduces to the Fisher information matrix in the case that φ coincides with the exponential function and $u_0 = 1$. Moreover, the right-hand side of (1) is invariant under reparametrization. The matrix (g_{ij}) can also be expressed as

$$g_{ij} = E_{\theta}^{"} \left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}} \right], \tag{2}$$

where

$$E_{\theta}''[\cdot] = \frac{\int_{T} (\cdot) \varphi''(f_{\theta}) d\mu}{\int_{T} u_{0} \varphi'(f_{\theta}) d\mu}.$$

Because the operations of integration with respect to μ and differentiation with respect to θ^i are commutative, we have

$$0 = \frac{\partial}{\partial \theta^{i}} \int_{T} p_{\theta} d\mu = \int_{T} \frac{\partial}{\partial \theta^{i}} \varphi(f_{\theta}) d\mu = \int_{T} \frac{\partial f_{\theta}}{\partial \theta^{i}} \varphi'(f_{\theta}) d\mu, \tag{3}$$

and

$$0 = \int_{T} \frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} \varphi'(f_{\theta}) d\mu + \int_{T} \frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}} \varphi''(f_{\theta}) d\mu. \tag{4}$$

Thus expression (2) follows from (4). In addition, expression (3) implies

$$E_{\theta}' \left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \right] = 0. \tag{5}$$

A consequence of (2) is the correspondence between the functions $\partial f_{\theta}/\partial \theta^{i}$ and the basis vectors $\partial/\partial \theta^{i}$. The inner product of vectors

$$X = \sum_{i} a^{i} \frac{\partial}{\partial \theta^{i}}$$
 and $Y = \sum_{i} b^{j} \frac{\partial}{\partial \theta^{j}}$

can be written as

$$g(X,Y) = \sum_{i,j} g_{ij} a^i b^j = \sum_{i,j} E_{\theta}'' \left[\frac{\partial f_{\theta}}{\partial \theta^i} \frac{\partial f_{\theta}}{\partial \theta^j} \right] a^i b^j = E_{\theta}'' [\widetilde{X}\widetilde{Y}], \tag{6}$$

where

$$\widetilde{X} = \sum_{i} a^{i} \frac{\partial f_{\theta}}{\partial \theta^{i}}$$
 and $\widetilde{Y} = \sum_{i} b^{j} \frac{\partial f_{\theta}}{\partial \theta^{j}}$.

As a result, the tangent space $T_{p_{\theta}}\mathcal{P}$ can be identified with $\widetilde{T}_{p_{\theta}}\mathcal{P}$, which is defined as the vector space spanned by $\partial f_{\theta}/\partial \theta^{i}$, equipped with the inner product $\langle \widetilde{X}, \widetilde{Y} \rangle_{\theta} = E''_{\theta}[\widetilde{X}\widetilde{Y}]$. By (5), if a vector \widetilde{X} belongs to $\widetilde{T}_{p_{\theta}}\mathcal{P}$, then $E'_{\theta}[\widetilde{X}] = 0$. Independent of the definition of (g_{ij}) , the expression in the right-hand side of (6) always defines a semi-inner product in $\widetilde{T}_{p_{\theta}}\mathcal{P}$.

2.1 Parametric φ -Families of Probability Distribution

Let $c: T \to \mathbb{R}$ be a measurable function such that $p := \varphi(c)$ is probability density in \mathcal{P}_{μ} . We take any measurable functions $u_1, \ldots u_n \colon T \to \mathbb{R}$ satisfying the following conditions:

- (i) $\int_T u_i \varphi'(c) d\mu = 0$, and
- (ii) there exists $\varepsilon > 0$ such that

$$\int_{T} \varphi(c + \lambda u_i) d\mu < \infty, \quad \text{for all } \lambda \in (-\varepsilon, \varepsilon).$$

Define $\Theta \subseteq \mathbb{R}^n$ as the set of all vectors $\theta = (\theta^i) \in \mathbb{R}^n$ such that

$$\int_{T} \varphi \left(c + \lambda \sum_{k=1}^{n} \theta^{i} u_{i} \right) d\mu < \infty, \quad \text{for some } \lambda > 1.$$

The elements of the parametric φ -family $\mathcal{F}_p = \{p(t;\theta) : \theta \in \Theta\}$ centered at $p = \varphi(c)$ are given by the one-to-one mapping

$$p(t;\theta) := \varphi\left(c(t) + \sum_{i=1}^{n} \theta^{i} u_{i}(t) - \psi(\theta) u_{0}(t)\right), \quad \text{for each } \theta = (\theta^{i}) \in \Theta.$$
 (7)

where the normalizing function $\psi \colon \Theta \to [0, \infty)$ is introduced so that expression (7) defines a probability distribution in \mathcal{P}_{μ} .

Condition (ii) is always satisfied if the set T is finite. It can be shown that the normalizing function ψ is also convex (and the set Θ is open and convex). Under conditions (i)–(ii), the family \mathcal{F}_p is a submanifold of a non-parametric φ -family. For the non-parametric case, we refer to [11, 10].

By the equalities

$$\frac{\partial f_{\theta}}{\partial \theta^{i}} = u_{i}(t) - \frac{\partial \psi}{\partial \theta^{i}}, \qquad -\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} = -\frac{\partial^{2} \psi}{\partial \theta^{i} \partial \theta^{j}},$$

we get

$$g_{ij} = \frac{\partial^2 \psi}{\partial \theta^i \partial \theta^j}.$$

In other words, the matrix (g_{ij}) is the Hessian of the normalizing function ψ .

For $\varphi(\cdot) = \exp(\cdot)$ and $u_0 = 1$, expression (7) defines a parametric exponential family of probability distributions \mathcal{E}_p . In exponential families, the normalizing function is recognized as the Kullback–Leibler divergence between p(t) and $p(t;\theta)$. Using this result, we can define the φ -divergence $\mathcal{D}_{\varphi}(\cdot \parallel \cdot)$, which generalizes the Kullback–Leibler divergence $\mathcal{D}_{\mathrm{KL}}(\cdot \parallel \cdot)$.

By (7) we can write

$$\psi(\theta)u_0(t) = \sum_{i=1}^n \theta^i u_i(t) + \varphi^{-1}(p(t)) - \varphi^{-1}(p(t;\theta)).$$

From condition (i), this equation yields

$$\psi(\theta) \int_T u_0 \varphi'(c) d\mu = \int_T [\varphi^{-1}(p) - \varphi^{-1}(p_\theta)] \varphi'(c) d\mu.$$

In view of $\varphi'(c) = 1/(\varphi^{-1})'(p)$, we get

$$\psi(\theta) = \frac{\int_{T} \frac{\varphi^{-1}(p) - \varphi^{-1}(p_{\theta})}{(\varphi^{-1})'(p)} d\mu}{\int_{T} \frac{u_{0}}{(\varphi^{-1})'(p)} d\mu} =: \mathcal{D}_{\varphi}(p \parallel p_{\theta}), \tag{8}$$

which defines the φ -divergence $\mathcal{D}_{\varphi}(p \parallel p_{\theta})$. Clearly, expression (8) can be used to extend the definition of $\mathcal{D}_{\varphi}(\cdot \parallel \cdot)$ to any probability distributions p and q in \mathcal{P}_{μ} .

3 α -Connections

We use the φ -divergence $\mathcal{D}_{\varphi}(\cdot \| \cdot)$ to define a pair of mutually dual connection in generalized statistical manifolds. Let $\mathcal{D} \colon M \times M \to [0, \infty)$ be a non-negative, differentiable function defined on a smooth manifold M, such that

$$\mathcal{D}(p \parallel q) = 0 \quad \text{if and only if} \quad p = q. \tag{9}$$

The function $\mathcal{D}(\cdot \| \cdot)$ is called a *divergence* if the matrix (g_{ij}) , whose entries are given by

$$g_{ij}(p) = -\left[\left(\frac{\partial}{\partial \theta^i}\right)_p \left(\frac{\partial}{\partial \theta^j}\right)_q \mathcal{D}(p \parallel q)\right]_{q=p},\tag{10}$$

is positive definite for each $p \in M$. Hence a divergence $\mathcal{D}(\cdot \| \cdot)$ defines a metric in M. A divergence $\mathcal{D}(\cdot \| \cdot)$ also induces a pair of mutually dual connections D and D^* , whose Christoffel symbols are given by

$$\Gamma_{ijk} = -\left[\left(\frac{\partial^2}{\partial \theta^i \partial \theta^j} \right)_p \left(\frac{\partial}{\partial \theta^k} \right)_q \mathcal{D}(p \parallel q) \right]_{q=p}$$
(11)

and

$$\Gamma_{ijk}^* = -\left[\left(\frac{\partial}{\partial \theta^k}\right)_p \left(\frac{\partial^2}{\partial \theta^i \partial \theta^j}\right)_q \mathcal{D}(p \parallel q)\right]_{q=p},\tag{12}$$

respectively. By a simple computation, we get

$$\frac{\partial g_{jk}}{\partial \theta^i} = \Gamma_{ijk} + \Gamma_{ikj}^*,$$

showing that D and D^* are mutually dual.

In Section 2.1, the φ -divergence between two probability distributions p and q in \mathcal{P}_{μ} was defined as

$$\mathcal{D}_{\varphi}(p \parallel q) := \frac{\int_{T} \frac{\varphi^{-1}(p) - \varphi^{-1}(q)}{(\varphi^{-1})'(p)} d\mu}{\int_{T} \frac{u_0}{(\varphi^{-1})'(p)} d\mu}.$$
 (13)

Because φ is convex, it follows that $\mathcal{D}_{\varphi}(p \parallel q) \geq 0$ for all $p, q \in \mathcal{P}_{\mu}$. In addition, if we assume that $\varphi(\cdot)$ is strictly convex, then $\mathcal{D}_{\varphi}(p \parallel q) = 0$ if and only if p = q. In a generalized statistical manifold $\mathcal{P} = \{p(t; \theta) : \theta \in \Theta\}$, the metric derived from the divergence $\mathcal{D}(q \parallel p) := \mathcal{D}_{\varphi}(p \parallel q)$ coincides with (1). Expressing the φ -divergence $\mathcal{D}_{\varphi}(\cdot \parallel \cdot)$

between p_{θ} and p_{ϑ} as

$$\mathcal{D}(p_{\theta} \parallel p_{\vartheta}) = E_{\vartheta}'[(f_{\vartheta} - f_{\theta})],$$

after some manipulation, we get

$$g_{ij} = -\left[\left(\frac{\partial}{\partial \theta^{i}}\right)_{p} \left(\frac{\partial}{\partial \theta^{j}}\right)_{q} \mathcal{D}(p \parallel q)\right]_{q=p}$$
$$= -E'_{\theta} \left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}}\right].$$

As a consequence, expression (13) defines a divergence on statistical manifolds.

Let $D^{(1)}$ and $D^{(-1)}$ denote the pair of dual connections derived from $\mathcal{D}_{\varphi}(\cdot \| \cdot)$. By (11) and (12), the Christoffel symbols $\Gamma_{ijk}^{(1)}$ and $\Gamma_{ijk}^{(-1)}$ are given by

$$\Gamma_{ijk}^{(1)} = E_{\theta}^{"} \left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} \frac{\partial f_{\theta}}{\partial \theta^{k}} \right] - E_{\theta}^{"} \left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} \right] E_{\theta}^{"} \left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{k}} \right]$$

$$(14)$$

and

$$\Gamma_{ijk}^{(-1)} = E_{\theta}'' \left[\frac{\partial^2 f_{\theta}}{\partial \theta^i \partial \theta^j} \frac{\partial f_{\theta}}{\partial \theta^k} \right] + E_{\theta}''' \left[\frac{\partial f_{\theta}}{\partial \theta^i} \frac{\partial f_{\theta}}{\partial \theta^j} \frac{\partial f_{\theta}}{\partial \theta^k} \right] \\
- E_{\theta}'' \left[\frac{\partial f_{\theta}}{\partial \theta^j} \frac{\partial f_{\theta}}{\partial \theta^k} \right] E_{\theta}'' \left[u_0 \frac{\partial f_{\theta}}{\partial \theta^i} \right] - E_{\theta}'' \left[\frac{\partial f_{\theta}}{\partial \theta^i} \frac{\partial f_{\theta}}{\partial \theta^k} \right] E_{\theta}'' \left[u_0 \frac{\partial f_{\theta}}{\partial \theta^j} \right], \tag{15}$$

where

$$E_{\theta}^{""}[\cdot] = \frac{\int_{T}(\cdot)\varphi^{""}(f_{\theta})d\mu}{\int_{T}u_{0}\varphi^{\prime}(f_{\theta})d\mu}.$$

Notice that in parametric φ -families, the Christoffel symbols $\Gamma_{ijk}^{(1)}$ vanish identically. Thus, in these families, the connection $D^{(1)}$ is flat.

Using the pair of mutually dual connections $D^{(1)}$ and $D^{(-1)}$, we can specify a family of α -connections $D^{(\alpha)}$ in generalized statistical manifolds. The Christoffel symbol of $D^{(\alpha)}$ is defined by

$$\Gamma_{ijk}^{(\alpha)} = \frac{1+\alpha}{2} \Gamma_{ijk}^{(1)} + \frac{1-\alpha}{2} \Gamma_{ijk}^{(-1)}.$$
 (16)

The connections $D^{(\alpha)}$ and $D^{(-\alpha)}$ are mutually dual, since

$$\frac{\partial g_{jk}}{\partial \theta^i} = \Gamma_{ijk}^{(\alpha)} + \Gamma_{ikj}^{(-\alpha)}.$$

For $\alpha=0$, the connection $D^{(0)}$, which is clearly self-dual, corresponds to the Levi–Civita connection ∇ . One can show that $\Gamma^{(0)}_{ijk}$ can be derived from the expression defining the

Christoffel symbols of ∇ in terms of the metric:

$$\Gamma_{ijk} = \sum_{m} \Gamma_{ij}^{m} g_{mk} = \frac{1}{2} \left(\frac{\partial g_{ki}}{\partial \theta^{j}} + \frac{\partial g_{kj}}{\partial \theta^{i}} - \frac{\partial g_{ij}}{\partial \theta^{k}} \right).$$

The connection $D^{(\alpha)}$ can be equivalently defined by

$$\Gamma_{ijk}^{(\alpha)} = \Gamma_{ijk}^{(0)} - \alpha T_{ijk},$$

where

$$T_{ijk} = \frac{1}{2} E_{\theta}^{"'} \left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}} \frac{\partial f_{\theta}}{\partial \theta^{k}} \right] - \frac{1}{2} E_{\theta}^{"} \left[\frac{\partial f_{\theta}}{\partial \theta^{k}} \frac{\partial f_{\theta}}{\partial \theta^{i}} \right] E_{\theta}^{"} \left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{j}} \right]$$
$$- \frac{1}{2} E_{\theta}^{"} \left[\frac{\partial f_{\theta}}{\partial \theta^{k}} \frac{\partial f_{\theta}}{\partial \theta^{j}} \right] E_{\theta}^{"} \left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{i}} \right] - \frac{1}{2} E_{\theta}^{"} \left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}} \right] E_{\theta}^{"} \left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{k}} \right]. \tag{17}$$

In the case that φ is the exponential function and $u_0 = 1$, equations (14), (15), (16) and (17) reduce to the classical expressions for statistical manifolds.

3.1 Parallel Transport

Let $\gamma \colon I \to M$ be a smooth curve in a smooth manifold M, with a connection D. A vector field V along γ is said to be parallel if $D_{d/dt}V(t)=0$ for all $t \in I$. Take any tangent vector V_0 at $\gamma(t_0)$, for some $t_0 \in I$. Then there exists a unique vector field V along γ , called the parallel transport of V_0 along γ , such that $V(t_0) = V_0$.

A connection D can be recovered from the parallel transport. Fix any smooth vectors fields X and Y. Given $p \in M$, define $\gamma \colon I \to M$ to be an integral curve of X passing through p. In other words, $\gamma(t_0) = p$ and $\frac{d\gamma}{dt} = X(\gamma(t))$. Let $P_{\gamma,t_0,t} \colon T_{\gamma(t_0)}M \to T_{\gamma(t)}M$ denote the parallel transport of a vector along γ from t_0 to t. Then we have

$$(D_X Y)(p) = \frac{d}{dt} P_{\gamma, t_0, t}^{-1}(Y(c(t))) \Big|_{t=t_0}.$$

For details, we refer to [5].

To avoid some technicalities, we assume that the set T is finite. In this case, we can consider a generalized statistical manifold $\mathcal{P} = \{p(t;\theta) : \theta \in \Theta\}$ for which $\mathcal{P} = \mathcal{P}_{\mu}$. The connection $D^{(1)}$ can be derived from the parallel transport

$$P_{q,p} \colon \widetilde{T}_q \mathcal{P} \to \widetilde{T}_p \mathcal{P}$$

given by

$$\widetilde{X} \mapsto \widetilde{X} - E'_{\theta}[\widetilde{X}]u_0,$$

where $p = p_{\theta}$. Recall that the tangent space $T_{p}\mathcal{P}$ can be identified with $\widetilde{T}_{p}\mathcal{P}$, the vector space spanned by the functions $\partial f_{\theta}/\partial \theta^{i}$, equipped with the inner product $\langle \widetilde{X}, \widetilde{Y} \rangle = E''_{\theta}[\widetilde{X}\widetilde{Y}]$, where $p = p_{\theta}$. We remark that $P_{q,p}$ does not depend on the curve joining q and p. As a result, the connection $D^{(1)}$ is flat. Denote by $\gamma(t)$ the coordinate curve given locally by $\theta(t) = (\theta^{1}, \dots, \theta^{i} + t, \dots, \theta^{n})$. Observing that $P_{\gamma(0),\gamma(t)}^{-1}$ maps the vector $\frac{\partial f_{\theta}}{\partial \theta^{j}}(t)$ to

$$\frac{\partial f_{\theta}}{\partial \theta^{j}}(t) - E'_{\theta(0)} \left[\frac{\partial f_{\theta}}{\partial \theta^{j}}(t) \right] u_{0},$$

we define the connection

$$\begin{split} \widetilde{D}_{\partial f_{\theta}/\partial \theta_{i}} \frac{\partial f_{\theta}}{\partial \theta_{j}} &= \frac{d}{dt} P_{\gamma(0),\gamma(t)}^{-1} \left(\frac{\partial f_{\theta}}{\partial \theta_{j}} (\gamma(t)) \right) \Big|_{t=0} \\ &= \frac{d}{dt} \left(\frac{\partial f_{\theta(t)}}{\partial \theta^{j}} - E'_{\theta(0)} \left[\frac{\partial f_{\theta(t)}}{\partial \theta^{j}} \right] u_{0} \right) \Big|_{t=0} \\ &= \frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} - E'_{\theta} \left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} \right] u_{0}. \end{split}$$

Let us denote by D the connection corresponding to \widetilde{D} , which acts on smooth vector fields in $T_p\mathcal{P}$. By this identification, we have

$$g\left(D_{\partial/\partial\theta_{i}}\frac{\partial}{\partial\theta_{j}},\frac{\partial}{\partial\theta_{k}}\right) = \left\langle \widetilde{D}_{\partial f_{\theta}/\partial\theta_{i}}\frac{\partial f_{\theta}}{\partial\theta_{j}},\frac{\partial f_{\theta}}{\partial\theta_{k}}\right\rangle$$

$$= E_{\theta}''\left[\frac{\partial^{2} f_{\theta}}{\partial\theta^{i}\partial\theta^{j}}\frac{\partial f_{\theta}}{\partial\theta^{k}}\right] - E_{\theta}'\left[\frac{\partial^{2} f_{\theta}}{\partial\theta^{i}\partial\theta^{j}}\right]E_{\theta}''\left[u_{0}\frac{\partial f_{\theta}}{\partial\theta^{k}}\right]$$

$$= \Gamma_{ijk}^{(1)},$$

showing that $D = D^{(1)}$.

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