# New Metric and Connections in Statistical Manifolds* 

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July 13, 2021


#### Abstract

We define a metric and a family of $\alpha$-connections in statistical manifolds, based on $\varphi$-divergence, which emerges in the framework of $\varphi$-families of probability distributions. This metric and $\alpha$-connections generalize the Fisher information metric and Amari's $\alpha$-connections. We also investigate the parallel transport associated with the $\alpha$-connection for $\alpha=1$.


## 1 Introduction

In the framework of $\varphi$-families of probability distributions [11, the authors introduced a divergence $\mathcal{D}_{\varphi}(\cdot \| \cdot)$ between probabilities distributions, called $\varphi$-divergence, that generalizes the Kullback-Leibler divergence. Based on $\mathcal{D}_{\varphi}(\cdot \| \cdot)$ we can define a new metric and connections in statistical manifolds. The definition of metrics or connections in statistical manifolds is a common subject in the literature [2, 3, 7]. In our approach, the metric and $\alpha$-connections are intrinsically related to $\varphi$-families. Moreover, they can be recognized as a generalization of the Fisher information metric and Amari's $\alpha$-connections [1, 4].

Statistical manifolds are equipped with the Fisher information metric, which is given in terms of the derivative of $l(t ; \theta)=\log p(t ; \theta)$. Another metric can be defined if the $\operatorname{logarithm} \log (\cdot)$ is replaced by the inverse of a $\varphi$-function $\varphi(\cdot)$ [11]. Instead of $l(t ; \theta)=\log p(t ; \theta)$, we can consider $f(t ; \theta)=\varphi^{-1}(p(t ; \theta))$. The manifold equipped with

[^0]this metric, which coincides with the metric derived from $\mathcal{D}_{\varphi}(\cdot \| \cdot)$, is called a generalized statistical manifold.

Using the $\varphi$-divergence $\mathcal{D}_{\varphi}(\cdot \| \cdot)$, we can define a pair of mutually dual connections $D^{(1)}$ and $D^{(-1)}$, and then a family of $\alpha$-connections $D^{(\alpha)}$. The connections $D^{(1)}$ and $D^{(-1)}$ corresponds to the exponential and mixture connections in classical information geometry. For example, in parametric $\varphi$-families, whose definition is found in Section 2.1. the connection $D^{(1)}$ is flat (i.e, its torsion tensor $T$ and curvature tensor $R$ vanish identically). As a consequence, a parametric $\varphi$-family admits a parametrization in which the Christoffel symbols $\Gamma_{i j k}^{(-1)}$ associated with $D^{(-1)}$ vanish identically. In addition, parametric $\varphi$-families are examples of Hessian manifolds [8].

The rest of the paper is organized as follows. In Section 2, we define the generalized statistical manifolds. Section 2.1 deals with parametric $\varphi$-families of probability distribution. In Section 3, $\alpha$-connections are introduced. The parallel transport associated with $D^{(1)}$ is investigated in Section 3.1.

## 2 Generalized Statistical Manifolds

In this section, we provide a definition of generalized statistical manifolds. We begin with the definition of $\varphi$-functions. Let $(T, \Sigma, \mu)$ be a measure space. In the case $T=\mathbb{R}$ (or $T$ is a discrete set), the measure $\mu$ is considered to be the Lebesgue measure (or the counting measure). A function $\varphi: \mathbb{R} \rightarrow(0, \infty)$ is said to be a $\varphi$-function if the following conditions are satisfied:
(a1) $\varphi(\cdot)$ is convex,
(a2) $\lim _{u \rightarrow-\infty} \varphi(u)=0$ and $\lim _{u \rightarrow \infty} \varphi(u)=\infty$.
Moreover, we assume that a measurable function $u_{0}: T \rightarrow(0, \infty)$ can be found such that, for each measurable function $c: T \rightarrow \mathbb{R}$ such that $\varphi(c(t))>0$ and $\int_{T} \varphi(c(t)) d \mu=1$, we have
(a3) $\int_{T} \varphi\left(c(t)+\lambda u_{0}(t)\right) d \mu<\infty$, for all $\lambda>0$.
The exponential function and the Kaniadakis' $\kappa$-exponential function [6] satisfy conditions (a1)-(a3) [11]. For $q \neq 1$, the $q$-exponential function $\exp _{q}(\cdot)$ 9] is not a $\varphi$ function, since its image is $[0, \infty)$. Notice that if the set $T$ is finite, condition (a3) is always satisfied. Condition (a3) is indispensable in the definition of non-parametric families of probability distributions [11].

A generalized statistical manifold is a family of probability distributions $\mathcal{P}=\{p(t ; \theta)$ : $\theta \in \Theta\}$, which is defined to be contained in

$$
\mathcal{P}_{\mu}=\left\{p \in L^{0}: p>0 \text { and } \int_{T} p d \mu=1\right\},
$$

where $L^{0}$ denotes the set of all real-valued, measurable functions on $T$, with equality $\mu$-a.e. Each $p_{\theta}(t):=p(t ; \theta)$ is given in terms of parameters $\theta=\left(\theta^{1}, \ldots, \theta^{n}\right) \in \Theta \subseteq \mathbb{R}^{n}$ by a one-to-one mapping. The family $\mathcal{P}$ is called a generalized statistical manifold if the following conditions are satisfied:
(P1) $\Theta$ is a domain (an open and connected set) in $\mathbb{R}^{n}$.
(P2) $p(t ; \theta)$ is a differentiable function with respect to $\theta$.
(P3) The operations of integration with respect to $\mu$ and differentiation with respect to $\theta^{i}$ commute.
(P4) The matrix $g=\left(g_{i j}\right)$, which is defined by

$$
\begin{equation*}
g_{i j}=-E_{\theta}^{\prime}\left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}}\right] \tag{1}
\end{equation*}
$$

is positive definite at each $\theta \in \Theta$, where $f_{\theta}(t)=f(t ; \theta)=\varphi^{-1}(p(t ; \theta))$ and

$$
E_{\theta}^{\prime}[\cdot]=\frac{\int_{T}(\cdot) \varphi^{\prime}\left(f_{\theta}\right) d \mu}{\int_{T} u_{0} \varphi^{\prime}\left(f_{\theta}\right) d \mu}
$$

Notice that expression (1) reduces to the Fisher information matrix in the case that $\varphi$ coincides with the exponential function and $u_{0}=1$. Moreover, the right-hand side of (11) is invariant under reparametrization. The matrix $\left(g_{i j}\right)$ can also be expressed as

$$
\begin{equation*}
g_{i j}=E_{\theta}^{\prime \prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}}\right] \tag{2}
\end{equation*}
$$

where

$$
E_{\theta}^{\prime \prime}[\cdot]=\frac{\int_{T}(\cdot) \varphi^{\prime \prime}\left(f_{\theta}\right) d \mu}{\int_{T} u_{0} \varphi^{\prime}\left(f_{\theta}\right) d \mu}
$$

Because the operations of integration with respect to $\mu$ and differentiation with respect to $\theta^{i}$ are commutative, we have

$$
\begin{equation*}
0=\frac{\partial}{\partial \theta^{i}} \int_{T} p_{\theta} d \mu=\int_{T} \frac{\partial}{\partial \theta^{i}} \varphi\left(f_{\theta}\right) d \mu=\int_{T} \frac{\partial f_{\theta}}{\partial \theta^{i}} \varphi^{\prime}\left(f_{\theta}\right) d \mu \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
0=\int_{T} \frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} \varphi^{\prime}\left(f_{\theta}\right) d \mu+\int_{T} \frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}} \varphi^{\prime \prime}\left(f_{\theta}\right) d \mu . \tag{4}
\end{equation*}
$$

Thus expression (2) follows from (4). In addition, expression (3) implies

$$
\begin{equation*}
E_{\theta}^{\prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{i}}\right]=0 \tag{5}
\end{equation*}
$$

A consequence of (2) is the correspondence between the functions $\partial f_{\theta} / \partial \theta^{i}$ and the basis vectors $\partial / \partial \theta^{i}$. The inner product of vectors

$$
X=\sum_{i} a^{i} \frac{\partial}{\partial \theta^{i}} \quad \text { and } \quad Y=\sum_{i} b^{j} \frac{\partial}{\partial \theta^{j}}
$$

can be written as

$$
\begin{equation*}
g(X, Y)=\sum_{i, j} g_{i j} a^{i} b^{j}=\sum_{i, j} E_{\theta}^{\prime \prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}}\right] a^{i} b^{j}=E_{\theta}^{\prime \prime}[\widetilde{X} \widetilde{Y}] \tag{6}
\end{equation*}
$$

where

$$
\widetilde{X}=\sum_{i} a^{i} \frac{\partial f_{\theta}}{\partial \theta^{i}} \quad \text { and } \quad \widetilde{Y}=\sum_{i} b^{j} \frac{\partial f_{\theta}}{\partial \theta^{j}}
$$

As a result, the tangent space $T_{p_{\theta}} \mathcal{P}$ can be identified with $\widetilde{T}_{p_{\theta}} \mathcal{P}$, which is defined as the vector space spanned by $\partial f_{\theta} / \partial \theta^{i}$, equipped with the inner product $\langle\widetilde{X}, \widetilde{Y}\rangle_{\theta}=E_{\theta}^{\prime \prime}[\widetilde{X} \widetilde{Y}]$. By (5), if a vector $\widetilde{X}$ belongs to $\widetilde{T}_{p_{\theta}} \mathcal{P}$, then $E_{\theta}^{\prime}[\widetilde{X}]=0$. Independent of the definition of $\left(g_{i j}\right)$, the expression in the right-hand side of (6) always defines a semi-inner product in $\widetilde{T}_{p_{\theta}} \mathcal{P}$.

### 2.1 Parametric $\varphi$-Families of Probability Distribution

Let $c: T \rightarrow \mathbb{R}$ be a measurable function such that $p:=\varphi(c)$ is probability density in $\mathcal{P}_{\mu}$. We take any measurable functions $u_{1}, \ldots u_{n}: T \rightarrow \mathbb{R}$ satisfying the following conditions:
(i) $\int_{T} u_{i} \varphi^{\prime}(c) d \mu=0$, and
(ii) there exists $\varepsilon>0$ such that

$$
\int_{T} \varphi\left(c+\lambda u_{i}\right) d \mu<\infty, \quad \text { for all } \lambda \in(-\varepsilon, \varepsilon)
$$

Define $\Theta \subseteq \mathbb{R}^{n}$ as the set of all vectors $\theta=\left(\theta^{i}\right) \in \mathbb{R}^{n}$ such that

$$
\int_{T} \varphi\left(c+\lambda \sum_{k=1}^{n} \theta^{i} u_{i}\right) d \mu<\infty, \quad \text { for some } \lambda>1
$$

The elements of the parametric $\varphi$-family $\mathcal{F}_{p}=\{p(t ; \theta): \theta \in \Theta\}$ centered at $p=\varphi(c)$ are given by the one-to-one mapping

$$
\begin{equation*}
p(t ; \theta):=\varphi\left(c(t)+\sum_{i=1}^{n} \theta^{i} u_{i}(t)-\psi(\theta) u_{0}(t)\right), \quad \text { for each } \theta=\left(\theta^{i}\right) \in \Theta . \tag{7}
\end{equation*}
$$

where the normalizing function $\psi: \Theta \rightarrow[0, \infty)$ is introduced so that expression (7) defines a probability distribution in $\mathcal{P}_{\mu}$.

Condition (ii) is always satisfied if the set $T$ is finite. It can be shown that the normalizing function $\psi$ is also convex (and the set $\Theta$ is open and convex). Under conditions (i)-(ii), the family $\mathcal{F}_{p}$ is a submanifold of a non-parametric $\varphi$-family. For the non-parametric case, we refer to [11, 10].

By the equalities

$$
\frac{\partial f_{\theta}}{\partial \theta^{i}}=u_{i}(t)-\frac{\partial \psi}{\partial \theta^{i}}, \quad-\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}}=-\frac{\partial^{2} \psi}{\partial \theta^{i} \partial \theta^{j}},
$$

we get

$$
g_{i j}=\frac{\partial^{2} \psi}{\partial \theta^{i} \partial \theta^{j}} .
$$

In other words, the matrix $\left(g_{i j}\right)$ is the Hessian of the normalizing function $\psi$.
For $\varphi(\cdot)=\exp (\cdot)$ and $u_{0}=1$, expression (7) defines a parametric exponential family of probability distributions $\mathcal{E}_{p}$. In exponential families, the normalizing function is recognized as the Kullback-Leibler divergence between $p(t)$ and $p(t ; \theta)$. Using this result, we can define the $\varphi$-divergence $\mathcal{D}_{\varphi}(\cdot \| \cdot)$, which generalizes the Kullback-Leibler divergence $\mathcal{D}_{\mathrm{KL}}(\cdot \| \cdot)$.

By (7) we can write

$$
\psi(\theta) u_{0}(t)=\sum_{i=1}^{n} \theta^{i} u_{i}(t)+\varphi^{-1}(p(t))-\varphi^{-1}(p(t ; \theta)) .
$$

From condition (i), this equation yields

$$
\psi(\theta) \int_{T} u_{0} \varphi^{\prime}(c) d \mu=\int_{T}\left[\varphi^{-1}(p)-\varphi^{-1}\left(p_{\theta}\right)\right] \varphi^{\prime}(c) d \mu
$$

In view of $\varphi^{\prime}(c)=1 /\left(\varphi^{-1}\right)^{\prime}(p)$, we get

$$
\begin{equation*}
\psi(\theta)=\frac{\int_{T} \frac{\varphi^{-1}(p)-\varphi^{-1}\left(p_{\theta}\right)}{\left(\varphi^{-1}\right)^{\prime}(p)} d \mu}{\int_{T} \frac{u_{0}}{\left(\varphi^{-1}\right)^{\prime}(p)} d \mu}=: \mathcal{D}_{\varphi}\left(p \| p_{\theta}\right) \tag{8}
\end{equation*}
$$

which defines the $\varphi$-divergence $\mathcal{D}_{\varphi}\left(p \| p_{\theta}\right)$. Clearly, expression (8) can be used to extend the definition of $\mathcal{D}_{\varphi}(\cdot \| \cdot)$ to any probability distributions $p$ and $q$ in $\mathcal{P}_{\mu}$.

## $3 \alpha$-Connections

We use the $\varphi$-divergence $\mathcal{D}_{\varphi}(\cdot \| \cdot)$ to define a pair of mutually dual connection in generalized statistical manifolds. Let $\mathcal{D}: M \times M \rightarrow[0, \infty)$ be a non-negative, differentiable function defined on a smooth manifold $M$, such that

$$
\begin{equation*}
\mathcal{D}(p \| q)=0 \quad \text { if and only if } \quad p=q \tag{9}
\end{equation*}
$$

The function $\mathcal{D}(\cdot \| \cdot)$ is called a divergence if the matrix $\left(g_{i j}\right)$, whose entries are given by

$$
\begin{equation*}
g_{i j}(p)=-\left[\left(\frac{\partial}{\partial \theta^{i}}\right)_{p}\left(\frac{\partial}{\partial \theta^{j}}\right)_{q} \mathcal{D}(p \| q)\right]_{q=p}, \tag{10}
\end{equation*}
$$

is positive definite for each $p \in M$. Hence a divergence $\mathcal{D}(\cdot \| \cdot)$ defines a metric in $M$. A divergence $\mathcal{D}(\cdot \| \cdot)$ also induces a pair of mutually dual connections $D$ and $D^{*}$, whose Christoffel symbols are given by

$$
\begin{equation*}
\Gamma_{i j k}=-\left[\left(\frac{\partial^{2}}{\partial \theta^{i} \partial \theta^{j}}\right)_{p}\left(\frac{\partial}{\partial \theta^{k}}\right)_{q} \mathcal{D}(p \| q)\right]_{q=p} \tag{11}
\end{equation*}
$$

and

$$
\begin{equation*}
\Gamma_{i j k}^{*}=-\left[\left(\frac{\partial}{\partial \theta^{k}}\right)_{p}\left(\frac{\partial^{2}}{\partial \theta^{i} \partial \theta^{j}}\right)_{q} \mathcal{D}(p \| q)\right]_{q=p} \tag{12}
\end{equation*}
$$

respectively. By a simple computation, we get

$$
\frac{\partial g_{j k}}{\partial \theta^{i}}=\Gamma_{i j k}+\Gamma_{i k j}^{*}
$$

showing that $D$ and $D^{*}$ are mutually dual.
In Section 2.1, the $\varphi$-divergence between two probability distributions $p$ and $q$ in $\mathcal{P}_{\mu}$ was defined as

$$
\begin{equation*}
\mathcal{D}_{\varphi}(p \| q):=\frac{\int_{T} \frac{\varphi^{-1}(p)-\varphi^{-1}(q)}{\left(\varphi^{-1}\right)^{\prime}(p)} d \mu}{\int_{T} \frac{u_{0}}{\left(\varphi^{-1}\right)^{\prime}(p)} d \mu} \tag{13}
\end{equation*}
$$

Because $\varphi$ is convex, it follows that $\mathcal{D}_{\varphi}(p \| q) \geq 0$ for all $p, q \in \mathcal{P}_{\mu}$. In addition, if we assume that $\varphi(\cdot)$ is strictly convex, then $\mathcal{D}_{\varphi}(p \| q)=0$ if and only if $p=q$. In a generalized statistical manifold $\mathcal{P}=\{p(t ; \theta): \theta \in \Theta\}$, the metric derived from the divergence $\mathcal{D}(q \| p):=\mathcal{D}_{\varphi}(p \| q)$ coincides with (1). Expressing the $\varphi$-divergence $\mathcal{D}_{\varphi}(\cdot \| \cdot)$
between $p_{\theta}$ and $p_{\vartheta}$ as

$$
\mathcal{D}\left(p_{\theta} \| p_{\vartheta}\right)=E_{\vartheta}^{\prime}\left[\left(f_{\vartheta}-f_{\theta}\right)\right],
$$

after some manipulation, we get

$$
\begin{aligned}
g_{i j} & =-\left[\left(\frac{\partial}{\partial \theta^{i}}\right)_{p}\left(\frac{\partial}{\partial \theta^{j}}\right)_{q} \mathcal{D}(p \| q)\right]_{q=p} \\
& =-E_{\theta}^{\prime}\left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}}\right] .
\end{aligned}
$$

As a consequence, expression (13) defines a divergence on statistical manifolds.
Let $D^{(1)}$ and $D^{(-1)}$ denote the pair of dual connections derived from $\mathcal{D}_{\varphi}(\cdot \| \cdot)$. By (11) and (12), the Christoffel symbols $\Gamma_{i j k}^{(1)}$ and $\Gamma_{i j k}^{(-1)}$ are given by

$$
\begin{equation*}
\Gamma_{i j k}^{(1)}=E_{\theta}^{\prime \prime}\left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right]-E_{\theta}^{\prime}\left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}}\right] E_{\theta}^{\prime \prime}\left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right] \tag{14}
\end{equation*}
$$

and

$$
\begin{align*}
\Gamma_{i j k}^{(-1)}= & E_{\theta}^{\prime \prime}\left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right]+E_{\theta}^{\prime \prime \prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right] \\
& -E_{\theta}^{\prime \prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{j}} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right] E_{\theta}^{\prime \prime}\left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{i}}\right]-E_{\theta}^{\prime \prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right] E_{\theta}^{\prime \prime}\left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{j}}\right] \tag{15}
\end{align*}
$$

where

$$
E_{\theta}^{\prime \prime \prime}[\cdot]=\frac{\int_{T}(\cdot) \varphi^{\prime \prime \prime}\left(f_{\theta}\right) d \mu}{\int_{T} u_{0} \varphi^{\prime}\left(f_{\theta}\right) d \mu}
$$

Notice that in parametric $\varphi$-families, the Christoffel symbols $\Gamma_{i j k}^{(1)}$ vanish identically. Thus, in these families, the connection $D^{(1)}$ is flat.

Using the pair of mutually dual connections $D^{(1)}$ and $D^{(-1)}$, we can specify a family of $\alpha$-connections $D^{(\alpha)}$ in generalized statistical manifolds. The Christoffel symbol of $D^{(\alpha)}$ is defined by

$$
\begin{equation*}
\Gamma_{i j k}^{(\alpha)}=\frac{1+\alpha}{2} \Gamma_{i j k}^{(1)}+\frac{1-\alpha}{2} \Gamma_{i j k}^{(-1)} . \tag{16}
\end{equation*}
$$

The connections $D^{(\alpha)}$ and $D^{(-\alpha)}$ are mutually dual, since

$$
\frac{\partial g_{j k}}{\partial \theta^{i}}=\Gamma_{i j k}^{(\alpha)}+\Gamma_{i k j}^{(-\alpha)}
$$

For $\alpha=0$, the connection $D^{(0)}$, which is clearly self-dual, corresponds to the Levi-Civita connection $\nabla$. One can show that $\Gamma_{i j k}^{(0)}$ can be derived from the expression defining the

Christoffel symbols of $\nabla$ in terms of the metric:

$$
\Gamma_{i j k}=\sum_{m} \Gamma_{i j}^{m} g_{m k}=\frac{1}{2}\left(\frac{\partial g_{k i}}{\partial \theta^{j}}+\frac{\partial g_{k j}}{\partial \theta^{i}}-\frac{\partial g_{i j}}{\partial \theta^{k}}\right) .
$$

The connection $D^{(\alpha)}$ can be equivalently defined by

$$
\Gamma_{i j k}^{(\alpha)}=\Gamma_{i j k}^{(0)}-\alpha T_{i j k},
$$

where

$$
\begin{align*}
& T_{i j k}=\frac{1}{2} E_{\theta}^{\prime \prime \prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right]-\frac{1}{2} E_{\theta}^{\prime \prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{k}} \frac{\partial f_{\theta}}{\partial \theta^{i}}\right] E_{\theta}^{\prime \prime}\left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{j}}\right] \\
&-\frac{1}{2} E_{\theta}^{\prime \prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{k}} \frac{\partial f_{\theta}}{\partial \theta^{j}}\right] E_{\theta}^{\prime \prime}\left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{i}}\right]-\frac{1}{2} E_{\theta}^{\prime \prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{i}} \frac{\partial f_{\theta}}{\partial \theta^{j}}\right] E_{\theta}^{\prime \prime}\left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right] . \tag{17}
\end{align*}
$$

In the case that $\varphi$ is the exponential function and $u_{0}=1$, equations (14), (15), (16) and (17) reduce to the classical expressions for statistical manifolds.

### 3.1 Parallel Transport

Let $\gamma: I \rightarrow M$ be a smooth curve in a smooth manifold $M$, with a connection $D$. A vector field $V$ along $\gamma$ is said to be parallel if $D_{d / d t} V(t)=0$ for all $t \in I$. Take any tangent vector $V_{0}$ at $\gamma\left(t_{0}\right)$, for some $t_{0} \in I$. Then there exists a unique vector field $V$ along $\gamma$, called the parallel transport of $V_{0}$ along $\gamma$, such that $V\left(t_{0}\right)=V_{0}$.

A connection $D$ can be recovered from the parallel transport. Fix any smooth vectors fields $X$ and $Y$. Given $p \in M$, define $\gamma: I \rightarrow M$ to be an integral curve of $X$ passing through $p$. In other words, $\gamma\left(t_{0}\right)=p$ and $\frac{d \gamma}{d t}=X(\gamma(t))$. Let $P_{\gamma, t_{0}, t}: T_{\gamma\left(t_{0}\right)} M \rightarrow T_{\gamma(t)} M$ denote the parallel transport of a vector along $\gamma$ from $t_{0}$ to $t$. Then we have

$$
\left(D_{X} Y\right)(p)=\frac{d}{d t} P_{\gamma, t_{0}, t}^{-1}\left(\left.Y(c(t))\right|_{t=t_{0}} .\right.
$$

For details, we refer to [5].
To avoid some technicalities, we assume that the set $T$ is finite. In this case, we can consider a generalized statistical manifold $\mathcal{P}=\{p(t ; \theta): \theta \in \Theta\}$ for which $\mathcal{P}=\mathcal{P}_{\mu}$. The connection $D^{(1)}$ can be derived from the parallel transport

$$
P_{q, p}: \widetilde{T}_{q} \mathcal{P} \rightarrow \widetilde{T}_{p} \mathcal{P}
$$

given by

$$
\widetilde{X} \mapsto \widetilde{X}-E_{\theta}^{\prime}[\widetilde{X}] u_{0},
$$

where $p=p_{\theta}$. Recall that the tangent space $T_{p} \mathcal{P}$ can be identified with $\widetilde{T}_{p} \mathcal{P}$, the vector space spanned by the functions $\partial f_{\theta} / \partial \theta^{i}$, equipped with the inner product $\langle\widetilde{X}, \widetilde{Y}\rangle=$ $E_{\theta}^{\prime \prime}[\tilde{X} \widetilde{Y}]$, where $p=p_{\theta}$. We remark that $P_{q, p}$ does not depend on the curve joining $q$ and $p$. As a result, the connection $D^{(1)}$ is flat. Denote by $\gamma(t)$ the coordinate curve given locally by $\theta(t)=\left(\theta^{1}, \ldots, \theta^{i}+t, \ldots, \theta^{n}\right)$. Observing that $P_{\gamma(0), \gamma(t)}^{-1}$ maps the vector $\frac{\partial f_{\theta}}{\partial \theta^{j}}(t)$ to

$$
\frac{\partial f_{\theta}}{\partial \theta^{j}}(t)-E_{\theta(0)}^{\prime}\left[\frac{\partial f_{\theta}}{\partial \theta^{j}}(t)\right] u_{0}
$$

we define the connection

$$
\begin{aligned}
\widetilde{D}_{\partial f_{\theta} / \partial \theta_{i}} \frac{\partial f_{\theta}}{\partial \theta_{j}} & =\frac{d}{d t} P_{\gamma(0), \gamma(t)}^{-1}\left(\left.\frac{\partial f_{\theta}}{\partial \theta_{j}}(\gamma(t))\right|_{t=0}\right. \\
& =\left.\frac{d}{d t}\left(\frac{\partial f_{\theta(t)}}{\partial \theta^{j}}-E_{\theta(0)}^{\prime}\left[\frac{\partial f_{\theta(t)}}{\partial \theta^{j}}\right] u_{0}\right)\right|_{t=0} \\
& =\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}}-E_{\theta}^{\prime}\left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}}\right] u_{0} .
\end{aligned}
$$

Let us denote by $D$ the connection corresponding to $\widetilde{D}$, which acts on smooth vector fields in $T_{p} \mathcal{P}$. By this identification, we have

$$
\begin{aligned}
g\left(D_{\partial / \partial \theta_{i}} \frac{\partial}{\partial \theta_{j}}, \frac{\partial}{\partial \theta_{k}}\right) & =\left\langle\widetilde{D}_{\partial f_{\theta} / \partial \theta_{i}} \frac{\partial f_{\theta}}{\partial \theta_{j}}, \frac{\partial f_{\theta}}{\partial \theta_{k}}\right\rangle \\
& =E_{\theta}^{\prime \prime}\left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right]-E_{\theta}^{\prime}\left[\frac{\partial^{2} f_{\theta}}{\partial \theta^{i} \partial \theta^{j}}\right] E_{\theta}^{\prime \prime}\left[u_{0} \frac{\partial f_{\theta}}{\partial \theta^{k}}\right] \\
& =\Gamma_{i j k}^{(1)},
\end{aligned}
$$

showing that $D=D^{(1)}$.

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[^0]:    *This work was partially funded by CNPq (Proc. 309055/2014-8).
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