Selection-based Approach to Cooperative Interval Games

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Abstract. Cooperative interval games are a generalized model of cooperative games in which the worth of every coalition corresponds to a closed interval representing the possible outcomes of its cooperation. Selections are all possible outcomes of the interval game with no additional uncertainty.

We introduce new selection-based classes of interval games and prove their characterization theorems and relations to existing classes based on the interval weakly better operator. We show new results regarding the core and imputations and examine a problem of equivalence for two different versions of the core, the main stability solution of cooperative games. Finally, we introduce the definition of strong imputation and strong core as universal solution concepts of interval games.

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1 Introduction

Uncertainty and inaccurate data are issues occurring very often in real-world situations. Therefore it is important to be able to make decisions even when the exact data are not available and some bounds on them are known.

In classical cooperative game theory, every group of players (*coalition*) knows the precise reward for their cooperation; in cooperative interval games, only the worst and the best possible outcome are known. Such situations can be naturally modeled with intervals encapsulating these outcomes. This model is especially useful if we have no additional assumption on probability distribution on this interval. In some sense, cooperative interval games get the best of both worlds. We count in all the possible outcomes, yet our model is sufficiently simple to analyze.

Cooperation under interval uncertainty was first considered by Branzei, Dimitrov and Tijs in 2003 to study bankruptcy situations [10] and later further extensively studied by Alparslan Gök in her Ph.D. thesis [1] and in other papers written together with Branzei et al. (see references in [9]).

However, their approach is almost exclusively aimed at interval solutions, that is on payoff distributions consisting of intervals and thus containing yet another uncertainty. This is in contrast with selections – possible outcomes of an interval game with no additional uncertainty. The selection- based approach was never systematically studied and not very much is known. This paper is trying to fix this and summarizes our results regarding the selection-based approach to interval games.

The paper has the following structure. Section 2 is a preliminary section that presents necessary definitions and facts on classical cooperative games, interval analysis, and cooperative interval games. Section 3 is devoted to new selectionbased classes of interval games. We consequently prove their characterizations and relations to existing classes. Section 4 focuses on the so-called core incidence problem which asks under which conditions are the selection core and the set of payoffs generated by the interval core equal. In Section 5, the definitions of strong core and strong imputation are introduced as new concepts. We show some simple results on the strong core, one of them being a characterization of games with the strong imputation and strong core. Finally, we conclude this paper with a summary of our results and possible directions for future research.

2 Preliminaries

2.1 On mathematical notation

- We will use \leq relation on real vectors. For every $x, y \in \mathbb{R}^N$ we write $x \leq y$ if $x_i \leq y_i$ holds for every $1 \leq i \leq N$.
- We do not use symbol \subset in this paper. Instead, \subseteq and \subsetneq are used for subset and proper subset, respectively, to avoid ambiguity.

2.2 Classical cooperative game theory

Comprehensive sources on classical cooperative game theory are for example [11,14,15,18]. For more information on applications, see e.g. [8,13,16]. Here we present only necessary background theory for studying interval games. We examine games with transferable utility (TU) and therefore by a cooperative game we mean a cooperative TU game.

Definition 1. (Cooperative game) A cooperative game is an ordered pair (N, v), where $N = \{1, 2, ..., n\}$ is a set of players and $v : 2^N \to \mathbb{R}$ is a characteristic function of the cooperative game. We further assume that $v(\emptyset) = 0$.

The set of all cooperative games with a player set N is denoted by G^N .

Subsets of N are called *coalitions* and N itself is called the *grand coalition*.

We often write v instead of (N, v), because we can easily identify a game with its characteristic function without loss of generality.

To further analyze players' gains, we will need a *payoff vector* which can be interpreted as a proposed distribution of rewards between players.

Definition 2. (Payoff vector) A payoff vector for a cooperative game (N, v) is a vector $x \in \mathbb{R}^N$ with x_i denoting the reward given to the *i*th player.

Definition 3. (Imputation) An imputation of $(N, v) \in G^N$ is a vector $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = v(N)$ and $x_i \ge v(\{i\})$ for every $i \in N$.

The set of all imputations of a given cooperative game (N, v) is denoted by I(v).

Definition 4. (Core) The core of $(N, v) \in G^N$ is the set

$$C(v) = \Big\{ x \in I(v); \ \sum_{i \in S} x_i \ge v(S), \forall S \subseteq N \Big\}.$$

There are many important classes of cooperative games. Here we show the most important ones.

Definition 5. (Monotonic game) A game (N, v) is monotonic if for every $T \subseteq S \subseteq N$ we have

$$v(T) \le v(S)$$

Informally, in monotonic games, bigger coalitions are stronger.

Definition 6. (Superadditive game) A game (N, v) is superadditive if for every $S, T \subseteq N, S \cap T = \emptyset$ we have

$$v(T) + v(S) \le v(S \cup T).$$

In a superadditive game, a coalition has no incentive to divide itself since together they will always achieve at least as much as separated.

Superadditive games are not necessarily monotonic. Conversely, monotonic games are not necessarily superadditive. However, these classes have a nonempty intersection. Check Caulier's paper [12] for more details on the relationship between these two classes.

Definition 7. (Additive game) A game (N, v) is additive if for every $S, T \subseteq N$, $S \cap T = \emptyset$ we have

$$v(T) + v(S) = v(S \cup T).$$

Observe that additive games are superadditive as well.

Another important type of game is a *convex game*.

Definition 8. (Convex game) A game (N, v) is convex if its characteristic function is supermodular. The characteristic function is supermodular if for every $S \subseteq T \subseteq N$,

 $v(T) + v(S) \le v(S \cup T) + v(S \cap T).$

Clearly, supermodularity implies superadditivity.

Convex games have many nice properties. We remind the most important one.

Theorem 1. (Shapley 1971 [20]) If a game (N, v) is convex, then its core is nonempty.

2.3 Interval analysis

Definition 9. (Interval) An interval X is a set

 $X := [\underline{X}, \overline{X}] = \{ x \in \mathbb{R} : \underline{X} \le x \le \overline{X} \}.$

with \underline{X} being the lower bound and \overline{X} being the upper bound of the interval.

From now on, by an interval, we mean a closed interval. The set of all real intervals is denoted by \mathbb{IR} .

The following definition (from [17]) shows how to do basic arithmetics with intervals.

Definition 10. For every $X, Y, Z \in \mathbb{IR}$ and $0 \notin Z$ define

$$\begin{split} X+Y &:= [\underline{X}+\underline{Y}, \overline{X}+\overline{Y}], \\ X-Y &:= [\underline{X}-\overline{Y}, \overline{X}-\underline{Y}], \\ X\cdot Y &:= [\min S, \max S], \ S = \{\underline{X}\overline{Y}, \overline{X}\underline{Y}, \underline{X}\underline{Y}, \overline{X}\overline{Y}\}, \\ X/Z &:= [\min S, \max S], \ S = \{\underline{X}/\overline{Z}, \overline{X}/\underline{Z}, \underline{X}/\overline{Z}, \overline{X}/\overline{Z}\}. \end{split}$$

2.4 Cooperative interval games

This section aims at presenting results on cooperative interval games necessary to grasp our contribution to theory.

Definition 11. (Cooperative interval game) A cooperative interval game is an ordered pair (N, w), where $N = \{1, 2, ..., n\}$ is a set of players and $w : 2^N \to \mathbb{IR}$ is the characteristic function of the cooperative game. We further assume that $w(\emptyset) = [0, 0]$.

The set of all interval cooperative games on a player set N is denoted by IG^N .

We often write w(i) instead of $w(\{i\})$.

Remark 1. Each cooperative interval game in which the characteristic function maps to degenerate intervals only can be associated with some classical cooperative game. The converse holds as well.

Definition 12. (Border games) For every $(N, w) \in IG^N$, border games $(N, \underline{w}) \in G^N$ (lower border game) and $(N, \overline{w}) \in G^N$ (upper border game) are given by $\underline{w}(S) = w(S)$ and $\overline{w}(S) = \overline{w(S)}$ for every $S \in 2^N$.

Definition 13. (Length game) The length game of $(N, w) \in IG^N$ is the game $(N, |w|) \in G^N$ with

$$|w|(S) = \overline{w}(S) - \underline{w}(S), \ \forall S \in 2^N.$$

The basic notion of our approach will be a selection and consequently a selection imputation and a selection core.

Definition 14. (Selection) A game $(N, v) \in G^N$ is a selection of $(N, w) \in IG^N$ if for every $S \in 2^N$ we have $v(S) \in w(S)$. Set of all selections of (N, w) is denoted by Sel(w).

Note that border games are particular examples of selections.

Definition 15. (Interval selection imputation) The set of interval selection imputations (or just selection imputations) of $(N, w) \in IG^N$ is defined as

$$\mathcal{SI}(w) = \bigcup \{ I(v) \mid v \in \operatorname{Sel}(w) \}.$$

Definition 16. (Interval selection core) The interval selection core (or just selection core) of $(N, w) \in IG^N$ is defined as

$$\mathcal{SC}(w) = \bigcup \{ C(v) \mid v \in \operatorname{Sel}(w) \}.$$

Alparslan Gök [1] choose an approach using a weakly better operator. That was inspired by [19].

Definition 17. (Weakly better operator \succeq) Interval I is weakly better than interval J ($I \succeq J$) if $\underline{I} \ge \underline{J}$ and $\overline{I} \ge \overline{J}$. Furthermore, $I \preceq J$ if and only if $\underline{I} \le \underline{J}$ and $\overline{I} \le \overline{J}$. Interval I is better than J ($I \succ J$) if and only if $I \succeq J$ and $I \neq J$.

Their definition of imputation and core is as follows.

Definition 18. (Interval imputation) The set of interval imputations of $(N, w) \in IG^N$ is defined as

$$\mathcal{I}(w) := \Big\{ (I_1, I_2, \dots, I_N) \in \mathbb{IR}^N \mid \sum_{i \in N} I_i = w(N), \ I_i \succeq w(i), \ \forall i \in N \Big\}.$$

Definition 19. (Interval core) An interval core of $(N, w) \in IG^N$ is defined as

$$\mathcal{C}(w) := \left\{ (I_1, I_2, \dots, I_N) \in \mathcal{I}(w) \mid \sum_{i \in S} I_i \succeq w(S), \ \forall S \in 2^N \setminus \{\emptyset\} \right\}.$$

An important difference between the definitions of interval and selection core and imputation is that selection concepts yield payoff vectors from \mathbb{R}^N , while \mathcal{I} and \mathcal{C} yield vectors from \mathbb{IR}^N .

(Notation) Throughout the papers on cooperative interval games, notation, especially of core and imputations, is not unified. It is, therefore, possible to encounter different notation from ours.

Also, in these papers, the selection core is called the core of interval game. We consider that confusing and that is why we use the term selection core instead. The term selection imputation is used because of its connection with the selection core.

The following classes of interval games have been studied earlier (see e.g. [2]).

Definition 20. (Size monotonicity) A game $(N, w) \in IG^N$ is size monotonic if for every $T \subseteq S \subseteq N$ we have

$$|w|(T) \le |w|(S).$$

That is, its length game is monotonic.

The class of size monotonic games on a player set N is denoted by $SMIG^N$.

As we can see, size monotonic games capture situations in which an interval uncertainty grows with the size of a coalition.

Definition 21. (Superadditive interval game) A game $(N, w) \in IG^N$ is a superadditive interval game if for every $S, T \subseteq N, S \cap T = \emptyset$,

$$w(T) + w(S) \preceq w(S \cup T),$$

and its length game is superadditive. We denote by SIG^N the class of superadditive interval games on a player set N.

We should be careful with the following analogy of a convex game since unlike for classical games, supermodularity is not the same as convexity.

Definition 22. (Supermodular interval game) An interval game (N, w) is supermodular interval if for every $S \subseteq T \subseteq N$ holds

$$w(T) + v(S) \preceq w(S \cup T) + w(S \cap T).$$

We get immediately that an interval game is supermodular interval if and only if its border games are convex.

Definition 23. (Convex interval game) An interval game (N, w) is convex interval if its border games and length game are convex.

We write CIG^N for a set of convex interval games on a player set N.

A convex interval game is supermodular as well but the converse does not hold in general. See [2] for characterizations of convex interval games and discussion on their properties.

3 Selection-based classes of interval games

We will now introduce new classes of interval games based on the properties of their selections. We think that it is a natural way to generalize special classes from classical cooperative game theory. Consequently, we show their characterizations and relation to classes from the preceding section.

Definition 24. (Selection monotonic interval game) An interval game (N, v) is selection monotonic if all its selections are monotonic games. The class of such games on a player set N is denoted by SeMIG^N.

Definition 25. (Selection superadditive interval game) An interval game (N, v) is selection superadditive if all its selections are superadditive games. The class of such games on a player set N is denoted by SeSIG^N.

Definition 26. (Selection convex interval game) An interval game (N, v) is selection convex if all its selections are convex games. The class of such games on a player set N is denoted by SeCIG^N.

We see that many properties persist. For example, a selection convex game is a selection superadditive as well. Selection monotonic and selection superadditive are not subsets of each other but their intersection is nonempty. Furthermore, the selection core of selection convex game is nonempty, which is an easy observation.

We will now show characterizations of these three classes and consequently show their relations to the existing classes presented in Section 2.4.

Theorem 2. An interval game (N, w) is selection monotonic if and only if for every $S, T \in 2^N, S \subsetneq T$

 $\overline{w}(S) \le \underline{w}(T).$

Proof. For the "only if" part, suppose that (N, w) is a selection monotonic and $\overline{w}(S) > \underline{w}(T)$ for some $S, T \in 2^N, S \subsetneq T$. Then selection (N, v) with $v(S) = \overline{w}(S)$ and $v(T) = \underline{w}(T)$ clearly violates monotonicity and we arrive at a contradiction.

Now for the "if" part. For any two subsets S, T of N, one of the situations $S \subsetneq T, T \subsetneq S$ or S = T occurs. For S = T, in every selection $v, v(S) \le v(S)$ holds. As for the other two situations, it is obvious that monotonicity cannot be violated as well since $v(S) \le \overline{w}(S) \le w(T) \le v(T)$.

Notice the importance of using $S \subsetneq T$ in the formulation of Theorem 2. That is because using of $S \subseteq T$ (thus allowing situation S = T) would imply $\overline{w}(S) \leq \underline{w}(S)$ for every S in selection monotonic game which is obviously not true in general. In characterizations of selection superadditive and selection convex games, a similar situation arises.

Theorem 3. An interval game (N, w) is selection superadditive if and only if for every $S, T \in 2^N$ such that $S \cap T = \emptyset$, $S \neq \emptyset$, $T \neq \emptyset$

$$\overline{w}(S) + \overline{w}(T) \le \underline{w}(S \cup T).$$

Proof. Similar to the proof of Theorem 2.

We give a characterization of selection convex games as well:

Theorem 4. An interval game (N, w) is selection convex if and only if for every $S, T \in 2^N$ such that $S \not\subseteq T, T \not\subseteq S, S \neq \emptyset, T \neq \emptyset$ holds

$$\overline{w}(S) + \overline{w}(T) \le \underline{w}(S \cup T) + \underline{w}(S \cap T).$$

Proof. Similar to proof of Proposition 2.

Now let us look at a relation with existing classes of interval games.

For selection monotonic and size monotonic games, their relation is obvious. For nontrivial games, i.e. games with the size of player set greater than one, a selection monotonic game is not necessarily size monotonic and vice versa.

Theorem 5. For every player set N with |N| > 1, the following assertions hold.

 $\begin{array}{ll} (i) & \operatorname{SeSIG}^{N} \not\subseteq \operatorname{SIG}^{N}. \\ (ii) & \operatorname{SIG}^{N} \not\subseteq \operatorname{SeSIG}^{N}. \\ (iii) & \operatorname{SeSIG}^{N} \cap \operatorname{SIG}^{N} \neq \emptyset. \end{array}$

Proof. In (i), we can construct the counterexample in the following way.

Let us construct game (N, w). For $w(\emptyset)$, the interval is given. Now for any nonempty coalition, set w(S) := [2|S| - 2, 2|S| - 1]. For any $S, T \in 2^N$ with S and T being nonempty and disjoint, the following holds with the fact that $|S| + |T| = |S \cup T|$ taken into account.

$$\overline{w}(S) + \overline{w}(T) = (2|S| - 1) + (2|T| - 1)$$
$$= 2|S \cup T| - 2$$
$$= \underline{w}(S \cup T)$$

So (N, w) is selection superadditive by Theorem 3. Its length game, however, is not superadditive since for any two nonempty coalitions with empty intersection $|w|(S) + |w|(T) = 2 \leq 1 = |w|(S \cup T)$ holds.

In (ii), we can construct the following counterexample (N, w'). Set w'(S) =[0, |S|] for any nonempty S. The lower border game is trivially superadditive. For the upper game, $\overline{w'}(S) + \overline{w'}(T) = |S| + |T| = |S \cup T| = \overline{w'}(S \cup T)$ for any S, T with empty intersection, so the upper game is superadditive. Observe that the length game is the same as the upper border game. This shows interval superadditivity.

However, (N, w') is clearly not selection superadditive because of nonzero upper bounds, zero lower bounds of nonempty coalitions and the characterization of $SeSIG^N$ taken into account.

(iii) Nonempty intersection can be argued easily by taking some superadditive game $(N, c) \in G^N$. Then we can define corresponding game $(N, d) \in IG^N$ with

$$d(S) = [c(S), c(S)], \quad \forall S \in 2^N.$$

Game (N, d) is selection superadditive since its only selection is (N, c). And it is superadditive interval game since border games are supermodular and length game is |w|(S) = 0 for every coalition, which trivially implies its superadditivity.

Theorem 6. For every player set N with |N| > 1, the following assertions hold.

 $\begin{array}{ll} (i) & \operatorname{SeCIG}^{N} \not\subseteq \operatorname{CIG}^{N}.\\ (ii) & \operatorname{CIG}^{N} \not\subseteq \operatorname{SeCIG}^{N}.\\ (iii) & \operatorname{SeCIG}^{N} \cap \operatorname{CIG}^{N} \neq \emptyset. \end{array}$

Proof. For (i), take a game (N, w) assigning to each nonempty coalition S interval $[2^{|S|} - 2, 2^{|S|} - 1]$. From Theorem 4, we get that for inequalities which must hold in order to meet necessary conditions of game to be selection convex, $|S| < |S \cup T|$ and $|T| < |S \cup T|$ must hold. That gives the following inequality:

$$\begin{split} \overline{w}(S) + \overline{w}(T) &\leq (2^{|S \cup T| - 1} - 1) + (2^{|S \cup T| - 1} - 1) \\ &= 2^{|S \cup T|} - 2 \\ &= \underline{w}(S \cup T) \\ &\leq \underline{w}(S \cup T) + \underline{w}(S \cap T) \end{split}$$

This concludes that (N, w) is selection convex. We see that the border games and the length game are convex too. To have a game so that it is selection convex and not convex interval, we can take (N, c) and set c(S) := w(S) for $S \neq N$ and $c(N) := [\underline{w}(N), \underline{w}(N)]$. Now the game (N, c) is still selection convex, but its length game is not convex and (N, v) is not a convex interval game, which is what we wanted.

In (ii), we can take a game (N, w') from the proof of Theorem 5(ii). From the fact that $|S| + |T| = |S \cup T| + |S \cap T|$, it is clear that $\overline{w'}$ is convex. The lower border game is trivially convex and the length game is the same as upper. However, for nonempty $S, T \in 2^N$ such that $S \not\subseteq T, T \not\subseteq S, S \neq \emptyset, T \neq \emptyset$, convex selection games characterization is clearly violated.

As for *(iii)*, we can use the same steps as in *(iii)* of Theorem 5 or we can use a game (N, w) from *(i)* of this theorem.

4 Core coincidence

In Alparslan-Gök's PhD thesis [1] and [7], the following question is suggested:

"A difficult topic might be to analyze under which conditions the set of payoff vectors generated by the interval core of a cooperative interval game coincides with the core of the game in terms of selections of the interval game."

We decided to examine this topic. We call it the *core coincidence problem*. This section shows our results.

We remind the reader that whenever we talk about a relation and maximum, minimum, maximal, minimal vectors, we mean the relation \leq on real vectors unless we say otherwise.

The main thing to notice is that while the interval core gives us a set of interval vectors, selection core gives us a set of real numbered vectors. To be able to compare them, we need to assign to a set of interval vectors a set of real vectors generated by these interval vectors. That is exactly what the following function gen does. **Definition 27.** The function gen : $2^{\mathbb{R}^N} \to 2^{\mathbb{R}^N}$ maps to every set of interval vectors a set of real vectors. It is defined as

$$gen(S) = \bigcup_{s \in S} \{ (x_1, x_2, \dots, x_n) \mid x_i \in s_i \}.$$

The core coincidence problem can be formulated as this: What are the necessary and sufficient conditions to satisfy $gen(\mathcal{C}(w)) = \mathcal{SC}(w)$?

The main results of this section are two theorems which can be seen as a partial step towards an answer to the core coincidence problem.

In the following text by mixed system, we mean a system of equalities and inequalities.

Theorem 7. For every interval game (M, w) we have $gen(\mathcal{C}(w)) \subseteq \mathcal{SC}(w)$.

Proof. For any $x \in \text{gen}(\mathcal{C}(w))$, the inequality $\underline{w}(N) \leq \sum_{i \in N} x_i \leq \overline{w}(N)$ obviously holds. Furthermore, x is in the core for any selection of the interval game (N, s) with s given by

$$s(S) = \begin{cases} \left[\sum_{i \in N} x_i, \sum_{i \in N} x_i\right] \text{ if } S = N, \\ \left[\underline{w}(S), \min(\sum_{i \in S} x_i, \overline{w}(S))\right] \text{ otherwise.} \end{cases}$$

Clearly, $\operatorname{Sel}(s) \subseteq \operatorname{Sel}(w)$ and $\operatorname{Sel}(s) \neq \emptyset$. Therefore $\operatorname{gen}(\mathcal{C}(w)) \subseteq \mathcal{SC}(w)$.

Theorem 8. (Core coincidence characterization) For every interval game (N, w)we have gen $(\mathcal{C}(w)) = \mathcal{SC}(w)$ if and only if for every $x \in \mathcal{SC}(w)$ there exist nonnegative vectors $l^{(x)}$ and $u^{(x)}$ such that

$$\sum_{i \in N} (x_i - l_i^{(x)}) = \underline{w}(N), \qquad (4.1)$$

$$\sum_{i \in N} (x_i + u_i^{(x)}) = \overline{w}(N), \tag{4.2}$$

$$\sum_{i \in S} (x_i - l_i^{(x)}) \ge \underline{w}(S), \ \forall S \in 2^N \setminus \{\emptyset\},$$
(4.3)

$$\sum_{i \in S} (x_i + u_i^{(x)}) \ge \overline{w}(S), \ \forall S \in 2^N \setminus \{\emptyset\}.$$
(4.4)

Proof. First, we observe that with Theorem 7 taken into account, we only need to take care of $gen(\mathcal{C}(w)) \supseteq \mathcal{SC}(w)$ to obtain equality.

For $gen(\mathcal{C}(w)) \supseteq \mathcal{SC}(w)$, suppose we have some $x \in \mathcal{SC}(w)$. For this vector, we need to find some interval $X \in \mathcal{C}(w)$ such that $x \in gen(X)$. This is equivalent to the task of finding two nonnegative vectors $l^{(x)}$ and $u^{(x)}$ such that

$$([x_1 - l_1^{(x)}, x_1 + u_1^{(x)}]), [x_2 - l_2^{(x)}, x_2 + u_2^{(x)}], \dots, [x_n - l_n^{(x)}, x_n + u_n^{(x)}]) \in \mathcal{C}(w).$$

From the definition of interval core, we can see that these two vectors have to satisfy exactly the mixed system (4.1) - (4.4). That completes the proof. \Box

Example 1. Consider an interval game with $N = \{1, 2\}$ and $w(\{1\}) = w(\{2\}) = [1, 3]$ and w(N) = [1, 4]. Then vector (2, 2) lies in the core of the selection with $v(\{1\}) = v(\{2\}) = 2$ and v(N) = 4. However, to satisfy equation (4.1), we need to have $\sum_{i \in N} l_i = 3$ which means that either l_1 or l_2 has to be greater than 1. That means we cannot satisfy (4.3) and we conclude that $gen(\mathcal{C}(w)) \neq \mathcal{SC}(w)$.

The following theorem shows that it suffices to check only minimal and maximal vectors of $\mathcal{SC}(w)$.

Theorem 9. For every interval game (N, w), if there exist vectors $q, r, x \in \mathbb{R}^N$ such that $q, r \in \text{gen}(\mathcal{C}(w))$ and $q_i \leq x_i \leq r_i$ for every $i \in N$, then $x \in \text{gen}(\mathcal{C}(w))$.

Proof. Let $l^{(r)}, u^{(r)}, l^{(q)}, u^{(q)}$ be the corresponding vectors in sense of Theorem 8. We need to find vectors $l^{(x)}$ and $u^{(x)}$ satisfying (4.1) - (4.4) of Theorem 8. Let's define vectors $dq, dr \in \mathbb{R}^N$:

 $dq_i = x_i - q_i,$ $dr_i = r_i - x_i.$

Finally, we define $l^{(x)}$ and $u^{(x)}$ in this way:

$$l_i^{(x)} = dq_i + l_i^{(q)},$$

 $u_i^{(x)} = dr_i + u_i^{(r)}$

We need to check that we satisfy (4.1) - (4.4) for x, $l^{(x)}$ and $u^{(x)}$ We will show only (4.2) since remaining ones can be done in a similar way.

$$\sum_{i \in N} (x_i - l_i^{(x)}) = \sum_{i \in N} (x_i - dq_i - l_i^{(q)})$$
$$= \sum_{i \in N} (x_i - x_i + q_i - l_i^{(q)})$$
$$= \sum_{i \in N} (q_i - l_i^{(q)})$$
$$= \underline{w}(N).$$

For games with additive border games (see Definition 7) we get the following result.

Theorem 10. For an interval game (N, w) with additive border games, the payoff vector $(\underline{w}(1), \underline{w}(2), \dots, \underline{w}(n)) \in \text{gen}(\mathcal{C}(w)).$

Proof. First, let us look at an arbitrary additive game (A, v_A) . From additivity condition and the fact that we can write any subset of A as a union of one-player sets we conclude that $v_A(A) = \bigcup_{i \in A} v_A(\{i\})$ for every coalition A. This implies that vector a with $a_i = v_A(\{i\})$ is in the core.

This argument can be applied to border games of (N, w). The vector $q \in \mathbb{R}^N$ with $q_i = \underline{w}(i)$ is an element of the core of (N, \underline{w}) and an element of $\mathcal{SC}(w)$.

For the vector q we want to satisfy the mixed system (4.1)-(4.4) of Theorem 8.

Take the vector l containing zeros only and the vector u with $u_i = |w|(i)$. From the additivity, we get that $\sum_{i \in N} q_i - l_i = \underline{w}(N)$ and $\sum_{i \in N} q_i + u_i = \overline{w}(N)$.

Additivity further implies that inequalities (4.3) and (4.4) hold for q, l and u. Therefore, q is an element of gen($\mathcal{C}(w)$).

Theorem 10 implies that for games with additive border games, we need to check the existence of vectors l and u from (4.1) - (4.4) of Theorem 8 for maximal vectors of \mathcal{SC} only. That follows from the fact that for any vector $y \in \mathcal{SC}(w)$ holds $(\underline{w}(1), \underline{w}(2), \ldots, \underline{w}(n)) \leq y$. In other words, $(\underline{w}(1), \underline{w}(2), \ldots, \underline{w}(n))$ is a minimum vector of $\mathcal{SC}(w)$.

5 Strong imputation and core

In this section, our focus will be on a new concept of *strong imputation* and *strong core*.

Definition 28. (Strong imputation) For a game $(N, w) \in IG^N$ a strong imputation is a vector $x \in \mathbb{R}^N$ such that x is an imputation for every selection of (N, w).

Definition 29. (Strong core) For a game $(N, w) \in IG^N$ the strong core is a set of vectors $x \in \mathbb{R}^N$ such that x is an element of the core of every selection of (N, w).

Strong imputation and strong core can be considered as somewhat "universal" solutions. We show the following three simple facts about the strong core.

Theorem 11. For every interval game with nonempty strong core, w(N) is a degenerate interval.

Proof. The theorem follows easily by the fact that an element c of strong core must be efficient for every selection and therefore $\sum_{i \in N} c_i = \underline{w}(N) = \overline{w}(N)$. \Box

This leads us to a characterization of games with nonempty strong core.

Theorem 12. An interval game (N, w) has a nonempty strong core if and only if w(N) is a degenerate interval and the upper game \overline{w} has a nonempty core.

Proof. The theorem follows from a combination of Theorem 11 and the fact that an element c of the strong core has to satisfy $\sum_{i \in S} c_i \geq v(S)$, $\forall v \in \text{Sel}(w)$, $\forall S \in 2^N \setminus \emptyset$. We see that this fact is equivalent to condition $\sum_{i \in S} c_i \geq \overline{w}(S)$, $\forall S \in 2^N \setminus \emptyset$. Proving an equivalence is then straightforward. We observe that we can easily derive a characterization of games with a nonempty strong imputation set.

The strong core also has the following important property.

Theorem 13. For every element c of the strong core of (N, w), $c \in gen(\mathcal{C}(w))$.

Proof. The vector c has to satisfy mixed system (4.1)-(4.4) of Theorem 8 for some $l, u \in \mathbb{IR}^N$. We show that $l_i = u_i = 0$ will achieve this.

Equations (4.1) and (4.2) are satisfied by taking Theorem 11 into account. Inequalities (4.3) and (4.4) are satisfied as the consequence of Theorem 12. \Box

The reason behind the using of name strong core and strong imputation comes from interval linear algebra, where strong solutions of an interval system are solutions for any realization (selection) of interval matrices A and b in Ax = b.

One could ask why we do not introduce a strong game as a game in which each of its selection has an nonempty core. This is because such games are already defined as *strongly balanced games* (see e.g. [4]).

6 Concluding remarks

Selections of an interval game are very important since they do not contain any additional uncertainty. On the top of that, selection-based classes and the strong core and imputation have the crucial property that although we deal with uncertain data, all possible outcomes preserve important properties. In case of selection classes it is preserving superadditivity, supermodularity etc. In case of the strong core it is an invariant of having particular stable payoffs in each selection. Furthermore, "weak" concepts like SC are important as well since if SC is empty, no selection has a stable payoff.

The importance of studying selection-based classes instead of the existing classes using \succeq operator can be further illustrated by the following two facts:

- Classes based on weakly better operator may contain games with selections that do not have any link with the defining property of their border games and consequently no link with the name of the class. For example, superadditive interval games may contain a selection that is not superadditive.
- Selection-based classes are not contained in corresponding classes based on weakly better operator. Therefore, the results on existing classes are not directly extendable to selection-based classes.

Our results provide an important tool for handling cooperative situations involving interval uncertainty which is a very common situation in various OR problems. Some specific applications of interval games were already examined. See [3,5,6] for applications to mountain situations, airport games, and forest situations, respectively. However, these papers do not use a selection-based approach and therefore to study implications of our approach to them can be a theme for future research. To further study properties of selection-based classes is a possible topic. One of the directions could be to introduce strictly selection convex games or decomposable games and examine them. Another fruitful direction can be extending of the definition of stable set to interval games using selections. For example, one could look at a union or an intersection of stable sets for each selection. Studying Shapley value and other concepts in interval games context may be interesting as well. Some of these problems are work in progress.

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