Extreme functions with an arbitrary number of slopes^{*}

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Abstract

For the one dimensional infinite group relaxation, we construct a sequence of extreme valid functions that are piecewise linear and such that for every natural number $k \ge 2$, there is a function in the sequence with k slopes. This settles an open question in this area regarding a universal bound on the number of slopes for extreme functions. The function which is the pointwise limit of this sequence is an extreme valid function that is continuous and has an infinite number of slopes. This provides a new and more refined counterexample to an old conjecture of Gomory and Johnson stating that all extreme functions are piecewise linear. These constructions are extended to obtain functions for the higher dimensional group problems via the sequential-merge operation of Dey and Richard.

1 Introduction

Let $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. The infinite group relaxation I_b is the set of functions $y : \mathbb{R}^n \to \mathbb{Z}_+$ having finite support (that is, $\{r \in \mathbb{R}^n : y(r) > 0\}$ is a finite set) satisfying

$$\sum_{r \in \mathbb{R}^n} ry(r) \in b + \mathbb{Z}^n.$$
(1.1)

A function $\pi : \mathbb{R}^n \to \mathbb{R}_+$ is valid for I_b if

$$\sum_{r \in \mathbb{R}^n} \pi(r) y(r) \ge 1, \text{ for every } y \in R_b(\mathbb{R}^n, \mathbb{Z}^n).$$
(1.2)

The set I_b has been referred to by multiple names in the literature, see, e.g., [4].

Valid functions for the infinite group relaxation were first introduced by Gomory and Johnson [11, 12] as means to obtain cutting planes for mixed-integer programs. This idea has recently culminated in the study of *cut-generating functions* which has become one of the central aspects of modern cutting plane theory. The surveys of Basu, Hildebrand, Köppe [4, 5] and Basu, Conforti, Di Summa [2] provide a comprehensive introduction to the subject and survey the recent advances.

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The most well known valid function is the Gomory mixed-integer (GMI) function, which is a valid function for n = 1. The GMI is defined as follows:

$$\phi(x) = \begin{cases} \frac{1}{b}x, & 0 \le x < b\\ \frac{1}{1-b} - \left(\frac{1}{1-b}\right)x, & b \le x < 1\\ \phi(x-j), & x \in [j, j+1), \ j \in \mathbb{Z} \setminus \{0\} \end{cases}$$
(1.3)

A valid function π is minimal if $\pi = \pi'$ for every valid function π' such that $\pi' \leq \pi$. The motivation for this definition is the following. Given valid functions π and π' , we say that π' dominates π if for every function $y : \mathbb{R}^n \to \mathbb{Z}_+$ with finite support satisfying the inequality $\sum_{r \in \mathbb{R}^n} \pi(r)y(r) < 1$, the function y also satisfies the inequality $\sum_{r \in \mathbb{R}^n} \pi'(r)y(r) < 1$. Observe that if π' dominates π , then π is redundant for describing I_b . Furthermore, if $\pi' \leq \pi$, then π' dominates π . Thus, if a valid function is not minimal, then it is redundant for describing I_b .

A function $\theta \colon \mathbb{R}^n \to \mathbb{R}$ is subadditive if $\theta(x) + \theta(y) \ge \theta(x+y)$ for all $x, y \in \mathbb{R}^n$. θ satisfies the symmetry condition if $\theta(x) + \theta(b-x) = 1$ for all $x \in \mathbb{R}^n$. Finally, θ is periodic modulo \mathbb{Z}^n if $\theta(x) = \theta(x+z)$ for all $x \in \mathbb{R}^n$ and $z \in \mathbb{Z}^n$.

THEOREM 1.1 (Gomory and Johnson [11]). A function $\pi \colon \mathbb{R}^n \to \mathbb{R}_+$ is a minimal valid function for I_b if and only if $\pi(z) = 0$ for all $z \in \mathbb{Z}^n$, π is subadditive, and π satisfies the symmetry condition. (These conditions imply that π is periodic modulo \mathbb{Z}^n and $\pi(b) = 1$.)

It is easy to check that the Gomory mixed-integer function defined above is subadditive and satisfies the symmetry condition. Therefore, by the above theorem, it is a minimal function.

Minimal functions are the ones that are not dominated by any other function. However a minimal function may be implied by the convex combinations of other valid functions. Gomory and Johnson define a valid function π to be *extreme* if $\pi = \pi_1 = \pi_2$ for every pair of valid functions π_1, π_2 such that $\pi = \frac{\pi_1 + \pi_2}{2}$. If π is a valid function which is extreme, then π is easily seen to be minimal. Therefore extremality is a stronger requirement. An even more stringent definition is that of a *facet*. For any valid function π , define $P(\pi) :=$ $\{y \in R_b(\mathbb{R}^n, \mathbb{Z}^n) : \sum_{r \in \mathbb{R}^n} \pi(r) y(r) = 1\}$. A valid function π is a *facet* if $P(\pi) \subseteq P(\pi')$ implies $\pi = \pi'$ for all valid functions π' . It can be verified that every facet is extreme [6, Lemma 1.3]. It was recently shown that continuous piecewise linear extreme functions are also facets; however, there exist discontinuous piecewise linear extreme functions which are not facets [18].

We will need a formal notion of piecewise linear functions which we introduce now. A regular polyhedral complex in \mathbb{R}^n is a collection of polyhedra P_j , $j \in J$ such that three conditions are satisfied: 1) $\mathbb{R}^n = \bigcup_{j \in J} P_j$, 2) for any $i, j \in J$, $P_i \cap P_j$ is a common face of P_i and P_j and also belongs to the collection, and 3) any bounded subset of \mathbb{R}^n intersects only finitely many polyhedra from the collection. We say a function $\theta : \mathbb{R}^n \to \mathbb{R}$ is piecewise linear if there is a regular polyhedral complex in \mathbb{R}^n such that θ is affine linear over the interior of each polyhedron in the complex. Note this definition allows for discontinuous piecewise linear functions. For a natural number k, we say that a piecewise linear function has k slopes if it has k distinct values for the gradient, where it exists.

THEOREM 1.2 (Gomory and Johnson [11]). Let $\pi \colon \mathbb{R} \to \mathbb{R}_+$ be a minimal valid function which is continuous, piecewise linear and has only 2 slopes. Then π is a facet (and therefore extreme).

In particular, the above theorem implies that the Gomory mixed-integer function is a facet.

For the one-dimensional problem, i.e., n = 1, extreme valid functions or facets that are piecewise linear and have few slopes received the largest number of hits in the shooting experiments [14] and seem to be the most useful in practice. Indeed Gomory and Johnson [13] conjectured that every valid function that is extreme is piecewise linear. This has been disproved by Basu et al. [1].

Minimal valid functions with 3 slopes are not always extreme. However, Gomory and Johnson constructed an extreme function that is piecewise linear with 3 slopes. It appears to be hard to construct extreme functions that are piecewise linear with many slopes. Indeed, until 2013, all known families of piecewise linear extreme functions had at most 4 slopes. This had led Dey and Richard to pose the question of constructing extreme functions with more than 4 slopes at a 2010 Aussois meeting [8]. In 2013, Hildebrand, in an unpublished result, constructed an extreme function that is piecewise linear with 5 slopes and very recently Köppe and Zhou [17] constructed an extreme function that is piecewise linear with 28 slopes. These functions were found through a clever computer search.

Köppe and Zhou [17] expressed the belief that there exist extreme functions that are piecewise linear and have an arbitrary number of slopes (this is also stated as Open Question 2.15 in the survey by Basu, Hildebrand and Köppe [4].) We prove this. More precisely, we show the following:

THEOREM 1.3. Let $b \in \mathbb{R} \setminus \mathbb{Z}$. For $k \geq 2$, there exists a facet (and therefore an extreme valid function) for I_b that is piecewise linear with k slopes.

The proof of Theorem 1.3 provided here is constructive. We define a sequence of functions $\{\pi_k\}_{k=2}^{\infty}$, where π_2 is the Gomory mixed-integer function, and π_3 is an instantiation of a construction of extreme functions that are piecewise linear and have 3 slopes provided by Gomory and Johnson [13]. We first prove some properties about each function π_k in Section 2. In Section 3 we use these properties to show that these functions are subadditive and satisfy the symmetry condition. Therefore each function π_k is a minimal valid function, as it satisfies the conditions of Theorem 1.1. Section 4 is devoted to the proof that each function π_k is a facet.

Our next result states that the function which is the pointwise limit of this sequence is an extreme function that is continuous and has an infinite number of slopes. The proof appears in Section 5.

THEOREM 1.4. Let $b \in \mathbb{R} \setminus \mathbb{Z}$. There exists a continuous function π_{∞} that is a facet (and therefore extreme) for I_b with an infinite number of slopes (i.e., values for the derivative of π_{∞}).

This also provides a different family of counterexamples to the Gomory-Johnson conjecture that all extreme functions are piecewise linear. In contrast, the previous family of counterexamples from [1] all have 2 slopes. Note that in Theorems 1.3 and 1.4, we may assume $b \in (0, 1)$ since extreme functions are periodic with respect to \mathbb{Z} . We give constructions to establish Theorems 1.3 and 1.4 with bin the interval $(0, \frac{1}{2}]$. One may obtain extreme functions for values of $b \in [\frac{1}{2}, 1)$ by reflecting the constructions about 0. This is an example of an automorphism introduced by Gomory and later used by Johnson (Theorem 8.2 in [16], see also Theorem A.1 in the Appendix).

We end the paper by using the *sequential-merge* operation invented by Dey and Richard [9] to construct facets for the *n*-dimensional infinite group relaxation (for any $n \ge 1$) with an arbitrary number of slopes. The idea is to use the sequential-merge operation iteratively on the facets constructed for Theorem 1.3 and the GMI function from (1.3). See Theorem 6.1 for a detailed statement.

2 A Construction of k-Slope Functions π_k

Let $b \in (0, \frac{1}{2}]$. Let π_2 be the Gomory mixed-integer function defined by (1.3).

In constructing π_k for $k \geq 3$, we use the following intervals:

$$\begin{split} I_1^k &:= [0, b(\frac{1}{8})^{k-2}], & I_2^k &:= [b(\frac{1}{8})^{k-2}, 2b(\frac{1}{8})^{k-2}], \\ I_3^k &:= [2b(\frac{1}{8})^{k-2}, b - 2b(\frac{1}{8})^{k-2}], & I_4^k &:= [b - 2b(\frac{1}{8})^{k-2}, b - b(\frac{1}{8})^{k-2}], \\ I_5^k &:= [b - b(\frac{1}{8})^{k-2}, b], & I_6^k &:= [b, 1). \end{split}$$

Given π_{k-1} , where $k-1 \ge 2$, define π_k to be

$$\pi_k(x) = \begin{cases} \left(\frac{2^{k-2}-b}{b-b^2}\right)x, & x \in I_1^k \\ \frac{4^{2-k}}{1-b} - \left(\frac{1}{1-b}\right)x, & x \in I_2^k \\ \frac{1-4^{2-k}}{1-b} - \left(\frac{1}{1-b}\right)x, & x \in I_4^k \\ \frac{1-2^{k-2}}{1-b} + \left(\frac{2^{k-2}-b}{b-b^2}\right)x, & x \in I_5^k \\ \pi_{k-1}(x), & x \in I_3^k \cup I_6^k \\ \pi_k(x-j), & x \in [j,j+1), \ j \in \mathbb{Z} \setminus \{0\} \end{cases}$$

PROPOSITION 2.1. Let $k \ge 2$. Then π_k is well-defined, continuous, nonnegative, and $\pi_k(x) = 0$ if and only if $x \in \mathbb{Z}$.

The proof of Proposition 2.1 is in the Appendix.

Figure 1 shows π_k for various values of k when $b = \frac{1}{2}$. The plots were generated using the help of a software package created by Hong, Köppe, and Zhou [15].

Observe that π_k is built recursively with the Gomory mixed-integer function as the base case. Intuitively, π_k is created by adding to π_{k-1} a perturbation on a small interval to the right of 0 and applying a symmetric perturbation on an interval to the left of b; the interval [b, 1) is kept intact. These small perturbations allow π_k to maintain much of the structure of π_{k-1} , but the number of distinct slopes is increased by one. We collect some useful properties of π_k in Propositions 2.2 and 2.3.

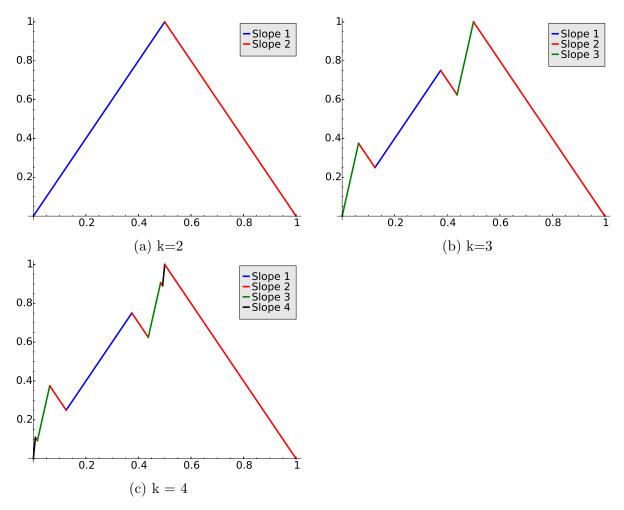


Figure 1: Plots of π_k for $k \in \{1, 2, 3\}$ and $b = \frac{1}{2}$.

PROPOSITION 2.2. Let $k \geq 3$. Then

- (i) $I_1^k \cup I_2^k \subsetneq I_1^{k-1}$ and $I_4^k \cup I_5^k \subsetneq I_5^{k-1}$.
- (ii) If $x \in I_3^k \cup I_6^k$, then $\pi_k(x) = \pi_{k-1}(x)$. If $x \in I_1^k \cup I_2^k$, then $\pi_k(x) \ge \pi_{k-1}(x)$. If $x \in I_4^k \cup I_5^k$, then $\pi_k(x) \le \pi_{k-1}(x)$.
- (iii) For any $x \in (0,1) \setminus \{b\}$, there exists some natural number N_x such that $x \in I_3^{N_x} \cup I_6^{N_x}$ and $\pi_k(x) = \pi_{N_x}(x)$ whenever $k \ge N_x$.

Proof.

(i) Observe that

$$b\left(\frac{1}{8}\right)^{k-3} = 8b\left(\frac{1}{8}\right)^{k-2} > 2b\left(\frac{1}{8}\right)^{k-2}$$
.

By the definitions of I_1^k, I_2^k and I_1^{k-1} , it follows that $I_1^k \cup I_2^k \subsetneq I_1^{k-1}$. A similar argument shows that $I_4^k \cup I_5^k \subsetneq I_5^{k-1}$.

(*ii*) Let $x \in [0, 1)$. If $x \in I_3^k \cup I_6^k$, then $\pi_k(x) = \pi_{k-1}(x)$ by definition. If $x \in I_1^k$, then from (*i*) it follows that $x \in I_1^{k-1}$. Note that

$$\left(\frac{2^{k-2}-b}{b-b^2}\right)x \ge \left(\frac{2^{k-3}-b}{b-b^2}\right)x ,$$

and so $\pi_k(x) \ge \pi_{k-1}(x)$. If $x \in I_2^k$, then again from (i), $x \in I_1^{k-1}$ and it follows that

$$\frac{4^{2-k}}{1-b} - \left(\frac{1}{1-b}\right) x = \left(\frac{1}{1-b}\right) \left(4^{2-k} - x\right) \\
\geq \left(\frac{1}{1-b}\right) \left(4^{2-k} - 2b\left(\frac{1}{8}\right)^{k-2}\right) \quad \text{since } x \in I_2^k \\
= \left(\frac{1}{b-b^2}\right) \left(2^{k-3} \left(2b\left(\frac{1}{8}\right)^{k-2}\right) - b\left(2b\left(\frac{1}{8}\right)^{k-2}\right)\right) \\
\geq \left(\frac{2^{k-3}-b}{b-b^2}\right) x \qquad \text{since } x \in I_2^k.$$

Hence $\pi_k(x) \ge \pi_{k-1}(x)$ on $I_1^k \cup I_2^k$. A similar argument shows that $\pi_k(x) \le \pi_{k-1}(x)$ on $I_4^k \cup I_5^k$.

(*iii*) Notice that $I_3^k \subseteq I_3^{k+1}$ for every natural number k, and as $k \to \infty$, I_3^k converges to (0, b). Thus, there exists some natural number N_x such that $x \in I_3^k \cup I_6^k$ for every natural number $k \ge N_x$. By the definition of π_k , if $k \ge N_x$, then $\pi_k(x) = \pi_{N_x}(x)$.

PROPOSITION 2.3. For each integer $k \geq 2$, the function π_k is piecewise linear and has k slopes taking values $-\frac{1}{1-b}$ and $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^k$. Moreover, if $k \geq 3$, then π_k has the k-2 slopes $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^{k-1}$ on I_3^k and the slope $-\frac{1}{1-b}$ on I_6^k .

Proof. We proceed by induction. For π_2 and π_3 , the result is readily verified by the definitions. So, assume that for $k-1 \ge 3$, π_{k-1} is piecewise linear with k-1 slopes and has the k-3slopes $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^{k-2}$ on I_3^{k-1} and the slope $-\frac{1}{1-b}$ on I_6^{k-1} . Consider π_k . The fact that π_k is piecewise linear follows from the definition of π_k , and

the induction hypothesis that π_{k-1} is piecewise linear.

It is left to consider the slope values of π_k . By Proposition 2.2 (*ii*), $\pi_k = \pi_{k-1}$ everywhere except $I_1^k \cup I_2^k$ and $I_4^k \cup I_5^k$, on which π_k takes on slope values $\frac{2^{k-2}-b}{b-b^2}$ and $-\frac{1}{1-b}$ by definition. Since $I_1^k \cup I_2^k \subsetneq I_1^{k-1}$ and $I_4^k \cup I_5^k \subsetneq I_5^{k-1}$ by Proposition 2.2 (i), it follows from the induction hypothesis that π_k also has slopes taking values $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^{k-1}$. Thus, π_k takes on slopes values $\left\{\frac{2^{i-2}-b}{b-b^2}\right\}_{i=2}^k$ and $-\frac{1}{1-b}$.

Finally, by definition, $\pi_k = \pi_{k-1}$ on $I_3^k \cup I_6^k$. Moreover, since π_k has slope values $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^k$, and π_{k-1} has slope values $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^{k-1}$, the only new slope in π_k is $\frac{2^{k-2}-b}{b-b^2}$, which only appears on $I_1^k \cup I_5^k$. Thus, π_k has the k-2 slopes $\{\frac{2^{i-2}-b}{b-b^2}\}_{i=2}^{k-1}$ on I_3^k and the slope $-\frac{1}{1-h}$ on I_6^k .

Proof of Minimality of π_k 3

In the proof of Theorem 1.3, we will show that π_k is a facet using the so-called Facet Theorem - see Theorem 4.2 in Section 4. Applying the Facet Theorem to π_k requires that π_k be a minimal valid function for I_b , which we verify in this section. Since by definition π_k is nonnegative, $\pi_k(0) = 0$, and π_k is periodic, by Theorem 1.1, it is sufficient to show that (a) $\pi_k(x) = \pi_k(b-x)$ for all $x \in [0,1)$, i.e. that π_k satisfies the symmetry condition, and (b) π_k is subadditive. We show (a) and (b) in Propositions 3.1 and 3.2, respectively.

PROPOSITION 3.1. π_k satisfies the symmetry condition for all $k \geq 2$.

Proof. We proceed by induction on k. The Gomory mixed-integer function is known to be minimal, and hence π_2 is symmetric. Assume π_{k-1} satisfies the symmetry condition for $k-1 \geq 2$ and consider $x \in [0,1)$. Observe that $x \in I_1^k$ if and only if $b-x \in I_5^k$. Therefore if $x \in I_1^k$, then

$$\pi_k(x) + \pi_k(b-x) = \left(\frac{2^{k-2}-b}{b-b^2}\right)x + \frac{1-2^{k-2}}{1-b} + \left(\frac{2^{k-2}-b}{b-b^2}\right)(b-x) = 1.$$

A similar argument can be used to show that π_k satisfies the symmetry condition on the intervals I_2^k and I_4^k . If $x \notin I_1^k \cup I_2^k \cup I_4^k \cup I_5^k$, then $b - x \notin I_1^k \cup I_2^k \cup I_4^k \cup I_5^k$, and so symmetry holds by induction.

PROPOSITION 3.2. π_k is subadditive for all $k \geq 2$.

Proof. We proceed by induction on k. Note that π_2 is subadditive, so assume π_{k-1} is subaddivide for $k-1 \geq 2$. By periodicity of π_k , it suffices to check $\pi_k(x) + \pi_k(y) \geq \pi_k(x+y)$ for all $x, y \in [0, 1)$ and $x \leq y$.

CLAIM. If $y \in I_6^k = [b, 1)$, then $\pi_k(x+y) \le \pi_k(x) + \pi_k(y)$.

Proof of Claim. Since π_k is piecewise linear and continuous by Propositions 2.1 and 2.3, we may integrate it over any bounded domain. A direct calculation shows

$$\pi_k(x+y) = \pi_k(x+(y-1)) \qquad \text{by periodicity of } \pi_k$$

$$= \pi_k(x) + \int_x^{x-(1-y)} \pi'_k(t) dt$$

$$= \pi_k(x) + \int_{x-(1-y)}^x -\pi'_k(t) dt$$

$$\leq \pi_k(x) + \int_y^1 -\pi'_k(t) dt$$

$$= \pi_k(x) - \pi_k(1) + \pi_k(y)$$

$$= \pi_k(x) + \pi_k(y) \qquad \text{since } \pi_k(1) = 0.$$

The inequality follows from Proposition 2.3, as the minimum value of the slope for π_k is $-\frac{1}{1-b}$ and this is the slope over the interval $[b, 1] \supseteq [y, 1]$. This concludes the proof of the claim. \diamond

By the above claim, it suffices to consider the case y < b. Since $b \leq \frac{1}{2}$, this implies that $x \leq y \leq x + y < 1$.

Case 1: $x+y \in I_1^k \cup I_2^k$. Note that the derivative π'_k is nonincreasing on $I_1^k \cup I_2^k \setminus \{b(\frac{1}{8})^{k-2}\}$. Thus,

$$\pi_k(x+y) = \pi_k(x) + \int_x^{x+y} \pi'_k(t) dt \le \pi_k(x) + \int_0^y \pi'_k(t) dt = \pi_k(x) + \pi_k(y).$$

Case 2: $x + y \in I_3^k$. Since $x, y \in I_1^k \cup I_2^k \cup I_3^k$ we have that

$$\pi_k(x) + \pi_k(y) \ge \pi_{k-1}(x) + \pi_{k-1}(y) \ge \pi_{k-1}(x+y) = \pi_k(x+y) ,$$

where the first inequality comes from Proposition 2.2 (ii), the second inequality comes from the induction hypothesis, and the final inequality comes again from Proposition 2.2 (ii).

Case 3: $x + y \in I_4^k \cup I_5^k$. If $y \in I_1^k \cup I_2^k \cup I_3^k$, then using the induction hypothesis and Proposition 2.2 (*ii*), it follows that

$$\pi_k(x) + \pi_k(y) \ge \pi_{k-1}(x) + \pi_{k-1}(y) \ge \pi_{k-1}(x+y) \ge \pi_k(x+y).$$

If $y \in I_4^k \cup I_5^k$, then $x \in [0, b-y]$ and $b-y \in I_1^k \cup I_2^k$. Hence, $x \in I_1^k \cup I_2^k$. Also, $b-(x+y) \in I_1^k \cup I_2^k$ since $x + y \in I_4^k \cup I_5^k$. Thus, we can apply Case 1 to the values x and b - (x + y) to obtain $\pi_k(b-y) \leq \pi_k(b-(x+y)) + \pi_k(x)$. Using this, we see that

$$\pi_k(x+y) = 1 - \pi_k(b - (x+y))$$
 by the symmetry property
$$\leq 1 - \pi_k(b-y) + \pi_k(x)$$

$$= \pi_k(y) + \pi_k(x)$$
 by the symmetry property.

Case 4: $x + y \in I_6^k$. π_k has a slope of $-\frac{1}{1-b}$ on the interval [b, x + y]. Moreover, by Proposition 2.3, this is the minimum slope that π_k admits. Therefore,

$$\pi_k(x+y) = \pi_k(b) + \int_b^{x+y} \pi'_k(t)dt \leq 1 + \int_{b-x}^y \pi'_k(t)dt = 1 + (\pi_k(y) - \pi_k(b-x)) = \pi_k(x) + \pi_k(y) ,$$

where the last equality follows by the symmetry of π_k .

4 π_k is a facet

By Proposition 2.3, in order to prove Theorem 1.3 it suffices to show the following result.

PROPOSITION 4.1. π_k is a facet for each $k \geq 2$.

We dedicate the remainder of the section to proving Proposition 4.1. To this end, given a function $\theta : \mathbb{R}^n \to \mathbb{R}$, define

$$E(\theta) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : \theta(x) + \theta(y) = \theta(x+y)\}.$$
(4.1)

Our proof of Proposition 4.1 is based on the *Facet Theorem*, which gives a sufficient condition for a function to be a facet [6, 11], and the *Interval Lemma*, which first appeared in [13], and was subsequently elaborated upon in [10, 9, 7, 3]; see also the survey [4, 5].

THEOREM 4.2 (Facet Theorem). Let $\pi : \mathbb{R}^n \to \mathbb{R}_+$ be a minimal valid function for I_b for some $b \in \mathbb{R}^n \setminus \mathbb{Z}^n$. Suppose that for every minimal function $\theta : \mathbb{R}^n \to \mathbb{R}_+$ satisfying $E(\pi) \subseteq E(\theta)$, it follows that $\theta = \pi$. Then π is a facet.

LEMMA 4.3 (Interval Lemma). Let U, V be nondegenerate closed intervals in \mathbb{R} . If $\theta : \mathbb{R} \to \mathbb{R}$ is bounded over U and V, and $U \times V \subseteq E(\theta)$, then θ is affine over U, V and U + V with the same slope.

We will often use the above lemma when θ is a minimal valid function. In this case θ is bounded, as $0 \le \theta \le 1$. We also say a function $\theta : \mathbb{R} \to \mathbb{R}$ is *locally bounded* if it is bounded on every compact interval.

PROPOSITION 4.4. Let $\theta : \mathbb{R} \to \mathbb{R}_+$ be such that $\theta(0) = 0$ and $\theta(x+z) = \theta(x) + \theta(z)$ for all $x \in \mathbb{R}$ and $z \in \mathbb{Z}$. Then θ is periodic, i.e., $\theta(x+z) = \theta(x)$ for all $x \in \mathbb{R}$ and $z \in \mathbb{Z}$.

Proof. It suffices to show that $\theta(z) = 0$ for all $z \in \mathbb{Z}$. This is true since $0 = \theta(0) = \theta(-z) + \theta(z)$ for all $z \in \mathbb{Z}$ and θ is nonnegative.

In the following Propositions 4.5, 4.6, 4.7, 4.8, we develop some tools towards proving facetness.

PROPOSITION 4.5. Let $k \geq 3$ and let π be a minimal valid function such that $\pi = \pi_k$ on I_6^k . Then for all locally bounded functions $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $E(\pi) \subseteq E(\theta)$ satisfying $\theta(0) = 0, \theta(b) = 1$, we must have $\theta = \pi = \pi_k$ on $I_6^k \cup \{1\}$.

Proof. Note that $I_6^k \cup \{1\} \equiv [\frac{1+b}{2}, 1] + [\frac{1+b}{2}, 1] \pmod{1}$. Since π is minimal, Theorem 1.1 implies π is periodic. Since $E(\pi) \subseteq E(\theta)$, Proposition 4.4 shows that θ is periodic. In particular, $\theta(1) = 0 = \pi(1)$ and $\theta(b) = 1 = \pi(b)$. Hence $\pi = \theta$ on the endpoints of $I_6^k \cup \{1\}$. Moreover, $x, y \in [\frac{1+b}{2}, 1]$ implies that

$$\pi(x) + \pi(y) = \left(\frac{1}{1-b} - \left(\frac{1}{1-b}\right)x\right) + \left(\frac{1}{1-b} - \left(\frac{1}{1-b}\right)y\right) \text{ since } \pi = \pi_k \text{ on } I_6^k$$
$$= \frac{1}{1-b} - \left(\frac{1}{1-b}\right)(x+y-1)$$
$$= \pi(x+y-1)$$
$$= \pi(x+y) \text{ by periodicity.}$$

Hence $[\frac{1+b}{2}, 1] \times [\frac{1+b}{2}, 1] \subseteq E(\pi) \subseteq E(\theta)$. Lemma 4.3 then implies that θ is affine over $I_6^k \cup \{1\}$. Since π is also affine over $I_6^k \cup \{1\}$ and $\pi = \theta$ on the endpoints of $I_6^k \cup \{1\}$, we must have $\pi = \theta$ on $I_6^k \cup \{1\}$.

PROPOSITION 4.6. Let $k \geq 3$ and let π be a minimal valid function such that $\pi = \pi_k$ on I_3^3 . Then for all locally bounded functions $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $E(\pi) \subseteq E(\theta)$ satisfying $\theta(\frac{b}{2}) = \frac{1}{2}$, we must have $\theta = \pi = \pi_k$ on I_3^3 .

Proof. Let $U = \begin{bmatrix} \frac{b}{4}, \frac{3b}{8} \end{bmatrix} \subseteq I_3^3$ and note that $U + U = \begin{bmatrix} \frac{b}{2}, \frac{3b}{4} \end{bmatrix} \subseteq I_3^3$. For $x, y \in U$, since $\pi = \pi_k$ on I_3^3 we see that

$$\pi(x) + \pi(y) = \frac{1}{b}x + \frac{1}{b}y = \frac{1}{b}(x+y) = \pi(x+y).$$

Hence $U \times U \subseteq E(\pi) \subseteq E(\theta)$. Using Lemma 4.3, θ is affine over $[\frac{b}{2}, \frac{3b}{4}]$. By assumption, $\theta(\frac{b}{2}) = \pi(\frac{b}{2}) = \frac{1}{2}$. Using this and $(\frac{b}{4}, \frac{b}{4}) \in E(\pi) \subseteq E(\theta)$, it follows that $\theta(\frac{b}{4}) = \pi(\frac{b}{4}) = \frac{1}{4}$. Since π satisfies the symmetry condition and $E(\pi) \subseteq E(\theta)$, θ also satisfies the symmetry condition. This implies $\theta(\frac{3b}{4}) = \pi(\frac{3b}{4}) = \frac{3}{4}$. Therefore, by the affine structure of θ and π over $[\frac{b}{2}, \frac{3b}{4}]$, it follows that $\theta = \pi$ on $[\frac{b}{2}, \frac{3b}{4}]$. The symmetric property of θ and π then yields $\theta = \pi$ on $[\frac{b}{4}, \frac{b}{2}]$ and thus on I_3^3 .

PROPOSITION 4.7. Let $k \geq 3$ and $j \in \{3, \ldots, k\}$. Let π be a minimal valid function such that $\pi = \pi_k$ on $I_2^j \cup I_4^j \cup I_6^j$. Then for all locally bounded functions $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $E(\pi) \subseteq E(\theta)$ and $\theta = \pi$ on $I_3^j \cup I_6^j \cup \{1\}$, we must have $\theta = \pi = \pi_k$ on $I_2^j \cup I_4^j$.

Proof. Let $U = \left[\frac{3}{2}b\left(\frac{1}{8}\right)^{j-2}, 2b\left(\frac{1}{8}\right)^{j-2}\right] \subseteq I_2^j$ and $V = \left[1 - \frac{1}{2}b\left(\frac{1}{8}\right)^{j-2}, 1\right] \subseteq I_6^j \cup \{1\}$. Observe that $U + V \equiv I_2^j$ (modulo 1). Moreover, $x \in U$ and $y \in V$ implies

$$\pi(x) + \pi(y) = \left(\frac{4^{2-j}}{1-b} - \left(\frac{1}{1-b}\right)x\right) + \left(\frac{1}{1-b} - \left(\frac{1}{1-b}\right)y\right) \text{ since } \pi = \pi_k \text{ on } I_2^j \cup I_6^j$$

= $\frac{4^{2-j}}{1-b} - \left(\frac{1}{1-b}\right)(x+y-1)$
= $\pi(x+y-1) = \pi(x+y)$ by periodicity.

Thus, $U \times V \subseteq E(\pi) \subseteq E(\theta)$, and by Lemma 4.3, π and θ are affine over I_2^j with the same slope as their corresponding slopes over V. Since $\theta = \pi = \pi_k$ over I_6^j and $V \subseteq I_6^j \cup \{1\}$, all three functions have the same slope over I_2^j . Since $\theta = \pi$ on I_3^j by assumption, it must be that $\theta\left(2b\left(\frac{1}{8}\right)^{j-2}\right) = \pi\left(2b\left(\frac{1}{8}\right)^{j-2}\right)$. Therefore, $\theta = \pi$ on I_2^j . Since π satisfies the symmetry

condition and $E(\pi) \subseteq E(\theta)$, θ also satisfies the symmetry condition. Using symmetry, we see that $\theta = \pi$ over I_4^j .

PROPOSITION 4.8. Let $k \geq 4$ and let $j \in \{3, \ldots, k-1\}$. Let π be a minimal valid function such that $\pi = \pi_k$ on $I_1^j \setminus \operatorname{int}(I_1^{j+1} \cup I_2^{j+1})$ and $I_5^j \setminus \operatorname{int}(I_4^{j+1} \cup I_5^{j+1})$. Then for all locally bounded functions $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $E(\pi) \subseteq E(\theta)$ and $\theta = \pi$ on $I_2^j \cup I_3^j \cup I_4^j \cup I_6^j$, we must have $\theta = \pi = \pi_k$ on $I_1^j \setminus \operatorname{int}(I_1^{j+1} \cup I_2^{j+1})$ and $I_5^j \setminus \operatorname{int}(I_4^{j+1} \cup I_5^{j+1})$.

Proof. By minimality, we have that $\pi(0) = \pi_k(0) = 0$. Since $E(\pi) \subseteq E(\theta)$, we have that $\pi(0) + \pi(0) = \pi(0)$ implies $\theta(0) + \theta(0) = \theta(0)$. This shows that $\theta(0) = 0 = \pi(0) = \pi_k(0)$. Now consider $I^* := I_1^j \setminus (\operatorname{int}(I_1^{j+1} \cup I_2^{j+1}) \cup \{0\}) = \left[2b\left(\frac{1}{8}\right)^{j-1}, b\left(\frac{1}{8}\right)^{j-2}\right]$. Let

$$U = \left[2b\left(\frac{1}{8}\right)^{j-1}, 4b\left(\frac{1}{8}\right)^{j-1}\right] \subseteq I^*.$$

Note that

$$U + U = \left[4b\left(\frac{1}{8}\right)^{j-1}, b\left(\frac{1}{8}\right)^{j-2}\right] \subseteq I^*$$

and $U \cup (U + U) = I^*$. A direct calculation shows that $I^* \subseteq I_3^m$ for all $m \ge j + 1$. Thus, by the definition of π_k , it follows that $\pi_k(x) = \pi_j(x) = (\frac{2^{j-2}-b}{b-b^2})x$ for all $x \in I^*$. Using this and the fact that $\pi = \pi_k$ over I^* , we see that

$$\pi(x) + \pi(y) = \pi_k(x) + \pi_k(x) = \left(\frac{2^{j-2} - b}{b - b^2}\right)x + \left(\frac{2^{j-2} - b}{b - b^2}\right)y$$
$$= \left(\frac{2^{j-2} - b}{b - b^2}\right)(x + y) = \pi_k(x + y) = \pi(x + y)$$

for $x, y \in U$, and so $U \times U \subseteq E(\pi) \subseteq E(\theta)$. By Lemma 4.3, θ is affine over U + U and U with the same slope, and thus affine over I^* . Similarly, π is affine over I^* .

Since $\theta = \pi$ on I_2^j , we have $\theta\left(b\left(\frac{1}{8}\right)^{j-2}\right) = \pi\left(b\left(\frac{1}{8}\right)^{j-2}\right)$. Also, since $2b\left(\frac{1}{8}\right)^{j-1}$, $4b\left(\frac{1}{8}\right)^{j-1} \in U$ and $U \times U \subseteq E(\pi) \subseteq E(\theta)$, we see that

$$4\theta \left(2b \left(\frac{1}{8}\right)^{j-1}\right) = 2\theta \left(2b \left(\frac{1}{8}\right)^{j-1} + 2b \left(\frac{1}{8}\right)^{j-1}\right)$$
$$= 2\theta \left(4b \left(\frac{1}{8}\right)^{j-1}\right) = \theta \left(4b \left(\frac{1}{8}\right)^{j-1} + 4b \left(\frac{1}{8}\right)^{j-1}\right)$$
$$= \theta \left(b \left(\frac{1}{8}\right)^{j-2}\right)$$
$$= \pi \left(b \left(\frac{1}{8}\right)^{j-2}\right) \qquad \text{since } \theta = \pi \text{ on } I_2^j$$
$$= 4\pi \left(2b \left(\frac{1}{8}\right)^{j-1}\right).$$

Thus, $\theta\left(2b\left(\frac{1}{8}\right)^{j-1}\right) = \pi\left(2b\left(\frac{1}{8}\right)^{j-1}\right)$, and so $\theta = \pi$ on the endpoints of I^* . Since both functions are affine on I^* , it follows that $\theta = \pi$ on I^* . Since π satisfies the symmetry condition and $E(\pi) \subseteq E(\theta)$, θ also satisfies the symmetry condition. Symmetry of θ and π yields that $\theta = \pi$ over $I_5^j \setminus \operatorname{int}(I_4^{j+1} \cup I_5^{j+1})$.

LEMMA 4.9. Let $k \geq 3$ and $j \in \{3, \ldots, k\}$. Let π be a minimal valid function such that $\pi = \pi_k$ on $I_3^j \cup I_6^j$. Then for all locally bounded functions $\theta : \mathbb{R} \to \mathbb{R}_+$ such that $E(\pi) \subseteq E(\theta)$ satisfying $\theta(0) = 0, \theta(b) = 1$, we must have $\theta = \pi = \pi_k$ on $I_3^j \cup I_6^j$.

Proof. By Proposition 4.5, we obtain $\theta = \pi$ on $I_6^k = I_6^j = I_6^3$. We prove $\theta = \pi$ on I_3^j by induction on j. For j = 3, the result follows from Proposition 4.6 (observe that $E(\pi) \subseteq E(\theta)$ implies that θ is symmetric and therefore $\theta(\frac{b}{2}) = \frac{1}{2}$). We assume the result holds for some jsuch that $3 \leq j \leq k-1$ and show that it holds for j+1. Observe that this assumption implies that $k \geq 4$. Note that $I_3^{j+1} \cup \{0, b\} = (I_1^j \setminus \operatorname{int}(I_1^{j+1} \cup I_2^{j+1})) \cup I_2^j \cup I_3^j \cup I_4^j \cup (I_5^j \setminus \operatorname{int}(I_4^{j+1} \cup I_5^{j+1}))$. By the induction hypothesis, $\theta = \pi$ on I_3^j . Since $\pi(0) = \pi(1) = 0$, it follows that $\pi(0) = \pi(1) = \pi(0) + \pi(1)$. Thus, since $E(\pi) \subseteq E(\theta)$, we have $0 = \theta(0) = \theta(0) + \theta(1) = \theta(1)$. By Proposition 4.7, $\theta = \pi$ on $I_2^j \cup I_4^j$. Using this, Proposition 4.8 implies that $\theta = \pi$ on $(I_1^j \setminus \operatorname{int}(I_1^{j+1} \cup I_2^{j+1})) \cup (I_5^j \setminus \operatorname{int}(I_4^{j+1} \cup I_5^{j+1}))$.

of Proposition 4.1. If k = 2, then the fact that π_k is a facet follows by Theorem 1.2. Consider the setting $k \geq 3$.

Let $\theta : \mathbb{R} \to \mathbb{R}_+$ be a minimal valid function for I_b such that $E(\pi_k) \subseteq E(\theta)$. Since θ is minimal, a consequence of Theorem 1.1 is that θ is locally bounded. Using $\pi = \pi_k$ in Lemma 4.9, it follows that $\theta = \pi_k$ on $I_3^k \cup I_6^k$. From Proposition 4.7 and again setting $\pi = \pi_k$, we obtain that $\theta = \pi_k$ on $I_2^k \cup I_4^k$. It is left to show that $\theta = \pi_k$ on I_1^k and I_5^k . Let $U = \left[0, \frac{b}{2} \left(\frac{1}{8}\right)^{k-2}\right]$ and observe that $U + U = \left[0, b \left(\frac{1}{8}\right)^{k-2}\right] = I_1^k$. It follows from

Let $U = \left[0, \frac{b}{2} \left(\frac{1}{8}\right)^{k-2}\right]$ and observe that $U + U = \left[0, b \left(\frac{1}{8}\right)^{k-2}\right] = I_1^k$. It follows from the definition of π_k that $\pi_k(x) + \pi_k(y) = \pi_k(x+y)$ for all $x, y, x+y \in I_1^k$, so $U \times U \subseteq E(\pi_k) \subseteq E(\theta)$. Since θ and π_k are minimal, $\theta(0) = \pi_k(0) = 0$. Also, since $\theta = \pi_k$ on I_2^k , $\theta\left(b\left(\frac{1}{8}\right)^{k-2}\right) = \pi_k\left(b\left(\frac{1}{8}\right)^{k-2}\right)$. Thus $\theta = \pi_k$ on the endpoints of I_1^k . Moreover, Lemma 4.3 implies that θ is affine over I_1^k . Since π_k is also affine over I_1^k and $\theta = \pi_k$ at the endpoints, we have $\theta = \pi_k$ on I_1^k . The fact that $\theta = \pi_k$ on I_5^k follows by symmetry. Therefore, $\theta = \pi_k$ everywhere. By Theorem 4.2, π_k is a facet.

of Theorem 1.3. By Proposition 2.3, the function π_k is piecewise linear and has k slopes. Every valid function that is a facet is also extreme. By Proposition 4.1, π_k is a facet (and therefore extreme). Thus, π_k proves the result.

5 Proof of Theorem 1.4

of Theorem 1.4. For $x \in [0,1) \setminus \{0,b\}$, let N_x be the natural number guaranteed by Proposition 2.2 (*iii*), that is $x \in I_3^{N_x} \cup I_6^{N_x}$ and $\pi_k(x) = \pi_{N_x}(x)$ whenever $k \ge N_x$. Define the function $\pi_{\infty}(x) : \mathbb{R} \to [0,1]$ by

$$\pi_{\infty}(x) = \begin{cases} 0, & x = 0\\ 1, & x = b\\ \pi_{N_{x}}(x), & x \in [0, 1) \setminus \{0, b\}\\ \pi_{\infty}(x - j), & x \in [j, j + 1), \ j \in \mathbb{Z} \setminus \{0\} \end{cases}.$$

We claim that the sequence $\{\pi_i\}_{i=2}^{\infty}$ converges uniformly to π_{∞} . To this end, let $\varepsilon > 0$. Choose a large enough $N \in \mathbb{N}$ such that $\frac{2^{4-3N}(2^N-4b)}{1-b} < \varepsilon$. Let $x \in \mathbb{R}$ and $k \ge N$. We consider cases on x.

Case 1: Assume $x \in I_1^k \cup I_2^k$. By Proposition 2.2 (i), $I_1^m \cup I_2^m \supseteq I_1^{m+1} \cup I_2^{m+1}$ for all $m \in \mathbb{N}$. For $m \ge k$, the maximum value of π_m on $I_1^m \cup I_2^m$ is $\frac{2^{4-3m}(2^m-4b)}{1-b} \le \frac{2^{4-3k}(2^k-4b)}{1-b} \le \frac{2^{4-3k}(2^k-4b)}{1-b} \le \frac{2^{4-3k}(2^k-4b)}{1-b}$. It follows that $0 \le \pi_\infty(x) \le \frac{2^{4-3N}(2^N-4b)}{1-b} < \varepsilon$. Similarly, $0 \le \pi_k(x) \le \varepsilon$. Thus, $|\pi_k(x) - \pi_\infty(x)| < \varepsilon$.

Case 2: Assume $x \in I_4^k \cup I_5^k$. Then $|\pi_{\infty}(x) - \pi_k(x)| < \varepsilon$ follows by the symmetry of each function in the sequence $\{\pi_i\}_{i=2}^{\infty}$ along with Case 1.

Case 3: Assume $x \in I_3^k \cup \overline{I}_6^k$. Note that $I_3^k \cup I_6^k \subseteq I_3^m \cup I_6^m$ for every $m \in \mathbb{N}$ such that $m \geq k$. Therefore, by definition of each π_m for $m \geq k$, we see that $\pi_m(x) = \pi_k(x)$ for all $m \geq k$. Hence $\pi_{\infty}(x) = \pi_k(x)$.

Case 4: Assume $x \in \{0, b\}$. Then $\pi_{\infty}(x) = \pi_k(x)$ follows directly by definition of $\pi_{\infty}(0)$ and $\pi_{\infty}(b)$.

Case 5: Assume that $x \in [j, j + 1)$ for $j \in \mathbb{Z} \setminus \{0\}$. Then using the periodicity of each function in $\{\pi_i\}_{i=2}^{\infty}$ and noting $N_x = N_{x-j}$, we obtain $|\pi_{\infty}(x) - \pi_k(x)| = |\pi_{\infty}(x-j) - \pi_k(x-j)| < \varepsilon$ by using Cases 1-4.

Since each π_k is minimal, by a standard limit argument, π_{∞} is minimal (Proposition 4 in [10], Proposition 6.1 in [5]). Also, since π_{∞} is the uniform limit of continuous functions, it too is continuous.

We next show that π_{∞} is a facet. Let θ be any minimal function such that $E(\pi_{\infty}) \subseteq E(\theta)$. If x = 0 or x = b, then $\pi_{\infty}(x) = \theta(x)$ by the minimality of π_{∞} and θ . So assume that $x \notin \{0, b\}$. Recall that $x \in I_3^{N_x} \cup I_6^{N_x}$. Observe that $\pi_{\infty} = \pi_{N_x}$ on $I_3^{N_x} \cup I_6^{N_x}$. Hence, by applying Lemma 4.9 with $k = j = N_x$ and $\pi = \pi_{\infty}$, we obtain that $\theta(x) = \pi_{\infty}(x)$. Therefore, $\theta = \pi_{\infty}$ everywhere. By Theorem 4.2, π_{∞} is a facet.

We finally verify that π_{∞} has infinitely many slopes. Note that for any $k \geq 3$, $\pi_{\infty} = \pi_k$ on $I_3^k \cup I_6^k$ and, by Proposition 2.3, π_k has k-1 different slopes on $I_3^k \cup I_6^k$.

6 Facets for higher dimensional group relaxations

One can ask if it is possible to find extreme functions with arbitrary number of slopes for the higher-dimensional infinite group relaxation. For $b \in \mathbb{R} \setminus \mathbb{Z}$, a trivial way to generalize to higher dimensions is to simply define $\pi_k^n : \mathbb{R}^n \to \mathbb{R}_+$ as $\pi_k^n(x_1, x_2, \ldots, x_n) = \pi_k(x_1)$ and $\pi_{\infty}^n : \mathbb{R}^n \to \mathbb{R}_+$ as $\pi_{\infty}^n(x_1, x_2, \ldots, x_n) = \pi_{\infty}(x_1)$. By Theorem 19.35 in [19], the functions π_k^n and π_{∞}^n are extreme for $I_{\tilde{b}}$ for $\tilde{b} \in \{b\} \times \{0\}^{n-1}$. However, one can ask whether there are more "non-trivial" examples. In particular, one can ask whether there exist genuinely n-dimensional extreme functions with arbitrary number of slopes for all $n \ge 1$. A function $\theta : \mathbb{R}^n \to \mathbb{R}$ is genuinely n-dimensional if there does not exist a linear map $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and a function $\theta' : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\theta = \theta' \circ T$. The construction of such a "non-trivial" facet is the main result in this section. We use the notation $\mathbf{1}_m$ to denote the vector of all ones in \mathbb{R}^m and $b\mathbf{1}_m$ to denote the vector in \mathbb{R}^m such that every component is equal to b.

THEOREM 6.1. Let $n, k \in \mathbb{N}$. For any $b \in \mathbb{R} \setminus \mathbb{Z}$, there exists a function $\Pi_k^n : \mathbb{R}^n \to \mathbb{R}_+$ such that Π_k^n has at least k slopes, is genuinely n dimensional, and is a facet (and thus extreme) for the n-dimensional infinite group relaxation I_{b1_n} .

We provide a constructive argument for the proof of Theorem 6.1 using the *sequential-merge* operation developed by Dey and Richard [9]. In particular, we employ Theorem 5 in [9], the assumptions of which will be proved throughout this section. The proof of Theorem 6.1 is the collection of these results and is presented at the end of the section. We begin with some definitions relating to sequential-merge.

Notation: Let $m \in \mathbb{N}$ and $x, y \in \mathbb{R}^m$. For $i \in \{1, \ldots, m\}$, we use the notation x_i to denote the *i*-th component of x. We define the vectors $\lfloor x \rfloor := (\lfloor x_1 \rfloor, \lfloor x_2 \rfloor, \ldots, \lfloor x_n \rfloor) \in \mathbb{R}^m$ and $x_{-1} := (x_2, \ldots, x_m) \in \mathbb{R}^{m-1}$. We use the notation $x \leq y$ to indicate that $x_i \leq y_i$ for each $i \in \{1, \ldots, m\}$.

Let $b \in [0,1)^n \setminus \{0\}$. The *lifting-space representation* of any function $\theta : \mathbb{R}^n \to \mathbb{R}$ is $[\theta]_b : \mathbb{R}^n \to \mathbb{R}$ defined by

$$[\theta]_b(x) := \sum_{i=1}^n x_i - \sum_{i=1}^n b_i \theta \left(x - \lfloor x \rfloor \right).$$

REMARK 6.2. The definition for lifting-space representation in [9] is given only for valid functions which in that context are periodic modulo \mathbb{Z}^n . Note that if θ is periodic, then $[\theta]_b(x) = \sum_{i=1}^n x_i - \sum_{i=1}^n b_i \theta(x).$

The group-space representation of any function $\psi: \mathbb{R}^n \to \mathbb{R}$ is $[\psi]_h^{-1}: \mathbb{R}^n \to \mathbb{R}$ defined by

$$[\psi]_b^{-1}(x) := \frac{\sum_{i=1}^n x_i - \psi(x)}{\sum_{i=1}^n b_i}.$$

A function $\psi : \mathbb{R}^n \to \mathbb{R}$ is called *superadditive* if $-\psi$ is subadditive. ψ is called *pseudo-periodic* if $\psi(x + e^i) = \psi(x) + 1$ for all standard unit vectors $e^i \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$.

We collect some useful facts above the above definitions below.

PROPOSITION 6.3. Let $b \in [0,1)^n \setminus \{0\}$.

- (i) If $\psi : \mathbb{R}^n \to \mathbb{R}$ is pseudo-periodic, then $[\psi]_b^{-1}$ is periodic modulo \mathbb{Z}^n .
- (ii) If π is a minimal valid function for I_b , then $[\pi]_b$ is superadditive and pseudo-periodic.
- (iii) If ψ is pseudo-periodic then $[[\psi]_b^{-1}]_b = \psi$.

Proof. If $\psi : \mathbb{R}^n \to \mathbb{R}$ is pseudo-periodic, then observe that for any $x \in \mathbb{R}^n$ and unit vector $e^i, \psi(x) = \psi((x - e^i) + e^i) = \psi(x - e^i) + 1$, i.e., $\psi(x - e^i) = \psi(x) - 1$. By iterating, we observe that $\psi(x + z) = \psi(x) + \sum_{i=1}^{n} z_i$ for all $x \in \mathbb{R}^n$ and $z \in \mathbb{Z}^n$. Therefore, $[\psi]_b^{-1}(x + z) = \sum_{i=1}^n \frac{\sum_{i=1}^n z_i - \psi(x) - \sum_{i=1}^n z_i}{\sum_{i=1}^n b_i} = \frac{\sum_{i=1}^n z_i - \psi(x)}{\sum_{i=1}^n b_i} = [\psi]_b^{-1}(x)$. This establishes (i).

(ii) follows from Proposition 3 in [9].

To establish (*iii*), first observe that by (*i*) above $[\psi]_b^{-1}$ is periodic modulo \mathbb{Z}^n . Therefore, by Remark 6.2, $[[\psi]_b^{-1}]_b(x) = \sum_{i=1}^n x_i - \sum_{i=1}^n b_i [\psi]_b^{-1}(x) = \sum_{i=1}^n x_i - \sum_{i=1}^n b_i \frac{\sum_{j=1}^n x_j - \psi(x)}{\sum_{j=1}^n b_j} = \psi(x).$

REMARK 6.4. The definition for group-space representation in [9] is given only for superadditive and pseudo-periodic ψ , and the domain of $[\psi]_b^{-1}(x)$ is defined to be $[0,1)^n$. We give the definition for more general functions and allow the domain of $[\psi]_b^{-1}(x)$ to be \mathbb{R}^n to make calculations easier. By Proposition 6.3 (i), restricted to pseudo-periodic functions, our definition of $[\psi]_b^{-1}(x)$ is simply a periodization of the definition from [9].

Let $b_1 \in (0,1)$ and $b_2 \in [0,1)^m \setminus \{0\}$. If $f : \mathbb{R} \to \mathbb{R}_+$ and $g : \mathbb{R}^m \to \mathbb{R}_+$ are minimal valid functions for I_{b_1} and I_{b_2} , respectively, then the sequential merge $f \diamond g : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is defined as

$$f \diamond g := [\psi]_{(b_1, b_2)}^{-1}$$

where $\psi : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ is the function $\psi(x_1, x_2) = [f]_{b_1}(x_1 + [g]_{b_2}(x_2)).$

REMARK 6.5. It is not true in general that if f, g are minimal, then $f \diamond g$ is minimal. However, when f, g are minimal valid functions, $f \diamond g$ is periodic modulo \mathbb{Z}^n . Indeed, since $f \diamond g := [\psi]_{(b_1,b_2)}^{-1}$, by Proposition 6.3 (i) it suffices to check that $\psi(x_1, x_2)$ defined above is pseudoperiodic when f, g are minimal. By Proposition 6.3 (ii), $[f]_{b_1}, [g]_{b_2}$ are both pseudo-periodic. Therefore, $\psi(x_1 + 1, x_2) = [f]_{b_1}(x_1 + 1 + [g]_{b_2}(x_2)) = [f]_{b_1}(x_1 + [g]_{b_2}(x_2)) + 1 = \psi(x_1, x_2) + 1$, using pseudo-periodicity of $[f]_{b_1}$. On the other hand, for any unit vector $e^i \in \mathbb{R}^m$, we have $\psi(x_1, x_2 + e^i) = [f]_{b_1}(x_1 + [g]_{b_2}(x_2 + e^i)) = [f]_{b_1}(x_1 + [g]_{b_2}(x_2) + 1) = [f]_{b_1}(x_1 + [g]_{b_2}(x_2)) + 1 =$ $\psi(x_1, x_2) + 1$, where the second equality uses pseudo-periodicity of $[g]_{b_2}$ and the last equality uses pseudo-periodicity of $[f]_{b_1}$.

In Dey and Richard's original definition from [9], the domain of $f \diamond g$ is defined as $[0,1) \times [0,1)^m$, and restricted to this domain, our definition is exactly the same as theirs. Thus, our definition over $\mathbb{R} \times \mathbb{R}^m$ is simply a periodization of Dey and Richard's definition for $f \diamond g$, when f, g are minimal functions. Since we will only apply the sequential merge operation on minimal valid functions, there is no discrepancy between the definition in [9] and our definition.

For the remainder of this section, we consider $b \in [1/2, 1)$. Although the specific construction of π_k provided in Section 2 uses $b \in (0, 1/2]$, creating π_k for $b \in [1/2, 1)$ can be done by defining $\pi_k(x) := \tilde{\pi}_k(1-x)$ for $x \in [0, 1]$ (and then enforcing periodicity by \mathbb{Z}), where $\tilde{\pi}_k$ is the function for I_{1-b} constructed in Section 2 (see also Theorem A.1).

Let ϕ denote the GMI function for I_b (defined in (1.3)). For $n \in \mathbb{N}$, $n \ge 2$, let $\Pi_k^n : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$\Pi_k^n(x_1,\ldots,x_n) := \pi_k \diamond (\phi \diamond (\ldots \diamond \phi) \ldots)) (x_1,\ldots,x_n)$$

where the sequential merge contains one copy of π_k and n-1 copies of ϕ . For $m \in \mathbb{N}$ and $m \geq 1$, let Φ_m denote $\phi \diamond (\phi \diamond (... \diamond \phi) ...)$, where there are m copies of ϕ in the sequential merge.

We require a couple of definitions (also taken from [9]) before we proceed with the proof of Theorem 6.1.

- 1. For $m \in \mathbb{N}$, a function $\theta : \mathbb{R}^m \to \mathbb{R}$ is *nondecreasing* if for all $x, y \in \mathbb{R}^m$, $x \leq y$ implies $\theta(x) \leq \theta(y)$.
- 2. For $m \in \mathbb{N}$ and a valid function $\pi : \mathbb{R}^m \to \mathbb{R}_+$, the set $E(\pi)$ defined in (4.1) is said to be unique up to scaling if for any continuous nonnegative function $\theta : \mathbb{R}^m \to \mathbb{R}_+$ satisfying $E(\pi) \subseteq E(\theta), \theta$ is a scaling of π , i.e. $\theta = \alpha \pi$ where $\alpha \in \mathbb{R}$.

REMARK 6.6. Dey and Richard note that every extreme function for I_b is unique up to scaling, see the top of page 6 in [9]. In particular, the GMI function $\phi = \pi_2$ is unique up to scaling.

As mentioned in Remark 6.5, $f \diamond g$ is not necessarily minimal even if f, g are both minimal. The following proposition gives conditions under which $f \diamond g$ is indeed minimal and will be useful in what follows.

PROPOSITION 6.7. [9, Proposition 7] Let $m \in \mathbb{N}$, $b^1 \in (0,1)$, and $b^2 \in [0,1)^m \setminus \{0\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a minimal function for I_{b^1} and $g : \mathbb{R}^m \to \mathbb{R}$ be a minimal function for I_{b^2} such that $[f]_{b^1}$ is nondecreasing. Then $f \diamond g$ is minimal for $I_{(b^1,b^2)}$.

To deduce that $f \diamond g$ is a *facet*, one needs additional assumptions. We now state the main theorem that guarantees facetness from the sequential-merge operation, due to Dey and Richard [9].

THEOREM 6.8. [9, Theorem 5] Let $m \in \mathbb{N}$, $b^1 \in (0,1)$, and $b^2 \in [0,1)^m \setminus \{0\}$. Let $f : \mathbb{R} \to \mathbb{R}$ be a minimal function for I_{b^1} and $g : \mathbb{R}^m \to \mathbb{R}$ be a minimal function for I_{b^2} such that the following hold:

- 1. f and g are piecewise linear, continuous functions,
- 2. $[f]_{b^1}$ and $[g]_{b^2}$ are both nondecreasing,
- 3. E(f) and E(g) are unique up to scaling, and
- 4. f and g are facets for their respective infinite group relaxations.

Then $f \diamond g$ is a facet for $I_{(b^1, b^2)}$.¹

We will prove that Π_k^n is a facet of I_{b1_n} by showing that π_k and Φ_{n-1} satisfy all the conditions of Theorem 6.8. We divide these into subsections to help organize our arguments.

6.1 Minimality of π_k, Φ_{n-1}

The minimality of π_k , $k \ge 2$ was established in Section 3; we concentrate on Φ_{n-1} .

PROPOSITION 6.9. Let $b \in [1/2, 1)$. The function $[\pi_k]_b$ is nondecreasing for every $k \ge 2$.

¹The definition of facet used in [9] is slightly different from our definition, and corresponds to what the authors in [4] refer to as *weak facet*. However, the proof in [9] works for the definition of facet used in this current manuscript. Moreover, we insist on f, g to be minimal valid functions, whereas Dey and Richard consider valid functions that are periodic modulo \mathbb{Z}^n , which is a slightly weaker hypothesis than minimality.

Proof. Let $x, y \in \mathbb{R}$ such that x < y. By the definition of the lifting-space representation of π_k and Remark 6.2, we see that

$$[\pi_k]_b(y) - [\pi_k]_b(x) = (y - b\pi_k(y)) - (x - b\pi_k(x)) = (y - x) - b(\pi_k(y) - \pi_k(x)).$$

If $[\pi_k]_b(y) < [\pi_k]_b(x)$, then $\frac{1}{b} < \frac{\pi_k(y) - \pi_k(x)}{y - x}$. However, this contradicts that the largest slope (and the only positive slope) in π_k is $\frac{1}{b}$ (this crucially uses the fact that we are using π_k with $b \in [1/2, 1)$). Thus, $[\pi_k]_b$ is nondecreasing.

PROPOSITION 6.10. Let $b \in [1/2, 1)$. Then Φ_m is minimal for I_{b1_m} for every $m \in \mathbb{N}$.

Proof. We proceed by induction on m. If m = 1, then $\Phi_m = \phi = \pi_2$ and the result follows from Theorem 1.1. So assume that Φ_m is minimal for $I_{b\mathbf{1}_m}$ for $m \in \mathbb{N}$, and consider Φ_{m+1} . Note that $\Phi_{m+1} = \phi \diamond \Phi_m$. From Proposition 6.9, $[\phi]_b = [\pi_2]_b$ is nondecreasing. Since ϕ and Φ_m are minimal by the induction hypothesis, Φ_{m+1} is minimal for $I_{b\mathbf{1}_{m+1}}$ by Proposition 6.7.

6.2 π_k and Φ_{n-1} are piecewise linear and continuous

 π_k is piecewise linear and continuous by Propositions 2.1 and 2.3. We analyze Φ_{n-1} . A nice formula for the sequential-merge procedure was stated in Proposition 5 of [9] and is applied to Φ_m and Π_k^n below.

PROPOSITION 6.11. Let $b \in [1/2, 1)$. For $m \in \mathbb{N}$ with $m \geq 2$ and $x \in \mathbb{R}^m$,

$$\Phi_m(x) = \frac{(m-1)\Phi_{m-1}(x_{-1}) + \phi\left(\sum_{i=1}^m x_i - (m-1)b\Phi_{m-1}(x_{-1})\right)}{m}.$$

For $k \in \mathbb{N}$ with $k \geq 2$ and $x \in \mathbb{R}^n$,

$$\Pi_k^n(x) = \frac{(n-1)\Phi_{n-1}(x_{-1}) + \pi_k \left(\sum_{i=1}^n x_i - (n-1)b\Phi_{n-1}(x_{-1})\right)}{n}.$$

We get the following corollary.

PROPOSITION 6.12. Let $b \in [1/2, 1)$. For $m \in \mathbb{N}$ with $m \geq 2$, Φ_m is piecewise linear and continuous. For $k \in \mathbb{N}$ with $k \geq 2$, The function Π_k^n is piecewise linear and continuous.

Proof. By Proposition 2.3, π_k and ϕ are piecewise linear functions. By Proposition 6.11 and since piecewise linear continuous functions are preserved under composition, the result follows by induction.

6.3 $[\pi_k]_b$ and $[\Phi_{n-1}]_{b\mathbf{1}_{n-1}}$ are nondecreasing

For $k \in \mathbb{N}$ with $k \ge 2$, $[\pi_k]_b$ is nondecreasing by Proposition 6.9. We analyze $[\Phi_m]_{b\mathbf{1}_m}$.

PROPOSITION 6.13. Let $b \in [1/2, 1)$. Then $[\Phi_m]_{b1_m}$ is nondecreasing for every $m \in \mathbb{N}$.

Proof. We prove it by induction on m. For m = 1, since $\phi = \pi_2$, it follows that $[\phi]_b$ is nondecreasing. Assume that $[\Phi_{m-1}]_{b\mathbf{1}_{m-1}}$ is nondecreasing and consider $[\Phi_m]_{b\mathbf{1}_m}$. Let $(x^1, x^2), (y^1, y^2) \in \mathbb{R} \times \mathbb{R}^{m-1}$ be such that $(x^1, x^2) \leq (y^1, y^2)$. Recall that $\Phi_m = \phi \diamond \Phi_{m-1} :=$ $[\psi]_{(b,b\mathbf{1}_{m-1})}^{-1}$ where $\psi(z_1, z_2) = [\phi]_b(z_1 + [\Phi_{m-1}]_{b\mathbf{1}_{m-1}}(z_2))$. As shown in Remark 6.5, ψ is pseudo-periodic since ϕ and Φ_{m-1} are minimal by Proposition 6.10. Therefore, applying Proposition 6.3 *(iii)*,

$$\begin{split} [\Phi_m]_{b\mathbf{1}_m}(x_1, x_2) &= [\phi]_b(x_1 + [\Phi_{m-1}]_{b\mathbf{1}_{m-1}}(x_2)) \\ &\leq [\phi]_b(y_1 + [\Phi_{m-1}]_{b\mathbf{1}_{m-1}}(y_2)) \\ &= [\Phi_m]_{b\mathbf{1}_m}(y_1, y_2), \end{split}$$

where the inequality holds because $[\phi]_b, [\Phi_{m-1}]_{b\mathbf{1}_{m-1}}$ are nondecreasing. Thus $[\Phi_m]_{b\mathbf{1}_m}$ is nondecreasing.

6.4 $E(\pi_k)$ and $E(\Phi_m)$ are unique up to scaling

PROPOSITION 6.14. Let $b \in [1/2, 1)$. For $m \in \mathbb{N}$, the sets $E(\pi_k)$ and $E(\Phi_m)$ are unique up to scaling.

Proof. First, we consider π_k . If k = 2, then by Remark 6.6 we have that $E(\pi_k)$ is unique up to scaling. So let $k \ge 3$ and let $\xi : \mathbb{R} \to \mathbb{R}_+$ be a continuous function such that $E(\pi_k) \subseteq E(\xi)$. We claim that $\xi = \xi(b)\pi_k$.

If $\xi(b) = 0$, then $\xi(x) + \xi(b - x) = 0$ for each $x \in \mathbb{R}$ since $E(\xi) \supseteq E(\pi_k)$. As ξ is nonnegative, this implies that $\xi(x) = 0$ for each $x \in \mathbb{R}$ and so $\xi = 0\pi_k$. Now suppose that $\xi(b) \neq 0$. It is sufficient to show that the function defined pointwise by $\tilde{\xi}(x) := \frac{1}{\xi(b)}\xi(-x)$ is equal to the function defined by $\tilde{\pi}_k(x) := \pi_k(-x)$. Recall that $\tilde{\pi}_k$ is extreme as discussed after Remark 6.5.

Observe that $E(\tilde{\pi}_k) \subseteq E(\tilde{\xi})$. Indeed, if $(x, y) \in E(\tilde{\pi}_k)$, then $\tilde{\pi}_k(x+y) = \tilde{\pi}_k(x) + \tilde{\pi}_k(y)$, and $\pi_k(-x-y) = \pi_k(-x) + \pi_k(-y)$ then follows from the definition of $\tilde{\pi}_k$. Since $E(\pi_k) \subseteq E(\xi)$, this implies $\tilde{\xi}(x+y) = \frac{1}{\xi(b)}\xi(-x-y) = \frac{1}{\xi(b)}\xi(-x) + \frac{1}{\xi(b)}\xi(-y) = \tilde{\xi}(x) + \tilde{\xi}(y)$. Hence $E(\tilde{\pi}_k) \subseteq E(\tilde{\xi})$. This observation has a few implications. First, it implies $\tilde{\xi}(0) + \tilde{\xi}(0) = \tilde{\xi}(0)$ and so $\tilde{\xi}(0) = 0$, and also that $\tilde{x}i(b) = 1$. Next, since $\tilde{\pi}_k$ is periodic, Proposition 4.4 implies that $\tilde{\xi}$ is periodic. Finally, since $\tilde{\xi}$ is continuous and periodic, it is locally bounded.

Using $\pi = \tilde{\pi}_k$ and $\theta = \tilde{\xi}$ in Lemma 4.9, it follows that $\tilde{\xi} = \tilde{\pi}_k$ on $I_3^k \cup I_6^k$. From Proposition 4.7 and again setting $\pi = \tilde{\pi}_k$ and $\theta = \tilde{\xi}$, we obtain that $\tilde{\xi} = \tilde{\pi}_k$ on $I_2^k \cup I_4^k$. It is left to show that $\tilde{\xi} = \tilde{\pi}_k$ on I_1^k and I_5^k .

left to show that $\tilde{\xi} = \tilde{\pi}_k$ on I_1^k and I_5^k . Let $U = \left[0, \frac{b}{2} \left(\frac{1}{8}\right)^{k-2}\right]$ and observe that $U + U = \left[0, b \left(\frac{1}{8}\right)^{k-2}\right] = I_1^k$. It follows from the definition of $\tilde{\pi}_k$ that $\tilde{\pi}_k(x) + \tilde{\pi}_k(y) = \tilde{\pi}_k(x+y)$ for all $x, y, x+y \in I_1^k$, so $U \times U \subseteq E(\tilde{\pi}_k) \subseteq E(\tilde{\xi})$. Recall that $\tilde{\xi}(0) = \tilde{\pi}_k(0) = 0$. Also, since $\tilde{\xi} = \tilde{\pi}_k$ on I_2^k , $\tilde{\xi} \left(b \left(\frac{1}{8}\right)^{k-2}\right) = \tilde{\pi}_k \left(b \left(\frac{1}{8}\right)^{k-2}\right)$. Thus $\tilde{\xi} = \tilde{\pi}_k$ on the endpoints of I_1^k . Moreover, Lemma 4.3 implies that $\tilde{\xi}$ is affine over I_1^k . Since $\tilde{\pi}_k$ is also affine over I_1^k and $\tilde{\xi} = \tilde{\pi}_k$ at the endpoints, we have $\tilde{\xi} = \tilde{\pi}_k$ on I_1^k . The fact that $\tilde{\xi} = \tilde{\pi}_k$ on I_5^k follows by symmetry (note that $\tilde{\xi}$ is also symmetric because $E(\tilde{\pi}_k) \subseteq E(\tilde{\xi})$). Therefore $\tilde{\xi} = \tilde{\pi}_k$ everywhere. Now consider Φ_m . Dey and Richard's proof of Theorem 6.8 shows that if E(f) and E(g) are unique up to scaling, then $E(f \diamond g)$ is also unique up to scaling. If m = 1, then $\Phi_m = \phi$. Since $\phi = \pi_2$, the set $E(\phi)$ is unique up to scaling by Remark 6.6. Now an induction argument shows that $E(\Phi_m)$ is unique up to scaling.

6.5 The proof of Theorem 6.1

PROPOSITION 6.15. Let $b \in [1/2, 1)$. For $m \in \mathbb{N}$, the function Φ_m is a facet for I_{b1_m} .

Proof. Using induction, the result is a consequence of Theorem 6.8; the assumptions of Theorem 6.8 are verified by the results of Propositions 6.10, 6.12, 6.13 and 6.14. \Box

The next few propositions argue that Π_k^n is genuinely *n* dimensional with at least *k* slopes. Note that, unlike the one dimensional setting in which exactly *k* slopes are attained, we are unsure of exactly how many slopes Π_k^n attains; all we can establish is that the number of slopes is greater than or equal to *k*.

PROPOSITION 6.16. Let $b \in [1/2, 1)$ and $m \in \mathbb{N}$. Then $\Phi_m(x) = 0$ if and only if $x \in \mathbb{Z}^m$. Also, for every $n, k \in \mathbb{N}$ such that $n, k \geq 2$, $\Pi_k^n(x) = 0$ if and only if $x \in \mathbb{Z}^n$.

Proof. We use induction on m. If m = 1, then $\Phi_m = \phi$ and the result follows from (1.3). So assume that $\Phi_t(x) = 0$ if and only if $x \in \mathbb{Z}^t$ for all $t \leq m$ with $m, t \in \mathbb{N}$. Using the formulas in Proposition 6.11 and the induction hypothesis, it directly follows that $\Phi_{m+1}(x) = 0$ for all $x \in \mathbb{Z}^{m+1}$. Let $x \in \mathbb{R}^{m+1} \setminus \mathbb{Z}^{m+1}$. By the induction hypothesis, if $x_{-1} \notin \mathbb{Z}^m$, then $\Phi_m(x_{-1}) > 0$, and since ϕ is nonnegative, $\Phi_{m+1}(x) > 0$ follows from Proposition 6.11. If $x_{-1} \in \mathbb{Z}^m$, then $\Phi_m(x_{-1}) = 0$ by the induction hypothesis, and $\sum_{i=1}^{m+1} x_i - mb\Phi_m(x_{-1}) = \sum_{i=1}^{m+1} x_i \notin \mathbb{Z}$. Again using the induction hypothesis, $\phi(\sum_{i=1}^{m+1} x_i - mb\Phi_m(x_{-1})) > 0$ and so $\Phi_{m+1}(x) > 0$ using the formula in Proposition 6.11.

For Π_k^n the result follows by applying the same argument as above and noting that $\pi_k(x) = 0$ if and only if $x \in \mathbb{Z}$; see Proposition 2.1.

PROPOSITION 6.17. Let $b \in [1/2, 1)$. The function $\prod_{k=1}^{n} n$ is genuinely n dimensional for every $n, k \in \mathbb{N}$ such that $n, k \geq 2$.

Proof. Assume to the contrary that Π_k^n is not genuinely n dimensional. Then there exist a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^{n-1}$ and a function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\Pi_k^n = \Psi \circ T$. Since T is linear with nontrivial kernel, there must exist $x \in \ker(T)$ such that $x \notin \mathbb{Z}^n$. It follows that

$$\Pi_k^n(x) = \Psi \circ T(x) = \Psi(0) = \Psi \circ T(0) = \Pi_k^n(0) = 0.$$

However, Proposition 6.16 implies that $x \in \mathbb{Z}^n$, which is a contradiction.

LEMMA 6.18. Let $b \in [1/2, 1)$ and $m \in \mathbb{N}$. Then $\Phi_m(x) = \frac{1}{mb} \sum_{i=1}^m x_i$ for all $x \in \mathbb{R}^m_+$ with $||x||_{\infty} < b$.

Proof. We proceed by induction on m. If m = 1, then $\Phi_m = \phi$ and the result follows by the definition of the GMI in (1.3). So let $m \ge 2$ and assume that $\Phi_{m-1}(x) = \frac{1}{(m-1)b} \sum_{i=1}^{m-1} x_i$ for all $x \in \mathbb{R}^{m-1}_+$ with $||x||_{\infty} < b$.

Let $x \in \mathbb{R}^m_+$ with $||x||_{\infty} < b$. Using Proposition 6.11 and the induction hypothesis, we see that

$$\Phi_m(x) = \frac{(m-1)\Phi_{m-1}(x_{-1}) + \phi\left(\sum_{i=1}^m x_i - (m-1)b\Phi_{m-1}(x_{-1})\right)}{m}$$
$$= \frac{\frac{1}{b}\sum_{i=2}^m x_i + \phi(x_1)}{m}.$$

Since $|x_1| < b$, we can apply the definition of the GMI to the previous equality and see

$$\Phi_m(x) = \frac{\frac{1}{b} \sum_{i=2}^m x_i + \frac{1}{b} x_1}{m} = \frac{1}{mb} \sum_{i=1}^m x_i,$$

as desired.

PROPOSITION 6.19. Let $b \in [1/2, 1)$. The function Π_k^n has at least k slopes for every $n, k \in \mathbb{N}$ such that $n, k \geq 2$.

Proof. By Theorem 1.3, π_k has k nondegenerate intervals $J_1, \ldots, J_k \subseteq \mathbb{R}$ such that π_k is affine over each J_i with slope $\sigma_i \in \mathbb{R}$, i.e., $\pi_k(x) = \sigma_i x + d_i$ for some $d_i \in \mathbb{R}$. Moreover, $\sigma_i \neq \sigma_j$ for $i \neq j$. For each $i = 1, \ldots, k$, let $R_i \subseteq \mathbb{R}^n$ be defined by

$$R_i := \{ x \in \mathbb{R}^n : x_1 \in J_i, \ x_{-1} \in B_{n-1} \},\$$

where $B_{n-1} = \{x \in \mathbb{R}^{n-1}_+ : \|x\|_{\infty} < b\}$. We claim that Π^n_k is affine over each R_i , and attains a different slope on each R_i .

In order to see that Π_k^n is affine over R_i , let $x \in R_i$. By Proposition 6.11, we have

$$\Pi_{k}^{n}(x) = \frac{(n-1)\Phi_{n-1}(x_{-1}) + \pi_{k}(\sum_{i=1}^{n} x_{i} - (n-1)b\Phi_{n-1}(x_{-1}))}{n}$$
$$= \frac{\frac{1}{b}\sum_{i=2}^{n} x_{i} + \pi_{k}(x_{1})}{n} \qquad \text{by Lemma 6.18}$$
$$= \frac{1}{bn}\sum_{i=2}^{n} x_{i} + \frac{\sigma_{i}x_{1} + d_{i}}{n} \qquad \text{since } x \in R_{i}$$
$$= \left(\frac{\sigma_{i}}{n}, \frac{1}{bn}\mathbf{1}_{n-1}\right) \cdot x + \frac{d_{i}}{n}.$$

Thus, $\Pi_k^n(x)$ is affine over R_i with gradient $(\frac{\sigma_i}{n}, \frac{1}{bn}\mathbf{1}_{n-1})$.

Since each σ_i is distinct for i = 1, ..., n, each gradient $(\frac{\sigma_i}{n}, \frac{1}{bn} \mathbf{1}_{n-1})$ is distinct. Note that as R_i is full dimensional, this vector is indeed a gradient. Hence, Π_k^n has at least k slopes, as desired.

of Theorem 6.1. Since facets are periodic with respect to \mathbb{Z}^n , we may assume that $b \in (0, 1)$. First, we prove the result for $b \in [1/2, 1)$. Sections 6.1, 6.2, 6.3, 6.4, and Propositions 4.1 and 6.15 establish that π_k and Φ_{n-1} satisfy the assumptions for Theorem 6.8. Thus, Π_k^n is a facet for I_{b1_n} . Proposition 6.17 shows that Π_k^n is genuinely n dimensional, and Proposition 6.19 shows that Π_k^n has at least k slopes. This gives the desired result.

Now, let $b \in (0, 1/2]$. By Theorem A.1 and the previous case of $b \in [1/2, 1)$, we obtain the desired result.

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A Appendix

THEOREM A.1. Let $n \geq 1$ be a natural number. A function $\theta : \mathbb{R}^n \to \mathbb{R}_+$ is minimal/extreme/facet for I_b when $b \in [0, 1/2]^n \setminus \{0\}$ if and only if $\tilde{\theta} : \mathbb{R}^n \to \mathbb{R}_+$ defined by $\tilde{\theta}(x) := \theta(-x)$ is minimal/extreme/facet for I_{1-b} , respectively, where $\mathbf{1} \in \mathbb{R}^n$ is the vector of all ones.

Proof. The result essentially follows by applying Theorem 8.2 in [16]. However, since that paper considers the so-called mixed-integer problem, while we are looking at the pure integer problem, we include a proof for completeness.

We show one direction as the other follows from swapping the roles of b and 1-b. Suppose that θ is minimal for I_b with $b \in [0, 1/2]^n \setminus \{0\}$. Define $\tilde{\theta}(x) := \theta(-x)$. We check that $\tilde{\theta}$ is minimal using Theorem 1.1. If $w \in \mathbb{Z}^n$ then so is -w, and therefore $\tilde{\theta}(w) = \theta(-w) = 0$ since θ is minimal. Let $x, y \in \mathbb{R}^n$ and note that

$$\tilde{\theta}(x+y) = \theta(-x-y) \le \theta(-x) + \theta(-y) = \tilde{\theta}(x) + \tilde{\theta}(y),$$

where the inequality follows from the subadditivity of θ . Hence θ is subadditive. Finally, let $r \in \mathbb{R}^n$ and note that

$$\tilde{\theta}(r) + \tilde{\theta}((1-b)-r) = \theta(-r) + \theta(b-(1-r)) = \theta(-r) + \theta(b-(-r)) = 1,$$

where the second equation follows from the periodicity of θ and the third equation from the symmetry of θ . Hence $\tilde{\theta}$ is symmetric about 1 - b. From Theorem 1.1, $\tilde{\theta}$ is minimal.

Now assume that θ is extreme. Let θ_1, θ_2 be valid for I_{1-b} such that $\tilde{\theta} = \frac{\theta_1 + \theta_2}{2}$. We claim that $\tilde{\theta}_i(r) := \theta_i(-r), i = 1, 2$, is a valid function for I_b . This would imply $\tilde{\theta} = \theta_1 = \theta_2$ from the extremality of θ . Let $y \in I_b$. Then $\tilde{y}(r) := y(-r) \in I_{1-b}$. Note that for i = 1, 2,

$$\sum_{r \in \mathbb{R}^n} \tilde{\theta}_i(r) y(r) = \sum_{r \in \mathbb{R}^n} \theta_i(-r) y(r) = \sum_{r \in \mathbb{R}^n} \theta_i(-r) \tilde{y}(-r) \ge 1,$$

since θ_i is valid for I_{1-b} .

The proof that θ is a facet if and only if $\tilde{\theta}$ is a facet is similar.

of Proposition 2.1. We prove this using induction on k. For k = 2, the result is easily verified using (1.3). So let $k \ge 3$ and assume that π_{k-1} is well-defined, nonnegative, and positive on $\mathbb{R} \setminus \mathbb{Z}$. First, we will show that π_k is well-defined at the points $\{b\left(\frac{1}{8}\right)^{k-2}, 2b\left(\frac{1}{8}\right)^{k-2}, b - 2b\left(\frac{1}{8}\right)^{k-2}, b - b\left(\frac{1}{8}\right)^{k-2}, b\}$, that is, we will show π_k is well-defined at the boundaries of the intervals on which it is defined. This will show that π_k is well-defined on [0, 1), and since π_k is periodic by definition, it will follow that π_k is well-defined everywhere. Note that

$$\frac{4^{2-k}}{1-b} - \left(\frac{1}{1-b}\right)b\left(\frac{1}{8}\right)^{k-2} = \left(\frac{2^{k-2}-b}{b-b^2}\right)b\left(\frac{1}{8}\right)^{k-2} = \left(\frac{2^{k-2}-b}{1-b}\right)\left(\frac{1}{8}\right)^{k-2} > 0, \quad (A.1)$$

where the inequality follows since $k \geq 3$ and $b \in (0, 1)$. Also, observe that

$$\frac{1-4^{2-k}}{1-b} - \left(\frac{1}{1-b}\right) \left(b-b\left(\frac{1}{8}\right)^{k-2}\right) = \frac{1-2^{k-2}}{1-b} + \left(\frac{2^{k-2}-b}{b-b^2}\right) \left(b-b\left(\frac{1}{8}\right)^{k-2}\right)$$
$$= \frac{1+b((\frac{1}{8})^{k-2}-1)-4^{2-k}}{1-b}$$
$$\ge \frac{1+\frac{1}{2}((\frac{1}{8})^{k-2}-1)-4^{2-k}}{1-b}$$
$$> 0, \qquad (A.2)$$

where the inequalities follow since $b \in (0, \frac{1}{2}]$ and $k \ge 3$. Equations (A.1) and (A.2) show that π_k is well-defined and positive at the points $b\left(\frac{1}{8}\right)^{k-2}$ and $b-b\left(\frac{1}{8}\right)^{k-2}$.

Observe that $b \in I_5^j \cap I_6^j$. Since $\frac{1-2^{k-2}}{1-b} + \left(\frac{2^{k-2}-b}{b-b^2}\right)b = 1$, it follows that π_k is well-defined and positive at b.

Notice that $2b\left(\frac{1}{8}\right)^{k-2} \in I_2^k \subseteq I_1^{k-1}$ by definition. Similarly, $b - 2b\left(\frac{1}{8}\right)^{k-2} \in I_4^k \subseteq I_5^{k-1}$. Therefore, by induction,

$$\frac{4^{2-k}}{1-b} - \left(\frac{1}{1-b}\right) 2b\left(\frac{1}{8}\right)^{k-2} = \pi_{k-1}\left(2b\left(\frac{1}{8}\right)^{k-2}\right) > 0,\tag{A.3}$$

and

$$\frac{1-4^{2-k}}{1-b} - \left(\frac{1}{1-b}\right) \left(b - 2b\left(\frac{1}{8}\right)^{k-2}\right) = \pi_{k-1} \left(b - 2b\left(\frac{1}{8}\right)^{k-2}\right) > 0, \qquad (A.4)$$

where the inequalities follows since π_{k-1} is nonnegative. Thus, π_k is well-defined and positive on $2b(\frac{1}{8})^{k-2}$ and $b - 2b(\frac{1}{8})^{k-2}$.

Continuity of π_k follows from the recursive piecewise definition and the confirmation above that the values are well-defined on the boundaries of the pieces.

We now show that π_k is nonnegative and $\pi_k(x) = 0$ if and only if $x \in \mathbb{Z}$. Let $x \in [0, 1)$. If $x \in I_3^k \cup I_6^k$, then $\pi_k(x) = \pi_{k-1}(x) > 0$ by the induction hypothesis. Observe that π_k is affine on the intervals I_1^k, I_2^k, I_4^k , and I_5^k . From Equations (A.1)-(A.4), π_k is positive on the endpoints of each of the latter intervals, except for when x = 0. Thus, if $x \in I_1^k \setminus \{0\} \cup I_2^k \cup I_4^k \cup I_5^k$, then $\pi_k(x) > 0$ and $\pi_k(0) = 0$. Finally, if $x \in \mathbb{R} \setminus [0, 1)$, then by the periodicity of π_k , it follows that $\pi_k(x) > 0$ and $\pi_k(x) = 0$ if $x \in \mathbb{Z}$.

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