Discrepancy Estimates for Acceptance-Rejection Samplers Using Stratified Inputs

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Abstract In this paper we propose an acceptance-rejection sampler using stratified inputs as diver sequence. We estimate the discrepancy of the points generated by this algorithm. First we show an upper bound on the star discrepancy of order $N^{-1/2-1/(2s)}$. Further we prove an upper bound on the q-th moment of the L_q -discrepancy $(\mathbb{E}[N^qL^q_{q,N}])^{1/q}$ for $2 \le q \le \infty$, which is of order $N^{(1-1/s)(1-1/q)}$. We also present an improved convergence rate for a deterministic acceptance-rejection algorithm using (t,m,s)-nets as driver sequence.

1 Introduction

Markov chain Monte Carlo (MCMC) sampling is a classical method widely used in simulation. Using a deterministic sequence as driver sequence in the MCMC procedure, known as Markov chain quasi-Monte Carlo (MCQMC) algorithm, shows potential to improve the convergence rate. Tribble and Owen [29] proved a consistency result for MCMC estimation for finite state spaces. A construction of weakly completely uniformly distributed sequences is also proposed. As a sequel to the work of Tribble, Chen [4] and Chen, Dick and Owen [5] demonstrated that MCQMC algorithms using a completely uniformly distributed sequence as driver sequence give a consistent result under certain assumptions on the update function and Markov chain. Further, Chen [4] also showed that MCQMC can achieve a convergence rate of $O(N^{-1+\delta})$ for any $\delta > 0$ under certain stronger assumptions, but he only showed the existence of a driver sequence.

In a different direction, L'Ecuyer, Lecot and Tuffin [21] proposed a randomized quasi-Monte Carlo method which simulates multiple Markov chains in parallel and randomly permutes the driver sequence in order to reduce variance. Garber and

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Choppin in [12] adapted low discrepancy point sets instead of random numbers in sequential Monte Carlo (SMC). They proposed a new algorithm, named sequential quasi-Monte Carlo (SQMC), through the use of a Hilbert space-filling curve. They constructed consistency and stochastic bounds based on randomized QMC point sets for this algorithm. More literature review about applying QMC to MCMC problems can be found in [5, Section 1] and the references therein.

In [10], jointly done with Rudolf, we prove upper bounds on the discrepancy for uniformly ergodic Markov chains driven by a deterministic sequence rather than independent random variables. We show that there exists a deterministic driver sequence such that the discrepancy of the Markov chain from the target distribution with respect to certain test sets converges with almost the usual Monte Carlo rate of $N^{-1/2}$. In the sequential work [9] done by Dick and Rudolf, they consider upper bounds on the discrepancy under the assumption that the Markov chain is variance bounding and the driver sequence is deterministic. In particular, they proved a better existence result, showing a discrepancy bound having a rate of convergence of almost N^{-1} under a stronger assumption on the update function, the so called anywhere-to-anywhere condition.

The acceptance-rejection algorithm is one of the widely used techniques for sampling from a distribution when direct simulation is not possible or expensive. The idea of this method is to determine a good choice of proposal density (also known as hat function), then sample from the proposal density with low cost. In particular, Devroye [6] gave a construction method of a proposal density for log-concave densities and Hörmann [17] proposed a rejection procedure, called transformed density rejection, to construct a proposal density. Detailed summaries of this technique and some extensions can be found in the monographs [3] and [18]. For many target densities, finding a good proposal density is difficult. An alternative approach to improve efficiency is to determine a better choice of sequences having the designated proposal density.

The deterministic acceptance-rejection algorithm has been discussed by Moskowitz and Caflisch [20], Wang [30, 31] and Nguyen and Ökten [22], where empirical evidence or a consistency result were given. Two measurements included therein are the empirical root mean square error (RMSE) and the empirical standard deviation. However, the discrepancy of samples is not directly investigated. Motivated by those papers, in [32] we investigated the discrepancy properties of points produced by a totally deterministic acceptance-rejection method. We proved that the discrepancy of samples generated by a QMC acceptance-rejection sampler is bounded from above by $N^{-1/s}$. A lower bound shows that for any given driver sequence, there always exists a target density such that the star discrepancy is at most $N^{-2/(s+1)}$.

In this work we first present an acceptance-rejection algorithm using stratified inputs as driver sequence. Stratified sampling is one of the variance reduction methods used in Monte Carlo sampling. More precisely, grid-based stratified sampling improves the RMSE to $N^{-1/2-1/s}$ for Monte Carlo, see for instance [26, Chapter 10]. In this paper, we are interested in the discrepancy properties of points produced by the acceptance-rejection method with stratified inputs as driver sequence. We obtain a convergence rate of the star-discrepancy of order $N^{-1/2-1/(2s)}$. Also an es-

timation of the L_q -discrepancy is considered for this setting. One would expect that the convergence rate which can be achieved using deterministic sampling methods also depends on properties of the target density function. One such property is the number of elementary intervals (for a precise definition see Definition 3 below) of a certain size needed to cover the graph of the density. We show that if the graph can be covered by a small number of elementary intervals, then an improved rate of convergence can be achieved using (t, m, s)-nets as driver sequence. In general, this strategy does not work with stratified sampling, unless one knows the elementary intervals explicitly.

The paper is organized as follows. In Section 2 we provide the needed notation and background. Section 3 introduces the proposed acceptance-rejection sampler using stratified inputs, where an existence upper bound on the star-discrepancy and an estimation of the L_q -discrepancy are given. Section 4 illustrates an improved rate of convergence when using (t, m, s)-nets as driver sequences.

2 Preliminaries

We are interested in the discrepancy properties of samples generated by the acceptance-rejection sampler. We consider the L_a -discrepancy and the star-discrepancy.

Definition 1 (L_q -discrepancy). Let $1 \le q \le \infty$ be a real number. For a point set P_N in $[0,1]^s$, the L_q -discrepancy is defined by

$$L_{q,N} = \left(\int_{[0,1]^s} \left| \frac{1}{N} \sum_{n=0}^{N-1} 1_{[\mathbf{0},\boldsymbol{t})}(\boldsymbol{x}_n) - \lambda([\mathbf{0},\boldsymbol{t})) \right|^q \mathrm{d}\boldsymbol{t} \right)^{1/q},$$

where $1_{[\mathbf{0},t)}(\mathbf{x}_n) = \begin{cases} 1, \text{ if } \mathbf{x}_n \in [\mathbf{0},t), \\ 0, \text{ otherwise.} \end{cases}$, $[\mathbf{0},t) = \prod_{j=1}^s [0,t_j)$ and λ is the Lebesgue measure, with the obvious modification for $q = \infty$. The $L_{\infty,N}$ -discrepancy is called the star-discrepancy which is also denoted by $D_N^*(P_N)$.

The acceptance-rejection algorithm accepts all points below the graph of the density function. In order to prove bounds on the discrepancy, we assume that the set below the graph of the density function admits a so-called Minkowski content. We introduce the Minkowski content in the following. For a set A we denote the boundary of A by ∂A .

Definition 2 (Minkowski content). For a set $A \subseteq [0,1]^s$, let

$$\mathcal{M}(\partial A) = \lim_{\varepsilon \to 0} \frac{\lambda((\partial A)_{\varepsilon})}{2\varepsilon},$$

where $(\partial A)_{\varepsilon} = \{ \boldsymbol{x} \in \mathbb{R}^s | || \boldsymbol{x} - \boldsymbol{y}|| \le \varepsilon \text{ for } \boldsymbol{y} \in \partial A \}$ and $|| \cdot ||$ denotes the Euclidean norm. If $\mathcal{M}(\partial A)$ (abbreviated as \mathcal{M}_A without causing confusion) exists and is finite, then ∂A is said to admit an (s-1)-dimensional Minkowski content.

3 Acceptance-Rejection Sampler Using Stratified Inputs

We now present the acceptance-rejection algorithm using stratified inputs.

Algorithm 1 Let the target density $\psi: [0,1]^{s-1} \to \mathbb{R}_+$ where $s \ge 2$, be given. Assume that there exists a constant $L < \infty$ such that $\psi(\mathbf{z}) \le L$ for all $\mathbf{z} \in [0,1]^{s-1}$. Let $A = \{\mathbf{z} \in [0,1]^s: \psi(z_1,\ldots,z_{s-1}) \ge Lz_s\}$ and assume that ∂A admits an (s-1)-dimensional Minkowski content.

- i) Let $M \in \mathbb{N}$ and let $\{Q_0, \dots, Q_{M-1}\}$ be a disjoint covering with of $[0,1]^s$ with Q_i of the form $\prod_{j=1}^s \left[\frac{c_j}{M^{1/s}}, \frac{c_j+1}{M^{1/s}}\right]$ with $0 \le c_j \le \lceil M^{1/s} \rceil 1$. Then $\lambda(Q_i) = 1/M$ for all $0 \le i \le M-1$. Generate a point set $P_M = \{x_0, \dots, x_{M-1}\}$ such that there is exactly one point of P_M uniformly distributed in each sub-cube Q_i .
- ii) Use the acceptance-rejection method for the points in P_M with respect to the density ψ , i.e. we accept the point \mathbf{x}_n if $\mathbf{x}_n \in A$, otherwise reject. Let $P_N^{(s)} = A \cap P_M = \{\mathbf{z}_0, \dots, \mathbf{z}_{N-1}\}$ be the sample set we accept.
- iii)Project the points we accepted $P_N^{(s)}$ onto the first (s-1) coordinates. Let $Y_N^{(s-1)} = \{\mathbf{y}_0, \dots, \mathbf{y}_{N-1}\}$ be the projections of the points $P_N^{(s)} = \{\mathbf{z}_0, \dots, \mathbf{z}_{N-1}\}$. iv) Return the point set $Y_N^{(s-1)}$.

3.1 Existence Result of Samples with Small Star Discrepancy

Here we present some results that we will use to prove an upper bound for the star discrepancy with respect to points generated by the acceptance-rejection sampler using stratified inputs. For any $0 \le \delta \le 1$, a set Γ of anchored boxes $[\mathbf{0}, \mathbf{x}] \subseteq [0, 1]^s$ is called a δ -cover of the set of anchored boxes $[\mathbf{0}, \mathbf{t}] \subseteq [0, 1]^s$ if for every point $\mathbf{t} \in [0, 1]^s$, there exist $[\mathbf{0}, \mathbf{x}], [\mathbf{0}, \mathbf{y}] \in \Gamma$ such that $[\mathbf{0}, \mathbf{x}] \subseteq [\mathbf{0}, \mathbf{t}] \subseteq [\mathbf{0}, \mathbf{y}]$ and $\lambda([\mathbf{0}, \mathbf{y}] \setminus [\mathbf{0}, \mathbf{x}]) \le \delta$. The following result on the size of the δ -cover is obtained from [13, Theorem 1.15].

Lemma 1. For any s and δ there exists a δ -cover of the set of anchored boxes $[\mathbf{0}, \mathbf{t}] \subseteq [0, 1]^s$ which has cardinality at most $(2e)^s (\delta^{-1} + 1)^s$.

By a simple generalization, the following result holds for our setting.

Lemma 2. Let $\psi: [0,1]^{s-1} \to \mathbb{R}_+$, where $s \ge 2$, be a function. Assume that there exists a constant $L < \infty$ such that $\psi(\mathbf{z}) \le L$ for all $\mathbf{z} \in [0,1]^{s-1}$. Let $A = \{\mathbf{z} \in [0,1]^s : \psi(z_1,\ldots,z_{s-1}) \ge Lz_s\}$ and $J_{\mathbf{t}}^* = ([\mathbf{0},\mathbf{t}) \times [0,1]) \cap A$. Let $(A,\mathcal{B}(A),\lambda)$ be a probability space where $\mathcal{B}(A)$ is the Borel σ -algebra of A. Define the set $\mathcal{A} \subset \mathcal{B}(A)$ of test sets by

$$\mathscr{A} = \{J_t^* : t \in [0,1]^{s-1}\}.$$

Then for any $\delta > 0$ there exists a δ -cover Γ_{δ} of \mathscr{A} with

$$|\Gamma_{\delta}| \leq (2e)^{s-1} (\delta^{-1} + 1)^{s-1}.$$

Proof. Let

$$\Gamma_{\delta} := \{ ([\mathbf{0}, \mathbf{x}] \times [0, 1]) \cap A, [\mathbf{0}, \mathbf{x}] \in \Gamma \},$$

where Γ is a δ -cover of the set of anchored boxes $[\mathbf{0}, \mathbf{t}] \subseteq [0, 1]^{s-1}$ with $|\Gamma| \le (2e)^{s-1}(\delta^{-1}+1)^{s-1}$. By Lemma 1 such a δ -cover Γ exists. For any set $J_t^* \in \mathscr{A}$, there exist $([\mathbf{0}, \mathbf{x}] \times [0, 1]) \cap A, ([\mathbf{0}, \mathbf{y}] \times [0, 1]) \cap A \in \Gamma_{\delta}$ such that

$$([\mathbf{0}, \mathbf{x}] \times [0, 1]) \cap A \subseteq J_t^* \subseteq ([\mathbf{0}, \mathbf{y}] \times [0, 1]) \cap A,$$

and

$$\lambda\left(\left(\left([\mathbf{0},\mathbf{y}]\times[0,1]\right)\cap A\right)\setminus\left(\left([\mathbf{0},\mathbf{x}]\times[0,1]\right)\cap A\right)\right)\leq\lambda\left([\mathbf{0},\mathbf{y}]\setminus[\mathbf{0},\mathbf{x}]\right)\leq\delta.$$

Hence Γ_{δ} forms a δ -cover of \mathscr{A} and $|\Gamma_{\delta}| = |\Gamma|$.

Lemma 3. Let the unnormalized density function $\psi : [0,1]^{s-1} \to \mathbb{R}_+$, with $s \ge 2$, be given. Assume that there exists a constant $L < \infty$ such that $\psi(\mathbf{z}) \le L$ for all $\mathbf{z} \in [0,1]^{s-1}$.

- Let $M \in \mathbb{N}$ and let the disjoint subsets Q_0, \ldots, Q_{M-1} be of the form $\prod_{i=1}^s \left[\frac{c_j}{M^{1/s}}, \frac{c_j+1}{M^{1/s}} \right]$ where $0 \le c_j \le \lceil M^{1/s} \rceil 1$. These sets form a disjoint covering of $[0,1]^s$ and each set Q_i satisfies $\lambda(Q_i) = 1/M$.
- Let

$$A = \{ \mathbf{z} \in [0,1]^s : \psi(z_1,\ldots,z_{s-1}) \ge Lz_s \}.$$

Assume that ∂A admits an (s-1)-dimensional Minkowski content \mathcal{M}_A .

• Let $J_t^* = ([\mathbf{0}, t) \times [0, 1]) \cap A$, where $t = (t_1, \dots, t_{s-1}) \in [0, 1]^{s-1}$.

Then there exists an $M_0 \in \mathbb{N}$ such that J_t^* at most intersects with $3s^{1/2}\mathcal{M}_A M^{1-1/s}$ subcubes Q_i for all $M \geq M_0$.

The result can be obtained utilizing a similar proof as in [14, Theorem 4.3]. For the sake of completeness, we repeat the proof here.

Proof. Since ∂A admits an (s-1)-dimensional Minkowski content, it follows that

$$\mathscr{M}_A = \lim_{arepsilon o 0} rac{\lambda\left((\partial A)_{arepsilon}
ight)}{2arepsilon} < \infty.$$

Thus by the definition of the limit, for any fixed $\vartheta > 2$, there exists ε_0 such that $\lambda((\partial A)_{\varepsilon}) \leq \vartheta \varepsilon \mathscr{M}_A$ whenever $\varepsilon \leq \varepsilon_0$.

Based on the form of the subcube given by $\prod_{i=1}^s \left[\frac{c_j}{M^{1/s}}, \frac{c_j+1}{M^{1/s}}\right]$, the largest diagonal length is $\sqrt{s}M^{-1/s}$. We can assume that $M > (\sqrt{s}/\varepsilon_0)^s$, then $\sqrt{s}M^{-1/s} =: \varepsilon < \varepsilon_0$ and $\bigcup_{i \in J} Q_i \subseteq (\partial A)_{\varepsilon}$, where J is the index set for the sets Q_i which satisfy $Q_i \cap A \neq \emptyset$. Therefore

$$|J| \leq \frac{\lambda((\partial A)_{\varepsilon})}{\lambda(O_i)} \leq \frac{\vartheta \varepsilon \mathcal{M}_A}{M^{-1}} = \sqrt{s}\vartheta \mathcal{M}_A M^{1-1/s}.$$

Without loss of generality, we can set $\vartheta = 3$, which completes the proof.

Remark 1. Ambrosio et al [1] found that for a closed set $A \subset \mathbb{R}^s$, if A has a Lipschitz boundary, then ∂A admits an (s-1)-dimensional Minkowski content. In particular, a convex set $A \subset [0,1]^s$ has an (s-1)-dimensional Minkowski content. Note that the surface area of a convex set in $[0,1]^s$ is bounded by the surface area of the unit cube $[0,1]^s$, which is 2s and it was also shown by Niederreiter and Wills [25] that 2s is best possible. It follows that the Minkowski content $\mathcal{M}_A \leq 2s$ when A is a convex set in $[0,1]^s$.

Lemma 4. Suppose that all the assumptions of Lemma 3 are satisfied. Let N be the number of points accepted by Algorithm 1. Then we have

$$M(\lambda(A) - 3s^{1/2}\mathcal{M}_A M^{-1/s}) \le N \le M(\lambda(A) + 3s^{1/2}\mathcal{M}_A M^{-1/s}).$$

Proof. The number of points we accept in Algorithm 1 is a random number since the driver sequence given by stratified inputs is random. Let $\mathbb{E}(N)$ be the expectation of N. The number of Q_i which have non-empty intersection with A is bounded by $l = 3s^{1/2}\mathcal{M}_A M^{1-1/s}$ from Lemma 3. Thus

$$\mathbb{E}[N] - l \le N \le \mathbb{E}[N] + l. \tag{1}$$

Further we have

$$\mathbb{E}[N] = \sum_{i=0}^{M-1} \frac{\lambda(Q_i \cap A)}{\lambda(Q_i)} = M\lambda(A). \tag{2}$$

Combining (1) and (2) and substituting $l = 3s^{1/2} \mathcal{M}_A M^{1-1/s}$, one obtains the desired result.

Before we start to prove the upper bound on the star-discrepancy, our method requires the well-known Bernstein-Chernoff inequality.

Lemma 5. [2, Lemma 2] Let $\eta_0, \ldots, \eta_{l-1}$ be independent random variables with $\mathbb{E}(\eta_i) = 0$ and $|\eta_i| \le 1$ for all $0 \le i \le l-1$. Denote by σ_i^2 the variance of η_i , i.e. $\sigma_i^2 = \mathbb{E}(\eta_i^2)$. Set $\beta = (\sum_{i=0}^{l-1} \sigma_i^2)^{1/2}$. Then for any $\gamma > 0$ we have

$$\mathbb{P}\Big(\big|\sum_{i=0}^{l-1}\eta_i\big|\geq\gamma\Big)\leq \begin{cases} 2e^{-\gamma/4}, & \text{if }\gamma\geq\beta^2,\\ 2e^{-\gamma^2/4\beta^2}, & \text{if }\gamma\leq\beta^2. \end{cases}$$

Theorem 1. Let an unnormalized density function $\psi:[0,1]^{s-1}\to\mathbb{R}_+$, with $s\geq 2$, be given. Assume that there exists a constant $L<\infty$ such that $\psi(\mathbf{z})\leq L$ for all $\mathbf{z}\in[0,1]^{s-1}$. Let $C=\int_{[0,1]^{s-1}}\psi(\mathbf{z})\,\mathrm{d}\mathbf{z}$ and let the graph under ψ be defined as

$$A = \{ \mathbf{z} \in [0,1]^s : \psi(z_1,\ldots,z_{s-1}) \ge Lz_s \}.$$

Assume that ∂A admits an (s-1)-dimensional Minkowski content \mathcal{M}_A . Then for all large enough N, with positive probability, Algorithm 1 yields a point set $Y_N^{(s-1)} \subseteq [0,1]^{s-1}$ such that

$$D_{N,\psi}^*(Y_N^{(s-1)}) \le \frac{s^{\frac{3}{4}}\sqrt{6\mathcal{M}_A}}{(2L)^{\frac{1}{2s}-\frac{1}{2}}C^{\frac{1}{2}-\frac{1}{2s}}} \frac{\sqrt{\log N}}{N^{\frac{1}{2}+\frac{1}{2s}}} + \frac{2C}{LN}.$$

Proof. Let $J_t^* = ([\mathbf{0}, t) \times [0, 1]) \cap A$, where $t = (t_1, \dots, t_{s-1})$. Using the notation from Algorithm 1, let y_n be the first s-1 coordinates of $z_n \in A$. For $n = 0, \dots, N-1$, we have

$$\sum_{n=0}^{M-1} 1_{J_{\boldsymbol{t}}^*}(\boldsymbol{x}_n) = \sum_{n=0}^{N-1} 1_{[\boldsymbol{0},\boldsymbol{t})}(\boldsymbol{y}_n).$$

Therefore

$$\left| \frac{1}{N} \sum_{n=0}^{N-1} 1_{[\mathbf{0},t)}(\mathbf{y}_n) - \frac{1}{C} \int_{[\mathbf{0},t)} \psi(\mathbf{z}) d\mathbf{z} \right| = \left| \frac{1}{N} \sum_{n=0}^{M-1} 1_{J_t^*}(\mathbf{x}_n) - \frac{1}{\lambda(A)} \lambda(J_t^*) \right|. \tag{3}$$

It is noted that

$$\left| \sum_{n=0}^{M-1} 1_{J_{t}^{*}}(\mathbf{x}_{n}) - \frac{N}{\lambda(A)} \lambda(J_{t}^{*}) \right| \leq \left| \sum_{n=0}^{M-1} 1_{J_{t}^{*}}(\mathbf{x}_{n}) - M\lambda(J_{t}^{*}) \right| + \left| \lambda(J_{t}^{*}) \left(M - \frac{N}{\lambda(A)} \right) \right|
\leq \left| \sum_{n=0}^{M-1} 1_{J_{t}^{*}}(\mathbf{x}_{n}) - M\lambda(J_{t}^{*}) \right| + \left| M\lambda(A) - N \right|
\leq \left| \sum_{n=0}^{M-1} 1_{J_{t}^{*}}(\mathbf{x}_{n}) - M\lambda(J_{t}^{*}) \right| + \left| M\lambda(A) - \sum_{n=0}^{M-1} 1_{A}(\mathbf{x}_{n}) \right|
\leq 2 \sup_{\mathbf{t} \in [0,1]^{s}} \left| \sum_{n=0}^{M-1} 1_{J_{t}^{*}}(\mathbf{x}_{n}) - M\lambda(J_{t}^{*}) \right|.$$
(4)

Let us associate with each Q_i , random points $x_i \in Q_i$ with probability distribution

$$\mathbb{P}(\mathbf{x}_i \in V) = \frac{\lambda(V)}{\lambda(Q_i)} = M\lambda(V),$$

for all measurable sets $V \subseteq Q_i$.

It follows from Lemma 3 that J_t^* at most intersect $l := 3s^{1/2} \mathcal{M}_A M^{1-1/s}$ sets Q_i . Therefore, J_t^* is representable as the disjoint union of sets Q_i entirely contained in J_t^* and the union of at most l pieces which are intersections of some sets Q_i and J_t^* , i.e.

$$J_{\boldsymbol{t}}^* = \bigcup_{i \in I} Q_i \cup \bigcup_{i \in J} (Q_i \cap J_{\boldsymbol{t}}^*),$$

where the index-set J has cardinality at most $\lceil 3s^{1/2}\mathcal{M}_A M^{1-1/s} \rceil$. Since for every Q_i , $\lambda(Q_i) = 1/M$ and Q_i contains exactly one element of $\{\mathbf{z}_1, \dots, \mathbf{z}_N\}$, the discrepancy of $\bigcup_{i \in J} Q_i$ is zero. Therefore, it remains to investigate the discrepancy of $\bigcup_{i \in J} (Q_i \cap J_t^*)$.

Since $\lambda(A) = C/L$ and $N \ge M(C/L - 3s^{1/2}\mathcal{M}_A M^{-1/s})$ by Lemma 4, we have $M \le 2LN/C$ for all $M > (6Ls^{1/2}\mathcal{M}_A/C)^s$. Consequently,

$$l = 3s^{1/2} \mathcal{M}_A M^{1-1/s} \le 3s^{1/2} (2L)^{1-1/s} C^{1/s-1} \mathcal{M}_A N^{1-\frac{1}{s}} = \Omega N^{1-1/s},$$

where $\Omega = 3s^{1/2}(2L)^{1-1/s}C^{1/s-1}\mathcal{M}_A$.

Let us define the random variable χ_i for $0 \le i \le l - 1$ as follows

$$\chi_i = \begin{cases} 1, & \text{if } \mathbf{z}_i \in Q_i \cap J_t^*, \\ 0, & \text{if } \mathbf{z}_i \notin Q_i \cap J_t^*. \end{cases}$$

By definition,

$$\left| \sum_{n=0}^{M-1} 1_{J_{t}^{*}}(\mathbf{x}_{n}) - M\lambda(J_{t}^{*}) \right| = \left| \sum_{i=0}^{l-1} \chi_{i} - M \sum_{i=0}^{l-1} \lambda(Q_{i} \cap J_{t}^{*}) \right|.$$
 (5)

Because of $\mathbb{P}(\chi_i = 1) = \lambda(Q_i \cap J_t^*)/\lambda(Q_i) = M\lambda(Q_i \cap J_t^*)$, we have

$$\mathbb{E}\chi_i = M\lambda(Q_i \cap J_t^*),\tag{6}$$

where $\mathbb{E}(\cdot)$ denotes the expected value. By (5) and (6),

$$\Delta_N(J_t^*; \boldsymbol{z}_1, \dots, \boldsymbol{z}_N) = \Big| \sum_{n=0}^{M-1} 1_{J_t^*}(\boldsymbol{x}_n) - M\lambda(J_t^*) \Big| = \Big| \sum_{i=0}^{L-1} (\chi_i - \mathbb{E}\chi_i) \Big|.$$
 (7)

Since the random variables χ_i for $0 \le i \le l-1$ are independent of each other, in order to estimate the sum $\sum_{i=0}^{l-1} (\chi_i - \mathbb{E}\chi_i)$ we are able to apply the classical Bernstein-Chernoff inequality of large deviation type. Let $\sigma_i^2 = \mathbb{E}(\chi_i - \mathbb{E}\chi_i)^2$ and set $\beta = (\sum_{i=1}^{l} \sigma_i^2)^{1/2}$. Let

$$\gamma = \theta l^{1/2} (\log N)^{1/2},$$

where θ is a constant depending only on the dimension s which will be fixed later. Without loss of generality, assume that $N \ge 3$.

Case 1: If $\gamma < \beta^2$, since $\beta^2 < l < \Omega N^{1-\frac{1}{s}}$, by Lemma 5 we obtain

$$\mathbb{P}\left(\Delta_{N}(J_{t}^{*}; \boldsymbol{z}_{1}, \dots, \boldsymbol{z}_{N}) \geq \theta l^{1/2} (\log N)^{1/2}\right)$$

$$= \mathbb{P}\left(\left|\sum_{i=1}^{l} (\chi_{i} - \mathbb{E}\chi_{i})\right| \geq \gamma\right) \leq 2e^{-\gamma^{2}/(4\beta^{2})} \leq 2N^{-\theta^{2}/4}.$$
(8)

Though the class of axis-parallel boxes is uncountable, it suffices to consider a small subclass. Based on the argument in Lemma 2, there is a 1/M-cover of cardinality $(2e)^{s-1}(M+1)^{s-1} \leq (2e)^{s-1}(2LN/C+1)^{s-1}$ for $M > M_0$ such that there exist $R_1, R_2 \in \Gamma_{1/M}$ having the properties $R_1 \subset J_t^* \subset R_2$ and $\lambda(R_2 \setminus R_1) \leq 1/M$. From this it follows that

$$\Delta_N(J_t^*; \boldsymbol{z}_1, \dots, \boldsymbol{z}_N) \leq \max_{i=1,2} \Delta(R_i; \boldsymbol{z}_1, \dots, \boldsymbol{z}_N) + 1,$$

see, for instance, [11, Lemma 3.1] and [16, Section 2.1]. This means that we can restrict ourselves to the elements of $\Gamma_{1/M}$.

In view of (8)

$$\mathbb{P}(\Delta(R_i; \mathbf{z}_1, \dots, \mathbf{z}_N) \geq \gamma) \leq |\Gamma_{1/M}| 2N^{-\frac{\theta^2}{4}} \leq 2N^{-\frac{\theta^2}{4}} (2e)^{s-1} (\frac{2LN}{C} + 1)^{s-1} < 1,$$

for $\theta = 2\sqrt{2s}$ and $N \ge \frac{8e}{C} + 2$.

Case 2: On the other hand, if $\gamma \ge \beta^2$, then by Lemma 5 we obtain

$$\mathbb{P}\left(\Delta(J_t^*; \mathbf{z}_1, \dots, \mathbf{z}_N) \ge \theta l^{1/2} (\log N)^{1/2}\right)
= \mathbb{P}\left(\left|\sum_{i=1}^{l} (\chi_i - \mathbb{E}\chi_i)\right| \ge \gamma\right) \le 2e^{-\frac{\theta l^{1/2} (\log N)^{1/2}}{4}}.$$
(9)

Similarly, using the 1/M-cover technique above, for $\theta = 2\sqrt{2s}$ and sufficiently large N we have

$$\begin{split} \mathbb{P}\big(\Delta(R_i; \pmb{z}_1, \dots, \pmb{z}_N) &\geq \gamma\big) &\leq |\Gamma_{1/M}| 2e^{-\frac{\theta l^{1/2} (\log N)^{1/2}}{4}} \\ &\leq 2e^{-\frac{\theta l^{1/2} (\log N)^{1/2}}{4}} (2e)^{s-1} \big(\frac{2LN}{C} + 1\big)^{s-1} < 1, \end{split}$$

where the last equation is satisfied for all large enough N.

By (3) and (4), we obtain that, with positive probability, Algorithm 1 yields a point set $Y_N^{(s-1)}$ such that

$$D_{N,\psi}^*(Y_N^{(s-1)}) \le \sqrt{2s}\Omega^{1/2}N^{-\frac{1}{2}-\frac{1}{2s}}(\log N)^{1/2} + 1/M.$$

By Lemma 1, we have $1/M \le 2C/(LN)$ for sufficiently large N. Thus the proof of Theorem 1 is complete.

3.2 Upper Bound on the L_q -discrepancy

In this section we prove an upper bound on the expected value of the L_q -discrepancy for $2 \le q \le \infty$.

Theorem 2. Let the unnormalized density function $\psi:[0,1]^{s-1}\to\mathbb{R}_+$ satisfy all the assumptions stated in Theorem 1. Let $Y_N^{(s-1)}$ be the samples generated by the acceptance-rejection sampler using stratified inputs. Then we have for $2\leq q\leq \infty$,

$$\left(\mathbb{E}[N^q L^q_{a,N}(Y^{(s-1)}_N)]\right)^{1/q} \leq (3s^{1/2}\mathcal{M}_A)^{1-1/q}(2LC^{-1})^{(1-1/s)(1-1/q)}N^{(1-1/s)(1-1/q)},$$

where \mathcal{M}_A is the (s-1)-dimensional Minkowski content and the expectation is taken with respect to the stratified inputs.

Proof. Let $J_t^* = ([0,t) \times [0,1]) \cap A$, where $t = (t_1, \dots, t_{s-1}) \in [0,1]^{s-1}$. Let

$$\xi_i(t) = 1_{Q_i \cap J_t^*}(\boldsymbol{x}_i) - \lambda(Q_i \cap J_t^*)/\lambda(Q_i),$$

where Q_i for $0 \le i \le M-1$ is the covering of $[0,1]^s$ with $\lambda(Q_i) = 1/M$. Then $\mathbb{E}(\xi_i(t)) = 0$ since we have $\mathbb{E}[1_{Q_i \cap J_t^*}(\mathbf{x}_i)] = M\lambda(Q_i \cap J_t^*)$. Hence

$$\begin{split} \mathbb{E}[\xi_i^2(t)] &= \mathbb{E}[(1_{Q_i \cap J_t^*}(\boldsymbol{x}_i) - M\lambda(Q_i \cap J_t^*))^2] \\ &= \mathbb{E}[1_{Q_i \cap J_t^*}(\boldsymbol{x}_i)] - 2M\lambda(Q_i \cap J_t^*) \mathbb{E}[1_{Q_i \cap J_t^*}(\boldsymbol{x}_i)] + M^2\lambda^2(Q_i \cap J_t^*) \\ &= M\lambda(Q_i \cap J_t^*)(1 - M\lambda(Q_i \cap J_t^*)) \\ &\leq M\lambda(Q_i \cap J_t^*) \leq 1. \end{split}$$

If $Q_i \subseteq J_t^*$ or if $Q_i \cap J_t^* = \emptyset$, we have $\xi_i(t) = 0$. We order the sets Q_i such that $Q_0, Q_1, \ldots, Q_{i_0}$ satisfy $Q_i \cap J_t^* \neq \emptyset$ and $Q_i \nsubseteq J_t^*$ (i.e. Q_i intersects the boundary of J_t^*) and the remaining sets Q_i either satisfy $Q_i \cap J_t^* = \emptyset$ or $Q_i \subseteq J_t^*$. Due to the fact that the density curve ψ at most intersects with $I := 3s^{1/2} \mathcal{M}(\partial A) M^{1-1/s}$ sets Q_i , if ∂A admits an (s-1)-dimensional Minkowski content, it follows that, for q=2,

$$\begin{split} \left(\mathbb{E}[N^{2}L_{2,N}^{2}(Y_{N}^{(s-1)})]\right)^{1/2} &= \left(\mathbb{E}\Big[\int_{[0,1]^{s}} \big|\sum_{i=0}^{M-1} \xi_{i}(\boldsymbol{t})\big|^{2} d\boldsymbol{t}\Big]\right)^{1/2} \\ &= \Big(\int_{[0,1]^{s}} \mathbb{E}\Big[\sum_{i=0}^{M-1} \xi_{i}(\boldsymbol{t})\Big]^{2} d\boldsymbol{t}\Big)^{1/2} \\ &= \Big(\int_{[0,1]^{s}} \sum_{i=0}^{l-1} \mathbb{E}[\xi_{i}(t)^{2}] d\boldsymbol{t}\Big)^{1/2} \leq l^{1/2}. \end{split}$$

Since $|\xi_i(t)| \le 1$, for $q = \infty$, we have

$$\begin{split} \sup_{P_{M} \subset [0,1]^{s}} |ND_{N}^{*}(Y_{N}^{(s-1)})| &= \sup_{P_{M} \subset [0,1]^{s}} \sup_{\boldsymbol{t} \in [0,1]^{s-1}} |\sum_{i=0}^{M-1} \xi_{i}(\boldsymbol{t})| = \sup_{P_{M} \subset [0,1]^{s}} \sup_{\boldsymbol{t} \in [0,1]^{s-1}} |\sum_{i=0}^{l-1} \xi_{i}(\boldsymbol{t})| \\ &\leq \sup_{P_{M} \in [0,1]^{s}} \sup_{\boldsymbol{t} \in [0,1]^{s-1}} \sum_{i=0}^{l-1} |\xi_{i}(\boldsymbol{t})| \leq l. \end{split}$$

Therefore, for $2 < q < \infty$,

$$\left(\mathbb{E}[N^q L_{q,N}^q(Y_N^{(s-1)})]\right)^{1/q} \le l^{1-1/q},$$

which is a consequence of the log-convexity of L_p -norms, i.e $||f||_{p_{\theta}} \le ||f||_{p_0}^{1-\theta} ||f||_{p_1}^{\theta}$, where $1/p_{\theta} = (1-\theta)/p_0 + \theta/p_1$. In our case, $p_0 = 2$ and $p_1 = \infty$.

Additionally, following from Lemma 4, we have $M \le 2LN/C$ whenever $M > (6Ls^{1/2}\mathcal{M}_A/C)^s$. Hence we obtain the desired result by substituting $l = 3s^{1/2}\mathcal{M}_A M^{1-1/s}$ and replacing M in terms of N.

Remark 2. It would also be interesting to obtain an upper bound for $1 \le q < 2$. See Heinrich [15] for a possible proof technique. We leave it as an open problem.

4 Improved Rate of Convergence for Deterministic Acceptance-Rejection Sampler

In this section, we prove a convergence rate of order $N^{-\alpha}$ for $1/s \le \alpha < 1$, where α depends on the target density ψ . See Corollary 1 below for details. For this result we use (t,m,s)-nets (see Definition 5 below) as inputs instead of stratified samples. The value of α here depends on how well the graph of ψ can be covered by certain rectangles (see Equation (10)). In practice this covering rate of order $N^{-\alpha}$ is hard to determine precisely, where α can range anywhere from 1/s to < 1, where α arbitrarily close to 1 can be achieved if ψ is constant. We also provide a simple example in dimension s=2 for which α can take on the values $\alpha=1-\ell^{-1}$ for $\ell\in\mathbb{N}, \ell>2$. See Example 1 for details.

We first establish some notation and some useful definitions and then obtain theoretical results. First we introduce the definition of (t, m, s)-nets in base b (see [8]) which we use as the driver sequence. The following fundamental definitions of elementary interval and fair sets are used to define a (t, m, s)-net in base b.

Definition 3. [b-adic elementary interval] Let $b \ge 2$ be an integer. An *s*-dimensional *b*-adic elementary interval is an interval of the form

$$\prod_{i=1}^{s} \left[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}} \right)$$

with integers $0 \le a_i < b^{d_i}$ and $d_i \ge 0$ for all $1 \le i \le s$. If d_1, \ldots, d_s are such that $d_1 + \cdots + d_s = k$, then we say that the elementary interval is of order k.

Definition 4 (fair sets). For a given set $P_N = \{x_0, x_1, ..., x_{N-1}\}$ consisting of N points in $[0,1)^s$, we say for a subset J of $[0,1)^s$ to be fair with respect to P_N , if

$$\frac{1}{N}\sum_{n=0}^{N-1}1_J(\boldsymbol{x}_n)=\lambda(J),$$

where $1_J(\mathbf{x}_n)$ is the indicator function of the set J.

Definition 5 ((t,m,s)-**nets in base b).** For a given dimension $s \ge 1$, an integer base $b \ge 2$, a positive integer m and an integer t with $0 \le t \le m$, a point set $Q_{m,s}$ of b^m points in $[0,1)^s$ is called a (t,m,s)-nets in base b if the point set $Q_{m,s}$ is fair with respect to all b-adic s-dimensional elementary intervals of order at most m-t.

We present the acceptance-rejection algorithm using (t, m, s)-nets as driver sequence.

Algorithm 2 Let the target density $\psi : [0,1]^{s-1} \to \mathbb{R}_+$, where $s \ge 2$, be given. Assume that there exists a constant $L < \infty$ such that $\psi(\mathbf{x}) \le L$ for all $\mathbf{x} \in [0,1]^{s-1}$. Let $A = \{\mathbf{z} \in [0,1]^s : \psi(z_1,\ldots,z_{s-1}) \ge Lx_s\}$. Suppose we aim to obtain approximately N samples from ψ .

- i) Let $M = b^m \ge \left\lceil N/(\int_{[0,1]^{s-1}} \psi(\mathbf{x})/Ld\mathbf{x}) \right\rceil$, where $m \in \mathbb{N}$ is the smallest integer satisfying this inequality. Generate a (t,m,s)-net $Q_{m,s} = \{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{b^m-1}\}$ in base b.
- ii) Use the acceptance-rejection method for the points $Q_{m,s}$ with respect to the density ψ , i.e. we accept the point \mathbf{x}_n if $\mathbf{x}_n \in A$, otherwise reject. Let $P_N^{(s)} = A \cap Q_{m,s} = \{\mathbf{z}_0, \dots, \mathbf{z}_{N-1}\}$ be the sample set we accept.
- iii)Project the points $P_N^{(s)}$ onto the first (s-1) coordinates. Let $Y_N^{(s-1)} = \{\mathbf{y}_0, \dots, \mathbf{y}_{N-1}\} \subseteq [0,1]^{s-1}$ be the projections of the points $P_N^{(s)}$.
- iv) Return the point set $Y_N^{(s-1)}$.

In the following we show that an improvement of the discrepancy bound for the deterministic acceptance-rejection sampler is possible. Let an unnormalized density function $\psi : [0,1]^{s-1} \to \mathbb{R}_+$, with $s \ge 2$, be given. Let again

$$A = \{ \boldsymbol{z} = (z_1, \dots, z_s) \in [0, 1]^s : \psi(z_1, \dots, z_{s-1}) \ge Lz_s \}$$

and $J_t^* = ([0, t) \times [0, 1]) \cap A$. Let ∂J_t^* denote the boundary of J_t^* and $\partial [0, 1]^s$ denotes the boundary of $[0, 1]^s$. For $k \in \mathbb{N}$ we define the covering number

$$\Gamma_{k}(\psi) = \sup_{\boldsymbol{t} \in [0,1]^{s}} \min\{v : \exists U_{1}, \dots, U_{v} \in \mathscr{E}_{k} : (\partial J_{\boldsymbol{t}}^{*} \setminus \partial [0,1]^{s}) \subseteq \bigcup_{i=1}^{v} U_{i},
U_{i} \cap U_{i'} = \emptyset \text{ for } 1 \le i < i' \le v\},$$
(10)

where \mathcal{E}_k is the family of elementary intervals of order k.

Lemma 6. Let $\psi : [0,1]^{s-1} \to [0,1]$ be an unnormalized target density and let the covering number $\Gamma_{m-t}(\psi)$ be given by (10). Then the discrepancy of the point set $Y_N^{(s-1)} = \{y_0, y_1, \dots, y_{N-1}\} \subseteq [0,1]^{s-1}$ generated by Algorithm 2 using a (t, m, s)-net in base b, for large enough N, satisfies

$$D_{N,\psi}^*(Y_N^{(s-1)}) \le 4C^{-1}b^t\Gamma_{m-t}(\psi)N^{-1},$$

where $C = \int_{[0,1]^{s-1}} \psi(z) dz$.

Proof. Let $\mathbf{t} \in [0,1]^s$ be given. Let $v = \Gamma_{m-t}(\psi)$ and U_1, \ldots, U_v be elementary intervals of order m-t such that $U_1 \cup U_2 \cup \cdots \cup U_v \supseteq (\partial J_t^* \setminus \partial [0,1]^s)$ and $U_i \cap U_{i'} = \emptyset$ for $1 \le i < i' \le v$. Let $V_1, \ldots, V_z \in \mathscr{E}_{m-t}$ with $V_i \subseteq J_t^*, V_i \cap V_{i'} = \emptyset$ for all $1 \le i < i' \le z$ and $V_i \cap U_i = \emptyset$ such that $\bigcup_{i=1}^z V_i \cup \bigcup_{i=1}^v U_i \supseteq J_t^*$. We define

$$\overline{W} = \bigcup_{i=1}^{z} V_i \cup \bigcup_{i=1}^{v} U_i$$

and

$$W^o = \bigcup_{i=1}^z V_i.$$

Then \overline{W} and W^o are fair with respect to the (t, m, s)-net, $W^o \subseteq J_t^* \subseteq \overline{W}$ and

$$\lambda(\overline{W}\setminus J_{\boldsymbol{t}}^*), \lambda(J_{\boldsymbol{t}}^*\setminus W^o) \leq \lambda(\overline{W}\setminus W^o) = \sum_{i=1}^{\nu} \lambda(U_i) = \sum_{i=1}^{\nu} b^{-m+t} = b^{-m+t} \Gamma_{m-t}(\psi).$$

The proof of the result now follows by the same arguments as the proofs in [32, Lemma 1&Theorem 1].

From Lemma 3 we have that if ∂A admits an (s-1)-dimensional Minkowski content, then

$$\Gamma_k(\psi) < c_s b^{(1-1/s)k}$$
.

This yields a convergence rate of order $N^{-1/s}$ in Lemma 6. Another known example is the following. Assume that ψ is constant. Since the graph of ψ can be covered by just one elementary interval of order m-t, this is the simplest possible case. The results from [24, Section 3] (see also [8, p. 184–190] for an exposition in dimensions s=1,2,3) imply that $\Gamma_k(\psi) \leq C_s k^{s-1}$ for some constant C_s which depends only on s. This yields the convergence rate of order $(\log N)^{s-1}N^{-1}$ in Lemma 6. Thus, in general, there are constants $c_{s,\psi}$ and $C_{s,\psi}$ depending only on s and ψ such that

$$c_{s,\psi}k^{s-1} \le \Gamma_k(\psi) \le C_{s,\psi}b^{(1-1/s)k},$$
 (11)

whenever the set ∂A admits an (s-1)—dimensional Minkowski content. This yields a convergence rate in Lemma 6 of order $N^{-\alpha}$ with $1/s \le \alpha < 1$, where the precise value of α depends on ψ . We obtain the following corollary.

Corollary 1. Let $\psi: [0,1]^{s-1} \to [0,1]$ be an unnormalized target density and let $\Gamma_k(\psi)$ be given by (10). Assume that there is a constant $\Theta > 0$ such that

$$\Gamma_k(\psi) \le \Theta b^{(1-\alpha)k} k^{\beta}$$
 for all $k \in \mathbb{N}$,

for some $1/s \le \alpha \le 1$ and $\beta \ge 0$. Then there is a constant $\Delta_{s,t,\psi} > 0$ which depends only on s,t and ψ , such that the discrepancy of the point set $Y_N^{(s-1)} = \{y_0,y_1,\ldots,y_{N-1}\} \subseteq [0,1]^{s-1}$ generated by Algorithm 2 using a (t,m,s)-net in base b, for large enough N, satisfies

$$D_{N,\psi}^*(Y_N^{(s-1)}) \le \Delta_{s,t,\psi} N^{-\alpha} (\log N)^{\beta}.$$

Example 1. To illustrate the bound in Corollary 1, we consider now an example for which we can obtain an explicit bound on $\Gamma_k(\psi)$ of order $b^{k(1-\alpha)}$ for $1/2 \le \alpha < 1$. For simplicity let s=2 and $\alpha=1-\ell^{-1}$ for some $\ell \in \mathbb{N}$ with $\ell \ge 2$. We define now a function $\psi_\ell: [0,1] \to [0,1]$ in the following way: let $x \in [0,1)$ have b-adic expansion

$$x = \frac{\xi_1}{b} + \frac{\xi_2}{b^2} + \frac{\xi_3}{b^3} + \cdots$$

where $\xi_i \in \{0, 1, ..., b-1\}$ and assume that infinitely many of the ξ_i are different from b-1. Then set

$$\psi_{\ell}(x) = \frac{\xi_1}{h^{l-1}} + \frac{\xi_2}{h^{2(l-1)}} + \frac{\xi_3}{h^{3(l-1)}} + \cdots$$

Let $t \in [0,1)$. In the following we define elementary intervals of order $k \in \mathbb{N}$ which cover $\partial J_t^* \setminus \partial [0,1]^2$. Assume first that k is a multiple of ℓ , then let $g = k/\ell$. Then we define the following elementary intervals of order $k = g\ell$:

$$\left[\frac{a_{1}}{b} + \dots + \frac{a_{g-1}}{b^{g-1}} + \frac{a_{g}}{b^{g}}, \frac{a_{1}}{b} + \dots + \frac{a_{g-1}}{b^{g-1}} + \frac{a_{g}+1}{b^{g}}\right) \times \left[\frac{a_{1}}{b^{\ell-1}} + \dots + \frac{a_{g-1}}{b^{(g-1)(\ell-1)}} + \frac{a_{g}}{b^{g(\ell-1)}}, \frac{a_{1}}{b^{\ell-1}} + \dots + \frac{a_{g-1}}{b^{(g-1)(\ell-1)}} + \frac{a_{g}+1}{b^{g(\ell-1)}}\right), (12)$$

where $a_1, \ldots, a_g \in \{0, 1, \ldots, b-1\}$ run through all possible choices such that

$$\frac{a_1}{h} + \dots + \frac{a_{g-1}}{h^{g-1}} + \frac{a_g + 1}{h^g} \le t.$$

The number of these choices for a_1, \ldots, a_g is bounded by b^g . Let

$$t = \frac{t_1}{h} + \dots + \frac{t_g}{h^g} + \frac{t_{g+1}}{h^{g+1}} + \dots$$

For integers $1 \le u \le g(\ell - 1)$ and $0 \le c_u < t_{g+u}$, we define the intervals

$$\left[\frac{t_1}{b} + \dots + \frac{t_{g+u-1}}{b^{g+u-1}} + \frac{c_u}{b^{g+u}}, \frac{t_1}{b} + \dots + \frac{t_{g+u-1}}{b^{g+u-1}} + \frac{c_u+1}{b^{g+u}}\right) \times \left[\frac{d_1}{b} + \dots + \frac{d_{g(\ell-1)-u}}{b^{g(\ell-1)-u}}, \frac{d_1}{b} + \dots + \frac{d_{g(\ell-1)-u}}{b^{g(\ell-1)-u}} + \frac{1}{b^{g(\ell-1)-u}}\right), \tag{13}$$

where $d_i = 0$ if $\ell \nmid i$, $d_i = t_{i/\ell}$ if $\ell \mid i$ and we set $\frac{d_1}{b} + \dots + \frac{d_g(\ell-1)-u}{b^{g(\ell-1)-u}} = 0$ if $u = g(\ell-1)$. Further we define the interval

$$\left[\frac{t_1}{b} + \dots + \frac{t_{g\ell}}{b^{g\ell}}, \frac{t_1}{b} + \dots + \frac{t_{g\ell}}{b^{g\ell}} + \frac{1}{b^{g\ell}}\right) \times [0, 1). \tag{14}$$

The intervals defined in (12), (13) and (14) cover $\partial J_t^* \setminus \partial [0,1]^2$. Thus we have

$$\Gamma_{g\ell}(\psi_{\ell}) \leq b^g + bg(\ell-1) + 1 \leq \ell b^g$$
.

For arbitrary $k \in \mathbb{N}$ we can use elementary intervals of order k which cover the same area as the intervals (12), (13) and (14). Thus we have at most $b^{\ell-1}$ times as many intervals and we therefore obtain

$$\Gamma_k(\psi_\ell) \le \ell b^{k/\ell+\ell-1}$$
.

Thus we obtain

$$\sup_{t \in [0,1]} \left| \frac{1}{N} \sum_{n=0}^{N-1} 1_{[0,t)}(y_n) - \frac{1}{C} \int_0^t \psi_{\ell}(z) dz \right| \le \Delta_{s,t,\psi} N^{-(1-\frac{1}{\ell})}.$$

Remark 3. In order to obtain similar results as in this section for stratified inputs rather than (t, m, s)—nets, one would have to use the elementary intervals U_1, \ldots, U_{ν} of order k which yield a covering of $\partial J_t^* \setminus \partial [0, 1]^s$ for all $t \in [0, 1]^{s-1}$. From this covering one would then have to construct a covering of $\partial A \setminus \partial [0, 1]^s$ and use this covering to obtain stratified inputs. Since such a covering is not easily available in general, we did not pursue this approach further.

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