# Subquadratic Algorithms for Succinct Stable Matching 

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December 21, 2016


#### Abstract

We consider the stable matching problem when the preference lists are not given explicitly but are represented in a succinct way and ask whether the problem becomes computationally easier and investigate other implications. We give subquadratic algorithms for finding a stable matching in special cases of natural succinct representations of the problem, the $d$-attribute, $d$-list, geometric, and single-peaked models. We also present algorithms for verifying a stable matching in the same models. We further show that for $d=\omega(\log n)$ both finding and verifying a stable matching in the $d$-attribute and $d$-dimensional geometric models requires quadratic time assuming the Strong Exponential Time Hypothesis. This suggests that these succinct models are not significantly simpler computationally than the general case for sufficiently large $d$.


## 1 Introduction

The stable matching problem has applications that vary from coordinating buyers and sellers to assigning students to public schools and residents to hospitals [21, 29, 37]. Gale and Shapley [17] proposed a quadratic time deferred acceptance algorithm for this problem which has helped clear matching markets in many real-world settings. For arbitrary preferences, the deferred acceptance algorithm is optimal and even verifying that a given matching is stable requires quadratic time [33, 38, 19]. This is reasonable since representing all participants' preferences requires quadratic space. However, in many applications the preferences are not arbitrary and can have more structure. For example, top doctors are likely to be universally desired by residency programs and students typically seek highly ranked schools. In these cases participants can represent their preferences succinctly. It is natural to ask whether the same quadratic time bounds apply with compact and structured preference models that have subquadratic representations. This will provide a more nuanced understanding of where the complexity lies: Is stable matching inherently complex, or is the complexity merely a result of the large variety of possible preferences? To this end, we

[^0]examine several restricted preference models with a particular focus on two originally proposed by Bhatnagar et al. [8], the $d$-attribute and $d$-list models. Using a wide range of techniques we provide algorithms and conditional hardness results for several settings of these models.

In the $d$-attribute model, we assume that there are $d$ different attributes (e.g. income, height, sense of humor, etc.) with a fixed, possibly objective, ranking of the men for each attribute. Each woman's preference list is based on a linear combination of the attributes of the men, where each woman can have different weights for each attribute. Some women may care more about, say, height whereas others care more about sense of humor. Men's preferences are defined analogously. This model is applicable in large settings, such as online dating systems, where participants lack the resources to form an opinion of every other participant. Instead the system can rank the members of each gender according to the $d$ attributes and each participant simply needs to provide personalized weights for the attributes. The combination of attribute values and weights implicitly represents the entire preference matrix. Bogomolnaia and Laslier [9] show that representing all possible $n \times n$ preference matrices requires $n-1$ attributes. Therefore it is reasonable to expect that when $d \ll n-1$, we could beat the worst case quadratic lower bounds for the general stable matching problem.

In the $d$-list model, we assume that there are $d$ different rankings of the men. Each women selects one of the $d$ lists as her preference list. Similarly, each man chooses one of $d$ lists of women as his preference list. This model captures the setting where members of one group (i.e. student athletes, sorority members, engineering majors) may all have identical preference lists. Mathematically, this model is actually a special case of the $d$-attribute model where each participant places a positive weight on exactly one attribute. However, its motivation is distinct and we can achieve improved results for this model.

Chebolu et al. prove that approximately counting stable matchings in the $d$-attribute model for $d \geq 3$ is as hard as the general case [12]. Bhatnagar et al. showed that sampling stable matchings using random walks can take exponential time even for a small number of attributes or lists but left it as an open question whether subquadratic algorithms exist for these models [8].

We show that faster algorithms exist for finding a stable matching in some special cases of these models. In particular, we provide subquadratic algorithms for the $d$-attribute model, where all values and weights are from a small set, and the one-sided $d$-attribute model, where one side of the market has only one attribute. These results show we can achieve meaningful improvement over the general setting for some restricted preferences.

While we only provide subquadratic algorithms to find stable matchings in special cases of the attribute model, we have stronger results concerning verification of stable matchings. We demonstrate optimal subquadratic stability testing algorithms for the $d$-list and boolean $d$-attribute settings as well as a subquadratic algorithm for the general $d$-attribute model with constant $d$. These algorithms provide a clear distinction between the attribute model and the general setting. Moreover, these results raise the question of whether verifying and finding a stable matching are equally hard problems for these restricted models, as both require quadratic time in the general case.

Additionally, we show that the stable matching problem in the $d$-attribute model for $d=$ $\omega(\log n)$ cannot be solved in subquadratic time under the Strong Exponential Time Hypothesis (SETH) [23, 25]. We show SETH-hardness for both finding and verifying a stable matching and for checking if a given pair is in any or all stable matchings, even when the weights and attributes are boolean. This adds the stable matching problem to a growing list of SETH-hard problems, including Fréchet distance [10], edit distance [6], string matching [1], $k$-dominating set [34], orthogonal vectors [42], and vector domination [24]. Thus the quadratic time hardness of the stable matching problem in the general case extends to the more restricted and succinct $d$-attribute model. This limits the
space of models where we can hope to find subquadratic algorithms.
We further present several results in related succinct preference areas. Single-peaked preferences are commonly used to model preferences in social choice theory because of their simplicity and because they often approximate reality. Essentially, single-peaked preferences require that everyone agree on a common spectrum along which all alternatives can be ranked. However, each individual may have a different ideal choice and prefers the "closest" alternatives. A typical example is the political spectrum where candidates fall somewhere between liberal and conservative. In this setting, voters tend to prefer the candidates that are closer to their own ideals. As explained below, these preferences can be succinctly represented. Bartholdi and Trick [7 present a subquadratic time algorithm for stable roommates (and stable matching) with narcissistic, single-peaked preferences. In the narcissistic case, the participants are located at their own ideals. This makes sense in some applications but is not always realistic. We provide a subquadratic algorithm to verify if a given matching is stable in the general single-peaked preference model. Chung uses a slightly different model of single-peaked preferences where a stable roommate matching always exists [13]. In this model the participants would rather be unmatched than matched with someone further away from their ideal than they are themselves, leading to incomplete preference lists.

We extend our algorithms and lower bounds for the attribute model to the geometric model where preference orders are formed according to euclidean distances among a set of points in multi-dimensional space. Arkin et al. [5] derive a subquadratic algorithm for stable roommates with narcissistic geometric preferences in constant dimensions. Our algorithms do not require the preferences to be narcissistic.

It is worth noting that all of our verification and hardness results apply to the stable roommates problem as well. This problem is identical to stable matching except we remove the bipartite distinction between the participants [21]. Unlike with bipartite stable matching, there need not always exist a stable roommate matching. However Irving discovered an algorithm that produces a stable matching or identifies that none exists in quadratic time [26]. Since finding a stable roommate matching is strictly harder than finding a stable matching, this is also optimal. Likewise, verification is equally hard for both stable roommates and stable matching, as we can simply duplicate every participant and treat the roommate matching as bipartite. Therefore, our results show that verification can be done more efficiently for the stable roommates problem when the preferences are succinct.

Finally, we address the issue of strategic behavior in these restricted models. It is often preferable for a market-clearing mechanism to incentivize truthful behavior from the participants so that the outcome faithfully captures the optimal solution. Particularly in matching markets, this objective complements the desire for a stable matching where participants have incentives to cooperate with the outcome. Roth [36] showed that there is no strategy proof mechanism to find a stable matching in the general preferences setting. Additionally, if a mechanism outputs the man-optimal stable matching, the women can manipulate it to obtain the woman-optimal solution by truncating their preference lists [36, 18]. Even if the women are required to rank all men, they can still achieve more preferable outcomes in some instances [41, 30]. However, in the $d$-attribute, $d$-list, single-peaked, and geometric preference models, there are considerably fewer degrees of freedom for preference misrepresentation. Nevertheless, we show that there is still no strategy proof mechanism to find a stable matching for any of these models with $d \geq 2$ and non-narcissistic preferences.

Dabney and Dean [14] study an alternative succinct preference representation where there is a canonical preference list for each side and individual deviations from this list are specified separately. They provide an adaptive $O(n+k)$ time algorithm for the special one-sided case, where $k$ is the number of deviations.

## 2 Summary of Results

Section 4.1 gives an $O\left(C^{2 d} n(d+\log n)\right)$ time algorithm for finding a stable matching in the $d$ attribute model if both the attributes and weights are from a set of size at most $C$. This gives a strongly subquadratic algorithm (i.e. $O\left(n^{2-\varepsilon}\right)$ for $\varepsilon>0$ ) if $d<\frac{1}{2 \log C} \log n$.

Section 4.2 considers an asymmetric case, where one side of the matching market has $d$ attributes, while the other side has a single attribute. We allow both the weights and attributes to be arbitrary real values. Our algorithm for finding a stable matching in this model has time complexity $\tilde{O}\left(n^{2-1 /\lfloor d / 2\rfloor}\right)$, which is strongly subquadratic for constant $d$.

In Section 5.1 we consider the problem of verifying that a given matching is stable in the $d$-attribute model with real attributes and weights. The time complexity of our algorithm is $\tilde{O}\left(n^{2-1 / 2 d}\right)$, which is again strongly subquadratic for constant $d$.

Section 5.2 gives an $O(d n)$ time algorithm for verifying a stable matching in the $d$-list model. This is linear in its input size and is therefore optimal.

In Section 5.3 we give a randomized $\tilde{O}\left(n^{2-1 / O\left(c \log ^{2}(c)\right)}\right)$ time algorithm for $d=c \log n$ for verifying a stable matching in the $d$-attribute model when both the weights and attributes are boolean. This algorithm is strongly subquadratic for $d=O(\log n)$.

In Section 6 we give a conditional lower bound for the three problems of finding and verifying a stable matching in the $d$-attribute model as well as the stable pair problem. We show that there is no strongly subquadratic algorithm for any of these problems when $d=\omega(\log n)$ assuming the Strong Exponential Time Hypothesis. For the stable pair problem we give further evidence that even nondeterminism does not give a subquadratic algorithm.

Finally in Section 7 we consider the related preference models of single-peaked and geometric preferences. We extend our algorithms and lower bounds for the attribute model to the geometric model and give an $O(n \log n)$ algorithm for verifying a stable matching with single-peaked preferences.

## 3 Preliminaries

A matching market consists of a set of men $M$ and a set of women $W$ with $|M|=|W|=n$. We further have a permutation of $W$ for every $m \in M$, and a permutation of $M$ for every $w \in W$, called preference lists. Note that representing a general matching market requires size $\Omega\left(n^{2}\right)$.

For a perfect bipartite matching $\mu$, a blocking pair with respect to $\mu$ is a pair ( $m, w$ ) $\notin \mu$ where $m \in M$ and $w \in W$, such that $w$ appears before $\mu(m)$ in $m$ 's preference list and $m$ appears before $\mu(w)$ in $w$ 's preference list. A perfect bipartite matching is called stable if there are no blocking pairs. In settings where ties in the preference lists are possible, we consider weakly stable matchings where $(m, w)$ is a blocking pair if and only if both strictly prefer each other to their partner. For simplicity, we assume all preference lists are complete though our results trivially extend to cases with incomplete lists.

Gale's and Shapley's deferred acceptance algorithm [17] works as follows. While there is an unmatched man $m$, have $m$ propose to his most preferred woman who has not already rejected him. A woman accepts a proposal if she is unmatched or if she prefers the proposing man to her current partner, leaving her current partner unmatched. Otherwise, she rejects the proposal. This process finds a stable matching in time $O\left(n^{2}\right)$.

A matching market in the $d$-attribute model consists of $n$ men and $n$ women as before. A participant $p$ has attributes $A_{i}(p)$ for $1 \leq i \leq d$ and weights $\alpha_{i}(p)$ for $1 \leq i \leq d$. For a man $m$ and woman $w, m$ 's value of $w$ is given by $\operatorname{val}_{m}(w)=\langle\alpha(m), A(w)\rangle=\sum_{i=1}^{d} \alpha_{i}(m) A_{i}(w)$. $m$ ranks the
women in decreasing order of value. Symmetrically, $w$ 's value of $m$ is $\operatorname{val}_{w}(m)=\sum_{i=1}^{d} \alpha_{i}(w) A_{i}(m)$. Note that representing a matching market in the $d$-attribute model requires size $O(d n)$. Unless otherwise specified, both attributes and weights can be negative.

A matching market in the $d$-list model is a matching market where both sides have at most $d$ distinct preference lists. Describing a matching market in this model requires $O(d n)$ numbers.

Throughout the paper, we use $\tilde{O}$ to suppress polylogarithmic factors in the time complexity.

## 4 Finding Stable Matchings

### 4.1 Small Set of Attributes and Weights

We first present a stable matching algorithm for the $d$-attribute model when the attribute and weight values are limited to a set of constant size. In particular, we assume that the number of possible values for each attribute and weight for all participants is bounded by a constant $C$.

```
Algorithm 1: Small Constant Attributes and Weights
    Group the women into sets \(S_{i}\) with a set for each of the \(C^{\prime}=O\left(C^{2 d}\right)\) types of women.
    \(\left(O\left(C^{d}\right)\right.\) possible attribute values and \(O\left(C^{d}\right)\) possible weight vectors.)
    Associate an empty min-heap \(h_{i}\) with each set \(S_{i}\).
    for each man \(m\) do
        Create \(m\) 's preference list of sets \(S_{i}\).
        \(\operatorname{index}(m) \leftarrow 1\)
    while there is a man \(m\) who is not in any heap do
        Let \(S_{i}\) be the index \((m)\) set on \(m\) 's list.
        if \(\left|h_{i}\right|<\left|S_{i}\right|\) then
            \(h_{i}\). insert ( \(m\) )
        else
            if \(\operatorname{val}_{S_{i}}(m)>\operatorname{val}_{S_{i}}\left(h_{i}\right.\).min \()\) then
                \(h_{i}\). delete_min()
                \(h_{i}\). insert ( \(m\) )
        index \((m) \leftarrow \operatorname{index}(m)+1\)
    for \(i=1\) to \(C^{\prime}\) do
        \(\mu \leftarrow \mu \bigcup\) Arbitrarily pair women in \(S_{i}\) with men in \(h_{i}\).
    return \(\mu\)
```

Theorem 1. There is an algorithm to find a stable matching in the d-attribute model with at most a constant $C$ distinct attribute and weight values in time $O\left(C^{2 d} n(d+\log n)\right)$.

Proof. Consider Algorithm [1 First observe that each man is indifferent between the women in a given set $S_{i}$ because each woman has identical attribute values. Moreover, the women in a set $S_{i}$ share the same ranking of the men, since they have identical weight vectors. Therefore, since we are looking for a stable matching, we can treat each set of women $S_{i}$ as an individual entity in a many to one matching where the capacity for each $S_{i}$ is the number of women it contains.

With these observations, the stability follows directly from the stability of the standard deferred acceptance algorithm for many-one stable matching. Indeed, each man proposes to the sets of women in the order of his preferences and each set of women tentatively accepts the best proposals, holding onto no more than the available capacity.

The grouping of the women requires $O\left(C^{2 d}+d n\right)$ time to initialize the groups and place each woman in the appropriate group. Creating the men's preference lists requires $O\left(d C^{2 d} n\right)$ time
to evaluate and sort the groups of women for every man. The while loop requires $O\left(C^{2 d} n(d+\right.$ $\log n)$ ) time since each man will propose to at most $C^{2 d}$ sets of women and each proposal requires $O(d+\log n)$ time to evaluate and update the heap. This results in an overall running time of $O\left(C^{2 d} n(d+\log n)\right)$.

As long as $d<\frac{1}{2 \log C} \log n$, the time complexity in Theorem $\square$ will be subquadratic. It is worth noting that the algorithm and proof actually do not rely on any restriction of the men's attribute and weight values. Thus, this result holds whenever one side's attributes and weight values come from a set of constant size.

### 4.2 One-Sided Real Attributes

In this section we consider a one-sided attribute model with real attributes and weights. In this model, women have $d$ attributes and men have $d$ weights, and the preference list of a man is given by the weighted sum of the women's attributes as in the two-sided attribute model. On the other hand there is only one attribute for the men. The women's preferences are thus determined by whether they have a positive or negative weight on this attribute. For simplicity, we first assume that all women have a positive weight on the men's attribute and show a subquadratic algorithm for this case. Then we extend it to allow for negative weights.

To find a stable matching when the women have a global preference list over the men, we use a greedy approach: process the men from the most preferred to the least preferred and match each man with the highest unmatched woman in his preference list. This general technique is not specific to the attribute model but actually works for any market where one side has a single global preference list. (e.g. [14] uses a similar approach for their algorithm.) The complexity lies in repeatedly finding which of the available women is most preferred by the current top man.

This leads us to the following algorithm: for every woman $w$ consider a point with $A(w)$ as its coordinates and organize the set of points into a data structure. Then, for the men in order of preference, query the set of points against a direction vector consisting of the man's weight and find the point with the largest distance along this direction. Remove that point and repeat.

The problem of finding a maximal point along a direction is typically considered in its dual setting, where it is called the ray shooting problem. In the ray shooting problem we are given $n$ hyperplanes and must maintain a data structure to answer queries. Each query consists of a vertical ray and the data structure returns the first hyperplane hit by that ray.

The relevant results are in Lemma 1 which follows from several papers for different values of $d$. For an overview of the ray shooting problem and related range query problems, see [3].

Lemma 1 ([22, 16, 2, 32]). Given an $n$ point set in $\mathbb{R}^{d}$ for $d \geq 2$, there is a data structure for ray shooting queries with preprocessing time $\tilde{O}(n)$ and query time $\tilde{O}\left(n^{1-1 /\lfloor d / 2\rfloor}\right)$. The structure supports deletions with amortized update time $\tilde{O}(1)$.

For $d=1$, queries can trivially be answered in constant time. We use this data structure to provide an algorithm when there is a global list for one side of the market.

Lemma 2. For $d \geq 2$ there is an algorithm to find a stable matching in the one-sided d-attribute model with real-valued attributes and weights in time $\tilde{O}\left(n^{2-1 /\lfloor d / 2\rfloor}\right)$ when there is a single preference list for the other side of the market.

Proof. For a man $m$, let $\operatorname{dim}(m)$ denote the index of the last non-zero weight, i.e. $\alpha_{\operatorname{dim}(m)+1}(m)=$ $\cdots=\alpha_{d}(m)=0$. We assume $\operatorname{dim}(m)>0$, as otherwise $m$ is indifferent among all women and we can pick any woman as $\mu(m)$. We assume without loss of generality $\alpha_{\operatorname{dim}(m)}(m) \in\{-1,1\}$. For
each $d^{\prime}$ such that $1 \leq d^{\prime} \leq d$ we build a data structure consisting of $n$ hyperplanes in $\mathbb{R}^{d^{\prime}}$. For each woman $w$, consider the hyperplanes

$$
\begin{equation*}
H_{d^{\prime}}(w)=\left\{x_{d^{\prime}}=\sum_{i=1}^{d^{\prime}-1} A_{i}(w) x_{i}-A_{d^{\prime}}(w)\right\} \tag{1}
\end{equation*}
$$

and for each $d^{\prime}$ preprocess the set of all hyperplanes according to Lemma 1, Note that $H_{d^{\prime}}(w)$ is the dual of the point $\left(A_{1}(w), \ldots, A_{d^{\prime}}(w)\right)$.

For a man $m$ we can find his most preferred partner by querying the $\operatorname{dim}(m)$-dimensional data structure. Let $s=\alpha_{\operatorname{dim}(m)}(m)$. Consider a ray $r(m) \in \mathbb{R}^{\operatorname{dim}(m)}$ originating at

$$
\begin{equation*}
\left(-\frac{\alpha_{1}(m)}{s}, \ldots,-\frac{\alpha_{\operatorname{dim}(m)-1}(m)}{s},-s \cdot \infty\right) \tag{2}
\end{equation*}
$$

in the direction $(0, \ldots, 0, s)$. If $\alpha_{\operatorname{dim}(m)}=1$ we find the lowest hyperplane intersecting the ray, and if $\alpha_{\operatorname{dim}(m)}=-1$ we find the highest hyperplane. We claim that the first hyperplane $r(m)$ hits corresponds to $m$ 's most preferred woman. Let woman $w$ be preferred over woman $w^{\prime}$, i.e. $\operatorname{val}_{m}(w)=\sum_{i=1}^{\operatorname{dim}(m)} A_{i}(w) \alpha_{i}(m) \geq \sum_{i=1}^{\operatorname{dim}(m)} A_{i}\left(w^{\prime}\right) \alpha_{i}(m)=\operatorname{val}_{m}\left(w^{\prime}\right)$. Since the ray $r(m)$ is vertical in coordinate $x_{d^{\prime}}$, it is sufficient to evaluate the right-hand side of the definition in equation 11. Indeed we have $\operatorname{val}_{m}(w) \geq \operatorname{val}_{m}\left(w^{\prime}\right)$ if and only if

$$
\begin{equation*}
\sum_{i=1}^{\operatorname{dim}(m)-1}-A_{i}(w) \frac{\alpha_{i}(m)}{s}-A_{\operatorname{dim}(m)}(w) \leq \sum_{i=1}^{\operatorname{dim}(m)-1}-A_{i}\left(w^{\prime}\right) \frac{\alpha_{i}(m)}{s}-A_{\operatorname{dim}(m)}\left(w^{\prime}\right) \tag{3}
\end{equation*}
$$

when $s=1$ and

$$
\begin{equation*}
\sum_{i=1}^{\operatorname{dim}(m)-1}-A_{i}(w) \frac{\alpha_{i}(m)}{s}-A_{\operatorname{dim}(m)}(w) \geq \sum_{i=1}^{\operatorname{dim}(m)-1}-A_{i}\left(w^{\prime}\right) \frac{\alpha_{i}(m)}{s}-A_{\operatorname{dim}(m)}\left(w^{\prime}\right) \tag{4}
\end{equation*}
$$

when $s=-1$.
Note that the query ray is dual to the set of hyperplanes with normal vector $\left(\alpha_{1}(m), \ldots, \alpha_{d}(m)\right)$.
Now we pick the highest man $m$ in the (global) preference list, consider the ray as above and find the first hyperplane $H_{\operatorname{dim}(m)}(w)$ hit by the ray. We then match the pair $(m, w)$, remove $H(w)$ from all data structures and repeat. Correctness follows from the correctness of the greedy approach when all women share the same preference list and the properties of the halfspaces proved above.

The algorithm preprocesses $d$ data structures, then makes $n$ queries and $d n$ deletions. The time is dominated by the $n$ ray queries each requiring time $\tilde{O}\left(n^{1-1 /\lfloor d / 2\rfloor}\right)$. Thus the total time complexity is bounded by $\tilde{O}\left(n^{2-1 /\lfloor d / 2\rfloor}\right)$, as claimed.

```
Algorithm 2: One-Sided Stable Matching
    // For points in \(P \in \mathbb{R}^{d}\) we use the notation \(\left(x_{1}, \ldots, x_{d}\right)\) to refer to its coordinates.
    Input: matching \(\mu\)
    for \(d^{\prime}=1\) to \(d\) do
        for each woman \(w\) do
            \(H(w) \leftarrow\left\{x_{d}=\sum_{i=1}^{d-1} A_{i}(w) x_{i}-A_{d^{\prime}}(w)\right\}\)
\(H_{d^{\prime}} \leftarrow H_{d^{\prime}} \cup H(w)\)
        \(H_{d^{\prime}}\).preprocess()
    for each man \(m\) in order of preference do
        \(s \leftarrow \alpha_{\operatorname{dim}(m)}(m)\)
        \(r(m) \leftarrow\left(-\frac{\alpha_{1}(m)}{s}, \ldots,-\frac{\alpha_{\operatorname{dim}(m)-1}(m)}{s}, \infty \cdot s\right)+t \cdot(0, \ldots, 0,-s)\)
        \(H(w) \leftarrow\) Query \(\left(H_{\operatorname{dim}(m)}, r(m)\right)\)
        \(\mu \leftarrow \mu \cup(m, w)\)
        for \(d^{\prime}=1\) to \(d\) do
            \(H_{d^{\prime}} \leftarrow H_{d^{\prime}}-H_{d^{\prime}}(w)\)
    return \(\mu\)
```

Note that for $d=1$ there is a trivial linear time algorithm for the problem.
We use the following lemma to extend the above algorithm to account for positive and negative weights for the women. It deals with settings where the women choose one of two lists $\left(\sigma_{1}, \sigma_{2}\right)$ as their preference lists over the men while the men's preferences can be arbitrary.

Lemma 3. Suppose there are $k$ women who use $\sigma_{1}$. If the top $k$ men in $\sigma_{1}$ are in the bottom $k$ places in $\sigma_{2}$, then the women using $\sigma_{1}$ will only match with those men and the $n-k$ women using $\sigma_{2}$ will only match with the other $n-k$ men in the woman-optimal stable matching.

Proof. Consider the operation of the woman-proposing deferred acceptance algorithm for finding the woman-optimal stable matching. Suppose the lemma is false so that at some point a woman using $\sigma_{1}$ proposed to one of the last $n-k$ men in $\sigma_{1}$. Let $w$ be the first such woman. $w$ must have been rejected by all of the top $k$, so at least one of those men received a proposal from a woman, $w^{\prime}$, using $\sigma_{2}$. However, since the top $k$ men in $\sigma_{1}$ are the bottom $k$ men in $\sigma_{2}, w^{\prime}$ must have been rejected by all of the top $n-k$ men in $\sigma_{2}$. But there are only $n-k$ women using $\sigma_{2}$, so one of the top $n-k$ men in $\sigma_{2}$ must have already received a proposal from a woman using $\sigma_{1}$. This is a contradiction because $w$ was the first woman using $\sigma_{1}$ to propose to one of the bottom $n-k$ men in $\sigma_{1}$ (which are the top $n-k$ men in $\sigma_{2}$ ).

We can now prove the following theorem where negative values are allowed for the women's weights.

Theorem 2. For $d \geq 2$ there is an algorithm to find a stable matching in the one-sided d-attribute model with real-valued attributes and weights in time $\tilde{O}\left(n^{2-1 /\lfloor d / 2\rfloor}\right)$.

Proof. Suppose there are $k$ women who have a positive weight on the men's attribute. Since the remaining $n-k$ women's preference list is the reverse, we can use Lemma 3 to split the problem into two subproblems. Namely, in the woman-optimal stable matching the $k$ women with a positive weight will match with the top $k$ men, and the $n-k$ women with a negative weight will match with the bottom $n-k$ men. Now the women in each of these subproblems all have the same list. Therefore we can use Lemma 2 to solve each subproblem. Splitting the problem into subproblems can be done in time $O(n)$ so the running time follows immediately from Lemma 2,

Table 1: Two-list preferences where no participant receives their top choice in the stable matching

| $\sigma_{1}$ | $\sigma_{2}$ | $\pi_{1}$ | $\pi_{2}$ |  | Man | List | Woman | List |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{1}$ | $m_{3}$ | $w_{1}$ | $w_{3}$ |  | $m_{1}$ | $\pi_{1}$ | $w_{1}$ | $\sigma_{2}$ |
| $m_{2}$ | $m_{5}$ | $w_{2}$ | $w_{5}$ |  | $m_{2}$ | $\pi_{1}$ | $w_{2}$ | $\sigma_{2}$ |
| $m_{3}$ | $m_{1}$ | $w_{3}$ | $w_{1}$ |  | $m_{3}$ | $\pi_{2}$ | $w_{3}$ | $\sigma_{1}$ |
| $m_{4}$ | $m_{4}$ | $w_{4}$ | $w_{4}$ |  | $m_{4}$ | $\pi_{1}$ | $w_{4}$ | $\sigma_{2}$ |
| $m_{5}$ | $m_{2}$ | $w_{5}$ | $w_{2}$ |  | $m_{5}$ | $\pi_{2}$ | $w_{5}$ | $\sigma_{1}$ |

As a remark, this "greedy" approach where we select a man, find his most preferred available woman, and permanently match him to her will not work in general. Table 1 describes a simple 2 list example where the unique stable matching is $\left\{\left(m_{1}, w_{2}\right),\left(m_{2}, w_{3}\right),\left(m_{3}, w_{5}\right),\left(m_{4}, w_{4}\right),\left(m_{5}, w_{1}\right)\right\}$. In this instance, no participant is matched with their top choice. Therefore, the above approach cannot work for this instance. This illustrates to some extent why the general case seems more difficult than the one-sided case.

An alternative model of a greedy approach that is based on work by Davis and Impagliazzo in [15] also will not work. In this model, an algorithm can view each of the lists and the preferences of the women. It can then (adaptively) choose an order in which to process the men. When processing a man, he must be assigned a partner (not necessarily his favorite available woman) once and for all, based only on his choice of preference list and the preferences of the previously processed men. This model is similar to online stable matching [28] except that it allows the algorithm to choose the processing order of the men. Using the preferences in Table 2 and minor modifications to them, we can show that no greedy algorithm of this type can successfully produce a stable matching. Indeed, the unique stable matching of the preference scheme below is $\mu=\left\{\left(m_{1}, w_{3}\right),\left(m_{2}, w_{1}\right),\left(m_{3}, w_{2}\right)\right\}$. However, changing the preference list for whichever of $m_{1}$ or $m_{2}$ is processed later will form a blocking pair with the stable partner of the other. If $m_{1}$ uses $\pi_{1},\left(m_{1}, w_{1}\right)$ blocks $\mu$ and if $m_{2}$ uses $\pi_{2},\left(m_{2}, w_{3}\right)$ blocks $\mu$. Therefore, no algorithm can succeed in assigning stable partners to these men without first knowing the preference list choice of all three.

Table 2: Two-list preferences where a greedy approach will not work

| $\sigma_{1}$ | $\sigma_{2}$ | $\pi_{1}$ | $\pi_{2}$ |  | Man | List | Woman | List |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{1}$ | $m_{2}$ | $w_{1}$ | $w_{3}$ |  | $m_{1}$ | $\pi_{2}$ | $w_{1}$ | $\sigma_{1}$ |
| $m_{2}$ | $m_{1}$ | $w_{2}$ | $w_{2}$ |  | $m_{2}$ | $\pi_{1}$ | $w_{2}$ | $\sigma_{2}$ |
| $m_{3}$ | $m_{3}$ | $w_{3}$ | $w_{1}$ |  | $m_{3}$ | $\pi_{1}$ | $w_{3}$ | $\sigma_{2}$ |

### 4.3 Strategic Behavior

As mentioned earlier, strategic behavior in the general preference setting allows for participants to truncate or rearrange their lists. However, in the $d$-attribute and $d$-list models, we assume that the attributes or lists are fixed, so that the only manipulation the participants are allowed is to misrepresent their weight vectors or which list they choose. Despite this limitation, there is still no strategy proof mechanism for finding a stable matching when $d \geq 2$.
Theorem 3. For $d \geq 2$ there is no strategy proof algorithm to find a stable matching in the d-list model.

Proof. Table 3 describes true preferences that can be manipulated by the women. Observe that there are two stable matchings: the man-optimal matching $\left\{\left(m_{1}, w_{1}\right),\left(m_{2}, w_{2}\right),\left(m_{3}, w_{3}\right),\left(m_{4}, w_{4}\right)\right\}$ and the woman-optimal matching $\left\{\left(m_{1}, w_{2}\right),\left(m_{2}, w_{3}\right),\left(m_{3}, w_{1}\right),\left(m_{4}, w_{4}\right)\right\}$. However, if $w_{2}$ used list $\sigma_{2}$ instead of $\sigma_{1}$, then there is a unique stable matching which is $\left\{\left(m_{1}, w_{2}\right),\left(m_{2}, w_{3}\right),\left(m_{3}, w_{1}\right),\left(m_{4}, w_{4}\right)\right\}$, the woman-optimal stable matching from the original preferences. Therefore, any mechanism that does not always output the woman optimal stable matching can be manipulated by the women to their advantage. By symmetry, any mechanism that does not always output the man-optimal matching could be manipulated by the men. Thus there is no strategy-proof mechanism for the $d$-list setting with $d \geq 2$.

Table 3: Two-list preferences that can be manipulated

| $\sigma_{1}$ | $\sigma_{2}$ | $\pi_{1}$ | $\pi_{2}$ |  | Man | List | Woman | List |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{1}$ | $m_{3}$ | $w_{1}$ | $w_{3}$ |  | $m_{1}$ | $\pi_{1}$ | $w_{1}$ | $\sigma_{2}$ |
| $m_{2}$ | $m_{1}$ | $w_{2}$ | $w_{1}$ |  | $m_{2}$ | $\pi_{1}$ | $w_{2}$ | $\sigma_{1}$ |
| $m_{3}$ | $m_{4}$ | $w_{3}$ | $w_{2}$ |  | $m_{3}$ | $\pi_{2}$ | $w_{3}$ | $\sigma_{1}$ |
| $m_{4}$ | $m_{2}$ | $w_{4}$ | $w_{4}$ |  | $m_{4}$ | $\pi_{2}$ | $w_{4}$ | $\sigma_{1}$ |

Since the $d$-list model is a special case of the $d$-attribute model, we immediately have the following result from Theorem 3.

Corollary 1. For $d \geq 2$ there is no strategy proof algorithm to find a stable matching in the d-attribute model.

Of course in the 1 -list setting there is a trivial unique stable matching. Moreover, in the one-sided $d$-attribute model our algorithm is strategy proof since the women are receiving the woman-optimal matching and each man receives his best available woman, so misrepresentation would only give him a worse partner.

## 5 Verification

We now turn to the problem of verifying whether a given matching is stable. While this is as hard as finding a stable matching in the general setting, the verification algorithms we present here are more efficient than our algorithms for finding stable matchings in the attribute model.

### 5.1 Real Attributes and Weights

In this section we adapt the geometric approach for finding a stable matching in the one-sided $d$ attribute model to the problem of verifying a stable matching in the (two-sided) $d$-attribute model. We express the verification problem as a simplex range searching problem in $\mathbb{R}^{2 d}$, which is the dual of the ray shooting problem. In simplex range searching we are given $n$ points and answer queries that ask for the number of points inside a simplex. In our case we only need degenerate simplices consisting of the intersection of two halfspaces. Simplex range searching queries can be done in sublinear time for constant $d$.

Lemma 4 (31]). Given a set of $n$ points in $\mathbb{R}^{d}$, one can process it for simplex range searching in time $O(n \log n)$, and then answer queries in time $\tilde{O}\left(n^{1-\frac{1}{d}}\right)$.

For $1 \leq d^{\prime} \leq d$ we use the notation $\left(x_{1}, \ldots, x_{d}, y_{1}, \ldots, y_{d^{\prime}-1}, z\right)$ for points in $\mathbb{R}^{d+d^{\prime}}$. We again let $\operatorname{dim}(w)$ be the index of $w$ 's last non-zero weight, assume without loss of generality $\alpha_{\operatorname{dim}(w)} \in\{-1,1\}$, and let $\operatorname{sgn}(w)=\operatorname{sgn}\left(\alpha_{\operatorname{dim}(w)}\right)$. We partition the set of women into $2 d$ sets $W_{d^{\prime}, s}$ for $1 \leq d^{\prime} \leq d$ and $s \in\{-1,1\}$ based on $\operatorname{dim}(w)$ and $\operatorname{sgn}(w)$. Note that if $\operatorname{dim}(w)=0$, then $w$ is indifferent among all men and can therefore not be part of a blocking pair. We can ignore such women.

For a woman $w$, consider the point

$$
\begin{equation*}
P(w)=\left(A_{1}(w), \ldots, A_{d}(w), \alpha_{1}(w), \ldots, \alpha_{\operatorname{dim}(w)-1}(w), \operatorname{val}_{w}(m)\right) \tag{5}
\end{equation*}
$$

where $m=\mu(w)$ is the partner of $w$ in the input matching $\mu$. For a set $W_{d^{\prime}, s}$ we let $P_{d^{\prime}, s}$ be the set of points $P(w)$ for $w \in W_{d^{\prime}, s}$. The basic idea is to construct a simplex for every man and query it against all sets $P_{d^{\prime}, s}$.

Given $d^{\prime}, s$, and a man $m$, let $H_{1}(m)$ be the halfspace $\left\{\sum_{i=1}^{d} \alpha_{i}(m) x_{i}>\operatorname{val}_{m}(w)\right\}$ where $w=$ $\mu(m)$. For $w^{\prime} \in W_{d^{\prime}, s}$ we have $P\left(w^{\prime}\right) \in H_{1}(m)$ if and only if $m$ strictly prefers $w^{\prime}$ to $w$. Further let $H_{2}(m)$ be the halfspace $\left\{\sum_{i=1}^{d^{\prime}-1} A_{i}(m) y_{i}+A_{d^{\prime}}(m) s>z\right\}$. For $w^{\prime} \in W_{d^{\prime}, s}$ we have $P\left(w^{\prime}\right) \in H_{2}(m)$ if and only if $w^{\prime}$ strictly prefers $m$ to $\mu\left(w^{\prime}\right)$. Hence $\left(m, w^{\prime}\right)$ is a blocking pair if and only if $P\left(w^{\prime}\right) \in H_{1}(m) \cap H_{2}(m)$.

Using Lemma 4 we immediately have an algorithm to verify a stable matching.
Theorem 4. There is an algorithm to verify a stable matching in the d-attribute model with realvalued attributes and weights in time $\tilde{O}\left(n^{2-1 / 2 d}\right)$

Proof. Partition the set of women into sets $W_{d^{\prime}, s}$ for $1 \leq d^{\prime} \leq d$ and $s \in\{-1,1\}$ and for $w \in W_{d^{\prime}, s}$ construct $P(w) \in \mathbb{R}^{d+d^{\prime}}$ as above. Then preprocess the sets according to Lemma (4) For each man $m$ query $H_{1}(m) \cap H_{2}(m)$ against the points in all sets. By the definitions of $H_{1}(m)$ and $H_{2}(m)$, there is a blocking pair if and only if for some man $m$ there is a point $P(w) \in H_{1}(m) \cap H_{2}(m)$ in one of the sets $P_{d^{\prime}, s}$.

The time to preprocess is $O(n \log n)$. There are $2 d n$ queries of time $\tilde{O}\left(n^{1-1 / 2 d}\right)$. Hence the whole process requires time $\tilde{O}\left(n^{2-1 / 2 d}\right)$ as claimed.

```
Algorithm 3: Verify Stable Matching with Reals
    // For points in \(P \in \mathbb{R}^{d+d^{\prime}}\) we use the notation \(\left(x_{1}, \ldots, x_{d}, y_{1}, \ldots, y_{d^{\prime}-1}, z\right)\) to refer to its
    coordinates.
    Input: matching \(\mu\)
    for each woman \(w\) do
        \(m \leftarrow \mu(w)\)
        \(P(w) \leftarrow\left(A_{1}(w), \ldots, A_{d}(w), \alpha_{1}(w), \ldots, \alpha_{d}(w), \operatorname{val}_{w}(m)\right)\)
        \(P_{\operatorname{dim}(w), \operatorname{sgn}(w)} \leftarrow W_{\operatorname{dim}(w), \operatorname{sgn}(w)} \cup P(w)\)
    for \(d^{\prime}=1\) to \(d\) and \(s \in\{-1,1\}\) do
        \(P_{d^{\prime}, s}\).preprocess()
        for each man \(m\) do
            \(w \leftarrow \mu(m)\)
            \(H_{1}(m) \leftarrow\left\{\sum_{i=1}^{d} \alpha_{i}(m) x_{i}>\operatorname{val}_{m}(w)\right\}\)
            \(H_{2}(m) \leftarrow\left\{\sum_{i=1}^{d^{\prime}-1} A_{i}(m) y_{i}+A_{d^{\prime}}(m) \cdot s>z\right\}\)
                if Query \(\left(P_{d^{\prime}, s}, H_{1}(m) \cap H_{2}(m)\right)>0\) then
                return \(\mu\) is not stable
    return \(\mu\) is stable
```


### 5.2 Lists

When there are $d$ preference orders for each side, and each participant uses one of the $d$ lists, we provide a more efficient algorithm. Here, assume $\mu$ is the given matching between $M$ and $W$. Let $\left\{\pi_{i}\right\}_{i=1}^{d}$ be the set of $d$ permutations on the women and $\left\{\sigma_{i}\right\}_{i=1}^{d}$ be the set of $d$ permutations on the men. Define $\operatorname{rank}(w, i)$ to be the position of $w$ in permutation $\pi_{i}$. This can be determined in constant time after $O(d n)$ preprocessing of the permutations. Let head $\left(\pi_{i}, j\right)$ be the first woman in $\pi_{i}$ who uses permutation $\sigma_{j}$ and $\operatorname{next}(w, i)$ be the next highest ranked woman after $w$ in permutation $\pi_{i}$ who uses the same permutation as $w$ or $\perp$ if no such woman exists. These can also be determined in constant time after $O(d n)$ preprocessing by splitting the lists into sublists, with one sublist for the women using each permutation of men. The functions rank, head, and next are defined analogously for the men.

```
Algorithm 4: Verify \(d\)-List Stable Matching
    for \(i=1\) to \(d\) do
        for \(j=1\) to \(d\) do
            \(w \leftarrow \operatorname{head}\left(\pi_{i}, j\right)\).
            \(m \leftarrow \operatorname{head}\left(\sigma_{j}, i\right)\).
            while \(m \neq \perp\) and \(w \neq \perp\) do
                if \(\operatorname{rank}(w, i)>\operatorname{rank}(\mu(m), i)\) then
                        \(m \leftarrow \operatorname{next}(m, j)\).
                else
                        if \(\operatorname{rank}(m, j)>\operatorname{rank}(\mu(w), j)\) then
                            \(w \leftarrow \operatorname{next}(w, i)\).
                                else
                        return \((m, w)\) is a blocking pair.
    return \(\mu\) is stable.
```

Theorem 5. There is an algorithm to verify a stable matching in the d-list model in $O(d n)$ time.
Proof. We claim that algorithm 4 satisfies the theorem. Indeed, if the algorithm returns a pair ( $m, w$ ) where $m$ uses $\pi_{i}$ and $w$ uses $\sigma_{j}$, then $(m, w)$ is a blocking pair because $w$ appears earlier in $\pi_{i}$ than $\mu(m)$ and $m$ appears earlier in $\sigma_{j}$ than $\mu(w)$.

On the other hand, suppose the algorithm returns that $\mu$ is stable but there is a blocking pair, $(m, w)$, where $m$ uses $\pi_{i}$ and $w$ uses $\sigma_{j}$. The algorithm considers permutations $\pi_{i}$ and $\sigma_{j}$ since it does not terminate early. Clearly if the algorithm evaluates $m$ and $w$ simultaneously when considering permutations $\pi_{i}$ and $\sigma_{j}$, it will detect that $(m, w)$ is a blocking pair. Therefore, the algorithm either moves from $m$ to next $(m, j)$ before considering $w$ or it moves from $w$ to next $(w, i)$ before considering $m$. In the former case, $\operatorname{rank}(\mu(m), i)<\operatorname{rank}\left(w^{\prime}, i\right)$ for some $w^{\prime}$ that comes before $w$ in $\pi_{i}$. Therefore $m$ prefers $\mu(m)$ to $w$. Similarly, in the latter case, $\operatorname{rank}(\mu(w), j)<\operatorname{rank}\left(m^{\prime}, i\right)$ for some $m^{\prime}$ that comes before $m$ in $\sigma_{j}$ so $w$ prefers $\mu(w)$ to $m$. Thus $(m, w)$ is not a blocking pair and we have a contradiction.

The for and while loops proceed through all men and women once for each of the $d$ lists in which they appear. Since at each step we are either proceeding to the next man or the next woman unless we find a blocking pair, the algorithm requires time $O(d n)$. This is optimal since the input size is $d n$.

### 5.3 Boolean Attributes and Weights

In this section we consider the problem of verifying a stable matching when the $d$ attributes and weights are restricted to boolean values and $d=c \log n$. The algorithm closely follows an algorithm for the maximum inner product problem by Alman and Williams [4]. The idea is to express the existence of a blocking pair as a probabilistic polynomial with a bounded number of monomials and use fast rectangular matrix multiplication to evaluate it. A probabilistic polynomial for a function $f$ is a polynomial $p$ such that for every input $x$

$$
\begin{equation*}
\operatorname{Pr}[f(x) \neq p(x)] \leq \frac{1}{3} \tag{6}
\end{equation*}
$$

We use the following tools in our algorithm. $\mathrm{THR}_{d}$ is the threshold function that outputs 1 if at least $d$ of its inputs are 1 .

Lemma 5 ([4]). There is a probabilistic polynomial for $\mathrm{THR}_{d}$ on $n$ variables and error $\varepsilon$ with degree $O(\sqrt{n \log (1 / \varepsilon)})$.

Lemma 6 ( 35,40$]$ ). There is a probabilistic polynomial for the disjunction of $n$ variables and error $\varepsilon$ with degree $O(\log (1 / \varepsilon))$
Lemma 7 ([43). Given a polynomial $P\left(x_{1}, \ldots, x_{m}, y_{1}, \ldots y_{m}\right)$ with at most $n^{0.17}$ monomials and two sets $X, Y \subseteq\{0,1\}^{m}$ with $|X|=|Y|=n$, we can evaluate $P$ on all pairs $(x, y) \in X \times Y$ in time $\tilde{O}\left(n^{2}+m \cdot n^{1.17}\right)$.

We construct a probabilistic polynomial that outputs 1 if there is a blocking pair. To minimize the degree of the polynomial, we pick a parameter $s$ and divide the men and women into sets of size at most $s$. The polynomial takes the description of $s$ men $m_{1}, \ldots, m_{s}$ and $s$ women $w_{1}, \ldots, w_{s}$ along with their respective partners as input, and outputs 1 if and only if there is a blocking pair ( $m_{i}, w_{j}$ ) among the $s^{2}$ pairs of nodes with high probability.

Lemma 8. Let $u$ be a large constant and $s=n^{1 / u c \log ^{2} c}$. There is a probabilistic polynomial with the following inputs:

- The attributes and weights of $s$ men, $A\left(m_{1}\right), \ldots, A\left(m_{s}\right), \alpha\left(m_{1}\right), \ldots, \alpha\left(m_{s}\right)$
- The attributes of the $s$ women that are matched with these men $A\left(\mu\left(m_{1}\right)\right), \ldots, A\left(\mu\left(m_{s}\right)\right)$
- The attributes and weights of $s$ women, $A\left(w_{1}\right), \ldots, A\left(w_{s}\right), \alpha\left(w_{1}\right), \ldots, \alpha\left(w_{s}\right)$
- The attributes of the $s$ men that are matched with these women $A\left(\mu\left(w_{1}\right)\right), \ldots, A\left(\mu\left(w_{s}\right)\right)$

The output of the polynomial is 1 if and only if there is a blocking pair with respect to the matching $\mu$ among the $s^{2}$ pairs in the input. The number of monomials is at most $n^{0.17}$ and the polynomial can be constructed efficiently.
Proof. A pair $\left(m_{i}, w_{j}\right)$ is a blocking pair if and only if $\left\langle\alpha\left(m_{i}\right), A\left(\mu\left(m_{i}\right)\right)\right\rangle<\left\langle\alpha\left(m_{i}\right), A\left(w_{j}\right)\right\rangle$ and $\left\langle\alpha\left(w_{j}\right), A\left(\mu\left(w_{j}\right)\right)\right\rangle<\left\langle\alpha\left(w_{j}\right), A\left(m_{i}\right)\right\rangle$. Rewriting

$$
\begin{equation*}
F(x, y, a, b):=\langle x, y\rangle<\langle a, b\rangle=\operatorname{THR}_{d+1}\left(\neg\left(x_{1} \wedge y_{1}\right), \ldots, \neg\left(x_{d} \wedge y_{d}\right), a_{1} \wedge b_{1}, \ldots, a_{d} \wedge b_{d}\right) \tag{7}
\end{equation*}
$$

we have a blocking pair if and only if

$$
\begin{equation*}
\bigvee_{\substack{i \in[1, s] \\ j \in[1, s]}}\left(F\left(\alpha\left(m_{i}\right), A\left(\mu\left(m_{i}\right)\right), \alpha\left(m_{i}\right), A\left(w_{j}\right)\right) \wedge F\left(\alpha\left(w_{j}\right), A\left(\mu\left(w_{j}\right)\right), \alpha\left(w_{j}\right), A\left(m_{i}\right)\right)\right) \tag{8}
\end{equation*}
$$

Note that we can easily adapt this algorithm to finding strongly blocking pairs by defining $F(x, y, a, b)$ as $\langle x, y\rangle \leq\langle a, b\rangle$.

Using Lemma 5 with $\varepsilon=\frac{1}{s^{3}}$ and Lemma 6 with $\varepsilon=1 / 4$ we get a probabilistic polynomial of degree $a \sqrt{d \log s}$ for some constant $a$ and error $1 / 4+1 / s<1 / 3$. Furthermore, since we are only interested in boolean inputs we can assume the polynomial to be multilinear. For large enough $u$ we have $2 d>a \sqrt{d \log (s)}$ (i.e. the degree is at most half of the number of variables) and the number of monomials is then bounded by $O\left(\left(s^{2}\left(\frac{4 d}{\sqrt{d \log (s)}}\right)\right)^{2}\right)$.

Simplifying the binomial coefficient we have

$$
\binom{4 d}{a \sqrt{d \log s}}=\binom{4 c \log n}{a \sqrt{\left(\log ^{2} n\right) / u \log ^{2} c}}=\binom{4 c \log n}{a \log n / \sqrt{u} \log c}
$$

Setting $\delta=a /(\sqrt{u} \log (c))$ we can upper bound this using Stirling's inequality by

$$
\binom{4 c \log n}{\delta \log n} \leq\left(\frac{(4 c \log n) \cdot e}{\delta}\right)^{\delta \log n}=n^{\delta \log (4 c e / \delta)}
$$

By choosing $u$ to be a large enough constant, we can make $\delta$ and the exponent arbitrarily small. The factor of $s^{2}$ only contributes a trivial constant to the exponent. Therefore we can bound the number of monomials by $n^{0.17}$.

Theorem 6. In the $d$-attribute model with $n$ men and women, and $d=c \log n$ boolean attributes and weights, there is a randomized algorithm to decide if a given matching is stable in time $\tilde{O}\left(n^{2-1 / O\left(c \log ^{2}(c)\right)}\right)$ with error probability at most $1 / 3$.

Proof. We again choose $s=n^{1 / u c \log ^{2} c}$ and construct the probabilistic polynomial as in Lemma 8, We then divide the men and women into $\left\lceil\frac{n}{s}\right\rceil$ groups of size at most $s$.

For a group of men $m_{1}, \ldots, m_{s}$ we let the corresponding input vector be

$$
A\left(m_{1}\right), \ldots, A\left(m_{s}\right), \alpha\left(m_{1}\right), \ldots, \alpha\left(m_{s}\right), A\left(\mu\left(m_{1}\right)\right), \ldots, A\left(\mu\left(m_{s}\right)\right)
$$

We set $X$ as the set of all input vectors for the $\left\lceil\frac{n}{s}\right\rceil$ groups. We define the set $Y$ symmetrically for the input vectors corresponding to the $\left\lceil\frac{n}{s}\right\rceil$ groups of women.

Using Lemma 7 we evaluate the polynomial on all pairs $x \in X, y \in Y$ in time

$$
\begin{equation*}
\tilde{O}\left(\left(\frac{n}{s}\right)^{2}+O(s d)\left(\frac{n}{s}\right)^{1.17}\right)=\tilde{O}\left(\left(\frac{n}{s}\right)^{2}\right)=\tilde{O}\left(n^{2-1 / O\left(c \log ^{2}(c)\right)}\right) \tag{9}
\end{equation*}
$$

The probability that the output is wrong for any fixed input pair is at most $1 / 3$. We repeat this process $O(\log n)$ times and take the threshold output for every pair of inputs, such that the error probability is at most $O\left(\frac{1}{n^{2}}\right)$ for any fixed pair of inputs. Using a union bound we can make the probability of error at most $1 / 3$ on any input.

## 6 Conditional Hardness

### 6.1 Background

The Strong Exponential Time Hypothesis has proved useful in arguing conditional hardness for a large number of problems. We show SETH-hardness for both verifying and finding a stable matching
in the $d$-attribute model, even if the weights and attributes are boolean. The main step of the proof is a reduction from the maximum inner product problem to the stable matching problem. The maximum inner product problem is known to be SETH-hard. We give the fine-grained reduction from CNFSAT to the vector orthogonality problem and from the vector orthogonality problem to the maximum inner product problem for the sake of completeness.

Definition 1 ([23, 25]). The Strong Exponential Time Hypothesis (SETH) stipulates that for each $\varepsilon>0$ there is a $k$ such that $k$-SAT requires time $\Omega\left(2^{(1-\varepsilon) n}\right)$.

Definition 2. For any $d$, the vector orthogonality problem is to decide if two input sets $U, V \subseteq \mathbb{R}^{d}$ with $|U|=|V|=n$ have a pair $u \in U, v \in V$ such that $\langle u, v\rangle=0$.

The boolean vector orthogonality problem is the variant where $U, V \subseteq\{0,1\}^{d}$.
Definition 3. For any d and input l, the maximum inner product problem is to decide if two input sets $U, V \subseteq \mathbb{R}^{d}$ with $|U|=|V|=n$ have a pair $u \in U, v \in V$ such that $\langle u, v\rangle \geq l$.

The boolean maximum inner product problem is the variant where $U, V \subseteq\{0,1\}^{d}$.
Lemma 9 ([25, 42, 4]). Assuming SETH, for any $\varepsilon>0$, there is a $c$ such that solving the boolean maximum inner product problem on $d=c \log n$ dimensions requires time $\Omega\left(n^{2-\varepsilon}\right)$.

Proof. The proof is a series of reductions from $k$-SAT to boolean inner product. By the Sparsification Lemma [25] we can reduce $k$-SAT to a subexponential number of $k$-SAT instances with at most $d=c_{k} n$ clauses, where $c_{k}$ does not depend on $n$. Hence, assuming SETH, for any $\varepsilon>0$, there is a $c$ such that CNFSAT with $c n$ clauses requires time $\Omega\left(2^{(1-\varepsilon) n}\right)$.

We reduce Cnfsat to the boolean vector orthogonality problem using a technique called Split and List. Divide the variable set into two sets $S, T$ of size $\frac{n}{2}$ and for each set consider all $N=2^{n / 2}$ assignments to the variables. For every assignment we construct a $d$-dimensional vector where the $i$ th position is 1 if and only if the assignment does not satisfy the $i$ th clause of the CNF formula. Let $U$ be the set of vectors corresponding to the assignments to $S$ and let $V$ be the set of vectors corresponding to $T$. A pair $u \in U, v \in V$ is orthogonal if and only if the corresponding assignment satisfies all clauses. An algorithm for boolean vector orthogonality in dimension $d=c n=2 c \log N$ and time $O\left(N^{2-\varepsilon}\right)=O\left(2^{(1-\varepsilon / 2) n}\right)$ would contradict SETH. Hence assuming SETH, for every $\varepsilon>0$ there is a $c$ such that the boolean vector orthogonality problem with $d=c \log N$ requires time $\Omega\left(N^{2-\varepsilon}\right)$.

Finally, we reduce the boolean vector orthogonality problem to the boolean maximum inner product problem by partitioning the set $U$ into sets $U_{i}$ for $0 \leq i \leq d$ where $U_{i}$ contains all vectors with Hamming weight $i$. Observe that a vector $v \in V$ is orthogonal to a vector $u \in U_{i}$ if and only if $\langle u, \neg v\rangle=i$, where $\neg v$ is the element-wise complement of $v$. Thus $U$ and $V$ have an orthogonal pair, if and only if there is an $i$ such that $U_{i}$ and $\neg V=\{\neg v \mid v \in V\}$ have a pair with inner product at least $i$. Therefore, for any $\varepsilon>0$ there is a $c$ such that the maximum inner product problem on $d=c \log N$ dimensions requires time $\Omega\left(N^{2-\varepsilon}\right)$ assuming SETH.

### 6.2 Finding Stable Matchings

In this subsection we give a fine-grained reduction from the maximum inner product problem to the problem of finding a stable matching in the boolean $d$-attribute model. This shows that the stable matching problem in the $d$-attribute model is SETH-hard, even if we restrict the attributes and weights to booleans.

Theorem 7. Assuming SETH, for any $\varepsilon>0$, there is a $c$ such that finding a stable matching in the boolean $d$-attribute model with $d=c \log n$ dimensions requires time $\Omega\left(n^{2-\varepsilon}\right)$.

Proof. The proof is a reduction from maximum inner product to finding a stable matching. Given an instance of the maximum inner product problem with sets $U, V \subseteq\{0,1\}^{d}$ where $|U|=|V|=n$ and threshold $l$, we construct a matching market with $n$ men and $n$ women. For every $u \in U$ we have a man $m_{u}$ with $A\left(m_{u}\right)=u$ and $\alpha\left(m_{u}\right)=u$. Similarly, for vectors $v \in V$ we have women $w_{v}$ with $A\left(w_{v}\right)=v$ and $\alpha\left(w_{v}\right)=v$. This matching market is symmetric in the sense that for $m_{u}$ and $w_{v}, \operatorname{val}_{m_{u}}\left(w_{v}\right)=\operatorname{val}_{w_{v}}\left(m_{u}\right)=\langle u, v\rangle$.

We claim that any stable matching contains a pair $\left(m_{u}, w_{v}\right)$ such that the inner product $\langle u, v\rangle$ is maximized. Indeed, suppose there are vectors $u \in U, v \in V$ with $\langle u, v\rangle \geq l$ but there exists a stable matching $\mu$ with $\left\langle u^{\prime}, v^{\prime}\right\rangle<l$ for all pairs $\left(m_{u^{\prime}}, w_{v^{\prime}}\right) \in \mu$. Then $\left(m_{u}, w_{v}\right)$ is clearly a blocking pair for $\mu$ which is a contradiction.

### 6.3 Verifying Stable Matchings

In this section we give a reduction from the maximum inner product problem to the problem of verifying a stable matching, showing that this problem is also SETH-hard.

Theorem 8. Assuming SETH, for any $\varepsilon>0$, there is a $c$ such that verifying a stable matching in the boolean $d$-attribute model with $d=c \log n$ dimensions requires time $\Omega\left(n^{2-\varepsilon}\right)$.

Proof. We give a reduction from maximum inner product with sets $U, V \subseteq\{0,1\}^{d}$ where $|U|=$ $|V|=n$ and threshold $l$. We construct a matching market with $2 n$ men and women in the $d^{\prime}$ attribute model with $d^{\prime}=d+2(l-1)$. Since $d^{\prime}<3 d$ the theorem then follows immediately from the SETH-hardness of maximum inner product.

For $u \in U$, let $m_{u}$ be a man in the matching market with attributes and weights $A\left(m_{u}\right)=$ $\alpha\left(m_{u}\right)=u \circ 1^{l-1} \circ 0^{l-1}$ where we use $\circ$ for concatenation. Similarly, for $v \in V$ we have a woman $w_{v}$ with $A\left(w_{v}\right)=\alpha\left(w_{v}\right)=v \circ 0^{l-1} \circ 1^{l-1}$. We further introduce dummy women $w_{u}^{\prime}$ for $u \in U$ with $A\left(w_{u}^{\prime}\right)=\alpha\left(w_{u}^{\prime}\right)=0^{d} \circ 1^{l-1} \circ 0^{l-1}$ and dummy men $m_{v}^{\prime}$ for $v \in V$ with $A\left(m_{v}^{\prime}\right)=\alpha\left(m_{v}^{\prime}\right)=$ $0^{d} \circ 0^{l-1} \circ 1^{l-1}$.

We claim that the matching consisting of pairs $\left(m_{u}, w_{u}^{\prime}\right)$ for all $u \in U$ and ( $m_{v}^{\prime}, w_{v}$ ) for all $v \in V$ is stable if and only if there is no pair $u \in U, v \in V$ with $\langle u, v\rangle \geq l$. For $u, u^{\prime} \in U$ we have $\operatorname{val}_{m_{u}}\left(w_{u^{\prime}}^{\prime}\right)=\operatorname{val}_{w_{u^{\prime}}^{\prime}}\left(m_{u}\right)=l-1$, and for $v, v^{\prime} \in V$ we have $\operatorname{val}_{w_{v}}\left(m_{v^{\prime}}^{\prime}\right)=\operatorname{val}_{m_{v^{\prime}}^{\prime}}\left(w_{v}\right)=l-1$. In particular, any pair in $\mu$ has (symmetric) value $l-1$. Hence there is a blocking pair with respect to $\mu$ if and only if there is a pair with value at least $l$. For $u \neq u^{\prime}$ and $v \neq v^{\prime}$ the pairs ( $m_{u}, w_{u^{\prime}}^{\prime}$ ) and $\left(w_{v}, m_{v^{\prime}}^{\prime}\right)$ can never be blocking pairs as their value is $l-1$. Furthermore for any pair of dummy nodes $w_{u}^{\prime}$ and $m_{v}^{\prime}$ we have $\operatorname{val}_{m_{v}^{\prime}}\left(w_{u}^{\prime}\right)=\operatorname{val}_{w_{u}^{\prime}}\left(m_{v}^{\prime}\right)=0$, thus no such pair can be a blocking pair either. This leaves pairs of real nodes as the only candidates for blocking pairs. For non-dummy nodes $m_{u}$ and $w_{v}$ we have $\operatorname{val}_{m_{u}}\left(w_{v}\right)=\operatorname{val}_{w_{v}}\left(m_{u}\right)=\langle u, v\rangle$ so $\left(m_{u}, w_{v}\right)$ is a blocking pair if and only if $\langle u, v\rangle \geq l$.


Figure 1: A representation of the reduction from maximum inner product to verifying a stable matching

### 6.4 Checking a Stable Pair

In this section we give a reduction from the maximum inner product problem to the problem of checking whether a given pair is part of any or all stable matchings, showing that these questions are SETH-hard when $d=c \log n$ for some constant $c$. For general preferences, both questions can be solved in time $O\left(n^{2}\right)$ [27, 20] and are known to require quadratic time [33, 38, 19].

Theorem 9. Assuming SETH, for any $\varepsilon>0$, there is a c such that determining whether a given pair is part of any or all stable matchings in the boolean $d$-attribute model with $d=c \log n$ dimensions requires time $\Omega\left(n^{2-\varepsilon}\right)$.

Proof. We again give a reduction from maximum inner product with sets $U, V \subseteq\{0,1\}^{d}$ where $|U|=|V|=n$ and threshold $l$. We construct a matching market with $2 n$ men and women in the $d^{\prime}$-attribute model with $d^{\prime}=7 d+7(l-1)+18$. Since $d^{\prime}<15 d$ the theorem then follows immediately from the SETH-hardness of maximum inner product.

For simplicity, we will first describe the preference scheme, then provide weight and attribute vectors that result in those preferences. For $u \in U$, let $m_{u}$ be a man in the matching market and for $v \in V$ we have a woman $w_{v}$. We also have $n-1$ dummy men $m_{i}: i=1 \ldots n-1$ and $n-1$ dummy women $w_{j}: j=1 \ldots n-1$. Finally, we have a special man $m^{*}$ and special woman $w^{*}$. This special pair is the one we will test for stability. Let the preferences be

$$
\begin{array}{rr}
m_{u}:\left\{w_{v}:\langle u, v\rangle \geq l\right\} \succ\left\{w_{j}\right\}_{j=1}^{n-1} \succ w^{*} \succ\left\{w_{v}:\langle u, v\rangle<l\right\} & \forall u \in U \\
m_{i}:\left\{w_{v}\right\} \succ\left\{w_{j}\right\}_{j=1}^{n-1} \succ w^{*} & \forall i \in\{1 \ldots n-1\} \\
m^{*}: w^{*} \succ\left\{w_{v}\right\} \succ\left\{w_{j}\right\}_{j=1}^{n-1} & \\
w_{v}:\left\{m_{u}:\langle u, v\rangle \geq l\right\} \succ\left\{m_{i}\right\}_{i=1}^{n-1} \succ m^{*} \succ\left\{m_{u}:\langle u, v\rangle<l\right\} & \forall v \in V \\
w_{j}:\left\{m_{u}\right\} \succ\left\{m_{i}\right\}_{i=1}^{n-1} \succ m^{*} & \forall j \in\{1 \ldots n-1\} \\
w^{*}:\left\{m_{i}\right\}_{i=1}^{n-1} \succ\left\{m_{u}\right\} \succ m^{*} &
\end{array}
$$

so that, for example, man $m_{u}$ corresponding to $u \in U$ will most prefer women $w_{v}$ for some $v \in V$ with $\langle u, v\rangle \geq l$ (in decreasing order of $\langle u, v\rangle$ ), then all of the dummy women (equally), then the special woman $w^{*}$, and finally the remaining women $w_{v}$ (in decreasing order of $\langle u, v\rangle$ ).

First suppose for some $\hat{u} \in U$ and $\hat{v} \in V$ we have $\langle\hat{u}, \hat{v}\rangle \geq l$ and let this be the pair with largest inner product. Now consider the deferred acceptance algorithm for finding the woman-optimal stable matching. First, $w_{\hat{v}}$ will propose to $m_{\hat{u}}$ and will be accepted. The dummy women will propose to the remaining men corresponding to $U$. Then any other woman $w_{v}$ will be accepted by either a dummy man or a man $m_{u}$, causing the dummy woman matched with him to move to a dummy man. In any case, all men besides $m^{*}$ are matched to a woman they prefer over $w^{*}$, so when she proposes to them, they will reject her. Thus $w^{*}$ will match with $m^{*}$. Since $w^{*}$ receives her least preferred choice in the woman optimal stable matching, $\left(m^{*}, w^{*}\right)$ is a pair in every stable matching.

Now suppose $\langle u, v\rangle<l$ for every $u \in U, v \in V$. Consider the deferred acceptance algorithm for finding the man-optimal stable matching. First, the dummy men will propose to the women corresponding to $V$ and will be accepted. Then every man $m_{u}$ will propose to the dummy women, but only $n-1$ of them can be accepted. The remaining one will propose to $w^{*}$. When $m^{*}$ proposes to $w^{*}$, she rejects him, causing him to eventually be accepted by the available woman $w_{v}$. Thus $m^{*}$ will not match with $w^{*}$ in any stable matching since she is his most preferred choice but he is not matched with her in the man-optimal stable matching, so $\left(m^{*}, w^{*}\right)$ is not a pair in any stable matching. Figure 2 demonstrates each of these cases.

Since the stable pair questions for whether $\left(m^{*}, w^{*}\right)$ are a stable pair in any or all stable matchings are equivalent with these preferences, this reduction works for both.

Finally, we claim the following vectors realize the preferences above for the attribute model. We leave it to the reader to verify this. As in our other hardness reductions, the weight and attribute vectors are identical for each participant.

| $m_{u}:$ | $u^{7}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{7(l-1)}$ | $\circ$ | $1^{6}$ | $\circ$ | $0^{6}$ | $\circ$ | $0^{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{i}:$ | $0^{7 d}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{6}$ | $\circ$ | $1^{6}$ | $\circ$ | $0^{6}$ |
| $m^{*}:$ | $0^{7 d}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{6}$ | $\circ$ | $0^{6}$ | $\circ$ | $1^{6}$ |
| $w_{v}:$ | $v^{7}$ | $\circ$ | $0^{7(l-1)}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{6}$ | $\circ$ | $1^{6}$ | $\circ$ | $\left(1 \circ 0^{5}\right)$ |
| $w_{j}:$ | $0^{7 d}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{7(l-1)}$ | $\circ$ | $1^{6}$ | $\circ$ | $\left(1^{5} \circ 0\right)$ | $\circ$ | $0^{6}$ |
| $w^{*}:$ | $0^{7 d}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{7(l-1)}$ | $\circ$ | $\left(1^{3} \circ 0^{3}\right)$ | $\circ$ | $\left(1^{4} \circ 0^{2}\right)$ | $\circ$ | $\left(1^{2} \circ 0^{4}\right)$ |



Figure 2: A representation of the reduction from maximum inner product to checking a stable pair
This reduction also has consequences on the existence of nondeterministic algorithms for the stable pair problem assuming the Nondeterministic Strong Exponential Time Hypothesis.
Definition 4 ([11). The Nondeterministic Strong Exponential Time Hypothesis (NSETH) stipulates that for each $\varepsilon>0$ there is a $k$ such that $k$-SAT requires co-nondeterministic time $\Omega\left(2^{(1-\varepsilon) n}\right)$.

In other words, the Nondeterministic Strong Exponential Time Hypothesis stipulates that for CNFSAT there is no proof of unsatisfiability that can be checked deterministically in time $\Omega\left(2^{(1-\varepsilon) n}\right)$.

Assuming NSETH, any problem that is SETH-hard at time $T(n)$ under deterministic reductions either require $T(n)$ time nondeterministically or co-nondeterministically, i.e. either there is no proof that an instance is true or there is no proof that an instance is false that can be checked in time faster than $T(n)$. Note that all reductions in this paper are deterministic. In particular, the maximum inner product problem does not have a $O\left(N^{2-\varepsilon}\right)$ co-nondeterministic time algorithm for any $\varepsilon>0$ assuming NSETH, since it has a simple linear time nondeterministic algorithm.

Since the reduction of Theorem 9 is a simple reduction that maps a true instance of maximum inner product to a true instance of the stable pair problem, we can conclude that the stable pair problem is also hard co-nondeterministically.

Corollary 2. Assuming NSETH, for any $\varepsilon>0$, there is a $c$ such that determining whether a given pair is part of any or all stable matchings in the boolean $d$-attribute model with $d=c \log n$ dimensions requires co-nondeterministic time $\Omega\left(n^{2-\varepsilon}\right)$.

We also have a reduction so that the given pair is stable in any or all stable matchings if and only if there is not a pair of vectors with large inner product. This shows that the stable pair problem is also hard nondeterministically.
Theorem 10. Assuming NSETH, for any $\varepsilon>0$, there is a $c$ such that determining whether a given pair is part of any or all stable matchings in the boolean d-attribute model with $d=c \log n$ dimensions requires nondeterministic time $\Omega\left(n^{2-\varepsilon}\right)$.

Proof. This reduction uses the same setup as the one in Theorem 9 except that we now have $n$ dummy men and women instead of $n-1$ and we slightly change the preferences as follows:

$$
\begin{array}{rlr}
m_{u}:\left\{w_{v}:\langle u, v\rangle \geq l\right\} \succ\left\{w_{j}\right\}_{j=1}^{n} \succ w^{*} \succ\left\{w_{v}:\langle u, v\rangle<l\right\} & \forall u \in U \\
m_{i}:\left\{w_{v}\right\} \succ \mathbf{w}^{*} \succ\left\{\mathbf{w}_{\mathbf{j}}\right\}_{\mathbf{j}=\mathbf{1}} & \forall i \in\{1 \ldots n\} \\
m^{*}: w^{*} \succ\left\{w_{v}\right\} \succ\left\{w_{j}\right\}_{j=1}^{n} & \\
w_{v}:\left\{m_{u}:\langle u, v\rangle \geq l\right\} \succ\left\{m_{i}\right\}_{i=1}^{n} \succ m^{*} \succ\left\{m_{u}:\langle u, v\rangle<l\right\} & \forall v \in V \\
w_{j}:\left\{m_{u}\right\} \succ\left\{m_{i}\right\}_{i=1}^{n} \succ m^{*} & \forall j \in\{1 \ldots n\} \\
w^{*}:\left\{m_{i}\right\}_{i=1}^{n} \succ\left\{m_{u}\right\} \succ m^{*} &
\end{array}
$$

First suppose for some $\hat{u} \in U$ and $\hat{v} \in V$ we have $\langle\hat{u}, \hat{v}\rangle \geq l$ and let this be the pair with largest inner product. Consider the deferred acceptance algorithm for finding the man-optimal stable matching. First, some of the men corresponding to $U$ will propose to the women corresponding to $V$ and at least $m_{\hat{u}}$ will be accepted by $w_{\hat{v}}$. The remaining men corresponding to $U$ will be accepted by dummy women. The dummy men will propose to the women corresponding to $V$ but not all can be accepted. These rejected dummy men will propose to $w^{*}$ who will accept one. Then when $m^{*}$ proposes to $w^{*}$ she will reject him, as will the women corresponding to $V$, so he will be matched with a dummy woman. Since $m^{*}$ and $w^{*}$ are not matched in the man optimal stable matching, $\left(m^{*}, w^{*}\right)$ is not a pair in any stable matching.

Now suppose $\langle u, v\rangle<l$ for every $u \in U, v \in V$ and consider the deferred acceptance algorithm for finding the woman-optimal stable matching. First, the dummy women will propose to the men corresponding to $U$ and will be accepted. Then every woman $w_{v}$ will propose to the dummy men and be accepted. Since every man besides $m^{*}$ is matched with a woman he prefers to $w^{*}$, when she proposes to them, she will be rejected, so she will pair with $m^{*}$. Since $w^{*}$ receives her least preferred choice in the woman optimal stable matching, $\left(m^{*}, w^{*}\right)$ is a pair in every stable matching. Figure 3 demonstrates each of these cases.

We can amend the vectors from Theorem 9 as follows so that they realize the changed preferences with the attribute model.

| $m_{u}:$ | $u^{7}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{7(l-1)}$ | $\circ$ | $1^{6}$ | $\circ$ | $0^{6}$ | $\circ$ | $0^{6}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{i}:$ | $0^{7 d}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{6}$ | $\circ$ | $1^{6}$ | $\circ$ | $0^{6}$ |
| $m^{*}:$ | $0^{7 d}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{6}$ | $\circ$ | $0^{6}$ | $\circ$ | $1^{6}$ |
| $w_{v}:$ | $v^{7}$ | $\circ$ | $0^{7(l-1)}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{6}$ | $\circ$ | $1^{6}$ | $\circ$ | $\left(1 \circ 0^{5}\right)$ |
| $w_{j}:$ | $0^{7 d}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{7(l-1)}$ | $\circ$ | $1^{6}$ | $\circ$ | $\left(\mathbf{1}^{3} \circ \mathbf{0}^{3}\right)$ | $\circ$ | $0^{6}$ |
| $w^{*}:$ | $0^{7 d}$ | $\circ$ | $1^{7(l-1)}$ | $\circ$ | $0^{7(l-1)}$ | $\circ$ | $\left(1^{3} \circ 0^{3}\right)$ | $\circ$ | $\left(1^{4} \circ 0^{2}\right)$ | $\circ$ | $\left(1^{2} \circ 0^{4}\right)$ |



Figure 3: A representation of the reduction from maximum inner product to checking a stable pair such that a true maximum inner product instance maps to a false stable pair instance

We would like to point out that the results on the hardness for (co-)nondeterministic algorithms do not apply to Merlin-Arthur (MA) algorithms, i.e. algorithms with access to both nondeterministic bits and randomness. Williams [44] gives fast MA algorithms for a number of SETH-hard problems, and the same techniques also yield a $O(d n)$ time MA algorithm for the verification of a stable matching in the boolean attribute model with $d$ attributes. We can obtain MA algorithms with time $O(d n)$ for finding stable matchings and certifying that a pair is in at least one stable matching by first nondeterministically guessing a stable matching.

## 7 Other Succinct Preference Models

In this section, we provide subquadratic algorithms for other succinct preference models, singlepeaked and geometric, which are motivated by economics.

### 7.1 One Dimensional Single-Peaked Preferences

Formally, we say the men's preferences over the women in a matching market are single-peaked if the women can be ordered as points along a line $\left(p\left(w_{1}\right)<p\left(w_{2}\right)<\cdots<p\left(w_{n}\right)\right)$ and for each man $m$ there is a point $q(m)$ and a binary preference relation $\succ_{m}$ such that if $p\left(w_{i}\right) \leq q(m)$ then $p\left(w_{i}\right) \succ_{m} p\left(w_{j}\right)$ for $j<i$ and if $p\left(w_{i}\right) \geq q(m)$ then $p\left(w_{i}\right) \succ_{m} p\left(w_{j}\right)$ for $j>i$. Essentially, each man prefers the women that are "closest" to his ideal point $q(m)$. One example of a preference relation for $m$ would be the distance from $q(m)$. If the women's preferences are also single-peaked then we say the matching market has single peaked preferences. Since these preferences only consist of the
$p$ and $q$ values and the preference relations for the participants, they can be represented succinctly as long as the relations require subquadratic space.

### 7.1.1 Verifying a Stable Matching for Single-Peaked Preferences

Here we demonstrate a subquadratic algorithm for verifying if a given matching is stable when the preferences of the matching market are single-peaked. We assume that the preference relations can be computed in constant time.

Theorem 11. There is an algorithm to verify a stable matching in the single-peaked preference model in $O(n \log n)$ time.

Proof. Let $p\left(m_{i}\right)$ be the point associated with man $m_{i}, q\left(m_{i}\right)$ be $m_{i}$ 's preference point, and $\succ_{m_{i}}$ be $m_{i}$ 's preference relation. The women's points are denoted analogously. We assume that $p\left(m_{i}\right)<$ $p\left(m_{j}\right)$ if and only if $i<j$ and the same for the women. Let $\mu$ be the given matching we are to check for stability.

First, for each man $m$, we compute the intervals along the line of women which includes all women $m$ strictly prefers to $\mu(m)$. If this interval is empty, $m$ is with his most preferred woman and cannot be involved in any blocking pairs so we can ignore him. For all nonempty intervals each endpoint is $p(w)$ for some woman $w$. We also compute these intervals for the women. Note that for any man $m$ and woman $w,(m, w)$ is a blocking pair for $\mu$ if and only if $m$ is in $w$ 's interval and $w$ is in $m$ 's interval.

We will process each of the women in order from $w_{1}$ to $w_{n}$ maintaining a balanced binary search tree of the men who prefer that woman to their partners. This will allow us to easily check if she prefers any of them by seeing if any elements in the tree are between the endpoints of her interval. Initially this tree is empty. When processing a woman $w$, we first add any man $m$ whose interval begins with $w$ to the search tree. Then we check to see if $w$ prefers any men in the tree. If so, we know the matching is not stable. Otherwise, we remove any man $m$ from the tree whose interval ends with $w$ and proceed to the next woman. Algorithm 5 provides pseudocode for this algorithm.

Computing the intervals requires $O(n \log n)$. Since we only insert each man into the tree at most once, maintaining the tree requires $O(n \log n)$. The queries also require $O(\log n)$ for each woman so the total time is $O(n \log n)$.

```
Algorithm 5: Single-Peaked Stable Matching Verification
    for each woman \(w\) do
        Create two empty lists w.begin and w.end.
        Use binary search to find the leftmost man \(m\) and rightmost man \(m^{\prime}\) that \(w\) prefers to
        \(\mu(w)\) if any. (Otherwise remove \(w\).)
        Let \(w \cdot s=p(m)\) and \(w \cdot t=p\left(m^{\prime}\right)\).
    for each man \(m\) do
        Use binary search to find the leftmost woman \(w\) and rightmost woman \(w^{\prime}\) that \(m\) prefers
        to \(\mu(m)\) if any. (Otherwise ignore \(m\).)
        Add \(m\) to \(w\).begin and \(w^{\prime}\).end.
    Initialize an empty balanced binary search tree \(T\).
    for \(i=1\) to \(n\) do
        for \(m \in w_{i}\).begin do
                \(T . \operatorname{insert}(p(m))\)
        if there are any points \(p(m)\) in \(T\) between \(w_{i}\).s and \(w_{i} . t\) then
            return \(\left(m, w_{i}\right)\) is a blocking pair.
        for \(m \in w_{i}\).end do
            \(T\). delete \((p(m))\)
    return \(\mu\) is stable.
```


### 7.1.2 Remarks on Finding a Stable Matching for Single-Peaked Preferences

The algorithm in [7] relies on the observation that there will always be a pair or participants who are each other's first choice with narcissistic single-peaked preferences. Thus a greedy approach where one such pair is selected and then removed works well. However, this is not the case when we remove the narcissistic assumption. In fact, as with the two-list case, Table 4 presents an example where no participant is matched with their top choice in the unique stable matching. Note that the preferences for the men and women are symmetric. The reader can verify that these preferences can be realized in the single-peaked preference model using the orderings $p\left(m_{1}\right)<$ $p\left(m_{2}\right)<p\left(m_{3}\right)<p\left(m_{4}\right)$ and $p\left(w_{1}\right)<p\left(w_{2}\right)<p\left(w_{3}\right)<p\left(w_{4}\right)$ and that the unique stable matching is $\left\{\left(m_{1}, w_{4}\right),\left(m_{2}, w_{2}\right),\left(m_{3}, w_{3}\right),\left(m_{4}, w_{1}\right)\right\}$ where no participant receives their first choice.

Table 4: Single-peaked preferences where no participant receives their top choice in the stable matching

| Man | Preference List | Woman | Preference List |
| :---: | :---: | :---: | :---: | :---: |
| $m_{1}$ | $w_{3} \succ w_{2} \succ w_{4} \succ w_{1}$ | $w_{1}$ | $m_{3} \succ m_{2} \succ m_{4} \succ m_{1}$ |
| $m_{2}$ | $w_{3} \succ w_{2} \succ w_{4} \succ w_{1}$ | $w_{2}$ | $m_{3} \succ m_{2} \succ m_{4} \succ m_{1}$ |
| $m_{3}$ | $w_{4} \succ w_{3} \succ w_{2} \succ w_{1}$ | $w_{3}$ | $m_{4} \succ m_{3} \succ m_{2} \succ m_{1}$ |
| $m_{4}$ | $w_{2} \succ w_{1} \succ w_{3} \succ w_{4}$ | $w_{4}$ | $m_{2} \succ m_{1} \succ m_{3} \succ m_{4}$ |

Also no greedy algorithm following the model inspired by [15] will succeed for single-peak preferences because the preferences in Table 2 can be realized in the single-peaked preference model using the orderings $p\left(m_{1}\right)<p\left(m_{2}\right)<p\left(m_{3}\right)$ and $p\left(w_{1}\right)<p\left(w_{2}\right)<p\left(w_{3}\right)$.

### 7.2 Geometric Preferences

We say the men's preferences over the women in a matching market are geometric in $d$ dimensions if each women $w$ is defined by a location $p(w)$ and for each man $m$ there is an ideal $q(m)$ such that $m$ prefers woman $w_{1}$ to $w_{2}$ if and only if $\left\|p(m)-q\left(w_{1}\right)\right\|_{2}^{2}<\left\|p(m)-q\left(w_{2}\right)\right\|_{2}^{2}$, i.e. $p\left(w_{1}\right)$ has smaller euclidean distance from the man's ideal than $p\left(w_{2}\right)$. If the women's preferences are also geometric we call the matching market geometric. We further call the preferences narcissistic if $p(x)=q(x)$ for every participant $x$. Our results for the attribute model extend to geometric preferences.

Note that one-dimensional geometric preferences are a special case of single-peaked preferences. As such, geometric preferences might be used to model preferences over political candidates who are given a score on several (linear) policy areas, e.g. protectionist vs. free trade and hawkish vs. dovish foreign policy.

Arkin et al. 5 also consider geometric preferences, but restrict themselves to the narcissistic case. Our algorithms do not require the preferences to be narcissistic, hence our model is more general. On the other hand, our lower bounds for large dimensions also apply to the narcissistic special case. While Arkin et al. take special care of different notions of stability in the presence of ties, we concentrate on weakly stable matchings. Although we restrict ourselves to the stable matching problem for the sake of presentation, all lower bounds and verification algorithms naturally extend to the stable roommate problem. Since all proofs in this section are closely related those for the attribute model, we restrict ourselves to proof sketches highlighting the main differences.

Theorem 1 extends immediately to the geometric case without any changes in the proof.
Corollary 3 (Geometric version of Theorem(1). There is an algorithm to find a stable matching in the d-dimensional geometric model with at most a constant $C$ distinct values in time $O\left(C^{2 d} n(d+\right.$ $\log n)$ ).

For the verification of a stable matching with real-valued vectors we use a standard lifting argument.

Corollary 4 (Geometric version of Theorem (4). There is an algorithm to verify a stable matching in the d-dimensional geometric model with real-valued locations and ideals in time $\tilde{O}\left(n^{2-1 / 2(d+1)}\right)$

Proof. Let $q \in \mathbb{R}^{d}$ be an ideal and let $a, b \in \mathbb{R}^{d}$ be two locations. Define $q^{\prime}, a^{\prime}, b^{\prime} \in \mathbb{R}^{d+1}$ as $q^{\prime}=\left(q_{1}, \ldots, q_{d},-1 / 2\right), a^{\prime}=\left(a_{1}, \ldots, a_{d}, \sum_{i=1}^{d} a_{i}^{2}\right)$ and $b^{\prime}=\left(b_{1}, \ldots, b_{d}, \sum_{i=1}^{d} b_{i}^{2}\right)$.

We have $\left\langle a^{\prime}, q\right\rangle=1 / 2 \sum_{i=1}^{d} q_{i}-1 / 2\|q-a\|_{2}^{2}$. Hence we get $\|q-a\|_{2}^{2}<\|q-b\|_{2}^{2}$ if and only if $\left\langle q^{\prime}, a^{\prime}\right\rangle>\left\langle q^{\prime}, b^{\prime}\right\rangle$, so we can reduce the stable matching problem in the $d$-dimensional geometric model to the $d+1$-attribute model.

For the boolean case, we can adjust the proof of Theorem 6 by using a threshold of parities instead of a threshold of conjunctions. The degree of the resulting polynomial remains the same.

Corollary 5 (Geometric version of Theorem (6). In the geometric model with $n$ men and women, with locations and ideals in $\{0,1\}^{d}$ with $d=c \log n$, there is a randomized algorithm to decide if a given matching is stable in time $\tilde{O}\left(n^{2-1 / O\left(c \log ^{2}(c)\right)}\right)$ with error probability at most $1 / 3$.

For lower bounds we reduce from the minimum Hamming distance problem which is SETH-hard with the same parameters as the maximum inner product problem [4]. The Hamming distance of two boolean vectors is exactly their squared euclidean distance, hence a matching market where the preferences are defined by Hamming distances is geometric.

Definition 5. For any d and input l, the minimum Hamming distance problem is to decide if two input sets $U, V \subseteq\{0,1\}^{d}$ with $|U|=|V|=n$ have a pair $u \in U, v \in V$ such that $\|u-v\|_{2}^{2}<l$.

Lemma 10 ([4). Assuming SETH, for any $\varepsilon>0$, there is a $c$ such that solving the minimum Hamming distance problem on $d=c \log n$ dimensions requires time $\Omega\left(n^{2-\varepsilon}\right)$.

For the hardness of finding a stable matching, the construction from Theorem 7 works without adjustments.

Corollary 6 (Geometric version of Theorem 7). Assuming SETH, for any $\varepsilon>0$, there is a $c$ such that finding a stable matching in the (boolean) d-dimensional geometric model with $d=c \log n$ dimensions requires time $\Omega\left(n^{2-\varepsilon}\right)$.

For the hardness of verifying a stable matching, the construction is as follows.
Corollary 7 (Geometric version of Theorem [8). Assuming SETH, for any $\varepsilon>0$, there is a c such that verifying a stable matching in the (boolean) d-dimensional geometric model with $d=c \log n$ dimensions requires time $\Omega\left(n^{2-\varepsilon}\right)$.

Proof. Let $U, V \subseteq\{0,1\}^{d}$ be the inputs to the minimum Hamming distance problem and let $l$ be the threshold.

For every $u \in U$, define a real man $m_{u}$ with both ideal and location as $u \circ 0^{l}$ and a dummy woman $w_{u}^{\prime}$ with ideal and location $u \circ 1^{l}$. Symmetrically for $v \in V$ define $w_{v}$ with $v \circ 0^{l}$ and $m_{v}^{\prime}$ with $v \circ 1^{l}$. The matching ( $m_{u}, w_{u}^{\prime}$ ) for all $u \in U$ and $\left(w_{v}, m_{v}^{\prime}\right)$ for all $v \in V$ is stable if and only if there is there is no pair $u, v$ with Hamming distance less than $l$.

The hardness results for checking a stable pair also translate to the geometric model. In particular, since both variants of the proof extend to the geometric model we have the same consequences for nondeterministic algorithms.

Corollary 8 (Geometric version of Theorem (9). Assuming SETH, for any $\varepsilon>0$, there is a $c$ such that determining whether a given pair is part of any or all stable matchings in the (boolean) $d$-dimensional geometric model with $d=c \log n$ dimensions requires time $\Omega\left(n^{2-\varepsilon}\right)$.

Proof. We again reduce from the minimum Hamming distance problem. We assume without loss of generality that $d$ is even and the threshold $l$ is exactly $d / 2+1$, i.e. the instance is true if and only if there are vectors $u, v$ with Hamming distance at most $d / 2$. We can reduce to this case from any other threshold by padding the vectors.

We use the same preference orders as in the $d$-attribute model. The following narcissistic instance realizes the preference order from Theorem 9 . For a vector $u \in\{0,1\}^{d}, \bar{u}$ denotes its component-wise complement.

$$
\begin{array}{lr}
m_{u}:(u \circ \bar{u} \circ u \circ \bar{u})^{3} \circ 000000000 \\
m_{i}: 0^{12 d} & \circ 100000000 \\
m^{*}: 0^{12 d} & \circ 001111111 \\
w_{v}:(v \circ \bar{v} \circ v \circ \bar{v})^{3} \circ 000000000 \\
w_{j}:\left(0^{2 d} \circ 1^{2 d}\right)^{3} & \circ 010000000 \\
w^{*}:\left(0^{2 d} \circ 1^{2 d}\right)^{3} & \circ 101110000
\end{array}
$$

Likewise the preference orders for Theorem 10 are achieved by the following vectors.

$$
\begin{array}{ll}
m_{u}:(u \circ \bar{u} \circ u \circ \bar{u})^{3} \circ 00000000000 \\
m_{i}: 0^{12 d} & \circ 10000000000 \\
m^{*}: 0^{12 d} & \circ 00111111100 \\
w_{v}:(v \circ \bar{v} \circ v \circ \bar{v})^{3} \circ 00000000000 \\
w_{j}:\left(0^{2 d} \circ 1^{2 d}\right)^{3} & \circ 01000000011 \\
w^{*}:\left(0^{2 d} \circ 1^{2 d}\right)^{3} & \circ 10111000000
\end{array}
$$

### 7.3 Strategic Behavior

With geometric and single-peaked preferences, we assume that the participants are not allowed to misrepresent their location points. Rather they may only misrepresent their preference ideal. As such, the results of this section do not apply when preferences are narcissistic.

Theorem 12. There is no strategy proof algorithm to find a stable matching in the geometric preference model.

Proof. We consider one-dimensional geometric preferences. Let the preference points and ideals be as given in Table 5 which yield the depicted preference lists. As in the proof for 3, there are two stable matchings: the man-optimal matching $\left\{\left(m_{1}, w_{3}\right),\left(m_{2}, w_{1}\right),\left(m_{3}, w_{2}\right)\right\}$ and the woman-optimal matching $\left\{\left(m_{1}, w_{3}\right),\left(m_{2}, w_{2}\right),\left(m_{3}, w_{1}\right)\right\}$. However, if $w_{2}$ changes her ideal to $5 / 3$ then her preference list is $m_{2} \succ m_{1} \succ m_{3}$. Now there is a unique stable matching which is $\left\{\left(m_{1}, w_{3}\right),\left(m_{2}, w_{2}\right),\left(m_{3}, w_{1}\right)\right\}$, the woman-optimal stable matching from the original preferences. Therefore, any mechanism that does not always output the woman optimal stable matching can be manipulated by the women to their advantage. Similarly, any mechanism that does not always output the man-optimal matching could be manipulated by the men in some instances. Thus there is no strategy-proof mechanism for geometric preferences.

Table 5: Geometric preferences that can be manipulated

| Man | Location $(p)$ | Ideal $(q)$ |  | Woman | Location $(p)$ | Ideal $(q)$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $m_{1}$ | 1 | $7 / 3$ |  | $w_{1}$ | 1 | 3 |
| $m_{2}$ | 2 | 1 |  | $w_{2}$ | 2 | $7 / 3$ |
| $m_{3}$ | 3 | $5 / 3$ |  | $w_{3}$ | 3 | 3 |
|  |  |  |  |  |  |  |
|  | Man | Preference List |  | Woman | Preference List |  |
|  | $m_{2} \succ w_{3} \succ w_{1}$ |  | $w_{1}$ | $m_{3} \succ m_{2} \succ m_{1}$ |  |  |
|  | $m_{2}$ | $w_{1} \succ w_{2} \succ w_{3}$ |  | $w_{2}$ | $m_{2} \succ m_{3} \succ m_{1}$ |  |
|  | $m_{3}$ | $w_{2} \succ w_{1} \succ w_{3}$ |  | $w_{3}$ | $m_{3} \succ m_{2} \succ m_{1}$ |  |

Since one-dimensional geometric preferences are a special case of single-peaked preferences the following corollary results directly from Theorem 12 .

Corollary 9. There is no strategy proof algorithm to find a stable matching in the single-peaked preference model.

## 8 Conclusion and Open Problems

We give subquadratic algorithms for finding and verifying stable matchings in the $d$-attribute model and $d$-list model. We also show that, assuming SETH, one can only hope to find such algorithms if the number of attributes $d$ is bounded by $O(\log n)$.

For a number of cases there is a gap between the conditional lower bound and the upper bound. Our algorithms with real attributes and weights are only subquadratic if the dimension is constant. Even for small constants our algorithm to find a stable matching is not tight, as it is not subquadratic for any $d=O(\log n)$. The techniques we use when the attributes and weights are small constants do not readily apply to the more general case.

There is also a gap between the time complexity of our algorithms for finding a stable matching and verifying a stable matching. It would be interesting to either close or explain this gap. On the one hand, subquadratic algorithms for finding a stable matching would demonstrate that the attribute and list models are computationally simpler than the general preference model. On the other hand, proving that there are no subquadratic algorithms would show a distinction between the problems of finding and verifying a stable matching in these settings which does not exist for the general preference model. Currently, we do not have a subquadratic algorithm for finding a stable matching even in the 2-list case, while we have an optimal algorithm for verifying a stable matching for $d$ lists. This 2 -list case seems to be a good starting place for further research.

Additionally it is worth considering succinct preference models for other computational problems that involve preferences to see if we can also develop improved algorithms for these problems. For example, the Top Trading Cycles algorithm [39] can be made to run in subquadratic time for $d$ attribute preferences (when $d$ is constant) using the ray shooting techniques applied in this paper to find participants' top choices.
Acknowledgment: We would like to thank Russell Impagliazzo, Vijay Vazirani, and the anonymous reviewers for helpful discussions and comments.

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[^0]:    *Part of this research was conducted while being affiliated with Max Planck Institute for Informatics, Saarbrücken, Germany and visiting the Simons Institute for the Theory of Computing, Berkeley.
    ${ }^{\dagger}$ This research was partially supported by the Army Research Office grant number W911NF-15-1-0253.
    ${ }^{\ddagger}$ This research is supported by the Simons Foundation and was partially conducted while visiting the Simons Institute for the Theory of Computing, Berkeley. This research is supported by NSF grant CCF-1213151 from the Division of Computing and Communication Foundations. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation. The conference version is available at Springer via http://dx.doi.org/10.1007/978-3-319-34171-2_21

