Algorithmic decidability of Engel's property for automaton groups

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Abstract. We consider decidability problems associated with Engel's identity $([\cdots [[x, y], y], \dots, y] = 1$ for a long enough commutator sequence) in groups generated by an automaton.

We give a partial algorithm that decides, given x, y, whether an Engel identity is satisfied. It succeeds, importantly, in proving that Grigorchuk's 2-group is not Engel.

We consider next the problem of recognizing Engel elements, namely elements y such that the map $x \mapsto [x, y]$ attracts to $\{1\}$. Although this problem seems intractable in general, we prove that it is decidable for Grigorchuk's group: Engel elements are precisely those of order at most 2.

Our computations were implemented using the package FR within the computer algebra system GAP.

1 Introduction

A law in a group G is a word $w = w(x_1, x_2, ..., x_n)$ such that $w(g_1, ..., g_n) = 1$ for all $g_1, ..., g_n \in G$; for example, commutative groups satisfy the law $[x_1, x_2] = x_1^{-1}x_2^{-1}x_1x_2$. A variety of groups is a maximal class of groups satisfying a given law; e.g. the variety of commutative groups (satisfying $[x_1, x_2]$) or of groups of exponent p (satisfying x_1^p).

Consider now a sequence $\mathscr{W} = (w_0, w_1, ...)$ of words in *n* letters. Say that (g_1, \ldots, g_n) almost satisfies \mathscr{W} if $w_i(g_1, \ldots, g_n) = 1$ for all *i* large enough, and say that *G* almost satisfies \mathscr{W} if all *n*-tuples from *G* almost satisfy \mathscr{W} . For example, *G* almost satisfies $(x_1, \ldots, x_1^{i_1}, \ldots)$ if and only if *G* is a torsion group.

The problem of deciding algorithmically whether a group belongs to a given variety has received much attention (see e.g. [17] and references therein); we consider here the harder problems of determining whether a group (respectively a tuple) almost satisfies a given sequence. This has, up to now, been investigated mainly for the torsion sequence above [12].

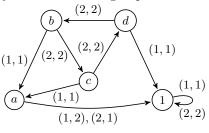


Fig. 1. The Grigorchuk automaton

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The first Grigorchuk group G_0 is an automaton group which appeared prominently in group theory, for example as a finitely generated infinite torsion group [14] and as a group of intermediate word growth [15]; see §2.1. It is the group of automatic transformations of $\{1, 2\}^{\infty}$ generated by the five states of the automaton from Figure 1, with input and output written as (in, out).

The Engel law is

$$E_c = E_c(x, y) = [x, y, \dots, y] = [\cdots [[x, y], y], \dots, y]$$

with c copies of 'y'; so $E_0(x, y) = x$, $E_1(x, y) = [x, y]$ and $E_c(x, y) = [E_{c-1}(x, y), y]$. See below for a motivation. Let us call a group (respectively a pair of elements) Engel if it almost satisfies $\mathscr{E} = (E_0, E_1, \ldots)$. Furthermore, let us call $h \in G$ an Engel element if (g, h) is Engel for all $g \in G$.

A concrete consequence of our investigations is:

Theorem 1. The first Grigorchuk group G_0 is not Engel. Furthermore, an element $h \in G_0$ is Engel if and only if $h^2 = 1$.

We prove a similar statement for another prominent example of automaton group, the *Gupta-Sidki group*, see Theorem 2.

Theorem 1 follows from a partial algorithm, giving a criterion for an element y to be Engel. This algorithm proves, in fact, that the element ad in the Grigorichuk group is not Engel. We consider the following restricted situation, which is general as far as the Engel property is concerned, see §2: an *automaton group* is a group G endowed with extra data, in particular with a family of self-maps called *states*, indexed by a set X and written $g \mapsto g@x$ for $x \in X$; it is *contracting* for the word metric $\|\cdot\|$ on G if there are constants $\eta < 1$ and C such that $\|g@x\| \leq \eta \|g\| + C$ holds for all $g \in G$ and all $x \in X$. Our aim is to solve the following **decision problems** in an automaton group G:

Engel(g,h) Given $g,h \in G$, does there exist $c \in \mathbb{N}$ with $E_c(g,h)$? **Engel**(h) Given $h \in G$, does Engel(g,h) hold for all $g \in G$?

The algorithm is described in §3. As a consequence,

Corollary 1. Let G be an automaton group acting on the set of binary sequences $\{1,2\}^*$, that is contracting with contraction coefficient $\eta < 1$. Then, for torsion elements h of order 2^e with $2^{2^e}\eta < 1$, the property Engel(h) is decidable.

The Engel property attracted attention for its relation to nilpotency: indeed a nilpotent group of class c satisfies E_c , and conversely among compact [21] and solvable [16] groups, if a group satisfies E_c for some c then it is locally nilpotent. Conjecturally, there are non-locally nilpotent groups satisfying E_c for some c, but this is still unknown. It is also an example of iterated identity, see [3, 7]. In particular, the main result of [3] implies easily that the Engel property is decidable in algebraic groups.

It is comparatively easy to prove that the first Grigorchuk group G_0 satisfies no law [1,20]; this result holds for a large class of automaton groups. In fact, if a group satisfies a law, then so does its profinite completion. In the class mentioned above, the profinite completion contains abstract free subgroups, precluding the existence of a law. No such arguments would help for the Engel property: the restricted product of all finite nilpotent groups is Engel, but the unrestricted product again contains free subgroups. This is one of the difficulties in dealing with iterated identities rather than identities.

If \mathfrak{A} is a nil algebra (namely, for every $a \in \mathfrak{A}$ there exists $n \in \mathbb{N}$ with $a^n = 0$) then the set of elements of the form $\{1 + a : a \in \mathfrak{A}\}$ forms a group $1 + \mathfrak{A}$ under the law (1+a)(1+b) = 1 + (a+b+ab). If \mathfrak{A} is defined over a field of characteristic p, then $1 + \mathfrak{A}$ is a torsion group since $(1+a)^{p^n} = 1$ if $a^{p^n} = 0$. Golod constructed in [13] non-nilpotent nil algebras \mathfrak{A} all of whose 2-generated subalgebras are nilpotent (namely, $\mathfrak{A}^n = 0$ for some $n \in \mathbb{N}$); given such an \mathfrak{A} , the group $1 + \mathfrak{A}$ is Engel but not locally nilpotent.

Golod introduced these algebras as means of obtaining infinite, finitely generated, residually finite (every non-trivial element in the group has a non-trivial image in some finite quotient), torsion groups. Golod's construction is highly non-explicit, in contrast with Grigorchuk's group for which much can be derived from the automaton's properties.

It is therefore of high interest to find explicit examples of Engel groups that are not locally nilpotent, and the methods and algorithms presented here are a step in this direction.

An important feature of automaton groups is their amenability to computer experiments, and even as in this case of rigorous verification of mathematical assertions; see also [18], and the numerous decidability and undecidability of the finiteness property in [2, 11, 19].

The proof of Theorem 1 relies on a computer calculation. It could be checked by hand, at the cost of quite unrewarding effort. One of the purposes of this article is, precisely, to promote the use of computers in solving general questions in group theory: the calculations performed, and the computer search involved, are easy from the point of view of a computer but intractable from the point of view of a human.

The calculations were performed using the author's group theory package FR, specially written to manipulate automaton groups. This package integrates with the computer algebra system GAP [8], and is freely available from the GAP distribution site

http://www.gap-system.org

2 Automaton groups

An automaton group is a finitely generated group associated with a Mealy automaton. We define a Mealy automaton \mathcal{M} as a graph such as that in Figure 1. It has a set of states Q and an alphabet X, and there are transitions between states with input and output labels in X, with the condition that, at every state, all labels appear exactly once as input and once as output on the set of outgoing transitions.

Every state $q \in Q$ of \mathscr{M} defines a transformation, written as exponentiation by q, of the set of words X^* by the following rule: given a word $x_1 \ldots x_n \in X^*$, there exists a unique path in the automaton starting at q and with input labels x_1, \ldots, x_n . Let y_1, \ldots, y_n be its corresponding output labels. Then declare $(x_1 \ldots x_n)^q = y_1 \ldots y_n$.

The action may also be defined recursively as follows: if there is a transition from $q \in Q$ to $r \in Q$ with input label $x_1 \in X$ and output label $y_1 \in X$, then $(x_1x_2...x_n)^q = y_1(x_2...,x_n)^r$.

By the *automaton group* associated with the automaton \mathcal{M} , we mean the group G of transformations of X^* generated by \mathcal{M} 's states. Note that all elements of G admit descriptions by automata; namely, a word of length n in G's generators is the transformation associated with a state in the n-fold product of the automaton of G. See [10] for the abstract theory of automata, and [9] for products more specifically.

The structure of the automaton \mathscr{M} may be encoded in an injective group homomorphism $\psi: G \to G^X \rtimes \operatorname{Sym}(X)$ from G to the group of G-decorated permutations of X. This last group — the wreath product of G with $\operatorname{Sym}(X)$ — is the group of permutations of X, with labels in G on each arrow of the permutation; the labels multiply as the permutations are composed. The construction of ψ is as follows: consider $q \in Q$. For each $x \in X$, let q@x denote the endpoint of the transition starting at q with input label x, and let x^{π} denote the output label of the same transition; thus every transition in the Mealy automaton gives rise to

$$(q) \xrightarrow{(x, x^{\pi})} (q@x)$$

The transformation π is a permutation of X, and we set

$$\psi(q) = \langle x \mapsto q@x \rangle \pi,$$

namely the permutation ε with decoration q@x on the arrow from x to x^{π} .

We generalize the notation q@x to arbitrary words and group elements. Consider a word $v \in X^*$ and an element $g \in G$; denote by v^g the image of v under g. There is then a unique element of G, written g@v, with the property

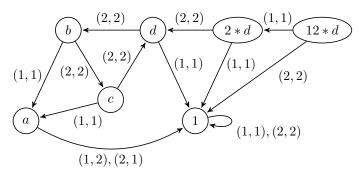
$$(v w)^g = (v^g) (w)^{g \otimes v}$$
 for all $w \in X^*$

We call by extension this element g@v the *state* of g at v; it is the state, in the Mealy automaton defining g, that is reached from g by following the path v as input; thus in the Grigorchuk automaton b@1 = a and b@222 = b. There is a reverse construction: by v * g we denote the permutation of X^* defined by

$$(v w)^{v*g} = v w^g, \qquad w^{v*g} = w \text{ if } w \text{ does not start with } v.$$

Given a word $w = w_1 \dots w_n \in X^*$ and a Mealy automaton \mathscr{M} of which g is a state, it is easy to construct a Mealy automaton of which w * g is a state: add a path of length n to \mathscr{M} , with input and output $(w_1, w_2), \dots, (w_n, w_n)$ along the

path, and ending at g. Complete the automaton with transitions to the identity element. Then the first vertex of the path defines the transformation w * g. For example, here is 12 * d in the Grigorchuk automaton:



Note the simple identities $(g@v_1)@v_2 = g@(v_1v_2), (v_1v_2) * g = v_1 * (v_2 * g)$, and (v * g)@v = g. Recall that we write conjugation in G as $g^h = h^{-1}gh$. For any $h \in G$ we have

$$(v*g)^{h} = v^{h} * (g^{h@v}).$$
(1)

An automaton group is called *regular weakly branched* if there exists a nontrivial subgroup K of G such that $\psi(K)$ contains K^X . In other words, for every $k \in K$ and every $v \in X^*$, the element v * k also belongs to K, and therefore to G. Abért proved in [1] that regular weakly branched groups satisfy no law.

In this text, we concentrate on the Engel property, which is equivalent to nilpotency for finite groups. In particular, if an automaton group G is to have a chance of being Engel, then its image under the map $G \to G^X \rtimes \operatorname{Sym}(X) \to$ $\operatorname{Sym}(X)$ should be a nilpotent subgroup of $\operatorname{Sym}(X)$. Since finite nilpotent groups are direct products of their *p*-Sylow subgroups, we may reduce to the case in which the image of G in $\operatorname{Sym}(X)$ is a *p*-group. A further reduction lets us assume that the image of G is an abelian subgroup of $\operatorname{Sym}(X)$ of prime order. We therefore make the following

Standing assumption 1 The alphabet is $X = \{1, ..., p\}$, and automaton groups are defined by embeddings $\psi: G \to G^p \rtimes \mathbb{Z}_{/p}$, with $\mathbb{Z}_{/p}$ the cyclic subgroup of $\operatorname{Sym}(X)$ generated by the cycle (1, 2, ..., p).

This is the situation considered in the Introduction.

We make a further reduction in that we only consider the Engel property for elements of finite order. This is not a very strong restriction: given h of infinite order, one can usually find an element $g \in G$ such that the conjugates $\{g^{h^n} : n \in \mathbb{Z}\}$ are independent, and it then follows that h is not Engel. This will be part of a later article.

2.1 Grigorchuk's first group

This section is not an introduction to Grigorchuk's first group, but rather a brief description of it with all information vital for the calculation in §4. For more details, see e.g. [5].

Fix the alphabet $X = \{1, 2\}$. The first Grigorchuk group G_0 is a permutation group of the set of words X^* , generated by the four non-trivial states a, b, c, d of the automaton given in Figure 1. Alternatively, the transformations a, b, c, d may be defined recursively as follows:

$$(1x_2...x_n)^a = 2x_2...x_n, (2x_2...x_n)^a = 1x_2...x_n, (1x_2...x_n)^b = 1a(x_2...x_n), (2x_2...x_n)^b = 2c(x_2...x_n), (1x_2...x_n)^c = 1a(x_2...x_n), (2x_2...x_n)^c = 2d(x_2...x_n), (1x_2...x_n)^d = 1x_2...x_n, (2x_2...x_n)^d = 2b(x_2...x_n)$$
(2)

which directly follow from d@1 = 1, d@2 = b, etc.

It is remarkable that most properties of G_0 derive from a careful study of the automaton (or equivalently this action), usually using inductive arguments. For example,

Proposition 1 ([14]). The group G_0 is infinite, and all its elements have order a power of 2.

The self-similar nature of G_0 is made apparent in the following manner:

Proposition 2 ([4, §4]). Define x = [a, b] and $K = \langle x, x^c, x^{ca} \rangle$. Then K is a normal subgroup of G_0 of index 16, and $\psi(K)$ contains $K \times K$.

In other words, for every $g \in K$ and every $v \in X^*$ the element v * g belongs to G_0 .

3 A semi-algorithm for deciding the Engel property

We start by describing a semi-algorithm to check the Engel property. It will sometimes not return any answer, but when it returns an answer then that answer is guaranteed correct. It is guaranteed to terminate as long as the contraction property of the automaton group G is strong enough.

Algorithm 1 Let G be a contracting automaton group with alphabet $X = \{1, ..., p\}$ for prime p, with the contraction property $||g@j|| \le \eta ||g|| + C$.

For $n \in p\mathbb{N}$ and $R \in \mathbb{R}$ consider the following finite graph $\Gamma_{n,R}$. Its vertex set is $B(R)^n \cup \{fail\}$, where B(R) denotes the set of elements of G of length at most R. Its edge set is defined as follows: consider a vertex (g_1, \ldots, g_n) in $\Gamma_{n,R}$, and compute

$$(h_1,\ldots,h_n) = (g_1^{-1}g_2,\ldots,g_n^{-1}g_1).$$

If h_i fixes X for all i, i.e. all h_i have trivial image in Sym(X), then for all $j \in \{1, \ldots, p\}$ there is an edge from (g_1, \ldots, g_n) to $(h_1@j, \ldots, h_n@j)$, or to fail if $(h_1@j, \ldots, h_n@j) \notin B(R)^n$. If some h_i does not fix X, then there is an edge from (g_1, \ldots, g_n) to (h_1, \ldots, h_n) , or to fail if $(h_1, \ldots, h_n) \notin B(R)^n$.

Given $g, h \in G$ with $h^n = 1$: Set $t_0 = (g, g^h, g^{h^2}, \ldots, g^{h^{n-1}})$. If there exists $R \in \mathbb{N}$ such that no path in $\Gamma_{n,R}$ starting at t_0 reaches fail, then Engel(g,h) holds if and only if the only cycle in $\Gamma_{n,R}$ reachable from passes through $(1, \ldots, 1)$.

If the contraction coefficient satisfies $2^n\eta < 1$, then it is sufficient to consider $R = (||g|| + n||h||)2^n C/(1-2^n\eta).$

- **Given** $n \in \mathbb{N}$: The Engel property holds for all elements of exponent n if and only if, for all $R \in \mathbb{N}$, the only cycle in $\Gamma_{n,R}$ passes through $(1, \ldots, 1)$. If the contraction coefficient satisfies $2^n \eta < 1$, then it is sufficient to consider $R = 2^n C/(1 - 2^n \eta)$.
- **Given G weakly branched and** $n \in \mathbb{N}$: If for some $R \in \mathbb{N}$ there exists a cycle in $\Gamma_{n,R}$ that passes through an element of $K^n \setminus 1^n$, then no element of G whose order is a multiple of n is Engel.

If the contraction coefficient satisfies $2^n\eta < 1$, then it is sufficient to consider $R = 2^n C/(1-2^n\eta)$.

We consider the graphs $\Gamma_{n,R}$ as subgraphs of a graph $\Gamma_{n,\infty}$ with vertex set G^n and same edge definition as the $\Gamma_{n,R}$.

We note first that, if G satisfies the contraction condition $2^{\eta} < 1$, then all cycles of $\Gamma_{n,\infty}$ lie in fact in $\Gamma_{n,2^nC/(1-2^n\eta)}$. Indeed, consider a cycle passing through (g_1,\ldots,g_n) with $\max_i ||g_i|| = R$. Then the cycle continues with $(g_1^{(1)},\ldots,g_n^{(1)})$, $(g_1^{(2)},\ldots,g_n^{(2)})$, etc. with $||g_i^{(k)}|| \le 2^k R$; and then for some $k \le n$ we have that all $g_i^{(k)}$ fix X; namely, they have a trivial image in Sym(X), and the map $g \mapsto g@j$ is an injective homomorphism on them. Indeed, let $\pi_1,\ldots,\pi_n,\pi_1^{(i)},\ldots,\pi_n^{(i)} \in \mathbb{Z}_{/p} \subset \text{Sym}(X)$ be the images of $g_1,\ldots,g_n,g_1^{(i)},\ldots,g_n^{(i)}$ respectively, and denote by $S \colon \mathbb{Z}_{/p}^n \to \mathbb{Z}_{/p}^n$ the cyclic permutation operator. Then $(\pi_1^{(n)},\ldots,\pi_n^{(n)}) = (S-1)^n(\pi_1,\ldots,\pi_n)$, and $(S-1)^n = \sum_j S^j \binom{n}{j} = 0$ since p|n and $S^n = 1$. Thus there is an edge from $(g_1^{(k)},\ldots,g_n^{(k)})$ to $(g_1^{(k+1)}@j,\ldots,g_n^{(k+1)}@j)$ with $||g_i^{(k+1)}@j|| \le \eta ||g_i^{(k)}|| + C \le \eta 2^n R + C$. Therefore, if $R > 2^n C/(1-2^n\eta)$ then $2^n \eta R + C < R$, and no cycle can return to (g_1,\ldots,g_n) .

Consider now an element $h \in G$ with $h^n = 1$. For all $g \in G$, there is an edge in $\Gamma_{n,\infty}$ from $(g, g^h, \ldots, g^{h^{n-1}})$ to $([g, h]^{@v}, [g, h]^h @v, [g, h]^{h^{n-1}} @v)$ for some word $v \in \{\varepsilon\} \sqcup X$, and therefore for all $c \in \mathbb{N}$ there exists $d \leq c$ such that, for all $v \in X^d$, there is a length-c path from $(g, g^h, \ldots, g^{h^{n-1}})$ to $(E_c(g, h) @v, \ldots, E_c(g, h)^{h^{n-1}} @v)$ in $\Gamma_{n,\infty}$.

We are ready to prove the first assertion: if $\operatorname{Engel}(g,h)$, then $E_c(g,h) = 1$ for some c large enough, so all paths of length c starting at $(g, g^h, \ldots, g^{h^{n-1}})$ end at $(1, \ldots, 1)$. On the other hand, if $\operatorname{Engel}(g,h)$ does not hold, then all long enough paths starting at $(g, g^h, \ldots, g^{h^{n-1}})$ end at vertices in the finite graph $\Gamma_{n,2^nC/(1-2^n\eta)}$ so must eventually reach cycles; and one of these cycles is not $\{(1, \ldots, 1)\}$ since $E_c(g, h) \neq 1$ for all c.

The second assertion immediately follows: if there exists $g \in G$ such that Engel(g, h) does not hold, then again a non-trivial cycle is reached starting

from $(g, g^h, \ldots, g^{h^{n-1}})$, and independently of g, h this cycle belongs to the graph $\Gamma_{n,2^nC/(1-2^n\eta)}$.

For the third assertion, let $\bar{k} = (k_1, \ldots, k_n) \in K^n \setminus 1^n$ be a vertex of a cycle in $\Gamma_{n,2^nC/(1-2^n\eta)}$. Consider an element $h \in G$ of order sn for some $s \in \mathbb{N}$. By the condition that #X = p is prime and the image of G in Sym(X) is a cyclic group, sn is a power of p, so there exists an orbit $\{v_1, \ldots, v_{sn}\}$ of h, so labeled that $v_i^h = v_{i-1}$, indices being read modulo sn. For $i = 1, \ldots, sn$ define

$$h_i = (h@v_1)^{-1} \cdots (h@v_i)^{-1},$$

noting $h_i(h@v_i) = h_{i-1}$ for all i = 1, ..., sn since $h^{sn} = 1$. Denote by 'i%n' the unique element of $\{1, ..., n\}$ congruent to *i* modulo *n*, and consider the element

$$g = \prod_{i=1}^{sn} \left(v_i * k_{i\%n}^{h_i} \right),$$

which belongs to G since G is weakly branched. Let $(k_1^{(1)}, \ldots, k_n^{(1)})$ be the next vertex on the cycle of \bar{k} . We then have, using (1),

$$[g,h] = g^{-1}g^h = \prod_{i=1}^{sn} \left(v_i * k_{i\%n}^{-h_i} \right) \prod_{i=1}^{sn} \left(v_{i-1} * k_{i\%n}^{h_i(h@v_i)} \right) = \prod_{i=1}^{sn} \left(v_i * (k_{i\%n}^{(1)})^{h_i} \right),$$

and more generally $E_c(g,h)$ and some of its states are read off the cycle of \bar{k} . Since this cycle goes through non-trivial group elements, $E_c(g,h)$ has a non-trivial state for all c, so is non-trivial for all c, and Engel(g,h) does not hold.

4 Proof of Theorem 1

The Grigorchuk group G_0 is contracting, with contraction coefficient $\eta = 1/2$. Therefore, the conditions of validity of Algorithm 1 are not satisfied by the Grigorchuk group, so that it is not guaranteed that the algorithm will succeed, on a given element $h \in G_0$, to prove that h is not Engel. However, nothing forbids us from running the algorithm with the hope that it nevertheless terminates. It seems experimentally that the algorithm always succeeds on elements of order 4, and the argument proving the third claim of Algorithm 1 (repeated here for convenience) suffices to complete the proof of Theorem 1.

Below is a self-contained proof of Theorem 1, extracting the relevant properties of the previous section, and describing the computer calculations as they were keyed in.

Consider first $h \in G_0$ with $h^2 = 1$. It follows from Proposition 1 that h is Engel: given $g \in G_0$, we have $E_{1+k}(g,h) = [g,h]^{(-2)^k}$ so $E_{1+k}(g,h) = 1$ for k larger than the order of [g,h].

For the other case, we start by a side calculation. In the Grigorchuk group G_0 , define x = [a, b] and $K = \langle x \rangle^{G_0}$ as in Proposition 2, consider the quadruple

$$A_0 = (A_{0,1}, A_{0,2}, A_{0,3}, A_{0,4}) = (x^{-2}x^{2ca}, x^{-2ca}x^2x^{2cab}, x^{-2cab}x^{-2}, x^2)$$

of elements of K, and for all $n \ge 0$ define

$$A_{n+1} = (A_{n,1}^{-1}A_{n,2}, A_{n,2}^{-1}A_{n,3}, A_{n,3}^{-1}A_{n,4}, A_{n,4}^{-1}A_{n,1})$$

Lemma 1. For all i = 1, ..., 4, the element $A_{9,i}$ fixes 111112, is non-trivial, and satisfies $A_{9,i}@111112 = A_{0,i}$.

Proof. This is proven purely by a computer calculation. It is performed as follows within GAP:

```
gap> LoadPackage("FR");
true
gap> AssignGeneratorVariables(GrigorchukGroup);;
gap> x2 := Comm(a,b)^2;; x2ca := x2^(c*a);; one := a^0;;
gap> A0 := [x2^-1*x2ca,x2ca^-1*x2*x2ca^b,(x2ca^-1)^b*x2^-1,x2];;
gap> v := [1,1,1,1,1,2];; A := A0;;
gap> for n in [1..9] do A := List([1..4],i->A[i]^-1*A[1+i mod 4]); od;
gap> ForAll([1..4],i->v^A[i]=v and A[i]<>one and State(A[i],v)=A0[i]);
true
```

Consider now $h \in G_0$ with $h^2 \neq 1$. Again by Proposition 1, we have $h^{2^e} = 1$ for some minimal $e \in \mathbb{N}$, which is furthermore at least 2. We keep the notation 'a%b' for the unique number in $\{1, \ldots, b\}$ that is congruent to a modulo b.

Let *n* be large enough so that the action of *h* on X^n has an orbit $\{v_1, v_2, \ldots, v_{2^e}\}$ of length 2^e , numbered so that $v_{i+1}^h = v_i$ for all *i*, indices being read modulo 2^e . For $i = 1, \ldots, 2^e$ define

$$h_i = (h@v_1)^{-1} \cdots (h@v_i)^{-1},$$

noting $h_i(h@v_i) = h_{i-1\%2^e}$ for all $i = 1, ..., 2^e$ since $h^{2^e} = 1$, and consider the element

$$g = \prod_{i=1}^{2^c} \left(v_i * A_{0,i\%4}^{h_i} \right),$$

which is well defined since $4|2^e$ and belongs to G_0 by Proposition 2. We then have, using (1),

$$[g,h] = g^{-1}g^h = \prod_{i=1}^{2^e} \left(v_i * A_{0,i\%4}^{-h_i} \right) \prod_{i=1}^{2^e} \left(v_{i-1\%2^e} * A_{0,i\%4}^{h_i(h@v_i)} \right) = \prod_{i=1}^{2^e} \left(v_i * A_{1,i}^{h_i} \right),$$

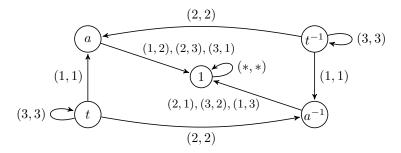
and more generally

$$E_c(g,h) = \prod_{i=1}^{2^c} (v_i * A_{c,i}^{h_i}).$$

Therefore, by Lemma 1, for every $k \ge 0$ we have $E_{9k}(g,h)@v_0(11112)^k = A_{0,1} \ne 1$, so $E_c(g,h) \ne 1$ for all $c \in \mathbb{N}$ and we have proven that h is not an Engel element.

5 Other examples

Similar calculations apply to the Gupta-Sidki group Γ . This is another example of infinite torsion group, acting on X^* for $X = \{1, 2, 3\}$ and generated by the states of the following automaton:



The transformations a, t may also be defined recursively by

$$\begin{aligned} (1v)^a &= 2v, & (2v)^a &= 3v, & (3v)^a &= 1v, \\ (1v)^t &= 1v^a, & (2v)^t &= 2v^{a^{-1}}, & (3v)^t &= 3v^t. \end{aligned}$$

The corresponding result is

Theorem 2. The only Engel element in the Gupta-Sidki group Γ is the identity.

We only sketch the proof, since it follows that of Theorem 1 quite closely. Analogues of Propositions 1 and 2 hold, with $[\Gamma, \Gamma]$ in the rôle of K. An analogue of Lemma 1 holds with $A_0 = ([a^{-1}, t], [a, t]^a, [t^{-1}, a^{-1}])$ and $A_{4,i}@122 = A_{0,i}$.

6 Closing remarks

It would be dishonest to withhold from the reader how I arrived at the examples given for the Grigorchuk and Gupta-Sidki groups. I started by small words g, hin the generators of G_0 , respectively Γ , and computed $E_c(g, h)$ for the first few values of c. These elements are represented, internally to FR, as Mealy automata. A natural measure of the complexity of a group element is the size of the minimized automaton, which serves as a canonical representation of the element.

For some choices of g, h the size increases exponentially with c, limiting the practicality of computer experiments. For others (such as $(g, h) = ((ba)^4 c, ad)$ for the Grigorchuk group), the size increases roughly linearly with c, making calculations possible for c in the hundreds. Using these data, I guessed the period p of the recursion (9 in the case of the Grigorchuk group), and searched among the states of $E_c(g, h)$ and $E_{c+p}(g, h)$ for common elements; in the example, I found such common states for c = 23. I then took the smallest-size quadruple of states that appeared both in $E_c(g, h)$ and $E_{c+p}(g, h)$ in the form of Lemma 1.

It was already shown by Bludov [6] that the wreath product $G_0^4 \rtimes D_4$ is not Engel. He gave, in this manner, an example of a torsion group in which a product of Engel elements is not Engel. Our proof is a refinement of his argument.

A direct search for elements $A_{0,1}, \ldots, A_{0,4}$ would probably not be successful, and has not yielded simpler elements than those given before Lemma 1, if one restricts them to belong to K; one can only wonder how Bludov found the quadruple (1, d, ca, ab), presumably without the help of a computer.

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Bibliography

- Miklós Abért, Group laws and free subgroups in topological groups, Bull. London Math. Soc. 37 (2005), no. 4, 525-534, available at arXiv:math.GR/0306364. MR2143732
- [2] Ali Akhavi, Ines Klimann, Sylvain Lombardy, Jean Mairesse, and Matthieu Picantin, On the finiteness problem for automaton (semi)groups, Internat. J. Algebra Comput. 22 (2012), no. 6, 1250052, 26, DOI 10.1142/S021819671250052X. MR2974106
- [3] Tatiana Bandman, Fritz Grunewald, and Boris Kunyavskiĭ, Geometry and arithmetic of verbal dynamical systems on simple groups, Groups Geom. Dyn. 4 (2010), no. 4, 607–655, DOI 10.4171/GGD/98. With an appendix by Nathan Jones. MR2727656 (2011k:14020)
- [4] Laurent Bartholdi and Rostislav I. Grigorchuk, On parabolic subgroups and Hecke algebras of some fractal groups, Serdica Math. J. 28 (2002), no. 1, 47–90, available at arXiv:math/9911206. MR1899368 (2003c:20027)
- [5] Laurent Bartholdi, Rostislav I. Grigorchuk, and Zoran Šunik, Branch groups, Handbook of algebra, Vol. 3, North-Holland, Amsterdam, 2003, pp. 989– 1112, DOI 10.1016/S1570-7954(03)80078-5, (to appear in print), available at arXiv:math/0510294. MR2035113 (2005f:20046)
- [6] Vasily V. Bludov, An example of not Engel group generated by Engel elements, A Conference in Honor of Adalbert Bovdi's 70th Birthday, November 18–23, 2005, Debrecen, Hungary, 2005, pp. 7–8.
- [7] Anna G. Erschler, Iterated identities and iterational depth of groups (2014), available at arxiv:math/1409.5953.
- [8] The GAP Group, GAP Groups, Algorithms, and Programming, Version 4.4.10, 2008.
- [9] Ferenc Gécseg, Products of automata, EATCS Monographs on Theoretical Computer Science, vol. 7, Springer-Verlag, Berlin, 1986. MR88b:68139b
- [10] Ferenc Gécseg and Béla Csákány, Algebraic theory of automata, Akademiami Kiado, Budapest, 1971.
- [11] Pierre Gillibert, The finiteness problem for automaton semigroups is undecidable, Internat. J. Algebra Comput. 24 (2014), no. 1, 1–9, DOI 10.1142/S0218196714500015. MR3189662
- [12] Thibault Godin, Ines Klimann, and Matthieu Picantin, On torsion-free semigroups generated by invertible reversible Mealy automata, Language and automata theory and applications, Lecture Notes in Comput. Sci., vol. 8977, Springer, Cham, 2015, pp. 328–339, DOI 10.1007/978-3-319-15579-1_25, (to appear in print). MR3344813
- [13] Evgueniĭ S. Golod, Some problems of Burnside type, Proc. Internat. Congr. Math. (Moscow, 1966), Izdat. "Mir", Moscow, 1968, pp. 284–289 (Russian). MR0238880 (39 #240)
- [14] Rostislav I. Grigorchuk, On Burnside's problem on periodic groups, Функционал. Анал. и Приложен. 14 (1980), no. 1, 53–54. English translation: Functional Anal. Appl. 14 (1980), 41–43. MR81m:20045
- [15] _____, On the Milnor problem of group growth, Dokl. Akad. Nauk SSSR 271 (1983), no. 1, 30–33. MR85g:20042
- [16] Karl W. Gruenberg, The Engel elements of a soluble group, Illinois J. Math. 3 (1959), 151–168. MR0104730 (21 #3483)

- [17] Marcel Jackson, On locally finite varieties with undecidable equational theory, Algebra Universalis 47 (2002), no. 1, 1–6, DOI 10.1007/s00012-002-8169-0. MR1901727 (2003b:08002)
- [18] Ines Klimann, Jean Mairesse, and Matthieu Picantin, Implementing computations in automaton (semi)groups, Implementation and application of automata, Lecture Notes in Comput. Sci., vol. 7381, Springer, Heidelberg, 2012, pp. 240–252, DOI 10.1007/978-3-642-31606-7_21, (to appear in print). MR2993189
- [19] Ines Klimann, The finiteness of a group generated by a 2-letter invertible-reversible Mealy automaton is decidable, 30th International Symposium on Theoretical Aspects of Computer Science, LIPIcs. Leibniz Int. Proc. Inform., vol. 20, Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2013, pp. 502–513. MR3090008
- [20] Yurii G. Leonov, On identities in groups of automorphisms of trees, Visnyk of Kyiv State University of T.G.Shevchenko 3 (1997), 37–44.
- [21] Yuri Medvedev, On compact Engel groups, Israel J. Math. 135 (2003), 147–156, DOI 10.1007/BF02776054. MR1997040 (2004f:20072)