On the hardness of switching to a small number of edges

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Abstract. Seidel's switching is a graph operation which makes a given vertex adjacent to precisely those vertices to which it was non-adjacent before, while keeping the rest of the graph unchanged. Two graphs are called switching-equivalent if one can be made isomorphic to the other one by a sequence of switches.
Jelínková et al. [DMTCS 13, no. 2, 2011] presented a proof that it is NP-complete to decide if the input graph can be switched to contain at most a given number of edges. There turns out to be a flaw in their proof. We present a correct proof.
Furthermore, we prove that the problem remains NP-complete even when restricted to graphs whose density is bounded from above by an arbitrary fixed constant. This partially answers a question of Matoušek and Wagner [Discrete Comput. Geom. 52, no. 1, 2014].
Keywords: Seidel's switching, Computational complexity, Graph density, Switching-minimal graphs, NP-completeness

1 Introduction

Seidel's switching is a graph operation which makes a given vertex adjacent to precisely those vertices to which it was non-adjacent before, while keeping the rest of the graph unchanged. Two graphs are called switching-equivalent if one can be made isomorphic to the other one by a sequence of switches. The class of graphs that are pairwise switching-equivalent is called a switching class.

Hage in his PhD thesis [4, p. 115, Problem 8.5] posed the problem to characterize the graphs that have the maximum (or minimum) number of edges in their switching class. We call such graphs *switching-maximal* and *switching-minimal*, respectively.

Some properties of switching-maximal graphs were studied by Kozerenko [7]. He proved that any graph with sufficiently large minimum degree is switching-maximal, and that the join of certain graphs is switching-maximal. Further, he

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gave a characterization of triangle-free switching-maximal graphs and of non-hamiltonian switching-maximal graphs.

It is easy to observe that a graph is switching-maximal if and only if its complement is switching-minimal. We call the problem to decide if a graph is switching-minimal SWITCH-MINIMAL.

Jelínková et al. [6] studied the more general problem SWITCH-FEW-EDGES – the problem of deciding if a graph can be switched to contain at most a certain number of edges. They presented a proof that the problem is NP-complete. Unfortunately, their proof is not correct. Specifically, Lemma 4.3 of [6], which claims to establish a reduction from the classical MAX-CUT problem to SWITCH-FEW-EDGES, is false. The claim of the lemma fails, e. g., on a graph G formed by two disjoint cliques of the same size.

In this paper, we provide a different proof of the NP-hardness of SWITCH-FEW-EDGES, based on a reduction from a restricted version of MAX-CUT. Furthermore, we strengthen this result by proving that for any $c > 0$, SWITCH-FEW-EDGES is NP-complete even if we require that the input graph has density at most c. We also prove that if the problem SWITCH-MINIMAL is co-NP-complete, then for any $c > 0$, the problem is co-NP-complete even on graphs with density at most c.

We thus partially answer a question of Matoušek and Wagner [10] posed in connection with properties of simplicial complexes – they asked if deciding switching-minimality was easy for graphs of bounded density. Our results also indicate that it might be unlikely to get an easy characterization of switching-minimal (or switching-maximal) graphs, which contributes to understanding Hage's question [4].

1.1 Formal definitions and previous results

Let G be a graph. Then the *Seidel's switch of a vertex subset* $A \subseteq V(G)$ is denoted by $S(G, A)$ and is defined by

$$S(G, A) = (V(G), E(G) \triangle \{xy : x \in A,\ y \in V(G) \setminus A\}).$$

It is the graph obtained from G by consecutive switching of the vertices of A.

We say that two graphs G and H are *switching-equivalent* (denoted by $G \sim H$) if there is a set $A \subseteq V(G)$ such that $S(G, A)$ is isomorphic to H. The set $[G] = \{S(G, A) : A \subseteq V(G)\}$ is called the *switching class* of G.

We say that a graph G is $(\leq k)$*-switchable* if there is a set $A \subseteq V(G)$ such that $S(G, A)$ contains at most k edges. Analogously, a graph G is $(\geq k)$*-switchable* if there is a set $A \subseteq V(G)$ such that $S(G, A)$ contains at least k edges.

It is easy to observe that a graph G is $(\leq k)$-switchable if and only if the complement $\overline{G}$ is $\left(\geq \binom{n}{2} - k\right)$-switchable. We may, therefore, focus on $(\leq k)$-switchability only.

We examine the following problems.

SWITCH-FEW-EDGES **Input:** A graph $G = (V, E)$, an integer k **Question:** Is G $(\leq k)$-switchable?

SWITCH-MINIMAL **Input:** A graph $G = (V, E)$ **Question:** Is G switching-minimal?

We say that a graph is *switching-reducible* if G is *not* switching-minimal, in other words, if there is a set $A \subseteq V(G)$ such that $S(G, A)$ contains fewer edges than G. For further convenience, we also define the problem SWITCH-REDUCIBLE.

SWITCH-REDUCIBLE **Input:** A graph $G = (V, E)$ **Question:** Is G switching-reducible?

Let $G = (V, E)$ be a graph. We say that a partition V_1, V_2 of V is a *cut* of G. For a cut V_1, V_2, the set of edges that have exactly one end-vertex in V_1 is denoted by $\text{cutset}(V_1)$, and the edges of $\text{cutset}(V_1)$ are called *cut-edges*. When there is no danger of confusion, we also say that a single subset $V_1 \subseteq V$ is a cut (meaning the partition V_1, $V \setminus V_1$).

1.2 Easy cases

In this subsection we present several results about easy special cases of the problems that we focus on. This complements our hardness results.

The following theorem was proved by Ehrenfeucht et al. [2] and also independently (in a slightly weaker form) by Kratochvíl [8].

Theorem 1. *Let $\mathcal{P}$ be a graph property that can be decided in time $\mathcal{O}(n^a)$ for an integer a. Let every graph with $\mathcal{P}$ contain a vertex of degree at most $d(n)$. Then the problem if an input graph is switching-equivalent to a graph with $\mathcal{P}$ can be decided in time $\mathcal{O}(n^{d(n)+1+\max(a,2)})$.*

The proof of Theorem 1 also gives an algorithm that works in the given time. Hence, it also provides an algorithm for SWITCH-FEW-EDGES: in a graph with at most k edges all vertex degrees are bounded by k. Hence, we can use $d(n) = k$ and $a = 2$ and get an $\mathcal{O}(n^{k+3})$-time algorithm. It was further proved by Jelínková et al. [6] that SWITCH-FEW-EDGES is fixed-parameter tractable; it has a kernel with $2k$ vertices, and there is an algorithm running in time $\mathcal{O}(2.148^k \cdot n + m)$, where m is the number of edges of the input graph. In Section 2, we provide a corrected NP-completeness proof.

The following proposition states a basic relation of switching-minimality and graph degrees.

Proposition 1 (folklore). *Every switching-minimal graph $G = (V, E)$ on n vertices has maximum degree at most $\lfloor (n-1)/2 \rfloor$.*

Proof. Clearly, if G contains a vertex v of degree greater than $\lfloor (n-1)/2 \rfloor$, then $S(G, \{v\})$ has fewer edges than G, showing that G is not switching-minimal. □

We remark that for a given graph G we can efficiently construct a switch whose maximum degree is at most $\lfloor (n-1)/2 \rfloor$; one by one, we switch vertices whose degree exceeds this bound (in this way, the number of edges is decreased in each step). However, the graph constructed by this procedure is not necessarily switching-minimal.

Let $e(A)$ denote the number of edges whose one vertex is in A and the other one in $V(G) \setminus A$. The next proposition is an equivalent formulation of Lemma 2.5 of Kozerenko [7], strengthening Proposition 1.

Proposition 2. *A graph G is switching-minimal if and only if for every $A \subseteq V(G)$, we have*

$$2e(A) \leq |A|(|V(G)| - |A|).$$

We derive the following consequence.

Proposition 3. *Let G be a graph with n vertices. If the maximum vertex degree in G is at most $\frac{n}{4}$, then G is switching-minimal.*

Proof. Let A be any subset of $V(G)$. We observe that $e(A) = e(V(G) \setminus A)$; hence we can assume without loss of generality that $|A| \leq n/2$, and thus $|V(G)| - |A| \geq n/2$.

Further, as $e(A) \leq \sum_{v \in A} \deg(v)$, we have that $e(A) \leq |A| \frac{n}{4}$. Hence, $2e(A) \leq |A|(|V(G)| - |A|)$, and the condition of Proposition 2 is fulfilled. □

Proposition 3 implies that Switch-Few-Edges and Switch-Minimal are trivially solvable in polynomial time for graphs on n vertices with maximum degree at most $\frac{n}{4}$.

We note that in Proposition 3, the bound $\frac{n}{4}$ in general cannot be improved, as shown by the example of a k-regular bipartite graph on n vertices with $k > \frac{n}{4}$. Such a graph is switching-equivalent to a $(\frac{n}{2} - k)$-regular bipartite graph, and therefore is not switching-minimal.

2 NP-Completeness of Switch-Few-Edges

Jelínková et al. [6] presented a proof that the problem Switch-Few-Edges is NP-complete. Unfortunately, there is an error in their proof. We present another proof here. The core of the original proof is a reduction from the Max-Cut problem. Our reduction works in a similar way. However, we need the following more special version of Max-Cut (we prove the NP-completeness of Large-Deg-Max-Cut in Section 3).

Large-Deg-Max-Cut
Input: A graph G with $2n$ vertices such that the minimum vertex degree of G is $2n-4$ and the complement of G does not contain triangles; an integer j
Question: Does there exist a cut V_1 of $V(G)$ with at least j cut-edges?

Proposition 4. *Let G be a graph. In polynomial time, we can find a graph G' such that $|V(G')| = 4|V(G)|$ and the following statements are equivalent for every integer j:*

(a) There is a cut in G with at least j cut-edges,
(b) there exists a set $A \subseteq V(G')$ such that $S(G', A)$ contains at most $|E(G')| - 16j$ edges.

Proof. We first describe the construction of the graph G'. For each vertex u of G we create a corresponding four-tuple $\{u', u'', u''', u''''\}$ of pairwise non-adjacent vertices in G'. An edge of G is then represented by a complete bipartite graph interconnecting the two four-tuples, and a non-edge in G is represented by 8 edges that form a cycle that alternates between the two four-tuples (see Fig. 1).

We remark that our construction of G' follows a similar idea as the construction in the attempted proof of Jelínková et al. [6], a notable difference being that in the original construction, a vertex of G was replaced by a pair of vertices of G' rather than a four-tuple.

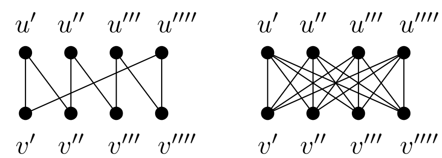

Fig. 1. The representation of non-edges and edges of G.

A vertex four-tuple in G' corresponding to a vertex of G is called an *o-vertex*. A pair of o-vertices corresponding to an edge of G is called an *o-edge* and a pair of o-vertices corresponding to an non-edge of G is called an *o-non-edge*. Where there is no danger of confusion, we identify o-vertices with vertices of G, o-edges with edges of G and o-non-edges with non-edges of G.

We now prove that the statements (a) and (b) are equivalent. First assume that there is a cut V_1 of $V(G)$ with j' cut-edges. Let V_1' be the set of vertices u', u'', u''', u'''' for all $u \in V_1$. We prove that $S(G', V_1')$ contains at most $|E(G')| - 16j'$ edges.

We say that a non-edge *crosses the cut* V_1 if the non-edge has exactly one vertex in V_1. It is clear that G' contains 16 edges per every o-edge and 8 edges per every o-non-edge. In $S(G', V_1')$, every o-edge corresponding to an edge that is not a cut-edge is unchanged by the switch and yields 16 edges. Similarly, every o-non-edge corresponding to a non-edge that does not cross the cut yields 8 edges.

Fig. 2 illustrates the switches of o-non-edges and o-edges that have exactly one end-o-vertex in V_1. We can see that every o-non-edge corresponding to a non-edge that crosses the cut yields 8 edges in $S(G', V_1')$, and that every o-edge corresponding to a cut-edge yields 0 edges. Altogether, $S(G', V_1')$ has $|E(G')| - 16j'$ edges, which we wanted to prove.

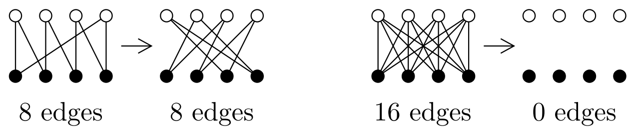

Fig. 2. Switches of an o-non-edge and of an o-edge.

Now assume that there exists a set $A \subseteq V(G')$ such that $S(G', A)$ contains at most $|E(G')| - 16j$ edges. We want to find a cut in G with at least j cut-edges.

We say that an o-vertex u of G' is *broken in* A if A contains exactly one, two or three vertices out of u', u'', u''', u''''; otherwise, we say that u is *legal in* A. We say that an o-edge or o-non-edge $\{u, v\}$ is *broken in* A if at least one of the o-vertices u, v is broken. Otherwise, we say that $\{u, v\}$ is *legal in* A.

If all vertices of G are legal in A, we say that A is *legal.* Legality is a desired property, because for a legal set A we can define a subset V_A of $V(G)$ such that

$$V_A = \{u \in V(G) : \{u', u'', u''', u''''\} \subseteq A\} .$$

The set V_A then defines a cut in G. If a set is not legal, we proceed more carefully to get a cut from it. For any vertex subset A, we say that a set A' is a *legalization* of A if A' is legal and if A' and A differ only on o-vertices that are broken in A.

We want to show that for every illegal set A, there exists its legalization A' such that the number of edges in $S(G', A')$ is not much higher than in $S(G', A)$. To this end, we give the Algorithm Legalize which for a set A finds such a legalization A'. During the run of the Algorithm, we keep a set A''. In the beginning we set $A'' := A$ and in each step we change A'' so that more o-vertices are legal.

We define some notions needed in the Algorithm. Let v be an o-vertex and consider the o-vertices that are adjacent to v (through an o-edge); we call them *o-neighbors* of v. The o-neighbors of v are four-tuples of vertices and some of those vertices are in A'', some of them are not. We define $\mathrm{dif}(v)$ as the number of such vertices that are in A'' minus the number of such vertices that are not in A''. (Note that $\mathrm{dif}(v)$ is always an even number, because the total number of vertices in o-neighbors is even. If all o-neighbors were legal, then $\mathrm{dif}(v)$ would be divisible by four.)

The Algorithm is given in Fig. 3. As in the last step the Algorithm legalizes all remaining broken o-vertices, it is clear that the set A'' output by the Algorithm is a legalization of A. We prove that $|E(S(G', A''))| - |E(S(G', A))| \leq 7$.

We need to introduce more terminology. A pair of vertices of G' which belong to the same o-vertex is called a *v-pair*. A pair of vertices of G' which belong to different o-vertices that are adjacent (in G) is called an *e-pair*. A pair of vertices of G' which belong to different o-vertices that are non-adjacent (in G) is called an *n-pair*. It is easy to see that any edge of G' or $S(G', A'')$ is either a v-pair, an e-pair or an n-pair. We call such edges *v-edges*, *e-edges* and *n-edges*, respectively.

Algorithm Legalize(A)
Set $A'' := A$; do the following while any of the cases applies.

Case 1. There exists a broken o-vertex v such that $|\mathrm{dif}(v)| \geq 4$. If $\mathrm{dif}(v) \geq 4$, set $A'' := A'' \setminus \{v', v'', v''', v''''\}$. Otherwise, set $A'' := A'' \cup \{v', v'', v''', v''''\}$.
Case 2. Case 1 does not apply and there exists an asymmetric broken vertex v such that v contains exactly one vertex from A'' and $\mathrm{dif}(v) = 2$. Set $A'' := A'' \setminus \{v', v'', v''', v''''\}$.
Case 3. Case 1 does not apply and there exists an asymmetric broken vertex v such that v contains exactly three vertices from A'' and $\mathrm{dif}(v) = -2$. Set $A'' := A'' \cup \{v', v'', v''', v''''\}$.
Case 4. None of Cases 1, 2, 3 applies and there exist two adjacent broken o-vertices u and v. Set

$$A_1 := (A'' \cup \{u', u'', u''', u''''\}) \setminus \{v', v'', v''', v''''\},$$

$$A_2 := (A'' \cup \{v', v'', v''', v''''\}) \setminus \{u', u'', u''', u''''\}.$$

If $|E(S(G', A_1))| < |E(S(G', A_2))|$ then set $A'' := A_1$, otherwise $A'' := A_2$.
Case 5. None of the above cases applies. Then legalize the remaining broken o-vertices arbitrarily, output A'' and STOP.

Fig. 3. The Algorithm Legalize.

We say that a broken o-vertex v is *asymmetric* if it contains an odd number of vertices of A''; we say that a broken o-vertex is *symmetric* if it contains two vertices out of A''.

To measure how the number of edges of $S(G', A'')$ changes during the run of the Algorithm, we define a variable $c(A'')$ which we call the *charge* of the graph $S(G', A'')$. Before the first step we set $c(A'') := |E(S(G', A))|$. After a step of the Algorithm, we update $c(A'')$ in the following way.

- For every v-pair or e-pair that was an edge of $S(G', A'')$ before the step and is no longer an edge of $S(G', A'')$ after the step, we decrease $c(A'')$ by one.
- For every v-pair or e-pair that was not an edge of $S(G', A'')$ before the step and that has become an edge of $S(G', A'')$ after the step, we increase $c(A'')$ by one.
- For every o-vertex v that was legalized in the step and is incident to an o-non-edge, we change $c(A'')$ in the following way:
  - if v is symmetric, we increase $c(A'')$ by 2.5 for every o-non-edge incident to v;
  - if v is asymmetric, we increase $c(A'')$ by 1.5 for every o-non-edge incident to v.

To explain the last two points, we observe how the number of n-edges increases after legalizing an o-vertex. By analyzing all cases of o-non-edges with one or two broken end-o-vertices (see Fig. 4), we get that there are four cases where the o-non-edges have less than 8 n-edges before legalization: either 6 or

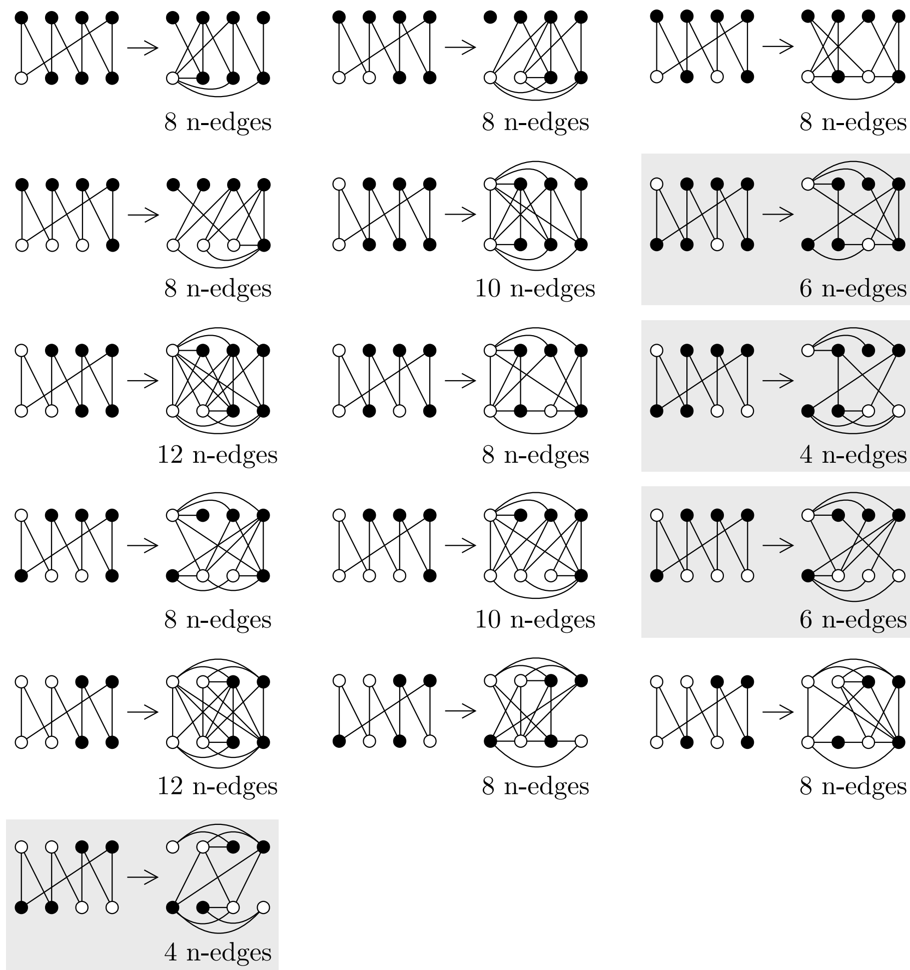

Fig. 4. All possible illegal switches of o-non-edges (up to symmetry). Vertices of A are marked in white and edges are as in G (left to the arrow) and as in $S(G', A)$ (right to the arrow). In the highlighted cases, the number of n-edges in $S(G', A)$ is lower than 8.

4 n-edges. In these cases, both end-o-vertices are broken. If there are only 4 n-edges, at least one of the end-o-vertices is symmetric. After one end-o-vertex is legalized, the number of n-edges increases by 2 or 4. When the second end-o-vertex is legalized, the number of n-edges does not increase for this particular o-non-edge.

After both end-o-vertices are legalized, the charge has been changed in the following way: if both end-o-vertices were symmetric, we have increased the charge by 5. If one of them was symmetric and the other one was asymmetric, we have increased the charge by 4. Finally, if both were asymmetric, we have increased the charge by 3. In all these cases, the increase is an upper bound on the number of contributed n-edges.

Further, every v-edge or e-edge that has appeared or disappeared during the run of the Algorithm is counted immediately after the corresponding step. Hence, we have proved the following Claim.

Claim 1 *At the end of the Algorithm we have that* $c(A'') \geq |E(S(G', A''))|$.

Next, we give an upper bound on the charge $c(A'')$.

Claim 2 *After every step of the Algorithm except for the last one, the charge* $c(A'')$ *is decreased. After the last step, the charge is increased by at most 7. Hence,* $c(A'') \leq |E(S(G', A))| + 7$.

To prove Claim 2, we count how the charge changes after each step of the Algorithm Legalize. We distinguish cases according to which the step was done.

Case 1. We may assume without loss of generality that $\mathrm{dif}(v) \geq 4$ (otherwise we swap the roles of A'' and $V(G') \setminus A''$. Further, v can be either symmetric or asymmetric; we first assume that v is symmetric (see Fig. 5). Then by its legalization the number of v-edges is decreased by 4.
As $\mathrm{dif}(v) \geq 4$, then among vertices in o-neighbors of v, there must be at least four more vertices belonging to A'' than those not belonging to A''. Thus, by removing any vertex of $\{v', v'', v''', v''''\}$ from A'' we reduce the number of e-edges by at least 4. As v contains two vertices out of $\{v', v'', v''', v''''\}$, we reduce the number of e-edges by at least 8.
For n-pairs that have one vertex inside v the charge is increased by at most $3 \cdot 2.5$, which is 7.5. To sum it up:
- For v-pairs the charge is decreased by 4,
- for e-pairs the charge is decreased by at least 8,
- for n-pairs the the charge is increased by at most 7.5.

Altogether, the charge is decreased by at least 4.5.
If the o-vertex v is asymmetric, then in an analogical way we have that for v-pairs the charge is decreased by 3, for e-pairs the charge is decreased by at least 4, and for n-pairs the the charge is increased by at most 4.5. Altogether, the charge is decreased by at least 2.5.

Case 2. The analysis is similar as above. We get -3 for v-pairs, -2 for e-pairs, and ≤ 4.5 for n-pairs. Altogether, the charge is decreased by at least 0.5.

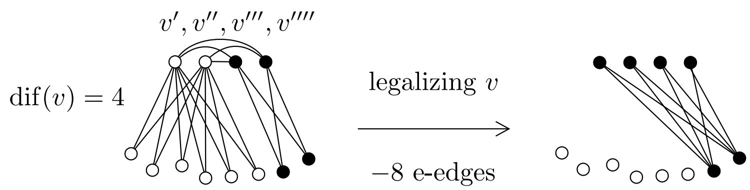

Fig. 5. A simplified illustration to the analysis of Case 1. Vertices of A'' are marked in white, and edges are as in $S(G', A'')$ before the step (left to the arrow) and after the step (right to the arrow).

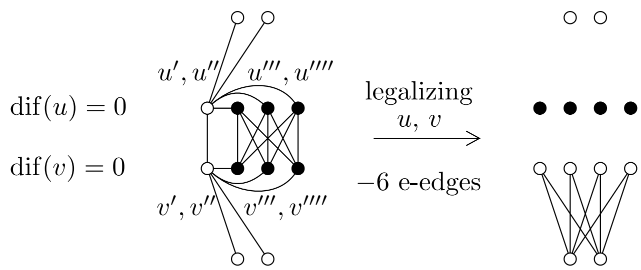

Fig. 6. A simplified illustration to the analysis of Case 4, I. Vertices of A'' are marked in white, and edges are as in $S(G', A'')$ before the step (left to the arrow) and after the step (right to the arrow).

Case 3. This case is symmetric to Case 2. Hence, the charge is decreased by at least 0.5 as well.

Case 4. In this case, when counting how the charge was changed because of e-pairs, we need to bound both the number of e-edges between a vertex in u and a vertex in v, and the number of e-edges between a vertex inside u or v and a vertex inside one of their other o-neighbors. This depends also on the values of $\mathrm{dif}(u)$ and $\mathrm{dif}(v)$.
We analyze four subcases of o-edges whose both end-o-vertices are broken – they are numbered as in Fig. 7.

I. First assume that $\mathrm{dif}(u) = 0$ and $\mathrm{dif}(v) = 0$ (see Fig. 6). We can see that vertices inside v contribute by -2 to $\mathrm{dif}(u)$. Hence, outside v, there must be two more vertices in o-neighbors of u that are in A'' than those not in A''. The same holds symmetrically for o-neighbors of v outside u.
We may without loss of generality assume that the Algorithm chose to set

$$A'' := (A'' \cup \{v', v'', v''', v''''\}) \setminus \{u', u'', u''', u''''\}.$$

Then, the number of e-pairs adjacent to both u and v is decreased by 10; the number of e-pairs adjacent to u and not to v is decreased by 2, and the number of e-pairs adjacent to v and not to u is

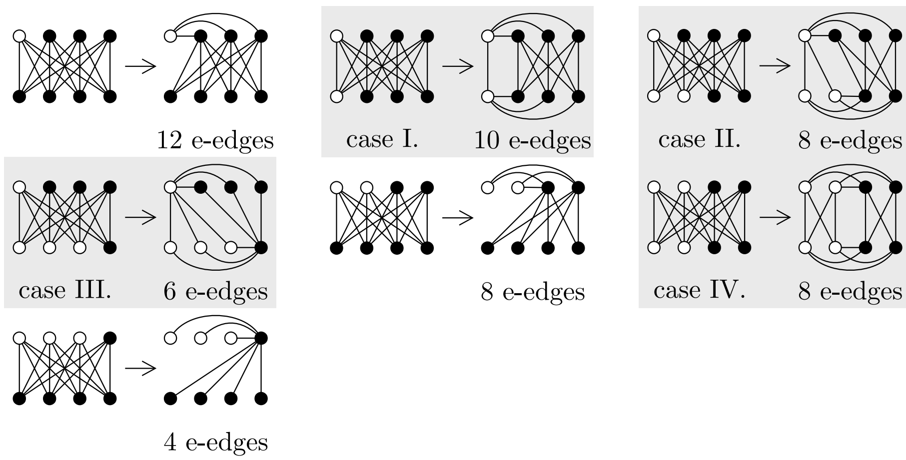

Fig. 7. All possible illegal switches of o-edges (up to symmetry). Vertices of A are marked in white and edges are as in G (left to the arrow) and as in $S(G', A)$ (right to the arrow). In the highlighted cases, both end-o-vertices are broken.

increased by 6. Altogether, the charge is decreased by 6 for e-pairs.

For v-pairs, the charge is decreased by 6, and for n-pairs, the charge is increased by at most $6 \cdot 1.5$. Altogether, the charge is decreased by at least 3.

It remains to analyze the cases when $\mathrm{dif}(u)$ and $\mathrm{dif}(v)$ are different. As neither Case 2 nor Case 3 applies, we know that none of $\mathrm{dif}(u)$, $\mathrm{dif}(v)$ is equal to 2.

By analogical ideas as above, we get that if one of $\mathrm{dif}(u)$, $\mathrm{dif}(v)$ is equal to 0 and the other one to -2, the charge is decreased by at least 9. If both $\mathrm{dif}(u)$, $\mathrm{dif}(v)$ are equal to -2, then the charge is decreased by at least 7.

II. As u is asymmetric and v is symmetric, we have that for n-pairs the charge is increased by $3 \cdot 2.5 + 3 \cdot 1.5$, which is 12. For v-pairs, the charge is decreased by $3 + 4$, which is 7.

We consider the case when the Algorithm chose to set

$$A'' := A'' \cup \{u', u'', u''', u''''\} \setminus \{v', v'', v''', v''''\}$$

(if we get a sufficient bound for this case, then the other case could only be better).

To count the decrease for e-pairs, we need to consider the values of $\mathrm{dif}(u)$ and $\mathrm{dif}(v)$. Assume that $\mathrm{dif}(u) = 0$. Then, outside v, there must be the same number of vertices in o-neighbors of u that are in A'' as those that are not in A''.

If $\mathrm{dif}(v) = 0$, then outside u there must be two more vertices in o-neighbors of v that are in A'' than those not in A''. Then for

e-pairs, the charge is decreased by 12. If $\mathrm{dif}(v) = 2$, then using analogous ideas we get that for e-pairs, the charge is decreased by 16. If $\mathrm{dif}(v) = -2$, we get 8.
Now assume that $\mathrm{dif}(u) = -2$. By considering the number of vertices in o-neighbors of u and v, we get that the charge decrease for e-pairs is either 14 (if $\mathrm{dif}(v) = -2$) or 18 (if $\mathrm{dif}(v) = 0$) or 22 (if $\mathrm{dif}(v) = 2$).
As Case 2 does not apply, we know that $\mathrm{dif}(u)$ is not equal to 2. Hence, we have considered all the cases, and the charge decrease for e-pairs is at least 8. Altogether, the charge is decreased by at least $-12 + 7 + 8$, which is 3.

III. As both u and v are asymmetric, we have that for n-pairs the charge is increased by $6 \cdot 1.5$, which is 9. For v-pairs, the charge is decreased by $3 + 3$, which is 6.
Again, we consider the case when the Algorithm chose to set

$$A'' := A'' \cup \{u', u'', u''', u''''\} \setminus \{v', v'', v''', v''''\}.$$

By using the same idea as above, we get that for e-pairs, the charge is decreased by 18 (if $\mathrm{dif}(u) = 0$ and $\mathrm{dif}(v) = 0$), or by 24 (if $\mathrm{dif}(u) = -2$ and $\mathrm{dif}(v) = 0$, or if $\mathrm{dif}(u) = 0$ and $\mathrm{dif}(v) = 2$), or by 30 (if $\mathrm{dif}(u) = -2$ and $\mathrm{dif}(v) = 2$).
As Case 2 does not apply, we know that $\mathrm{dif}(u)$ cannot be 2 and $\mathrm{dif}(v)$ cannot be -2. Hence, we have considered all the cases and for e-pairs, the charge is decreased by at least 18. Altogether, the charge is decreased by at least $-9 + 6 + 18$, which is 15.

IV. As both u and v are symmetric, we have that for n-pairs the charge is increased by $6 \cdot 2.5$, which is 15. For v-pairs, the charge is decreased by $4 + 4$, which is 8.
Without loss of generality, we consider only cases when $\mathrm{dif}(u) \leq \mathrm{dif}(v)$ (the other cases are symmetric). Thus, we may limit ourselves again to the case when the Algorithm chose to set

$$A'' := A'' \cup \{u', u'', u''', u''''\} \setminus \{v', v'', v''', v''''\}.$$

If $\mathrm{dif}(u) = \mathrm{dif}(v)$, then we easily check that the charge decrease for e-pairs is 8. If $\mathrm{dif}(u) = 0$ and $\mathrm{dif}(v) = 2$ then the charge decrease for e-pairs is 12. If $\mathrm{dif}(u) = -2$ and $\mathrm{dif}(v) = 0$ then the decrease is 12, and if $\mathrm{dif}(u) = -2$ and $\mathrm{dif}(v) = 2$ then the decrease is 16.
Altogether, the charge decrease for e-pairs is at least 8, and the total decrease is at least $-15 + 8 + 8$, which is 1.

Case 5. If Case 5 applies, then all remaining broken o-vertices must be pairwise non-adjacent (because Case 4 does not apply). Hence, there must be at most two broken o-vertices left (otherwise, there would be a triangle in the complement of the input graph, which would contradict the assumptions). Further, each of these o-vertices has $\mathrm{dif} = 0$, because all

its o-neighbors are legal and Case 1 does not apply. Thus, the charge change for e-pairs due to this last step is 0.
To count the charge change for n-pairs and v-pairs, we analyze the five cases (one or two o-vertices, symmetric or asymmetric). If there is one symmetric o-vertex left, then the charge increase for n-pairs is $3 \cdot 2.5$ and the decrease for v-pairs is 4, hence the total increase is $7.5 - 4$, which is 3.5. If there is one asymmetric o-vertex, then the total increase is $3 \cdot 1.5 - 3$, which is 1.5.
If there are two broken o-vertices left and both are asymmetric, then the total increase is $6 \cdot 1.5 - 6$, which is 3. If one of them is symmetric and the other one is asymmetric we get 5; if both are symmetric, we get 7. Altogether, we get that the charge is increased by at most 7.

We have proved Claim 2. Further, by Claim 1 and Claim 2 we have that $|E(S(G', A''))| \leq |E(S(G', A))| + 7$, and hence A'' is the sought legalization of A.

We continue the proof of Proposition 4. We have already argued that a legal set A'' defines a subset $V_{A''}$ of $V(G)$, and hence a cut in G. Assume that $\mathrm{cutset}(V_{A''})$ has j' edges. From the proof of the first implication of Proposition 4 we know that the number of edges in $S(G', A'')$ can be expressed as $|E(G')| - 16j'$.

On the other hand, we have proved that the number of edges in $S(G', A'')$ is at most $|E(G')| - 16j + 7$. We get that $|E(G')| - 16j' \leq |E(G')| - 16j + 7$, and hence $j' \geq j - 7/16$. As both j and j' are integers, we have that $j' \geq j$. Hence, $\mathrm{cutset}(V_{A''})$ has at least j edges, and Proposition 4 is proved. □

Theorem 2. Switch-Few-Edges *is NP-complete.*

Proof. Theorem 3 in the next section gives the NP-completeness of Large-Deg-Max-Cut. Further, by Proposition 4, an instance (G, j) of Large-Deg-Max-Cut can be transformed into an instance (G', j') of Switch-Few-Edges such that there is a cut in G with at least j cut-edges if and only if G' is $(\leq j')$-switchable. The transformation works in polynomial time.

Finally, it is clear that the problem Switch-Few-Edges is in NP. □

3 The NP-Completeness of Large-Deg-Max-Cut

Let G be a graph with $2n$ vertices. A *bisection of* G is a partition S_1, S_2 of $V(G)$ such that $|S_1| = |S_2| = n$ (hence, a bisection is a special case of a cut). The size of $\mathrm{cutset}(S_1)$ is called the *size* of the bisection S_1, S_2. A *minimum bisection of* G is a bisection of G with minimum size.

Garey et al. [3] proved that, given a graph G and an integer b, the problem to decide if G has a bisection of size at most b is NP-complete (by a reduction of Max-Cut). Their formulation is slightly different from ours – two distinguished vertices must be each in one part of the partition, and the input graph does not have to be connected. However, their reduction from Max-Cut (see [3, pages 242–243]) produces only connected graphs as instances of the bisection problem, and it is immediate that the two distinguished vertices are not important in the proof. Hence, their proof gives also the NP-completeness of the following version of the problem.

> CONNECTED-MIN-BISECTION
> **Input:** A connected graph G with $2n$ vertices, an integer b
> **Question:** Is there a bisection S_1, S_2 of $V(G)$ such that cutset(S_1) contains at most b edges?

From the NP-completeness of MIN-BISECTION, Bui et al. [1] proved the NP-completeness of MIN-BISECTION restricted to 3-regular graphs (as a part of a more general result, see [1, proof of Theorem 2]). We use their result to prove the NP-completeness of LARGE-DEG-MAX-CUT.

> LARGE-DEG-MAX-CUT
> **Input:** A graph G with $2n$ vertices such that the minimum vertex degree of G is $2n-4$ and the complement of G is connected and does not contain triangles; an integer j
> **Question:** Does there exist a cut V_1 of G with at least j cut-edges?

Lemma 1. *Let G be a connected 3-regular graph on $2n$ vertices. Let b be the size of the minimum bisection in G and let c be the size of the maximum cut in $\overline{G}$. Then $b = n^2 - c$.*

Proof. Let S_1, S_2 be a minimum bisection in G and let b be the size of the bisection. In $\overline{G}$, the partition S_1, S_2 yields a cut with $n^2 - b$ cut-edges. Hence, $c \geq n^2 - b$.

On the other hand, let V_1, V_2 be a maximum cut in $\overline{G}$ for which the sizes of V_1 and V_2 are as close as possible. If $|V_1| = |V_2| = n$, the partition $|V_1|$, $|V_2|$ gives a bisection in G of size $n^2 - c$, hence $b \leq n^2 - c$ and we are done.

Otherwise, assume that $|V_1| = n - k$ and $|V_2| = n + k$ for a $k \geq 1$. As the graph G is connected, there is a vertex v in V_2 that has at least one neighbor in V_1. We set $V_1' = V_1 \cup \{v\}$ and $V_2' = V_2 \setminus \{v\}$.

The vertex v has at least one neighbor in V_1. Hence, in G, there is at least one edge between v and V_1, and in $\overline{G}$, there are at most $n - k - 1$ cut-edges adjacent to v.

Further, v has at most two neighbors in V_2 and at least $n+k-3$ non-neighbors in V_2. Hence, in the partition V_1', V_2', there will be at least $n + k - 3$ cut-edges adjacent to v. Cut edges that are not adjacent to v are the same in V_1, V_2 as in V_1', V_2'.

Altogether, $|\text{cutset}(V_1')| - |\text{cutset}(V_1)| \geq n + k - 3 - (n - k - 1) \geq 2k - 2 \geq 0$. Hence, the partition V_1', V_2' has smaller difference of the sizes of the two parts while the size of the cut is not smaller, which is a contradiction with the choice of V_1, V_2. □

Theorem 3. LARGE-DEG-MAX-CUT *is NP-complete.*

Proof. Let (G, b) be an instance of Connected-Min-Bisection. We use the construction of Bui et al. [1, proof of Theorem 2]. Their first step is to construct from an instance (G, b) of Min-Bisection a 3-regular graph G^* such that G has a minimum bisection of size b if and only if G^* has a minimum bisection of size b. Further, it is immediate from their construction that G^* contains no triangles, and if G is connected, then G^* is connected as well. Moreover, G^* has an even number of vertices.

We see that $\overline{G^*}$ fulfills the conditions of an instance of Large-Deg-Max-Cut. By Lemma 1 we know that G^* has a minimum bisection of size b if and only if $\overline{G^*}$ has a maximum cut of size $m^2 - b$.

Altogether, G has a minimum bisection of size b if and only if $\overline{G^*}$ has a maximum cut of size $m^2 - b$. Hence, $(\overline{G^*}, m^2 - b)$ is an equivalent instance of Large-Deg-Max-Cut. To finish the proof that Large-Deg-Max-Cut is NP-complete, we observe that Large-Deg-Max-Cut is in NP. ◻

4 Switching of Graphs with Bounded Density

The *density* of a graph G is defined as

$$D(G) = \frac{|E(G)|}{\binom{|V(G)|}{2}} = \frac{2|E(G)|}{|V(G)|(|V(G)| - 1)}.$$

In connection with properties of simplicial complexes, Matoušek and Wagner [10] asked if deciding switching-minimality was easy for graphs of bounded density. We give a partial negative answer by proving that the problem Switch-Few-Edges stays NP-complete even for graphs of density bounded by an arbitrarily small constant. This is in contrast with Proposition 3, which shows that any graph G with maximum degree at most $|V(G)|/4$ is switching-minimal. The core of our argument is the following Proposition.

Proposition 5. *Let G be a graph, let k be an integer, and let c be a fixed constant in $(0, 1)$. In polynomial time, we can find a graph G' and an integer k' such that*

1. *$D(G') \leq c$,*
2. *G' is $(\leq k')$-switchable if and only if G is $(\leq k)$-switchable,*
3. *G' is switching-minimal if and only if G is switching-minimal, and*
4. *$|V(G')| = O(|V(G)|)$.*

Proof. Let $n = |V(G)|$ and let $N = \max\left\{n, \left\lceil \frac{3n}{4c} \right\rceil\right\}$. We construct the graph G' in the following way (see also Fig. 8). Let $V = V(G)$. Then

$$V(G') = V \cup Y \cup Z,$$

where Y is a set of N vertices and Z is a set of N more vertices, and

$$E(G') = \{\{v_1, v_2\} \colon v_1 \in Y, v_2 \in V\} \cup E(G).$$

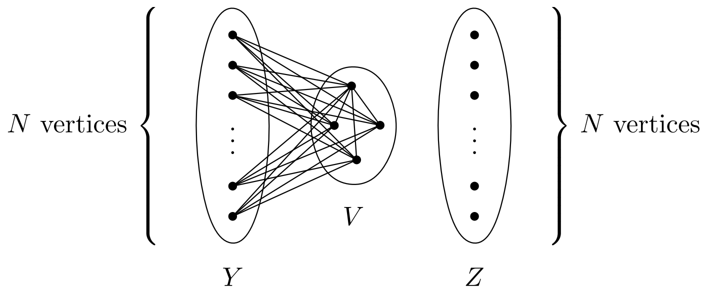

Fig. 8. The graph G'.

We prove that G' fulfills the conditions of Proposition 5. It is easy to see that Condition 4 holds and that G' can be obtained in polynomial time. We prove that Conditions 2 and 3 hold, too.

Assume that G is switching-reducible, i. e., there exists a set $A \subseteq V$ such that $S(G, A)$ contains fewer edges than G. Let us count the number of edges in $S(G', A)$.

It is easy to see that if we switch a subset of V in G', the number of edges whose one endpoint is outside V is unchanged, and the number of edges with both endpoints outside V remains zero. We also observe that $S(G', A)[V]$ (the induced subgraph of $S(G', A)$ on the vertex subset V) is equal to $S(G, A)$. Hence, $S(G', A)$ has fewer edges than G', showing that G' is switching-reducible.

Moreover, if $S(G, A)$ has l edges for an integer l, then $S(G', A)$ has $l + nN$ edges. Thus, if G is $(\leq k)$-switchable, we have that G' is $(\leq k+nN)$-switchable.

Now assume that G' is switching-reducible, i. e., there exists a set $A \subseteq V(G')$ such that $S(G', A)$ has fewer edges than G'. If $A \subseteq V$, we have that $S(G, A)$ has fewer edges than G, and Condition 3 is satisfied. On the other hand, if $A \nsubseteq V$, we use the following Claim.

Claim 3 *Let A be a subset of $V(G')$ and let $A' = A \cap V$. Then the number of edges in $S(G', A')$ is less than or equal to the number of edges in $S(G', A)$.*

To prove the claim, we fix a set $A \subseteq V(G')$. We may assume that $|A \cap (Y \cup Z)| \leq |Y \cup Z|/2 = N$, otherwise we replace A by its complement $\overline{A} = V(G') \setminus A$ (note that $S(G', A \cap V)$ has the same number of edges as $S(G', \overline{A} \cap V)$).

Define the sets $A' = A \cap V$ and $A'' = A \setminus A' = A \cap (Y \cup Z)$. Let $G'_1 = S(G', A')$ and $G'_2 = S(G', A)$. Note that $G'_2 = S(G'_1, A'')$. To prove the claim, we need to show that G'_1 has at most as many edges as G'_2.

In G'_2, every vertex of A'' is adjacent to every vertex of $(Y \cup Z) \setminus A''$, whereas no such pair is adjacent in G'_1. This means that $|E(G'_2) \setminus E(G'_1)| \geq |A''|(|Y| + |Z| - |A''|) \geq |A''|N$, where we used the fact that A'' has size at most N.

On the other hand, an edge belonging to G'_1 but not to G'_2 must necessarily connect a vertex from A'' with a vertex from V. Therefore, $|E(G'_1) \setminus E(G'_2)| \leq$

$|A''|n$. Combining these estimates, we get

$$\begin{aligned}|E(G_2')| - |E(G_1')| &= |E(G_2') \setminus E(G_1')| - |E(G_1') \setminus E(G_2')| \\ &\geq |A''|N - |A''|n \\ &\geq 0.\end{aligned}$$

This proves the claim. As a consequence of Claim 3, if G' is switching-reducible, then it can be reduced by switching a set $A' \subseteq V$. The same set A' then reduces G, and Condition 3 of the Proposition holds. Analogically, if G' can be switched to contain L edges for an integer L, then G can be switched to contain $L - nN$ edges. Hence, we have proved Condition 2 with $k' = k + nN$.

It remains to check Condition 1. By definition, the density of G' is

$$\begin{aligned}D(G') &= \frac{2|E(G')|}{(2N+n)(2N+n-1)} \\ &\leq \frac{2\left(\binom{n}{2} + nN\right)}{(2N+n)(2N+n-1)} \\ &\leq \frac{n^2 + 2nN}{4N^2} \\ &\leq \frac{3nN}{4N^2} = \frac{3n}{4N} \\ &\leq c.\end{aligned}$$

This completes the proof. □

Proposition 5 allows us to state a stronger version of Theorem 2 for the special case of graphs with bounded density.

Theorem 4. *For every $c > 0$, the problem* SWITCH-FEW-EDGES *is NP-complete for graphs of density at most c.*

Proof. As shown by Proposition 5, a general instance (G, k) of SWITCH-FEW-EDGES can be transformed into an equivalent instance (G', k') of density at most c. Since SWITCH-FEW-EDGES is NP-complete on general instances by Theorem 2, it remains NP-complete on instances of density at most c. □

5 Concluding Remarks

5.1. We have been trying to prove that the problem SWITCH-REDUCIBLE is NP-complete (and hence, SWITCH-MINIMAL is co-NP-complete). We have not yet succeeded. However, if it is true, then Proposition 5 gives the following analogue of Theorem 4 even for these problems.

Proposition 6. *If the problem* SWITCH-REDUCIBLE *is NP-complete, then for every $c > 0$, the problem* SWITCH-REDUCIBLE *is NP-complete for graphs of density at most c, and the problem* SWITCH-MINIMAL *is co-NP-complete for graphs of density at most c.*

5.2. Lindzey [9] noticed that it is possible to speed-up several graph algorithms using switching to a lower number of edges – he obtained up to super-polylogarithmic speed-ups of algorithms for diameter, transitive closure, bipartite maximum matching and general maximum matching. However, he focuses on switching digraphs (with a definition somewhat different to Seidel's switching in undirected graphs), where the situation is in sharp contrast with our results – a digraph with the minimum number of edges in its switching-class can be found in $O(n + m)$ time.

5.3. It has been observed before (cf. e.g. [2]) that for a graph property $\mathcal{P}$, the complexity of deciding $\mathcal{P}$ is independent on the complexity of deciding if an input graph can be switched to a graph possessing the property $\mathcal{P}$. Switching to few edges thus adds another example of a polynomially decidable property (counting the edges is easy) whose switching version is hard. Previously known cases are the NP-hardness of deciding switching-equivalence to a regular graph [8] and deciding switching-equivalence to an H-free graph for certain specific graphs H [5].

5.4. Let $d > 0$ be a constant. What can we say about the complexity of Switch-Reducible and Switch-Few-Edges on graphs of maximum degree at most dn? If $d \leq \frac{1}{4}$, the two problems are trivial by Proposition 3. On the other hand, for $d \geq \frac{1}{2}$ the restriction on maximum degree becomes irrelevant, since any switching-minimal graph has maximum degree at most $\frac{n}{2}$ by Proposition 1. For any $d \in (\frac{1}{4}, \frac{1}{2})$, the complexity of the two problems on instances of maximum degree at most dn is open.

References

1. T. N. Bui, S. Chaudhuri, F. T. Leighton, and M. Sipser. Graph bisection algorithms with good average case behavior. *Combinatorica*, 7(2):171–191, 1987.
2. Andrzej Ehrenfeucht, Jurriaan Hage, Tero Harju, and Grzegorz Rozenberg. Complexity Issues in Switching of Graphs. In Hartmut Ehrig, Gregor Engels, Hans-Jörg Kreowski, and Grzegorz Rozenberg, editors, *Theory and Application to Graph Transformations*, volume 1764 of *LNCS*, pages 59–70. Springer, Heidelberg, 2000.
3. M. R. Garey, D. S. Johnson, and L. Stockmeyer. Some simplified NP-complete graph problems. *Theoretical Computer Science*, 1(3):237 – 267, 1976.
4. Jurriaan Hage. *Structural Aspects Of Switching Classes.* PhD thesis, Leiden Institute of Advanced Computer Science, 2001.
5. Eva Jelínková and Jan Kratochvíl. On Switching to H-Free Graphs. *Journal of Graph Theory*, 75(4):387–405, 2014.
6. Eva Jelínková, Ondřej Suchý, Petr Hliněný, and Jan Kratochvíl. Parameterized Problems Related to Seidel's Switching. *Discrete Mathematics and Theoretical Computer Science*, 13(2):19–42, 2011.
7. Sergiy Kozerenko. On graphs with maximum size in their switching classes. *Commentationes Mathematicae Universitatis Carolinae*, 56(1):51–61, 2015.
8. Jan Kratochvíl. Complexity of Hypergraph Coloring and Seidel's Switching. In Hans L. Bodlaender, editor, *WG*, volume 2880 of *LNCS*, pages 297–308. Springer Verlag, 2003.

9. Nathan Lindzey. Speeding up Graph Algorithms via Switching Classes. In *Proceedings of IWOCA 2014: 25th International Workshop on Combinatorial Algorithms*, 2015. To appear. Preprint available online at http://arxiv.org/abs/1408.4900.
10. Jiří Matoušek and Uli Wagner. On Gromov's method of selecting heavily covered points. *Discrete Comput. Geom.*, 52(1):1–33, July 2014.