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Convex Independence in Permutation Graphs

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Abstract. A set C of vertices of a graph is P_3-convex if every vertex outside C has at most one neighbor in C. The convex hull $\sigma(A)$ of a set A is the smallest P_3-convex set that contains A. A set M is convexly independent if for every vertex $x \in M$, $x \notin \sigma(M-x)$. We show that the maximal number of vertices that a convexly independent set in a permutation graph can have, can be computed in polynomial time.

1 Introduction

Popular models for the spread of disease and of opinion are graph convexities. The P_3-convexity is one such convexity, and it is defined as follows.

Definition 1. *A set* S *of vertices in a graph* G *is* P_3*-convex if every vertex outside* S *has at most one neighbor in* S*.*

The P_3-convexity will be the only convexity studied in this paper, so from now on we use the term convex, instead of P_3-convex. For a set A of vertices we let $\sigma(A)$ denote its convex hull, that is, the smallest convex set that contains A.[3]

For a set of points A in $\mathbb{R}^d$ and a point x in its Euclidean convex hull, there exists a set $F \subseteq A$ of at most $d+1$ points such that $x \in \sigma(F)$, ie, x is in the Euclidean convex hull of F. This is Carathéodory's theorem. For convexities in graphs one defines the Carathéodory number as the smallest number k such that, for any set A of vertices, and any vertex $x \in \sigma(A)$, there exists a set $F \subseteq A$ with $|F| \leqslant k$ and $x \in \sigma(F)$. For a set S, let

$$\partial(S) = \sigma(S) \setminus \bigcup_{x \in S} \sigma(S-x). \tag{1}$$

A set is irredundant if $\partial(S) \neq \varnothing$. Duchet showed that the Carathéodory number is the maximal cardinality of an irredundant set.

[3] In his classic paper, Duchet defines a graph convexity as a collection of 'convex' subsets of a (finite) set V that contains $\varnothing$ and V, and that is closed under intersections, and that, furthermore, has the property that each convex subset induces a connected subgraph. This last condition is, here, omitted.

Definition 2. *A set* S *is convexly independent if*

$$\text{for all } x \in S, \quad x \notin \sigma(S - x). \tag{2}$$

Notice that, if a set is convexly independent then so is every subset of it (since σ is a closure operator).

It appears that there is no universal notation for the maximal cardinality of a convexly independent set.[4] In this paper we denote it by $\beta_c(G)$. Every irredundant set is convexly independent, thus the convex-independence number $\beta_c(G)$ is an upperbound for the Carathéodory number. For example, for paths P_n with n vertices, and for cycles C_n with n vertices, we have equality;

$$\beta_c(P_n) = 2 \cdot \left\lfloor \frac{n}{3} \right\rfloor + (n \bmod 3) \quad \text{and} \quad \beta_c(C_n) = \beta_c(P_{n-1}). \tag{3}$$

Other examples, for which the Carathéodory number equals the convex independence number, are leafy trees, which are trees with at most one vertex of degree two. It is easy to check, that

$$T \text{ is a leafy tree} \quad \Rightarrow \quad \beta_c(T) = \text{the number of leaves in } T. \tag{4}$$

Examples for which the Carathéodory number is strictly less than the convex-independence number are disconnected graphs. If S is an irredundant set then $\sigma(S)$ is necessarily connected. However, $\beta_c(G)$ is the sum of the convexly independence numbers of G's components. Notice also that $\beta_c(P_6) = 4$, but there exists a maximum convexly independent set S for which $\sigma(S) = S$ and is disconnected, and, thence, redundant.

A set S is a 2-packing if it is an independent set in G^2, that is, no two vertices of S are adjacent or have a common neighbor. Every 2-packing S is convexly independent, as $\sigma(S) = S$. For splitgraphs with minimal degree at least two, a maximal convex-independent set is a 2-packing, unless it has only two vertices. It follows that computing the convexly independence number is NP-complete for splitgraphs (it is Karp's SET PACKING, problem 4). For biconnected chordal graphs (including the splitgraphs mentioned above), every vertex is in the convex hull of any set of two vertices at distance at most two. Thus, the Carathéodory number for those is two.

Ramos et al show that computing the convexly independence number remains NP-complete for bipartite graphs, and they show that it is polynomial for trees and for threshold graphs.

The intersection graph of a collection of straight line segments, with endpoints on two parallel (horizontal) lines, is called a permutation graph. Dushnik and Miller characterize them as the comparability graphs for which the complement is a comparability graph as well. In this paper we show that the convexly independence number of permutation graphs is computable in polynomial time.

[4] Ramos et al call it the 'rank' of the graph, but this word has been used for so many different concepts that it has lost all meaning.

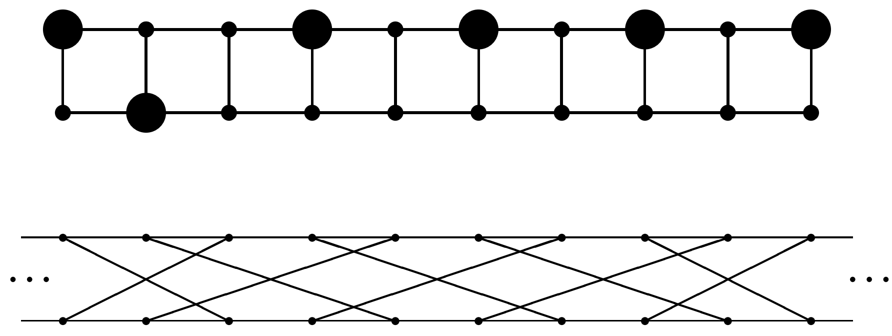

Fig. 1. The heavy dots specify an irredundant set S with $\sigma(S) = V$. The second figure shows the intersection model, ie, the 'permutation diagram,' for ladders. This example shows that the Carathéodory number for biconnected permutation graphs is unbounded.

This seems a good time to do a warm-up; let's have a close look at convex independence in cographs.

2 Convex independence in cographs

Definition 3. *A graph is a cograph if it has no induced* P_4*, the path with 4 vertices.*

Cographs are characterized by the property that every induced subgraph is disconnected or else, its complement is disconnected. In other words, cographs allow a complete decomposition by joins and unions. It follows that cographs are permutation graphs, as also this class is closed under joins and unions.

Ramos et al analyze the convex-independence number for threshold graphs. Threshold graphs are the graphs without induced P_4, C_4 and $2K_2$, hence, threshold graphs are properly contained in the class of cographs. In the following theorem we extend their results.

Theorem 1. *There exists a linear-time algorithm to compute the convex-independence number of cographs.*

Proof. Let G be a cograph. First assume that G is a union of two smaller cographs, G_1 and G_2. In that case, the convex-independence number of G is the sum of $\beta_c(G_1)$ and $\beta_c(G_2)$, that is,

$$G = G_1 \oplus G_2 \quad \Rightarrow \quad \beta_c(G) = \beta_c(G_1) + \beta_c(G_2). \tag{5}$$

Now, assume that G is a join of two smaller cographs G_1 and G_2. In that case, every vertex of G_1 is adjacent to every vertex of G_2. Let S be a convex-independent set. If S has at least one vertex in G_1 and at least one vertex in G_2, then $|S| = 2$, since $G[S]$ cannot have an induced P_3 or K_3.

Consider a convex-independent set $S \subseteq V(G_1)$. Assume that $|S| > 1$ and that $|V(G_2)| \geqslant 2$. Then, any two vertices of $S \cap V(G_1)$ generate $V(G_2) \subseteq \sigma(S)$, and, in turn, $V(G)$ is in their convex hull. This implies that S cannot have any other vertices, that is,

$$|V(G_2)| \geqslant 2 \quad \Rightarrow \quad |S| \leqslant 2.$$

Next, assume

$$\boxed{S \subseteq V(G_1) \quad \text{and} \quad |V(G_2)| = 1.}$$

Say u is in the singleton $V(G_2)$, that is, u is a universal vertex. Let $C_1, \dots, C_t$ be the components of G_1. We claim that

$$|S \cap C_i| \leqslant \min\{\,2,\ |C_i|\,\}.$$

To see that, assume that $|C_i| \geqslant 2$. Then, since $G[C_i]$ is a connected cograph, $G[C_i]$ is the join of two cographs, say with vertex sets A and B. If S has three vertices in A, then each of them is in the convex hull of the other two, since $B \cup \{u\}$ is contained in their common neighborhood, and this set contains at least two vertices.

Assume S has vertices in at least two different components of G_1. Assume furthermore that one component C_i has at least two vertices of S, say p and q. Let ζ be a vertex of S in another component. Then $u \in \sigma(\{p, \zeta\})$, because $[p, u, \zeta]$ is an induced P_3.

The induced subgraph $G[C_i]$ is a join of two smaller cographs, say with vertex sets A and B. If p and q are both in A, then $q \in \sigma(S - q)$, since p and u generate $B \subset \sigma(S)$, and $B \cup \{u\}$ contains two neighbors of q. If $p \in A$ and $q \in B$, then q has two neighbors in $\sigma(S - q)$, namely p and u. Thus, again, $q \in \sigma(S - q)$.

In fine, either each component of G_1 contains one vertex of S, or else $|S| \leqslant 2$.

This proves the theorem. □

3 Monadic second-order logic

In this section we show that the maximal cardinality of a convex-independent set is computable in linear time for graphs of bounded treewidth or rankwidth. To do that, we show that there is a formulation of the problem in monadic second-order logic. The claim then follows from Courcelle's theorem.

By definition, a set of vertices $W \subseteq V$ is convex if

$$\forall_{x \in V}\ x \notin W \quad \Rightarrow \quad |N(x) \cap W| \leqslant 1. \tag{6}$$

Let $S \subseteq V$. To formulate that a set $W = \sigma(S)$ we formulate that

1. $S \subseteq W$, and
2. W satisfies (6), and
3. For all W' for which the previous two conditions hold, $W \subseteq W'$.

Finally, a set S is convexly independent if

$$\forall_{x \in V} \; x \in S \quad \Rightarrow \quad x \notin \sigma(S - x). \tag{7}$$

Actually, to show that $x \notin \sigma(S - x)$ it is sufficient to formulate that (for every vertex $x \in S$) there is a set W_x such that

$$W_x \text{ is convex} \quad \text{and} \quad S \setminus \{x\} \subseteq W_x \quad \text{and} \quad x \notin W_x. \tag{8}$$

The formulas (6)—(8) show that convex independence can be formulated in monadic second-order logic (without quantification over subsets of edges). By Courcelle's theorem we obtain the following.

Theorem 2. *There exists a linear-time algorithm to compute the convex-independence number for graphs of bounded treewidth or rankwidth.*

Remark 1. Notice that also the Carathéodory number is expressible in monadic second-order logic.

3.1 Trees

Let T be a tree with n vertices and maximal degree Δ. Ramos et al present an involved algorithm, that runs in $O(n \log \Delta)$ time, to compute a convexly independent set. By Theorem 2, there exists a linear-time algorithm that accomplishes this. We propose a different algorithm.

Theorem 3. *There exists a linear-time algorithm that computes the convexly independence number of trees.*

Proof. Let T be a tree. Decompose T into a minimal number of maximal, vertex-disjoint, leafy trees, $F_1, \ldots, F_s$, and a collection of paths. The endpoints of the paths are separate leaves of the trees F_i or pendant vertices. By Equation (4), each leafy tree F_i has a maximum convexly independent set consisting of its leaves. The convexly independence numbers of the connecting paths are given by Equation (3). Notice that each path has a maximum, convexly independent set that contains the two endpoints. □

4 The convex-independence number of permutation graphs

In this section we show that there exists a polynomial-time algorithm to compute the convex-independence number of permutation graphs.

In the following discussion, let G be a permutation graph with a fixed permutation diagram. We refer to S as a generic convex-independent set in G.

Definition 4. *Let S be a convex-independent set in G. Let $x \in V \setminus S$. A* <u>*2-path*</u> *connecting x to S is a sequence of vertices*

$$\Delta = [s_1, s_2, x_1, \dots, x] \tag{9}$$

in which every vertex has two neighbors that appear earlier in the sequence, or else it is in S.

Lemma 1.

$$x \in \sigma(S) \quad \Leftrightarrow \quad \text{there is a 2-path connecting } x \text{ to } S. \tag{10}$$

Proof. Following Duchet, let $I(x, y)$ be the 'interval function' of the P_3-convexity, that is, for two vertices x and y, $I(x, y)$ is $\{x, y\}$ plus the set of vertices that are adjacent to both x and y.[5] For a set S, we let

$$I^0(S) = S \quad \text{and} \quad I^{k+1}(S) = I(I^k(S) \times I^k(S)). \tag{11}$$

Then,

$$\sigma(S) = \bigcup_{k \in \mathbb{N} \cup \{0\}} I^k(S). \tag{12}$$

In other words, a vertex is in $I^{k+1}(S)$ if it is in $I^k(S)$, or else it has two neighbors in $I^k(S)$. This is expressed by the existence of a 2-path.

This proves the lemma. □

Lemma 2. *Each component of $G[S]$, ie, the subgraph induced by S, is a single vertex or an edge. In a permutation diagram for G, there is a linear left-to-right ordering of the components of $G[S]$.*

Proof. Since S is convexly independent, $G[S]$ cannot contain K_3 or P_3. Thus each component of $G[S]$ is an edge or a vertex.

Fix a permutation diagram for G. The line segments that correspond to the vertices of S form a permutation diagram for $G[S]$. Each component of $G[S]$ is a connected part of the diagram, and the left-to-right ordering of the connected parts in the diagram yields a total ordering of the components of $G[S]$. □

[5] Duchet proved that, for interval convexities, the Carathéodory number is the smallest integer $k \in \mathbb{N}$ such that every $(k+1)$-set is redundant. Thus, the fixed-parameter Carathéodory number is polynomial.

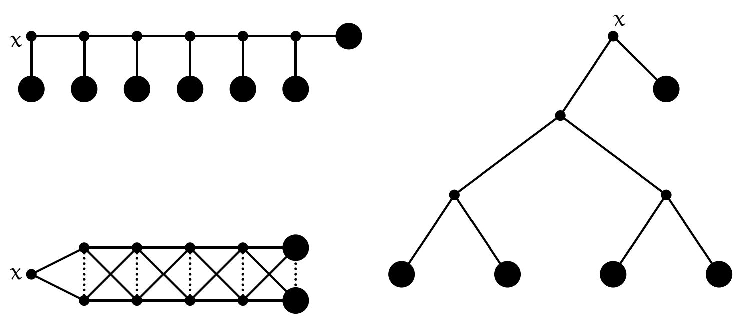

Fig. 2. The figure shows three examples of 2-paths from a vertex x to a set S. The heavy dots represent vertices of S. Notice, however, that the binary tree is not a permutation graph (since it has an asteroidal triple). The second example is a simple path in which each vertex, except x, is replaced by a twin. Permutation graphs are closed under creating twins, so, since paths are permutation graphs, this second example is so also.

Definition 5. *The last component of* S *is the rightmost component in the linear ordering as specified in Lemma 2.*

We say that a vertex $\notin S$ is to the right of the last component if its line segment appears to the right of the last component, ie, the endpoints of the line segment, on the top line and bottom line of the diagram, appear to the right of the endpoints of the last component.

Definition 6. *The* <u>*border*</u> *of* $\sigma(S)$ *is the set of the two rightmost endpoints, on the top line and bottom line of the permutation diagram, that are endpoints of line segments corresponding to vertices of* $\sigma(S)$*.*

We say that a line segment is to the left of the border if both its endpoints are left of the appertaining endpoints that constitute the border.

Lemma 3. *If the elements of the border of* $\sigma(S)$ *are the endpoints of a single line segment, then this is the line segment of a vertex in* S*.*

Proof. Let x be the vertex whose line segment has endpoints that form the border of $\sigma(S)$. Assume that $x \notin S$. Then, by definition, there is a 2-path Δ from x to S and all vertices of the 2-path are in $\sigma(S)$. Since $x \notin S$, it has two neighbors that appear earlier in Δ. The line segments of the two neighbors are crossing the line segment of x, and so there must be endpoints of $\sigma(S)$ that appear to the right of the endpoints of x. This is a contradiction. □

Lemma 4. *For every vertex in* $\sigma(S) \setminus S$ *there exist two vertex-disjoint paths to* S *with all vertices in* $\sigma(S)$*.*

Proof. We may assume that $|S| > 1$, otherwise $\sigma(S) = S$ and the claim is void. For convenience, add edges to the graph such that S becomes a clique and remove the vertices that are not in $\sigma(S)$. Assume that some vertex of $\sigma(S) \setminus S$ is separated from S by a cutvertex c. Since S is a clique, $S - c$ is contained in one component C_1 of $\sigma(S) - c$ and some vertices of $\sigma(S) \setminus S$ are in some other component C_2. Consider all 2-paths from vertices in C_2 to S. Let x be a vertex that is in C_2 with a shortest 2-path to S. Then x must have two neighbors that appear earlier in the 2-path. But that is impossible, since there is only one candidate, namely c. $\square$

Lemma 5. *Let the line segment of a vertex x be to the right of the last component of S. Then $x \in \sigma(S)$ if and only if x's line segment is to the left of the border.*

Proof. By Lemma 3, we may assume that the border corresponds to two adjacent vertices a and b. By definition of the border, $x \in \sigma(S)$ implies that both of x's endpoints are left of the border elements. If x is adjacent to both a and b then x has two neighbors in $\sigma(S)$, and so, x itself is in $\sigma(S)$. Assume that $x \notin N(a)$. Since $a \in \sigma(S) \setminus S$, by Lemma 4, there are two vertex-disjoint paths from a to S. Since the line segment of x is between a and the last component of S, each path must contain a neighbor of x. This implies that $x \in \sigma(S)$. $\square$

Our algorithm performs a dynamic programming on feasible last components of S and the border of $\sigma(S)$. By Lemma 5, the last element of S and the border supply sufficient information to decide whether a 'new last component,' to the right of the previous last component, has a vertex in $\sigma(S)$ or not.

Let $S^* = S \cup X$, where X is either a single vertex or an edge, and assume that X has no vertex $x \in X \cap \sigma(S^* - x)$.

Remark 2. Notice that, if $|X| = 1$, then it may be adjacent to one vertex appertaining the border. (For an example, see the figure below.) When $|X| = 2$, both vertices of X must be to the right of the border, otherwise, one element of X is adjacent to an element of $\sigma(S)$ and to the other element of X, which would make S^* convexly dependent. In any case, given the border, it is easy to check, algorithmically, the feasibility of a new component X.

To guarantee that S^* is a convex-independent set (with last component X), we need to check that, for any $s \in S$, $s \notin \sigma(S^* - s)$. The figure below shows that the last component of S and the border of $\sigma(S)$ do not convey sufficient information to guarantee that S^* is convexly independent.

In the following, let L be the last component of S, let X be a feasible, 'new,' last component of $G[S^*]$, to the right of L, where $S^* = S \cup X$. (Of course, both L and X are either single vertices or edges, and no vertex of L is adjacent to any vertex of X.) The feasibility of X is defined so that:

$$\forall_{x \in X} \; x \notin \sigma(S^* - x). \tag{13}$$

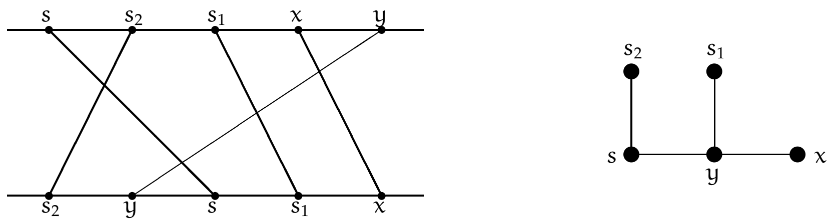

Fig. 3. The figure shows $s \in \sigma(S^* - s)$, where $S = \{s, s_1, s_2\}$, $S^* = \{s, s_1, s_2, x\}$, $\sigma(S) = \{s, s_1, s_2, y\}$ and $\sigma(S^* - s) = \{s_1, s_2, x, y_2, s\}$. Notice that $x \notin \sigma(S)$ (x is not left of the border; the border of $\sigma(S)$ has the endpoint of y on the top line and the endpoint of s_1 on the bottom line).

In our final theorem, below, we prove that it is sufficient to maintain a constant amount of information, to enable the algorithm to check the convexly independence of S^*.

First we define a partial 2-path. We consider two cases, namely where $|X| = 1$ and where $|X| = 2$. To define it, we 'simulate' X by an auxiliary vertex, or an auxiliary true twin, that we place immediately to the right of L.

Definition 7. *When $|X| = 1$, add one vertex s' with a line segment whose endpoints are immediately to the right of the rightmost endpoint of L on the top line and the rightmost endpoint of L on the bottom line. When $|X| = 2$, then replace the vertex s' above by a true twin s'_1 and s'_2. Let $X' = \{s'\}$ when $|X| = 1$ and $X' = \{s'_1, s'_2\}$ when $|X| = 2$. Finally, let $S' = S \cup X'$. A partial 2-path from $u \in S$ to $S^* - u$ is a 2-path from u to $S' - u$, from which S' is removed.*

Theorem 4. *There exists a polynomial-time algorithm to compute a convexly independent set of maximal cardinality in permutation graphs.*

Proof. Consider a vertex $u \in S$ for which $u \in \sigma(S^* - u)$. We may assume that $u \notin L$, because L is available to the algorithm and so, it is easy to check the condition for elements of L. Then there is a 2-path $\Delta = [s_1, s_2, \ldots, u]$ from u to $S^* - u$. If no vertex of X is in this 2-path, then $u \in \sigma(S - u)$, which contradicts our assumption that S is convexly independent. We may assume that at least one of s_1 and s_2 is an element of X.

Since Δ contains two vertex-disjoint paths from u to $S^* - u$, at least one of these paths must contain some vertex of $N(L)$. Partition the vertices of $N(L)$ in two parts. One part contains those vertices that have their endpoint on the top line to the right of L, and the other part contains those vertices that have their endpoint on the bottom line to the right of L. We claim that both parts are totally ordered by set-inclusion of their neighborhoods in the component of $G - N[L]$ that contains X. To see that, consider two elements a and b of $N(L)$. Say that a and b both have an endpoint on the top line, to the right of L. If that endpoint of a is to the left of the endpoint of b, then every neighbor of a in the component of $G - N[L]$ that contains X is also a neighbor of b.

We store subsets with two vertices, y_1 and y_2 in $N(L)$, for which there is a partial 2-path from some vertex $u \in S$ to y_1 and y_2. It is sufficient to store only those two vertices y_1 and y_2 that they have a maximal neighborhood. In other words, we choose y_1 and y_2 such that their endpoints on the top line and bottom line are furthest to the right, or, if they are both in the same part, the two that have a maximal neighborhood.

One other possibility is, that a 2-path from u to $\sigma(S^* - u)$ has only one vertex $y \in N(L)$ on a path from u to X. Of those 2-paths, We also store the element y, with a largest neighborhood in the component of $G - N[L]$ that contains X.

To check if S^* is convexly independent it is now sufficient to check if one of the partial paths to y_1 and y_2, or to the single element y, extend to X.

This proves that there is a dynamic programming algorithm to compute a maximum convexly independent set. □

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