

An algebra of synchronous atomic steps^{*}

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Abstract. This research started with an algebra for reasoning about rely/guarantee concurrency for a shared memory model. The approach taken led to a more *abstract algebra of atomic steps*, in which atomic steps synchronise (rather than interleave) when composed in parallel. The algebra of rely/guarantee concurrency then becomes an interpretation of the more abstract algebra. Many of the core properties needed for rely/guarantee reasoning can be shown to hold in the abstract algebra where their proofs are simpler and hence allow a higher degree of automation. Moreover, the realisation that the synchronisation mechanisms of standard process algebras, such as CSP and CCS/SCCS, can be interpreted in our abstract algebra gives evidence of its unifying power. The algebra has been encoded in Isabelle/HOL to provide a basis for tool support.

1 Introduction

Our goal is to provide better methods for deriving concurrent programs from abstract specifications, and to provide tool support for compositional reasoning about their correctness. The rely/guarantee approach of Jones [Jon81,Jon83] achieves compositionality by abstracting the interference a process can tolerate from and inflict on its environment. A *rely* condition r is a binary relation between states that represents an assumption bounding the interference that a process p can tolerate from its environment. If the environment fails to meet its obligation r , p may deviate from its specification and show erratic behaviour (i.e. abort). A *guarantee* condition g is the corresponding notion that bounds the interference inflicted on its environment by p . For a system of parallel processes to function correctly, each process's guarantee must imply the rely of every other parallel process. These concepts can be captured uniformly (and hence the manipulation of process terms kept simple) in a framework in which both the steps of a process and the steps of its environment are explicitly represented.

The semantic model for rely/guarantee reasoning suggested by Aczel is one such framework [Acz83,dR01]. In this model, parallel composition *synchronises* a program step of one process with an environment step of another, to give a program step of their composition. Aczel's approach, of insisting each step of one process is synchronised with a step of the other process, differs from the commonly used approach of interleaving atomic steps of processes (except when they communicate), e.g. CCS [Mil89],

^{*} This work is supported by Australian Research Council (ARC) Discovery Project DP130102901 and the UK EPSRC 'Taming Concurrency' research grant.

CSP [Hoa85] and ACP [BK84,BK85]. Aczel’s approach is closer to Milner’s Synchronous CCS (SCCS) [Mil89, Section 9.3] and Meije (the calculus at the basis of the synchronous programming language Esterelle) [BC85].

Our methodology is to develop a refinement calculus for concurrent programs that lifts rely and guarantee conditions to commands³ [JHC15,HJC14] (from parameters to the notion of correctness). That allows algebraic reasoning about concurrent programs in a rely/guarantee style. To this end we have designed a *Concurrent Refinement Algebra* (CRA) to support the rely/guarantee approach [Hay16]. In exploring the laws in CRA, we discovered that atomic steps have specific algebraic properties that can be captured in an *abstract algebra of atomic steps* which is embedded in CRA.

The abstract algebra of atomic steps delivers a range of useful properties for manipulating process terms. For example, based on the notion of atomic steps the parallel composition of processes can be simplified as follows

$$(a ; c) \parallel (b ; d) = (a \parallel b) ; (c \parallel d) , \quad (1)$$

where a and b are atomic steps and c and d are arbitrary processes. Note that the above equivalence does not hold if a and b are arbitrary processes. For an interleaving operator $\parallel\parallel$ the corresponding law is the more complicated:

$$(a ; c) \parallel\parallel (b ; d) = a ; (c \parallel\parallel b ; d) \sqcap b ; (a ; c \parallel\parallel d) . \quad (2)$$

In (1), parallel composition of two atomic steps a and b gives an atomic step $a \parallel b$, where the interpretation of $a \parallel b$ depends on the particular model. As a consequence, the algebra can be applied to a range of models. For example, as well as allowing an Aczel-trace model to support shared variable concurrency, communication in process algebras such as CSP and CCS/SCCS can be interpreted in the abstract algebra and hence it provides a foundation for a range of concurrency models.

Kleene Algebra with Tests (KAT) by Kozen [Koz97] combines Kleene algebra (the algebra of regular expressions [Con71]) with a Boolean sub-algebra representing tests. KAT supports sequential programs with conditionals and finite iterations (partial correctness). The *Demonic Refinement Algebra* (DRA) of von Wright [vW04] generalises Kozen’s work to support possibly infinite iteration and with that the concept of aborting behaviour. The approach used in this paper is based on that of von Wright in order to faithfully capture Jones’ theory, in particular his rely condition.

Concurrent Kleene Algebra (CKA) [HMSW11] adds a parallel operator to Kleene algebra to support sequential and parallel programs. Prisacariu’s *Synchronous Kleene Algebra* (SKA) [Pri10] extends Kleene algebra with a synchronous parallel operator similar to that in Milner’s SCCS [Mil89]. Like Milner he proposes a specific interpretation of the parallel composition of atomic steps. In contrast to both CKA and SKA, our *Concurrent Refinement Algebra* [Hay16], which we use as a basis for this work, adds a parallel operator to the sequential algebra DRA (rather than Kleene algebra).

The major contribution of this paper is an *algebra of atomic steps* which introduces a synchronous parallel operator for atomic steps. The interpretation of two atomic steps acting in parallel, however, is left open, hence allowing a range of different models

³ We use the terms *command*, *program* and *process* synonymously.

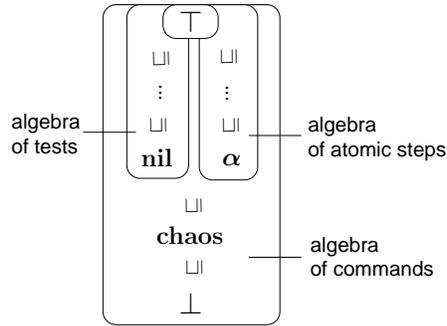


Fig. 1. The Concurrent Refinement Algebra and its sub-algebras

(including those of Milner and Prisacariu). Further, atomic steps are treated as a Boolean sub-algebra (similar to the way in which tests are treated as a Boolean sub-algebra in KAT). Hence the Concurrent Refinement Algebra (CRA) contains both a sub-algebra of tests and a sub-algebra of atomic steps (as illustrated in Figure 1 via their lattices). Separating out these sub-algebras enables one to prove properties that are specific to atomic steps using the full power of a Boolean algebra. This raises the level of support for reasoning about programs provided by our algebra, as well as the level of automation that is possible for the mechanised proof support by the theorem prover Isabelle.

To build the algebra, we start in Section 2 with CRA for reasoning about commands in general. Commands include a sub-lattice of tests (Section 3) as well as a second sub-lattice of atomic steps (Section 4), the novel contribution of this paper. Section 5 gives an interpretation of the abstract algebra based on Aczel’s trace model. A simplified treatment of relies and guarantees is outlined in Section 6. Section 7 illustrates how the communication models of CCS, CSP and SCCS can be interpreted in our abstract algebra of atomic steps.

2 Concurrent Refinement Algebra

A Concurrent Refinement Algebra (CRA) is defined as the following structure

$$(\mathcal{C}, \sqcap, \sqcup, ;, \parallel, \perp, \top, \text{nil}, \text{skip})$$

where the carrier set \mathcal{C} is the set of *commands*. Sequential composition ($;$) has higher precedence than parallel (\parallel), which has higher precedence than \sqcap and \sqcup , which have equal precedence.

Commands form a complete distributive lattice $(\mathcal{C}, \sqcap, \sqcup, \perp, \top)$ with *nondeterministic choice* as the lattice meet ($c \sqcap d$), and *conjunction* of commands as the lattice join ($c \sqcup d$). The top of the lattice \top is the infeasible command (called “magic” in the refinement calculus) and the bottom of the lattice \perp is the command that aborts. The partial order defined on commands is the refinement relation $c \sqsubseteq d$ meaning c is refined (or implemented by) d . For any commands $c, d \in \mathcal{C}$, $c \sqsubseteq d \hat{=} (c \sqcap d) = c$, and hence $\perp \sqsubseteq c \sqsubseteq \top$. We refer to this as the *refinement lattice* (see Figure 1). Note that since

CRA is a *refinement algebra* it uses \sqsubseteq as partial order instead of Kozen's \geq and hence our lattice of commands is the dual of Kozen's lattice (i.e., \sqcap in CRA matches \sqcup in KAT, and \sqcup in CRA matches \sqcap in KAT). Given commands form a complete lattice, for any monotone function least/greatest fixed points are well defined. In particular, fixed points are used to define iteration operators below.

Sequential composition of commands $(c ; d)$ is associative and has identity **nil**. As an abbreviation, the sequential composition operator may be elided. Sequential composition has both \top and \perp as left (but not right) annihilators⁴, i.e. $\top c = \top$ and $\perp c = \perp$. It distributes over arbitrary choices on the right (3),

$$(\sqcap C) d = \sqcap_{c \in C} (c d). \quad (3)$$

The iteration of a command is inductively defined as $c^0 = \mathbf{nil}$ and $c^{i+1} = c c^i$. More general iteration operators are captured via greatest (ν) and least (μ) fixed points: $c^* \hat{=} \nu x. \mathbf{nil} \sqcap c x$ for finite iteration zero or more times, and $c^\omega \hat{=} \mu x. \mathbf{nil} \sqcap c x$ for finite or possibly infinite iteration. Infinite iteration is defined as $c^\infty = c^\omega \top$. The unfolding laws (4) and (5) result from the fixed point definitions for iterations, and (6) follows from (4) and the definition of c^∞ which also justifies (8). Law (7) follows from (6) by induction.

$$c^\omega = \mathbf{nil} \sqcap c c^\omega \quad (4) \qquad c^\infty = c^i c^\infty \quad (7)$$

$$c^* = \mathbf{nil} \sqcap c c^* \quad (5) \qquad c^\infty d = c^\infty \quad (8)$$

$$c^\infty = c c^\infty \quad (6)$$

Some models also distribute sequential composition over non-empty choices on the left (9) (i.e., in refinement calculus terms the operator is conjunctive).

$$D \neq \{\} \Rightarrow c (\sqcap D) = \sqcap_{d \in D} (c d) \quad (9)$$

This axiom is not assumed to generally hold in CCS and CSP but it holds for our relational model in Section 5 and is required to show laws (10) and (11). Laws (12) and (13) follow from (10), (8) and (6).

$$c^\omega = c^* \sqcap c^\infty \quad (10) \qquad c^\omega d = c^* d \sqcap c^\infty \quad (12)$$

$$c^* = \sqcap_{i \in \mathbb{N}} c^i \quad (11) \qquad c c^\omega d = c c^* d \sqcap c^\infty \quad (13)$$

Parallel composition of commands is associative, commutative, has the identity **skip**, and \top serves as an annihilator: $c \parallel \top = \top$. Parallel distributes over non-deterministic choice of any set of commands, $c \parallel (\sqcap D) = \sqcap_{d \in D} (c \parallel d)$. Note the identities for sequential and parallel composition, **nil** and **skip** respectively, differ. However, they are related by **skip** \sqsubseteq **nil** and **nil** \parallel **nil** = **nil**.

3 The Boolean sub-algebra of tests

Tests are special commands that are used to model conditionals and loops and hence form an essential construct when reasoning about programs. Assume t is a test, $\neg t$ is its

⁴ Here our approach based on DRA differs from approaches based on Kleene algebra, like CKA and SKA, in which \top is also a right annihilator.

negation, and c and d are commands, an abstract algebraic representation of conditionals and while loops for sequential programs is given by

$$\mathbf{if } t \mathbf{ then } c \mathbf{ else } d \hat{=} tc \sqcap \neg td \quad \text{and} \quad \mathbf{while } t \mathbf{ do } c \hat{=} (tc)^\omega \neg t$$

Blikle [Bli78] used this style of representation of programs in a relational algebra and [GM93] and [vW04] in the refinement calculus. Kozen [Koz97] provided a more abstract *Kleene Algebra with Tests* (KAT) as a framework for reasoning about programs. Kleene algebra is the algebra of regular expressions, where for the interpretation as programs, alternation becomes non-deterministic choice with unit \top , concatenation becomes sequential composition with unit **nil**, and iteration becomes finite iteration of commands. Tests, in Kozen's approach, form a Boolean sub-algebra within the Kleene algebra.

We follow this construction for the Concurrent Refinement Algebra. That means in CRA tests form a subset of commands for which a negation operator \neg is defined. This results in an extended algebra

$$(\mathcal{C}, \mathcal{B}, \sqcap, \sqcup, ;, \parallel, \perp, \top, \mathbf{nil}, \mathbf{skip}, \neg)$$

where the additional carrier set \mathcal{B} is the set of *test commands* ($\mathcal{B} \subseteq \mathcal{C}$). As in Kozen's work, tests form a Boolean algebra $(\mathcal{B}, \sqcap, \sqcup, \neg, \top, \mathbf{nil})$ which is a *sub-lattice* of commands (see Figure 1).

The sub-lattice of tests shares its top element (the false test) with the top of the lattice of commands, \top , but does not share its bottom element, the true test, that instead corresponds to the command **nil**, that has no effect and immediately terminates. Tests are closed under lattice meet and join, as well as sequential and parallel composition as both are defined via the join operator \sqcup on commands. For any t and t' in \mathcal{B} ,

$$tt' = t \sqcup t' \quad (14) \quad t \parallel t' = t \sqcup t' \quad (15)$$

where the join of two test acts as logical conjunction. Property (15) can be generalised to the following interchange axiom. For any commands c and d in \mathcal{C} and any tests t and t' in \mathcal{B} the following hold.

$$(tc) \parallel (t'd) = (t \sqcup t')(c \parallel d) \quad (16) \quad (tc) \sqcup (t'd) = (t \sqcup t')(c \sqcup d) \quad (17)$$

A range of useful laws follow from this axiomatisation that help simplifying program terms involving tests.

Tests also give rise to the concept of *assertions* (preconditions) [vW04,Sol07]. The assertion corresponding to a test t is a command which terminates if the test holds and aborts if the test does not hold, i.e., **assert** $t = t \sqcap \neg t \perp$.

4 Abstract atomic steps

This section gives an abstract algebra for the subset of commands that correspond to atomic steps. This algebra delivers core properties of atomic steps (that do not hold for commands in general) under only a few assumptions about the form of atomic steps. Atomic steps are closed under parallel composition but the parallel composition of atomic steps, $a \parallel b$, is left uninterpreted. Lifting these properties to the level

of an abstract algebra results in simpler proofs and allows for their reuse in different interpretations. Section 5 forms an interpretation of the atomic step algebra that corresponds to Aczel's program and environment steps and defines parallel composition of atomic steps in detail. Section 7 on the other hand, uses the atomic step algebra to capture CCS-style as well as CSP-style communication of events, which resides in a very different domain.

In the same manner that tests form a sub-lattice of commands, the set of atomic steps, \mathcal{A} , forms a sub-lattice of commands which is a Boolean algebra and shares the lattice meet and join of commands (see Figure 1). The top of the sub-lattice is the same as the top of the command lattice (\top) but the bottom of the sub-lattice is the new command α , that can be thought of as the non-deterministic choice between all possible atomic steps. In fact, tests and atomic steps share only one element (\top) and hence

$$\alpha \sqcup \mathbf{nil} = \top . \quad (18)$$

The term *step* is used exclusively for an atomic step. Steps are closed under lattice meet and join as well as parallel composition (but not sequential composition). As for commands, the meet corresponds to non-deterministic choice, $a \sqcap b$, and can behave as either a or b . The join of two steps, $a \sqcup b$, can be thought of as a step that both a and b agree to do. (In Section 5 this corresponds to the intersection of the sets of primitive steps a and b can make.)

Because \mathcal{A} forms a Boolean algebra, all of the laws of Boolean algebra are available to manipulate combinations of steps not involving sequential composition. The theorem prover Isabelle directly supports forming such an interpretation and hence the theory of Boolean algebra can be re-used for \mathcal{A} . This is a significant saving, as the laws of Boolean algebra do not need to be reproven.

In addition, atomic steps are assumed to have an *identity*, \mathcal{E} , of parallel composition, giving the following axiom.

$$a \parallel \mathcal{E} = a \quad (19)$$

Prefixing a command c with \mathcal{E} , i.e. $\mathcal{E}c$, allows the process to wait one step before behaving as c , and $\mathcal{E}^\omega c$ allows it to wait any number of steps (including 0). The step \mathcal{E} can be interpreted as a placeholder for one step taken by its environment.

Besides laws for reasoning about atomic steps in isolation, one needs laws that allow reasoning about their interaction with non-atomic commands. A small set of additional axioms is used as the basis of these laws. The approach taken to handling parallel composition is not the usual interleaving of steps, rather each step of one process must synchronise with a step of the other process. If a and b cannot synchronise then $a \parallel b$ is infeasible (\top). For steps a and b , and any commands c and d , we assume the following axioms.

$$ac \parallel bd = (a \parallel b) (c \parallel d) \quad (20) \qquad ac \parallel \mathbf{nil} = \top \quad (22)$$

$$ac \sqcup bd = (a \sqcup b) (c \sqcup d) \quad (21) \qquad ac \sqcup \mathbf{nil} = \top \quad (23)$$

The interchange axioms (20) and (21) become refinements from left to right if a and b are allowed to be arbitrary commands (which corresponds to the weak interchange law in CKA [HMSW11]). The abstract algebra does not define the details of parallel composition of pairs of steps. (See the relational interpretation of the algebra in Section 5

for one example of defining parallel composition of atomic steps.) The command, **nil**, that terminates immediately without making any steps whatsoever cannot synchronise with a process that makes at least one step, i.e. (22) and (23).

The *negation* operator (!) for atomic steps satisfies the following axioms of a Boolean algebra. Steps a and $!a$ have no common behaviour (24) and $!a$ has all the step behaviours that a does not have (25).

$$a \sqcup !a = \top \quad (24) \quad a \sqcap !a = \alpha \quad (25)$$

Note that negation for tests (\neg) differs from negation for atomic steps (!) as we have $\neg \top = \mathbf{nil}$ but $! \top = \alpha$. The inclusion of a negation operator on steps allows one to define an equivalent of an assertion for steps on the abstract level. For any step a define,

$$\mathbf{assume} a \hat{=} a \sqcap (!a) \perp . \quad (26)$$

The command **assume** a behaves as a and terminates, or as $!a$ and aborts. It represents an assumption that step a occurs in the sense that any other step allows any behaviour to occur after that step. It provides the basis for rely conditions because they specify assumptions about the environment's behaviour (see Section 6).

4.1 Canonical representation of commands

If the primitive commands of our language are tests, atomic steps and \perp , and all other commands are built from these primitives using the operators of the language, then initially, a command may either terminate immediately, abort or perform some atomic step. That leads to the canonical representation theorem, in which c can terminate if some test t succeeds, abort if some test t' succeeds, or performs some step a_i followed by some command c_i , for some $i \in I$.

Theorem 1 (canonical-representation). *Any command c can be expressed in the following form*

$$c = t \sqcap t' \perp \sqcap \prod_{i \in I} a_i c_i$$

where t and t' are tests, and for any i in some (possibly empty) index set I , a_i is an atomic step not equal to \top , and c_i is a command.

The proof is conducted by structural induction over commands. Note that if c cannot terminate immediately, t is \top . If c cannot abort, t' is \top . If c cannot perform any step, $I = \{\}$. A similar theorem can be found in [Pri10] for SKA.

Because \mathcal{E} is the identity of parallel for a single step, \mathcal{E}^ω acts as the identity of any sequence of steps and hence \mathcal{E}^ω is the identity of parallel, i.e. **skip** = \mathcal{E}^ω .

Lemma 1 (atomic-identity-iteration). $\mathcal{E}^\omega \parallel c = c$

The proof makes use of Theorem 1 to express c in canonical form (the proof is included in the appendix of [HCM⁺16]).

4.2 Properties of iterations of atomic steps

In addition to defining programming statements such as while loops, iterators are used to build specifications from atomic steps. For instance commands corresponding to

Jones' rely and guarantee concepts are constructed as iterations of relatively straightforward commands that make assumptions about the steps of the environment and constrain the steps of the program, respectively (see Section 6). Below we provide some useful properties of iterations of atomic steps.

Because **nil** performs no steps, if it is run in parallel with a (possibly) finite iteration, the composition cannot perform any steps but can terminate and hence equals **nil**. If **nil** is run in parallel with an infinite iteration, the combination cannot perform any steps but cannot terminate, and hence equals the infeasible command \top .

Lemma 2 (atomic-iteration-nil).

$$a^* \parallel \mathbf{nil} = \mathbf{nil} \quad a^\omega \parallel \mathbf{nil} = \mathbf{nil} \quad a^\infty \parallel \mathbf{nil} = \top$$

Proof. The properties follow from axiom (22) using unfolding of the iterations (i.e. $a^* = \mathbf{nil} \sqcap a a^*$, $a^\omega = \mathbf{nil} \sqcap a a^\omega$ and $a^\infty = a a^\infty$). \square

For the following lemmas, let a and b be atomic steps, and c and d any commands. Axiom (20) can be extended to iteration i times as given in the following lemma, which is proven by induction on i .

Lemma 3 (atomic-iteration-power). $a^i c \parallel b^i d = (a \parallel b)^i (c \parallel d)$

Choosing c and d to both be **nil** gives the corollary that $a^i \parallel b^i = (a \parallel b)^i$.

For all further lemmas in this sub-section, we assume that sequential composition is conjunctive (9) and hence that properties (10) and (11) hold. Two useful properties are the following.

$$a^* \parallel b^* = (a \parallel b)^* \quad (27) \quad a^\infty \parallel b^\infty = (a \parallel b)^\infty \quad (28)$$

Property (27) can be proven using the property that non-deterministic choice over an arbitrary set distributes over parallel. A proof of (28) would follow straightforwardly if the supremum over an arbitrary set (or even a chain) distributed over parallel, however, that distribution property does not hold in general. We take property (28) as an axiom because it does hold in our intended model. Whether this axiom is independent of the other axioms in our algebra is an open question.

Property (27) holds for atomic steps a and b but is only a refinement from left to right if a and b are replaced by arbitrary commands. Property (27) can be generalised to the following lemma where we take into account that the number of iterations of a and b might be the same, or there are more iterations of a than b (and hence the additional iterations of a are in parallel with the start of d), or the symmetric case when there are more occurrences of b than a .

Lemma 4 (atomic-iteration-finite).

$$a^* c \parallel b^* d = (a \parallel b)^* ((c \parallel d) \sqcap (c \parallel b b^* d) \sqcap (a a^* c \parallel d))$$

Isabelle/HOL proofs of these lemmas have been completed. They may be also found in the appendix of [HCM⁺16]. Choosing c and d to both be **nil** gives (27) as a corollary.

An infinite iteration in parallel with an initial finite iteration matches the finite iteration as well as what follows it.

Lemma 5 (atomic-iteration-finite-infinite). $a^* c \parallel b^\infty = (a \parallel b)^* (c \parallel b^\infty)$

Lemma 4 can be extended to initial iterations that are either finite or infinite.

Lemma 6 (atomic-iteration-either).

$$a^\omega c \parallel b^\omega d = (a \parallel b)^\omega ((c \parallel d) \sqcap (c \parallel b b^\omega d) \sqcap (a a^\omega c \parallel d))$$

Choosing c and d to both be **nil** gives the corollary that $a^\omega \parallel b^\omega = (a \parallel b)^\omega$.

To see the relationship to an interleaving operator, for any step a , define an *action* as $\langle a \rangle = \mathcal{E}^\omega a \mathcal{E}^\omega$, then properties of $\langle a \rangle$ can be proven using properties of the abstract algebra. For example, one can derive the following lemma.

Lemma 7 (atomic-interleaving). $\langle a \rangle \parallel \langle b \rangle = \langle a \parallel b \rangle \sqcap \langle a \rangle \langle b \rangle \sqcap \langle b \rangle \langle a \rangle$

If a and b cannot synchronise (i.e. $a \parallel b = \top$) then $\langle a \rangle \parallel \langle b \rangle = \langle a \rangle \langle b \rangle \sqcap \langle b \rangle \langle a \rangle$ which echoes the following property of an interleaving operator: $a \parallel b = a b \sqcap b a$. Hence by including an identity, \mathcal{E} , for parallel with an atomic step, one can represent interleaving properties in the synchronising algebra albeit in a more complex form. This approach was used by Milner in Synchronous CCS [Mil83] to allow the encoding of the better-known process algebra CCS. Our identity element takes on a similar role, although we lift it to a command as opposed to a transition event as in Milner's operational semantics. The advantage of the synchronising algebra is that one can represent both synchronising events and interleaving events in the one theory. By using separate program and environment events, such a theory supports the rely/guarantee approach of Jones for reasoning about concurrent programs.

5 Relational atomic steps

This section examines an interpretation of the abstract atomic step algebra \mathcal{A} in terms of Aczel's program and environment state transitions.⁵ The resulting *relational atomic steps* are used to define guarantees and relies in Section 6. This interpretation assumes that sequential composition is conjunctive (9).

Given a state space Σ and a binary relation $r \in \mathbb{P}(\Sigma \times \Sigma)$, the command $\pi(r)$ can take an atomic *program* step from state σ to σ' for any pair of states (σ, σ') in r . Similarly, $\epsilon(r)$ is a command that can perform any environment step from state σ to σ' whenever $(\sigma, \sigma') \in r$.

$$\pi : \mathbb{P}(\Sigma \times \Sigma) \rightarrow \mathcal{A} \qquad \epsilon : \mathbb{P}(\Sigma \times \Sigma) \rightarrow \mathcal{A}$$

The commands $\pi(\emptyset)$ and $\epsilon(\emptyset)$ are infeasible, i.e., $\pi(\emptyset) = \epsilon(\emptyset) = \top$. The images of π and ϵ are disjoint except when the relation is empty, i.e. for all r_1 and r_2 ,

$$\pi(r_1) \sqcup \epsilon(r_2) = \top. \tag{29}$$

Together π and ϵ form a sub-lattice of commands with two further sub-lattices: all the $\pi(r)$ commands form a sub-lattice and all the $\epsilon(r)$ commands form a sub-lattice.

⁵ A semantic model for this interpretation may be found in [CHM16].

The functions π and ϵ are injective, i.e. different relations map to different commands, and union of relations maps to a non-deterministic choice between the mappings of the relations and intersection maps to the supremum in the command ordering.

$$r_1 = r_2 \Leftrightarrow \pi(r_1) = \pi(r_2) \quad (30) \quad \pi(r_1 \cup r_2) = \pi(r_1) \sqcap \pi(r_2) \quad (31)$$

$$\pi(r_1 \cap r_2) = \pi(r_1) \sqcup \pi(r_2) \quad (32)$$

If $r_1 \subseteq r_2$, then $\pi(r_1) \sqcap \pi(r_2) = \pi(r_1 \cup r_2) = \pi(r_2)$, and therefore $\pi(r_2) \sqsubseteq \pi(r_1)$. Similar laws hold for ϵ steps.

In this interpretation one can instantiate the test command from Section 3 as $\tau(p)$ for $p \in \mathbb{P} \Sigma$, which succeeds and terminates immediately if p holds but is \top otherwise, e.g. $\tau(\emptyset) = \top$ and $\tau(\Sigma) = \mathbf{nil}$. As in the refinement calculus, a precondition command $\{p\}$ can then be defined as **assert** $\tau(p)$, which equals $\tau(p) \sqcap \tau(\neg p) \perp$, and hence terminates immediately if p holds but aborts otherwise, e.g. $\{\emptyset\} = \perp$ and $\{\Sigma\} = \mathbf{nil}$.

6 Relies and guarantees

The rely/guarantee approach of Jones [CJ07] makes use of a rely condition, r , a binary relation on states that expresses an assumption that every step made by the environment of the process satisfies r between its before and after states. Complementing that, all processes in its environment have a guarantee condition, g , a binary relation on states that expresses that every program step made by the process satisfies g . For each process, its guarantee condition must imply the rely conditions of all the processes in its environment. This section encodes guarantees and relies using the abstract algebra of atomic steps.

6.1 The guarantee command

For a process to ensure a guarantee g , every atomic program (π) step made by the program must satisfy g . A guarantee puts no constraints on the environment of the process. A guarantee command, **guar** g , is defined in terms of the iteration of a single step guarantee, (π -**restrict** g), defined as follows.

$$(\pi\text{-restrict } g) \hat{=} \pi(g) \sqcap \mathcal{E} \quad \mathbf{guar } g \hat{=} (\pi\text{-restrict } g)^\omega$$

A command c with a guarantee of g enforced on every program step could possibly be expressed as (**guar** g) $\sqcup c$, but that turns out to be too strong a requirement because it masks any aborting behaviour of c because the guarantee never aborts, (**guar** g) $\sqcup \perp = (\mathbf{guar } g)$. Instead, the weak conjunction operator is used.

Weak conjunction on commands, $\mathring{\cap}$, behaves like \sqcup unless one of its operands aborts in which case we have $c \mathring{\cap} \perp = \perp$. The operator is associative, commutative and idempotent, and satisfies $c \mathring{\cap} (\prod D) = (\prod_{d \in D} c \mathring{\cap} d)$ for any non-empty set of commands D . For any commands c and d , steps a and b , and tests t and t' weak conjunction satisfies the following axioms. (Note the similarities between (36) and (21), (37) and (23) and (38) and (28).)

$$c \mathbin{\frown} \perp = \perp \quad (33) \quad (ac) \mathbin{\frown} (bd) = (a \mathbin{\frown} b) (c \mathbin{\frown} d) \quad (36)$$

$$a \mathbin{\frown} b = a \sqcup b \quad (34) \quad (ac) \mathbin{\frown} \mathbf{nil} = \top \quad (37)$$

$$t \mathbin{\frown} t' = t \sqcup t' \quad (35) \quad a^\infty \mathbin{\frown} b^\infty = (a \mathbin{\frown} b)^\infty \quad (38)$$

Hence $a \mathbin{\frown} \alpha = a \sqcup \alpha = a$, i.e. α is the atomic step identity of weak conjunction. More generally, $\mathbf{chaos} \hat{=} \alpha^\omega$ is the identity of weak conjunction for any sequence of atomic steps. The following lemma (and its proof) is similar to the corollary of Lemma 6.

Lemma 8 (atomic-iteration-conjunction). $a^\omega \mathbin{\frown} b^\omega = (a \mathbin{\frown} b)^\omega$

A command c with a guarantee g is represented by $(\mathbf{guar} \ g) \mathbin{\frown} c$. In the theory of Jones, a guarantee on a process may be strengthened. That is reflected by the fact that if $g_1 \subseteq g_2$, then $\pi(g_2) \sqsubseteq \pi(g_1)$ and hence $(\pi\text{-restrict} \ g_2) \sqsubseteq (\pi\text{-restrict} \ g_1)$. A process that must satisfy both guarantee g_1 and guarantee g_2 , must satisfy $g_1 \cap g_2$ because

$$\begin{aligned} & (\pi\text{-restrict} \ g_1) \mathbin{\frown} (\pi\text{-restrict} \ g_2) \\ &= (\pi(g_1) \sqcap \mathcal{E}) \mathbin{\frown} (\pi(g_2) \sqcap \mathcal{E}) \\ &= (\pi(g_1) \mathbin{\frown} \pi(g_2)) \sqcap (\pi(g_1) \mathbin{\frown} \mathcal{E}) \sqcap (\mathcal{E} \mathbin{\frown} \pi(g_2)) \sqcap (\mathcal{E} \mathbin{\frown} \mathcal{E}) \\ &= \pi(g_1 \cap g_2) \sqcap \mathcal{E} \\ &= (\pi\text{-restrict} \ (g_1 \cap g_2)) \end{aligned}$$

The weak conjunction of a possibly infinite iteration of atomic steps distributes over the sequential composition of commands c and d .

Lemma 9 (atomic-infinite-distribution). $a^\omega \mathbin{\frown} (cd) = (a^\omega \mathbin{\frown} c) (a^\omega \mathbin{\frown} d)$

The proof uses the canonical representation of a command (Theorem 1) and can be found in the appendix of [HCM⁺16]. As a consequence guarantees distribute over a sequence of commands.

$$(\mathbf{guar} \ g) \mathbin{\frown} (cd) = ((\mathbf{guar} \ g) \mathbin{\frown} c) ((\mathbf{guar} \ g) \mathbin{\frown} d)$$

6.2 The rely command

A rely condition r represents an assumption about environment steps. If any environment step does not satisfy r , i.e. a step that refines $\epsilon(\bar{r})$, the process may do anything, which can be represented by it aborting. Any other step is allowed. The rely command is defined in terms of a single step assumption, itself defined in terms of the abstract command **assume** (26) as follows.

$$\begin{aligned} (\epsilon\text{-assm} \ r) & \hat{=} \mathbf{assume}(!\epsilon(\bar{r})) &= !\epsilon(\bar{r}) \sqcap \epsilon(\bar{r}) \perp \\ \mathbf{rely} \ r & \hat{=} (\epsilon\text{-assm} \ r)^\omega \end{aligned}$$

An environment assumption is placed on a command c by placing the assumption on every step of c , i.e. $(\mathbf{rely} \ r) \mathbin{\frown} c$. A command c with rely r and guarantee g is expressed as $(\mathbf{rely} \ r) \mathbin{\frown} (\mathbf{guar} \ g) \mathbin{\frown} c$, for which every program step is required to satisfy g unless an

environment step does not satisfy r , in which case it aborts. Here using weak conjunction ($\text{\textcircled{m}}$) rather than the lattice join (\sqcup) is essential to prevent the guarantee masking the possible aborting behaviour of the rely. Because $\mathbf{assume} a \text{\textcircled{m}} \mathbf{assume} b = \mathbf{assume}(a \sqcup b)$, combining environment assumptions gives

$$\begin{aligned} (\epsilon\text{-assm } r_1) \text{\textcircled{m}} (\epsilon\text{-assm } r_2) &= \mathbf{assume}(!\epsilon(\overline{r_1}) \sqcup !\epsilon(\overline{r_2})) = \mathbf{assume}(!\epsilon(\overline{r_1 \cap r_2})) \\ &= (\epsilon\text{-assm}(r_1 \cap r_2)) . \end{aligned}$$

From Lemma 9 and Theorem 1, a rely can be distributed over a sequential composition (the proof is included in the appendix of [HCM⁺16]).

$$(\mathbf{rely } r) \text{\textcircled{m}} (c d) = (\mathbf{rely } r \text{\textcircled{m}} c) (\mathbf{rely } r \text{\textcircled{m}} d)$$

6.3 Rely/Guarantee Logic

Rely/guarantee reasoning is traditionally formulated in terms of a quintuple $\{p, r\}c\{g, q\}$, which extends Hoare logic with the rely r and guarantee g to handle concurrency. The quintuple states that every step of c satisfies g and that it terminates and establishes the postcondition q , provided it is executed from an initial state satisfying p and interference from the environment is bounded by r . This quintuple is interpreted in our logic as the following refinement.⁶

$$\{p\} ((\mathbf{rely } r) \text{\textcircled{m}} (\mathbf{guar } g) \text{\textcircled{m}} [q]) \sqsubseteq c$$

This demonstrates the application of the algebra to reasoning about shared data. As well as being able to express any law presented in terms of quintuples, we are able to reason about the component commands separately, e.g., strengthening a guarantee g does not involve p , r and q .

7 Abstract communication in process algebras

In the process algebra domain, processes communicate via a set of synchronisation events, in contrast to processes in a shared memory concurrency model which interleave operations on state. We may build a core process algebra from the basic operators, with the addition of a set of atomic program steps $\pi(a)$ that model a process engaging in the corresponding abstract event $a \in \mathit{Event}$, where Event includes at least the silent event ι . The basic properties of this language are those of the underlying algebra but we do not assume conjunctivity of sequential composition (9) in order to be consistent with CCS.

Similarly to notation introduced in Section 4.2 we define

$$\langle a \rangle \hat{=} \mathcal{E}^\omega \pi(a) \mathcal{E}^\omega \tag{39}$$

This models process engaging in event a (note that we drop the ‘ π ’ tag from the $\langle a \rangle$ notation) preceded and succeeded by steps of the environment, similar to *asynchronising*

⁶ We use the syntax of Morgan’s specification command $[q]$ [Mor88] whose definition is omitted for space reasons. It represents any sequence of atomic steps that establishes q between its initial and final states. See [CHM16] for details.

in Synchronous CCS [Mil83] (discussed in [Mil89]). This is the building block of event based languages: we interpret both prefixing in CCS ($a.p$) and CSP ($a \rightarrow p$) as $\langle a \rangle p$. We extend the core algebra to give two types of abstract interprocess communication: CCS-style binary synchronisation (achieved by restricting the program) and CSP-style multi-way synchronisation (achieved in-part by restricting the environment).

7.1 Communication in CCS

The main point of difference with the rely-guarantee algebra is that program steps representing events can combine into a single program step (communication). Interactions with \mathcal{E} remain the same as in the abstract algebra. In CCS each non-silent event a has a complementary event \bar{a} . A program step $\pi(a)$ and its corresponding complementary program step $\pi(\bar{a})$ may synchronise to become a silent step, $\pi(a) \parallel \pi(\bar{a}) = \pi(\iota)$, and hence using an instantiation of Lemma 7,

$$\langle a \rangle \parallel \langle \bar{a} \rangle = \langle \iota \rangle \sqcap \langle a \rangle \langle \bar{a} \rangle \sqcap \langle \bar{a} \rangle \langle a \rangle . \quad (40)$$

As such, events may synchronise *or* interleave. In CCS the restriction operator $p \setminus A$, where A is a set of *Events*, may be employed to exclude the final two interleaving options and hence force processes to synchronise and generate a silent step. It may be defined straightforwardly using join (\sqcup) to forbid events in A , where we use the abbreviation $\pi(A) \hat{=} \prod_{a \in A} \pi(a)$ and note that $! \pi(A) = \pi(\bar{A}) \sqcap \mathcal{E}$.

$$p \setminus A \hat{=} p \sqcup (! \pi(A))^\omega \quad (41)$$

Hence, by (40) and (41), $(\langle a \rangle \parallel \langle \bar{a} \rangle) \setminus \{a, \bar{a}\} = \langle \iota \rangle$.

7.2 Communication in CSP

To achieve CSP-style multi-way communication, a process p prevents its environment from communicating via an event in p 's alphabet until p is ready. We introduce a step $\epsilon(a)$, where $\mathcal{E} \sqsubseteq \epsilon(a)$ for all $a \in \text{Event}$. Its interactions through the parallel operator are defined (in a different way to CCS) below; all other combinations of atomic steps result in \top .

$$\pi(a) \parallel \pi(a) = \pi(a) \quad \text{for } a \neq \iota \quad \pi(a) \parallel \epsilon(a) = \pi(a) \quad \epsilon(a) \parallel \epsilon(a) = \epsilon(a)$$

Fundamental to CSP is the notion of a process's *alphabet*, the set of events via which it may communicate and in particular upon which the environment may not independently synchronise. Here we explicitly associate an alphabet $A \subseteq \text{Event}$ with process p by the syntax $A:p$, defined by,

$$A:p \hat{=} p \sqcup (! \epsilon(A))^\omega \quad (42)$$

where analogously to program steps we define $\epsilon(A) \hat{=} \prod_{a \in A} \epsilon(a)$. Note the similarity to CCS's restriction operator (41) but here it is the environment that is restricted, rather than the program.

In an early formulation by Hoare [Hoa85] every process p implicitly has an alphabet A associated with it (A is sometimes syntactically deduced from p). In formulations

such as Roscoe’s [Ros98] the alphabets are not associated with processes but are instead made explicit on the parallel operator. We may define alphabetised parallel straightforwardly as $p_1 \parallel_A p_2 \hat{=} (A:p_1) \parallel (A:p_2)$. Each side of the parallel composition prevents the other from taking a unilateral program step on events in A by restricting its environment. Some of the basic communication properties from CSP follow from the above definitions and the atomic algebra, for instance, recalling that CSP’s prefixing operator $a \rightarrow p \hat{=} \langle a \rangle p$, for any $a \in A$, $(a \rightarrow p_1 \parallel_A a \rightarrow p_2) = a \rightarrow (p_1 \parallel_A p_2)$.

The *hiding* operator of CSP, p/A , affects program steps, renaming events in A to silent events. Hiding distributes over sequential and choice (but not parallel); its relationship with atomic steps is $b/A = \begin{cases} \pi(\iota) & \text{if } b \text{ is of the form } \pi(a) \text{ and } a \in A \\ b & \text{otherwise} \end{cases}$.

7.3 Communication in SCCS

Synchronous CCS (SCCS) [Mil83,Mil89] is a process algebra designed to be as minimal as possible in terms of operators. It includes event prefix, disjunction (nondeterministic choice), composition (corresponding to our parallel), and restriction similar to that of CCS (41). SCCS events may be structured from a finite set of “particles”, forming a commutative group $(Event, \mathbf{1}, \times, ^{-1})$. Every event is the product of particles: for instance, the step a is an event $(a^1 \times b^0 \times c^0 \times \dots)$. The silent (or waiting) event $\mathbf{1}$ is event identity, and fulfils a similar role to that of \mathcal{E} in our algebra. The complement of event a is simply a^{-1} and hence the product of an event and its complement, $a^1 \times a^{-1}$, naturally equals $\mathbf{1}$.

The key aspect of SCCS is its simple definition of parallel composition in terms of product: for atomic steps a and b , $a \parallel b = a \times b$. An event process $\langle a \rangle$ is defined as $\mathbf{1}^\omega a \mathbf{1}^\omega$, which has the effect of *asynchronising* the event, preserving Lemma 7. Milner shows that CCS can be encoded in SCCS through the addition of asynchronising actions defined through the operational semantics; in an algebraic setting the $\mathbf{1}$ s are made explicit in the processes. Note that in this model there is no distinction between silent steps and environment steps: in SCCS both are $\mathbf{1}$, whereas in CCS the former is $\pi(\iota)$.

8 Related Work

Our Concurrent Refinement Algebra (CRA) (Section 2) compares to Concurrent Kleene Algebra (CKA) [HMSW11] in that both extend a sequential algebra to allow for reasoning about parallel composition. Synchronous Kleene Algebra (SKA) [Pri10] is also based on Kleene Algebra but, unlike CKA, it adds tests and a synchronous parallel operator based on that of Milner’s SCCS [Mil83]. Both CKA and SKA are based on Kleene algebra and hence only support finite iteration and partial correctness. In comparison, our CRA supports general fixed points and hence recursion and both finite and infinite iteration. The richer structure of DRA contains a sub-lattice of commands below **chaos** (see Fig. 1) that includes assertions (and hence preconditions in the relational interpretation) and assumptions (and hence rely commands), and allows the weak conjunction operator, \mathfrak{m} , to be distinguished from strong conjunction, \sqcup . All these constructs are needed to faithfully represent rely/guarantee theory.

CKA is also applied to rely/guarantee rules [HMSW11] but they define a Jones-style 5-tuple (as in Section 6.3) in terms of two separate refinement conditions, whereas in our approach the existing (single) refinement relation can be used directly. In Jones' theory, a guarantee has to be satisfied only from initial states satisfying the precondition of the program, and further, if its rely condition is broken by the environment, the program can abort. However, in the CKA framework, the guarantee has to always be maintained by the program, irrespective of what the initial state is and how the environment is behaving; that over restricts the set of possible implementations. Our theory faithfully reflects Jones' approach.

Our algebra of atomic steps makes use of a synchronous parallel operator similar to that in SCCS [Mil89] and in SKA [Pri10] but it differs in two ways:

- instead of atomic actions being separate from commands (as in SCCS and SKA), they are treated as a sub-algebra within CRA and
- while both SCCS and SKA explicitly define composition of atomic steps (their \times operator), our parallel operator is used directly on atomic steps (because they are commands) and its definition is left open.

9 Conclusion

This paper presents an abstract algebra of atomic steps for concurrent programs. It is a Boolean algebra that is embedded as a sub-lattice into our Concurrent Refinement Algebra in a similar way as tests are embedded in Kleene algebras. As for tests, a range of useful laws can be derived for atomic steps within this abstract algebra (e.g., on iteration and distributivity), despite the fact that the interpretation of the parallel composition of two atomic steps is left open.

This construction simplifies many essential laws and their proofs, as most supporting lemmas almost come for free on this abstract level. Accordingly, the mechanisation of the theory within the theorem prover Isabelle is lean and achieved a high degree of automation. As the Concurrent Refinement Algebra was conceived to support reasoning with relies and guarantees this simplification is of particular benefit in our laws for rely and guarantee commands.

A further gain of the generic shape of the abstract algebra lies in its potential for reuse. We have demonstrated this by instantiating our abstract algebra with two quite different styles of communication, a synchronous model (as in SKA [Pri10] and SCCS) versus an interleaving model (as in CCS and CSP). For both styles the abstract algebra of atomic steps proves to be suitable.

The concept of sub-algebras in our Concurrent Refinement Algebra is also applicable to assertions and assumptions. Assertions form a Boolean algebra with **nil** as top element and \perp as bottom element whereas step assumptions form a Boolean algebra with top element α and bottom $\alpha \perp$. Both inherit the laws on Boolean algebras similarly to tests and atomic steps. Future work will investigate these structures and will extend our theories accordingly.

The relationship between CCS and CSP has been explored in several papers [Bro83,vG97] including augmenting the operational rules of CSP so that the failures-divergences model (FDR) is respected in CCS [HH10]. Future work is to apply a more

algebraic approach to the relationships between well known process algebras (especially ACP [BK84]).

Acknowledgements. This work has benefited from input from Cliff Jones and Kim Solin.

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10 Proofs for inspection

For all lemmas we assume a and b to be atomic steps and c and d arbitrary commands. Furthermore, all lemmas except Lemmas 1 and 9 assume that the sequential operator is conjunctive (9) as this property is used within the proofs.

Lemma 4 (atomic-iteration-finite)

$$a^* c \parallel b^* d = (a \parallel b)^* ((c \parallel d) \sqcap (c \parallel b b^* d) \sqcap (a a^* c \parallel d))$$

Proof. The proof relies on (11), i.e., $a^* = \prod_{i \in \mathbb{N}} a^i$. The notation $\prod_{i,j \in \mathbb{N}}^{i < j} c_{i,j}$ stands for the choice of $c_{i,j}$ over all natural numbers i and j , such that $i < j$.

$$\begin{aligned}
& a^* c \parallel b^* d \\
&= (\prod_{i \in \mathbb{N}} a^i c) \parallel (\prod_{j \in \mathbb{N}} b^j d) = \prod_{i \in \mathbb{N}, j \in \mathbb{N}} (a^i c \parallel b^j d) \\
&= \prod_{i \in \mathbb{N}} (a^i c \parallel b^i d) \sqcap \prod_{i,j \in \mathbb{N}}^{i < j} (a^i c \parallel b^i b^{j-i} d) \sqcap \prod_{i,j \in \mathbb{N}}^{i > j} (a^i a^{i-j} c \parallel b^j d) \\
&= \prod_{i \in \mathbb{N}} (a \parallel b)^i (c \parallel d) \sqcap \prod_{i,k \in \mathbb{N}}^{k > 0} (a \parallel b)^i (c \parallel b^k d) \sqcap \prod_{j,k \in \mathbb{N}}^{k > 0} (a \parallel b)^j (a^k c \parallel d) \\
&= (a \parallel b)^* (c \parallel d) \sqcap (\prod_{i \in \mathbb{N}} (a \parallel b)^i) \prod_{k \in \mathbb{N}}^{k > 0} (c \parallel b^k d) \\
&\quad \sqcap (\prod_{j \in \mathbb{N}} (a \parallel b)^j) \prod_{k \in \mathbb{N}}^{k > 0} (a^k c \parallel d) \\
&= (a \parallel b)^* ((c \parallel d) \sqcap (c \parallel \prod_{k \in \mathbb{N}}^{k > 0} b^k d)) \sqcap (\prod_{k \in \mathbb{N}}^{k > 0} a^k c \parallel d) \\
&= (a \parallel b)^* ((c \parallel d) \sqcap (c \parallel b b^* d) \sqcap (a a^* c \parallel d))
\end{aligned}$$

□

Lemma 5 (atomic-iteration-finite-infinite)

$$a^* c \parallel b^\infty = (a \parallel b)^* (c \parallel b^\infty)$$

Proof. Note that, by unfolding law (7), $b^\infty = b^i b^\infty$ for any $i \in \mathbb{N}$. The proof uses also Lemma 3.

$$\begin{aligned}
a^* c \parallel b^\infty &= (\prod_{i \in \mathbb{N}} a^i c) \parallel b^\infty \\
&= \prod_{i \in \mathbb{N}} (a^i c \parallel b^\infty) \\
&= \prod_{i \in \mathbb{N}} (a^i c \parallel b^i b^\infty) \\
&= \prod_{i \in \mathbb{N}} (a \parallel b)^i (c \parallel b^\infty) \\
&= (a \parallel b)^* (c \parallel b^\infty)
\end{aligned}$$

□

Lemma 6 (atomic-iteration-either)

$$a^\omega c \parallel b^\omega d = (a \parallel b)^\omega ((c \parallel d) \sqcap (c \parallel b b^\omega d) \sqcap (a a^\omega c \parallel d))$$

Proof. Note that, by (10) and (8), $a^\omega = a^* \sqcap a^\infty$ and $a^\infty c = a^\infty$, and by (13), $a a^* c \sqcap a^\infty = a a^\omega c$. The proof uses also (12), and Lemmas 4 and 5, and (28), i.e. $a^\infty \parallel b^\infty = (a \parallel b)^\infty$.

$$\begin{aligned}
 & a^\omega c \parallel b^\omega d \\
 &= (a^* \sqcap a^\infty) c \parallel (b^* \sqcap b^\infty) d \\
 &= (a^* c \parallel b^* d) \sqcap (a^* c \parallel b^\infty) \sqcap (a^\infty \parallel b^* d) \sqcap (a^\infty \parallel b^\infty) \\
 &= (a \parallel b)^* ((c \parallel d) \sqcap (c \parallel b b^* d) \sqcap (a a^* c \parallel d)) \sqcap \\
 &\quad (a \parallel b)^* (c \parallel b^\infty) \sqcap (a \parallel b)^* (a^\infty \parallel d) \sqcap (a \parallel b)^\infty \\
 &= (a \parallel b)^* ((c \parallel d) \sqcap ((c \parallel b b^* d) \sqcap (c \parallel b^\infty))) \sqcap \\
 &\quad ((a a^* c \parallel d) \sqcap (a^\infty \parallel d)) \sqcap (a \parallel b)^\infty \\
 &= (a \parallel b)^* ((c \parallel d) \sqcap (c \parallel (b b^* d \sqcap b^\infty))) \sqcap ((a a^* c \sqcap a^\infty) \parallel d) \sqcap (a \parallel b)^\infty \\
 &= (a \parallel b)^* ((c \parallel d) \sqcap (c \parallel b b^\omega d) \sqcap (a a^\omega c \parallel d)) \sqcap (a \parallel b)^\infty \\
 &= (a \parallel b)^\omega ((c \parallel d) \sqcap (c \parallel b b^\omega d) \sqcap (a a^\omega c \parallel d))
 \end{aligned}$$

□

Lemma 1 (atomic-identity-iteration) $\mathcal{E}^\omega \parallel c = c$

Proof. Theorem 1 states that c can be represented as $t \sqcap t' \perp \sqcap \prod_{i \in I} a_i c_i$. The proof is via structural induction and hence we assume $\mathcal{E}^\omega \parallel c_i = c_i$, for all $i \in I$.

$$\begin{aligned}
 \mathcal{E}^\omega \parallel c &= \mathcal{E}^\omega \parallel (t \sqcap t' \perp \sqcap \prod_{i \in I} a_i c_i) \\
 &= (\mathcal{E}^\omega \parallel t) \sqcap (\mathcal{E}^\omega \parallel t' \perp) \sqcap (\mathcal{E}^\omega \parallel \prod_{i \in I} a_i c_i) \\
 &= t \sqcap t' \perp \sqcap \prod_{i \in I} (\mathcal{E}^\omega \parallel a_i c_i) \\
 &= t \sqcap t' \perp \sqcap \prod_{i \in I} (\mathcal{E} \mathcal{E}^\omega \parallel a_i c_i) \\
 &= t \sqcap t' \perp \sqcap \prod_{i \in I} a_i (\mathcal{E}^\omega \parallel c_i) \\
 &= t \sqcap t' \perp \sqcap \prod_{i \in I} a_i c_i \quad \text{by inductive hypothesis} \\
 &= c
 \end{aligned}$$

Note that $\mathcal{E}^\omega \parallel t = t (\mathcal{E}^\omega \parallel \mathbf{nil}) = t \mathbf{nil} = t$ and $\mathcal{E}^\omega \parallel t' \perp = t' (\mathcal{E}^\omega \parallel \perp) = t' \perp$ because $\mathcal{E}^\omega \parallel \perp \sqsubseteq \mathcal{E} \parallel \perp = \perp$ and hence $\mathcal{E}^\omega \parallel \perp = \perp$.

□

Lemma 7 (atomic-interleaving)

$$\langle a \rangle \parallel \langle b \rangle = \langle a \parallel b \rangle \sqcap \langle a \rangle \langle b \rangle \sqcap \langle b \rangle \langle a \rangle$$

Proof. The proof uses Lemmas 6 and 1 and $\mathcal{E}^\omega \mathcal{E}^\omega = \mathcal{E}^\omega$.

$$\begin{aligned}
& \langle a \rangle \parallel \langle b \rangle \\
&= (\mathcal{E}^\omega a \mathcal{E}^\omega) \parallel (\mathcal{E}^\omega b \mathcal{E}^\omega) \\
&= (\mathcal{E}^\omega (a \mathcal{E}^\omega \parallel b \mathcal{E}^\omega) \cap (\mathcal{E}^\omega (a \mathcal{E}^\omega \parallel \mathcal{E} \mathcal{E}^\omega b \mathcal{E}^\omega)) \cap (\mathcal{E}^\omega (\mathcal{E} \mathcal{E}^\omega a \mathcal{E}^\omega \parallel b \mathcal{E}^\omega))) \\
&= (\mathcal{E}^\omega (a \parallel b) \mathcal{E}^\omega) \cap (\mathcal{E}^\omega a (\mathcal{E}^\omega \parallel \mathcal{E}^\omega b \mathcal{E}^\omega)) \cap (\mathcal{E}^\omega b (\mathcal{E}^\omega a \mathcal{E}^\omega \parallel \mathcal{E}^\omega)) \\
&= (\mathcal{E}^\omega (a \parallel b) \mathcal{E}^\omega) \cap (\mathcal{E}^\omega a \mathcal{E}^\omega b \mathcal{E}^\omega) \cap (\mathcal{E}^\omega b \mathcal{E}^\omega a \mathcal{E}^\omega) \\
&= \langle a \parallel b \rangle \cap \langle a \rangle \langle b \rangle \cap \langle b \rangle \langle a \rangle
\end{aligned}$$

□

Lemma 9 (atomic-infinite-distribution)

$$a^\omega \pitchfork (c d) = (a^\omega \pitchfork c) (a^\omega \pitchfork d)$$

Proof. Using Theorem 1 we may assume $c = t \cap t' \perp \cap \prod_{i \in I} b_i c_i$. The proof follows by induction on command c , i.e. assume the lemma holds for c_i .

$$\begin{aligned}
& a^\omega \pitchfork (c d) \\
&= a^\omega \pitchfork ((t \cap t' \perp \cap \prod_{i \in I} b_i c_i) d) \\
&= a^\omega \pitchfork (t d \cap t' \perp \cap \prod_{i \in I} b_i c_i d) \\
&= (a^\omega \pitchfork t d) \cap (a^\omega \pitchfork t' \perp) \cap (a^\omega \pitchfork \prod_{i \in I} b_i c_i d) \\
&= t (a^\omega \pitchfork d) \cap t' \perp \cap (a^\omega \pitchfork \prod_{i \in I} b_i c_i d) \\
&= t (a^\omega \pitchfork d) \cap t' \perp \cap (a^\omega \pitchfork d) \cap \prod_{i \in I} (a \pitchfork b_i) (a^\omega \pitchfork c_i d) \\
&= (t \cap t' \perp) (a^\omega \pitchfork d) \cap \prod_{i \in I} (a \pitchfork b_i) (a^\omega \pitchfork c_i) (a^\omega \pitchfork d) \\
&= (t \cap t' \perp) (a^\omega \pitchfork d) \cap \prod_{i \in I} (a a^\omega \pitchfork b_i c_i) (a^\omega \pitchfork d) \\
&= (t \cap t' \perp \cap \prod_{i \in I} (a^\omega \pitchfork b_i c_i)) (a^\omega \pitchfork d) \\
&= ((a^\omega \pitchfork t) \cap (a^\omega \pitchfork t' \perp) \cap (a^\omega \pitchfork \prod_{i \in I} b_i c_i)) (a^\omega \pitchfork d) \\
&= (a^\omega \pitchfork (t \cap t' \perp \cap \prod_{i \in I} b_i c_i)) (a^\omega \pitchfork d) \\
&= (a^\omega \pitchfork c) (a^\omega \pitchfork d)
\end{aligned}$$

□

Lemma 10. For any test t , $a^\omega \pitchfork t = t$

Proof.

$$\begin{aligned}
 & a^\omega \pitchfork t \\
 &= (\mathbf{nil} \sqcap aa^\omega) \pitchfork t \\
 &= (\mathbf{nil} \pitchfork t) \sqcap (aa^\omega \pitchfork t) \\
 &= t \sqcap t(aa^\omega \pitchfork \mathbf{nil}) = t \sqcap t\top = t
 \end{aligned}$$

□

Lemma 11. For any test t , $a^\omega \pitchfork t\perp = t\perp$

Proof.

$$\begin{aligned}
 & a^\omega \pitchfork t\perp \\
 &= (\mathbf{nil} \sqcap aa^\omega) \pitchfork t\perp \\
 &= (\mathbf{nil} \pitchfork t\perp) \sqcap (aa^\omega \pitchfork t\perp) \\
 &= t(\mathbf{nil} \pitchfork \perp) \sqcap t(aa^\omega \pitchfork \perp) = t\perp \sqcap t\perp = t\perp
 \end{aligned}$$

□

Lemma 12 (rely-distribution).

$$(\mathbf{rely} \ r) \pitchfork (c \ d) = ((\mathbf{rely} \ r) \pitchfork c) ((\mathbf{rely} \ r) \pitchfork d)$$

Proof. We base the proof on the more general concept of *assumptions*. Using the definition of environment assumptions, $(\epsilon\text{-assm} \ r) = \mathbf{assume}(!\epsilon(\bar{r})) = !\epsilon(\bar{r}) \sqcap \epsilon(\bar{r})\perp$ and relies, $\mathbf{rely} \ r \hat{=} (\epsilon\text{-assm} \ r)^\omega$, we can deduce

$$\exists a \bullet \mathbf{rely} \ r = (\mathbf{assume} \ a)^\omega = a^\omega \sqcap a^\omega !a\perp$$

Furthermore, using Theorem 1 we may assume $c = t \sqcap t' \perp \sqcap \prod_{i \in I} b_i c_i$.

$$\begin{aligned}
 & (\mathbf{assume} \ a)^\omega \pitchfork (c \ d) \\
 &= (a^\omega \sqcap a^\omega !a\perp) \pitchfork (c \ d) \\
 &= a^\omega (\mathbf{nil} \sqcap !a\perp) \pitchfork (c \ d) \\
 &= \quad \mathbf{assume} \ x \hat{=} \mathbf{nil} \sqcap !a\perp \text{ and the canonical form of } c \\
 & \quad a^\omega x \pitchfork (t \sqcap t' \perp \sqcap \prod_{i \in I} b_i c_i) d \\
 &= \quad \text{distribute } \sqcap \text{ over } \pitchfork \text{ and left-distributivity of sequential} \\
 & \quad a^\omega x \pitchfork t d \sqcap a^\omega x \pitchfork t' \perp \sqcap (\prod_{i \in I} a^\omega x \pitchfork b_i c_i d)
 \end{aligned}$$

$$\begin{aligned}
&= \text{by unfolding of } a^\omega \text{ using (4)} \\
&t(a^\omega x \pitchfork d) \sqcap t' \perp \sqcap \left(\prod_{i \in I} \boxed{(aa^\omega x \sqcap x) \pitchfork b_i c_i d} \right) \\
&\quad (aa^\omega x \sqcap x) \pitchfork b_i c_i d \\
&= \text{by definition of } x \\
&\quad (a \pitchfork b_i)(a^\omega x \pitchfork c_i d) \sqcap (!a \pitchfork b_i)(\perp \pitchfork c_i d) \\
&= \text{by induction assumption: } a^\omega x \pitchfork c_i d = (a^\omega x \pitchfork c_i)(a^\omega x \pitchfork d) \\
&\quad (a \pitchfork b_i)(a^\omega x \pitchfork c_i)(a^\omega x \pitchfork d) \sqcap (!a \pitchfork b_i) \perp \\
&= \text{with (33) and } \perp \text{ left annihilator} \\
&\quad (a \pitchfork b_i)(a^\omega x \pitchfork c_i)(a^\omega x \pitchfork d) \sqcap (!a \perp \pitchfork b_i c_i)(a^\omega x \pitchfork d) \\
&= \text{with (37)} \\
&\quad (aa^\omega x \pitchfork b_i c_i)(a^\omega x \pitchfork d) \sqcap ((!a \perp \sqcap \mathbf{nil}) \pitchfork b_i c_i)(a^\omega x \pitchfork d) \\
&= \text{with (3)} \\
&\quad ((aa^\omega x \pitchfork b_i c_i) \sqcap (!a \perp \sqcap \mathbf{nil}) \pitchfork b_i c_i)(a^\omega x \pitchfork d) \\
&= \text{distribute } \pitchfork \text{ over } \sqcap, \text{ with definition of } x \text{ and (4)} \\
&\quad (a^\omega x \pitchfork b_i c_i)(a^\omega x \pitchfork d) \\
&= t(a^\omega x \pitchfork d) \sqcap t' \perp (a^\omega x \pitchfork d) \sqcap \left(\prod_{i \in I} \boxed{(a^\omega x \pitchfork b_i c_i)(a^\omega x \pitchfork d)} \right) \\
&= \text{distribute of sequential over } \sqcap \\
&\quad (t \sqcap t' \perp \sqcap (a^\omega x \pitchfork \prod_{i \in I} b_i c_i))(a^\omega x \pitchfork d) \\
&= \text{by Lemmas 10 and 11} \\
&\quad ((a^\omega x \pitchfork t) \sqcap (a^\omega x \pitchfork t' \perp) \sqcap (a^\omega x \pitchfork \prod_{i \in I} b_i c_i))(a^\omega x \pitchfork d) \\
&= \text{with Theorem 1} \\
&\quad (a^\omega x \pitchfork c)(a^\omega x \pitchfork d)
\end{aligned}$$

□