An Upper Bound on Burning Number of Graphs

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Abstract

The burning number $b(G)$ of a graph G was introduced by Bonato, Janssen, and Roshanbin [Lecture Notes in Computer Science 8882 (2014)] for measuring the speed of the spread of contagion in a graph. They proved for any connected graph G of order n, $b(G) \leq 2\lceil\sqrt{n}\rceil - 1$, and conjectured that $b(G) \leq \lceil\sqrt{n}\rceil$. In this paper, we proved $b(G) \leq \lceil\frac{-3+\sqrt{24n+33}}{4}\rceil$, which is roughly $\frac{\sqrt{6}}{2}\sqrt{n}$. We also settled the following conjecture of Bonato-Janssen-Roshanbin: $b(G)b(\bar{G}) \leq n+4$ provided both G and $\bar{G}$ are connected.

1 Introduction

The burning number of a graph was introduced by Bonato-Janssen-Roshanbin [3, 4, 10]. It is related to the contact processes on graphs such as the Firefighter problem [6, 8, 9]. In the paper [3, 4], Bonato-Janssen-Roshanbin considered a graph process which they called *burning*. At the beginning of the process, all vertices are *unburned*. During each round, one may choose an unburned vertex and change its status to *burned*. At the same time, each of the vertices that are already burned, will remain burned and spread to all of its neighbors and change their status to burned. A graph is called *k-burnable* if it can be burned in at most k steps. The *burning number* of a graph G, denoted by $b(G)$, is the minimum number of rounds necessary to burn all vertices of the graph. For example, $b(K_n) = 2$, $b(P_4) = 2$, and $b(C_5) = 3$. In the paper [4], they proved $b(P_n) = \lceil n^{1/2}\rceil$. Based on this result, Bonato-Janssen-Roshanbin [4] made the following conjecture.

Conjecture 1: *for any connected graph G of order n, $b(G) \leq \lceil n^{1/2}\rceil$.*

Bonato-Janssen-Roshanbin [3, 4] proved $b(G) \leq 2\lceil n^{1/2}\rceil - 1$. The previous best known bound is due to Bonato et al. [7]:

$$b(G) \leq \left(\sqrt{\frac{32}{19}} + o(1)\right)\sqrt{n}.$$

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In this paper, we improved the upper bound of $b(G)$ as follows.

Theorem 1. *If G is a connected graph of order n, then*

$$b(G) \leq \left\lceil \frac{-3+\sqrt{24n+33}}{4} \right\rceil.$$

In the paper [4], Bonato, Janssen, and Roshanbin also considered Nordhaus-Gaddum Type problem on the burning number. Let $\bar{G}$ be the complement graph of the graph G. In [4], they proved $b(G)+b(\bar{G}) \leq n+2$ and $b(G)b(\bar{G}) \leq 2n$. Both bounds are tight and are achieved by the complete graph and its complement. When both graphs G and $\bar{G}$ are connected, they proved $b(G)+b(\bar{G}) \leq 3\lceil n^{1/2}\rceil - 1$ and $b(G)b(\bar{G}) \leq n+6$ for all graph G_n of order $n \geq 6$. The following conjecture has been made in [4]:

Conjecture 2: *If both G and $\bar{G}$ are connected graphs of order n, then $b(G)b(\bar{G}) \leq n+4$.*

Using Theorem 1, we settled this conjecture positively.

Theorem 2. *If both G and $\bar{G}$ are connected graphs of order n, then*

$$b(G)b(\bar{G}) \leq n+4.$$

The equality holds if and only if $G = C_5$.

2 Notations and Lemmas

For each positive integer k, let $[k]$ denote the set $\{1, 2, \ldots, k\}$. A graph $G = (V, E)$ consists of a set of vertices V and edges E. The *order* of G, dented by $|G|$, is the number of vertices in G. A graph G is called *connected* if for any two vertices there is a path connecting them. In this paper, we always assume that G is a connected graph. The *distance* between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest path from u to v in graph G. The *eccentricity* of a vertex v is the maximum distance between v and any other vertex in G. The maximum eccentricity is the *diameter* $D(G)$ while the minimum eccentricity is the *radius* $r(G)$. The *center* of G is the set of vertices of eccentricity equal to the radius.

For any nonnegative integer k and a vertex u, the *k-th closed neighborhood* of u is the set of vertices whose distance from u is at most k; denoted by $N_k[u]$. From the definition, a graph G is k-burnable if there is a *burning sequence* $v_1, \ldots, v_k$ of vertices such that

$$V \subset \cup_{i=1}^{k} N_{k-i}[v_i] \tag{1}$$

$$\forall i, j \in [k] \colon d(x_i, x_j) \geq j - i. \tag{2}$$

The burning number $b(G)$ is the smallest integer k such that G is k-burnable. It has been shown that Condition (2) is redundant for the definition of burning number $b(G)$ (see Lemma 1 of [7]). It is often convenient to rewrite Condition (1) by relabeling the vertices in the burning sequence as follows:

$$V \subset \cup_{i=1}^{k} N_{i-1}[v_i]. \tag{3}$$

This leads the following generalization, which is very useful for the purpose of induction. For a set (or multiset) A of k positive integers $a_1, a_2, \ldots, a_k$ (not necessarily all distinct), we say a graph G is *A-burnable*, if there exist k vertices $v_1, v_2, \ldots, v_k$ such that $G \subseteq \cup_{i=1}^{k} N_{a_i-1}[v_i]$. Under this terminology, the burning number $b(G)$ is the least k so that G is $[k]$-burnable.

A *tree* is an acyclic connected graph. For any tree T, it is well-known that the center of T consists of either one vertex or two vertices. If the center of T consists of one vertex, then $D(T) = 2r(T)$; otherwise, $D(T) = 2r(T) - 1$. (See [2].)

A *rooted tree* is a tree with one vertex r designated as the *root*. The *height* of a rooted tree is the eccentricity of the root. In a rooted tree, the *parent* of a vertex is the vertex connected to it on the path to the root. A *child* of a vertex v is a vertex of which v is the parent. A *descendent* of any vertex v is any vertex which is either the child of v or is (recursively) the descendent of any of the children of v. A *leaf* vertex is a vertex with degree 1 but not equal to the root. The *subtree rooted at v* is the induced subgraph on the set of v and its all descendents. The important observation is that if a subtree rooted at v is pruned from the whole tree, the remaining part (if non-empty) is still a tree. This observation is very useful for induction.

A *spanning* tree of graph G is a subtree of G that covers all vertices of G. In the papers [3, 4], Bonato, Janssen, and Roshanbin proved

$$b(G) = \min\{b(T) : T \text{ is a spanning subtree of } G\}. \tag{4}$$

Thus, it is sufficient to only consider the burning number $b(T)$ for tree T.

First we prove a simple lemma, which illustrates the idea of the induction.

Lemma 1. *Let $A = \{a_1, a_2, \ldots, a_k\}$ be a set of k nonnegative integers. If a tree T has order at most $\sum_{i=1}^{k} a_i + \max\{a_i : 1 \leq i \leq k\} - 1$, then T is A-burnable.*

Proof. With loss of generality, we can assume that $a_1 \geq a_2 \geq \cdots \geq a_k$. We will use induction on k. Initial case: $k = 1$, $A = \{a_1\}$. We need to prove that if a tree T has at most $2a_1 - 1$ vertices, then T is A-burnable. Note that

$$r(T) \leq \frac{D(T)+1}{2} \leq \frac{n}{2} \leq a_1 - \frac{1}{2}.$$

Since the radius $r(T)$ is an integer, we must have $r(T) \leq a_1 - 1$. Thus T is $\{a_1\}$-burnable.

Now we assume the statement holds for any set of $k-1$ integers. For any A of k integers $a_1 \geq a_2 \geq \cdots \geq a_k > 0$ and any tree T with at most $2a_1 + a_2 + \cdots + a_k - 1$, we will prove that T is A-burnable. Pick an arbitrary vertex r as the root of T. Let h be the height of this rooted tree. If $h \leq a_1 - 1$, then $V(T) \subset N_{a_1-1}(r)$. I.e., T is $\{a_1\}$-burnable. Thus T is A-burnable.

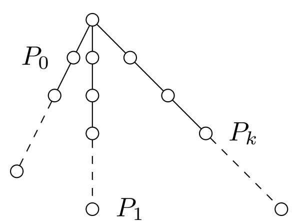
Now we assume $h \geq a_1$. Select a leaf vertex u such that $d(r, u) = h$. Let v_k be the vertex on the ru-path such that the distance $d(u, v_k) = a_k - 1$. (This is possible since $h \geq a_1 > a_k - 1$. Let T_1 be the subtree rooted at v_k, and $T_2 := T \setminus T_1$ be the remaining subtree. Notice that $|T_1| \geq a_k$. Thus,

$$\begin{aligned} |T_2| &= |T| - |T_1| \\ &\leq 2a_1 + a_2 + \cdots + a_k - 1 - a_k \\ &= 2a_1 + a_2 + \cdots + a_{k-1} - 1. \end{aligned}$$

By inductive hypothesis, T_2 is $\{a_1, a_2, \ldots, a_{k-1}\}$-burnable. Thus, there exists $k-1$ vertices $v_1, v_2, \ldots, v_{k-1}$ such that $T_2 \subseteq \cup_{i=1}^{k-1} N_{a_i - 1}[v_i]$. Also, notice $T_1 \subseteq N_{a_k - 1}[v_k]$. Therefore, $T \subseteq \cup_{i=1}^{k-1} N_{a_i - 1}[v_i]$. The proof of the lemma is finished. □

Remark 1. *The bound in Lemma 1 is tight.*

Proof. Consider the following example: for any positive integer a, let $a_1 = a_2 = \cdots = a_k = a$, i.e. A is a multiset consisting of k a's. Now we will construct a tree T as following. First construct $k+1$ disjoint paths $P_0, P_1, \ldots, P_k$ with each of order a. Create tree T by connecting one endpoint of $P_1, P_2, \ldots, P_k$ to the same endpoint of P_0 (see figure below).



The tree T has order $(k+1)a$, which is just one more than the amount of vertices in Lemma 1. Now we show T is not A-burnable. Otherwise, there exists $v_1, v_2, \ldots, v_k$ such that T is covered by $\cup_{i=1}^{k} N_a[v_i]$. By Pigeon-hole principle, one of the paths $P_0, P_1, \ldots, P_k$ will not contain $v_1, v_2, \ldots, v_k$, and the leaf vertex on this path is in any $N_{a-1}[v_i]$. Thus, T is not A-burnable. □

The following corollary is a slight improvement of Theorem 7 of [7].

Corollary 1. *For any connected graph G, $b(G) \leq \frac{-3+\sqrt{8n+17}}{2} \approx \sqrt{2n} - \frac{3}{2}$.*

Proof. Let $A = \{k, k-1, \cdots, 1\}$. By Lemma 1, any Tree of order $n \leq (\sum_{i=1}^{k} i) + k - 1 = \frac{k^2+3k-2}{2}$ is A-burnable. Solving k we get $k \leq \frac{-3+\sqrt{8n+17}}{2}$. Thus, $b(T) \leq \frac{-3+\sqrt{8n+17}}{2}$. By Equation (4), the same bound holds true for $b(G)$. □

3 Proof of Theorems 1 and 2

We have seen that Lemma 1 is sharp when all a_i's are equal. The improvement can be made when a_i's are distinct. Let $g(A)$ be a function of A so that any tree T with order at most $g(A)$ is A-burnable. In the proof of Lemma 1, we show that

$$g(A) \leq g(A \setminus \{a_k\}) + a_k.$$

The idea is to show a recursive bound

$$g(A) \leq \max_{1 \leq i \leq k-1} \{g(A \setminus \{a_i\}) + a_i\} + \left\lfloor \frac{k-1}{3} \right\rfloor$$

where k is the number of (distinct) elements in A. We first prove the following Lemma.

Lemma 2. *For any $k-1$ distinct positive integers $a_1 < a_2 < \cdots < a_{k-1}$, there exists an a_i such that $2\lfloor \frac{k-1}{3} \rfloor \leq a_i \leq a_{k-1} - \lfloor \frac{k-1}{3} \rfloor$.*

Proof. Let $j = \lfloor \frac{k-1}{3} \rfloor$ and $A = \{a_1, a_2, \ldots, a_{k-1}\}$. Divide $[1, a_{k-1}]$ into 3 intervals:

$$[1, 2j-1] \cup [2j, a_{k-1} - j] \cup [a_{k-1} - j + 1, a_{k-1}].$$

There are at most $2j-1$ elements of A in the first interval. There are at most j elements of A in the last interval. Since $3j - 1 < k - 1$, there exists at least one element of A in the middle interval. Call this element a_i. □

Lemma 3. *For all integer $k \geq 1$.*

$$\sum_{i=1}^{k} \left\lfloor \frac{i-1}{3} \right\rfloor = \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor.$$

Proof. For $k = 3s$, we have

$$\sum_{i=1}^{k} \left\lfloor \frac{i-1}{3} \right\rfloor = 3\sum_{j=1}^{s}(j-1) = \frac{3s(s-1)}{2} = \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor.$$

For $k = 3s + 1$, we have

$$\sum_{i=1}^{k} \left\lfloor \frac{i-1}{3} \right\rfloor = 3\sum_{j=1}^{s}(j-1) + s = \frac{3s(s-1)}{2} + s = \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor.$$

For $k = 3s + 2$, we have

$$\sum_{i=1}^{k} \left\lfloor \frac{i-1}{3} \right\rfloor = 3\sum_{j=1}^{s}(j-1) + 2s = \frac{3s(s-1)}{2} + 2s = \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor.$$

□

Theorem 3. *Let A be a set of k distinct positive integers $a_1 < a_2 < \cdots < a_k$. If a tree T has order at most*

$$\left(\sum_{i=1}^{k} a_i\right) + a_k - 1 + \left\lfloor \frac{k^2 - 3k + 2}{6} \right\rfloor.$$

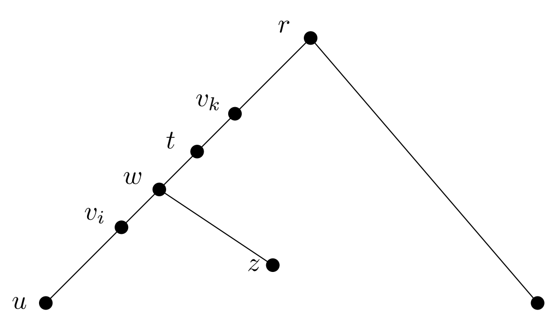
then T is A-burnable.

Proof. Let $f(k) := \lfloor \frac{k^2-3k+2}{6} \rfloor$. By Lemma 3, we have $f(k) = f(k-1) + \lfloor \frac{k-1}{3} \rfloor$. Now we use induction on k.

Initial case $k = 1$: $A = \{a_1\}$. by Lemma 1, if a tree T has order at most $2a_1 - 1$, then T is $\{a_1\}$-burnable. The statement holds true for $k = 1$ since $f(1) = 0$.

Now assume this statement holds true for any set of $k - 1$ distinct positive integers. Consider the case $A = \{a_1, a_2, \ldots, a_k\}$. We need to prove that if a tree T has order at most $a_1 + a_2 + \cdots + 2a_k - 1 + f(k)$ then T is A-burnable.

Let $j = \lfloor \frac{k-1}{3} \rfloor$. By Lemma 1, there exists a_i that satisfies $2j \le a_i \le a_{k-1} - j$. Choose an arbitrary root r and view T as a rooted tree. Let u be the leaf vertex which has the farthest distance away from the root r. If $d(r, u) \le a_k - 1$, then $V(T) \subset N_{a_k-1}(r)$; thus T is A-burnable. So, we can assume $d(r, u) \ge a_k$. We will name three vertices v_i, t, v_k on the ru-path such that $d(u, v_i) = a_i - 1$, $d(u, t) = a_i - 1 + j$, and $d(u, v_k) = a_{k-1}$. Let T_1 be the subtree rooted at t. There are two cases:



Case 1: $T_1 \subseteq N_{a_i-1}[v_i]$. Let $T_2 = T \setminus T_1$. Notice $|T_1| \ge a_i + j$. Then,

$$\begin{aligned} |T_2| &\le |T| - |T_1| \\ &\le a_1 + a_2 + \cdots + 2a_k - 1 + f(k) - (a_i + j) \\ &= a_1 + a_2 + \cdots + \hat{a}_i + \cdots + 2a_k - 1 + f(k-1). \end{aligned}$$

By inductive hypothesis, T_2 is $(A \setminus \{a_i\})$-burnable. Thus, T is A-burnable.

Case 2: $T_1 \not\subseteq N_{a_i-1}[v_i]$. Then there is a vertex $z \in T_1$ such that $d(v_i, z) \ge a_i$. Let w be the closest vertex on the path rt to z. Observe that w is not in the subtree rooted at v_i. Thus, w is between v_i and t. We have

$$d(w, z) = d(v_i, z) - d(v_i, w) \ge a_i - d(w, v_i) \ge a_i - d(v_i, t) \ge a_i - j \ge j.$$

The last inequality uses Lemma 2 for the choice of a_i.

Let v_k be a vertex on the path from u to the root with distance $d(u, v_k)$. Let T_3 be the subtree rooted at v_k and let $T_4 := T \setminus T_3$ be the remaining subtree. We have that $|T_3| \geq a_{k-1} + d(w, z) \geq a_{k-1} + j$.

$$\begin{aligned}|T_4| &\leq |T| - |T_3| \\ &\leq a_1 + a_2 + \cdots + 2a_k - 1 + f(k) - (a_{k-1} + j) \\ &= a_1 + a_2 + \cdots + a_{k-2} + 2a_k - 1 + f(k-1).\end{aligned}$$

By inductive hypothesis, T_4 is $(A\backslash\{a_{k-1}\})$-burnable. Clearly, T_3 is $\{a_{k-1}\}$-burnable. Putting together, T is A-burnable.

The inductive proof is finished. $\square$

Proof. Proof of Theorem 1 Let $A = (1, 2, \ldots, k)$. Applying Theorem 3, any tree of n vertices is $[k]$-burnable if

$$n \leq 1 + 2 + \cdots + k + k - 1 + \left\lfloor \frac{(k^2 - 3k + 2)}{6} \right\rfloor = \left\lfloor \frac{2k^2 + 3k - 2}{3} \right\rfloor.$$

Note that $\left\lfloor \frac{2k^2+3k-2}{3} \right\rfloor$ equals to $\frac{2k^2+3k-3}{3}$ if k is divisible by 3; equals to $\frac{2k^2+3k-2}{3}$ otherwise. In either case, G is $[k]$-burnable if $n \leq \frac{2k^2+3k-3}{3}$. Solving for k, we have $k \geq \frac{-3+\sqrt{24n+33}}{4}$. Since k is an integer, we can take ceiling on the bound of k. Thus for any tree T of n vertices,

$$b(T) \leq \left\lceil \frac{-3 + \sqrt{24n + 33}}{4} \right\rceil.$$

By equation (4), the same bound holds for all connected graphs G. $\square$

Lemma 4. *If G is connected and the radius satisfies $r(G) \geq 3$, then the complement $\bar{G}$ is also connected and $r(\bar{G}) \leq 2$.*

Proof. Since $r(G) \geq 3$, there exists a pair of vertex (u, v) with distance at least 3. Let S be the set of all neighbors of v in the graph G. For any vertex not in $S \cup \{v\}$, it is directly connected to v in the complement graph $\bar{G}$. For any vertex x in S, both xu and uv are edges of $\bar{G}$. Thus, the complement graph $\bar{G}$ has radius at most 2. $\square$

Proof of Theorem 2: By Lemma 4, either $r(G)$ or $r(\bar{G})$ is at most 2. Without loss of generality, we can assume $r(\bar{G}) \leq 2$, which implies $b(\bar{G}) \leq 3$. We have the following cases.

case 1 $n \leq 4$. Since both G and $\bar{G}$ are connected, the only graph G that can exist is the path P_4. In this case $G = \bar{G} = P_4$. Note, $b(P_4) = 2$. This satisfies

$$b(G) \cdot b(\bar{G}) = 4 < n + 4.$$

case 2 $n \geq 5$. By Theorem 1, $b(G_n) \leq \left\lceil \frac{-3+\sqrt{24n+33}}{4} \right\rceil$.

$$b(G) \cdot b(\bar{G}) \leq 3 \cdot \left\lceil \frac{-3+\sqrt{24n+33}}{4} \right\rceil.$$

Now we show this bound is at most $n+4$. When $n = 5, 6, 7, 8$, $\lceil \frac{-3+\sqrt{24n+33}}{4} \rceil = 3$, so $3 \cdot 3 = 9 \leq n + 4$. It holds for $n = 5, 6, 7$.

Now we assume $n \geq 9$, we use $\lceil \frac{-3+\sqrt{24n+33}}{4} \rceil \leq \frac{-3+\sqrt{24n+33}}{4} + 1$. It is sufficient to show

$$\frac{-3+\sqrt{24n+33}}{4} + 1 < n + 4.$$

A simple calculation yields $0 < n^2 - 7n - 8$. This true is for all $n \geq 9$.

From above argument, the equality holds only when $n = 5$ and $b(G) = b(\bar{G}) = 3$. Now assume $n = 5$. If G contains a vertex v of degree 3 or 4, then $b(G) \leq 2$ since we $N[v]$ can covers at least 4 vertices. Thus all degrees of G are at most 2. For the same reason, all degrees of $\bar{G}$ are at most 2. This implies that all degrees in G and in $\bar{G}$ are exactly 2. Since both G and $\bar{G}$ are connected and $n = 5$, the only possible case is $G = \bar{G} = C_5$. $\square$

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