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# A rigid cone in the truth-table degrees with jump

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**Abstract**

The automorphism group of the truth-table degrees with order and jump is fixed on the set of degrees above the fourth jump, $\mathbf{0}^{(4)}$.

## Contents

# 1 Introduction

A *cone* in a partial order $(D, \leq)$ is a set of the form $D(\geq a) := \{x \in D : x \geq a\}$ for some $a \in D$. A subset of $S$ of $D$ is *rigid* if it is fixed under the action of the automorphism group $\mathrm{Aut}(D, \leq)$, i.e., for each $x \in S$ and each $\pi \in \mathrm{Aut}(D, \leq)$, $\pi(x) = x$. We will also be interested in the case of structures $(D, \leq, \mathfrak{j})$ where $\mathfrak{j}$ is a unary function on $D$. In that case, rigidity of $S \subseteq D$ is defined with respect to $\mathrm{Aut}(D, \leq, \mathfrak{j})$ rather than $\mathrm{Aut}(D, \leq)$.

It is not known whether the structure of the Turing degrees is rigid, but it is known [JS] that the structure of the Turing degrees with jump contains a rigid cone. This is shown by applying a jump inversion theorem and results on initial segments. Here we show that also the structure of truth-table degrees with jump $(\mathcal{D}_{tt}, \leq, \mathfrak{j})$ contains a rigid cone. For definitions relating to initial segments we refer the reader to the author's doctoral dissertation [KH2], survey article [KH1], and forthcoming article [KH3].

Our main result is that each automorphism of the truth-table degrees with jump is equal to the identity on the cone above $\mathbf{0}^{(4)}$. This contrasts with the results of Anderson [A] that each automorphism of the truth-table degrees (not necessarily jump invariant) is equal to the identity on some cone, and each automorphism that preserves $\mathbf{0}^{(3)}$ and $\mathbf{0}^{(5)}$ is equal to the identity on the cone above $\mathbf{0}^{(5)}$. It is still open whether non-trivial automorphisms of these structures exist at all.

# 2 Steps of the proof

In this section we describe the global structure of the proof of our main theorem 2.8; further recursion-theoretic and lattice-theoretic details are given in the subsequent sections.

**Definition 2.1.** *In the tt-degrees we denote the order by $\leq$. If $\mathbf{x}$, $\mathbf{y}$ are tt-degrees, we say that $\mathbf{x} \equiv_T \mathbf{y}$ if for some $X \in \mathbf{x}$ and $Y \in \mathbf{y}$, we have $X \equiv_T Y$.*

The following theorem is due to Mohrherr [M3].

**Theorem 2.2.** *Let $n \geq 1$ and $\mathbf{a} \geq \mathbf{0}^{(n)}$. Then for some $\mathbf{b}$, $\mathbf{a} = \mathbf{b}^{(n)}$.*

**Proposition 2.3.** *For each $\mathbf{g}$, $[\mathbf{0}, \mathbf{g}]$ is $\Sigma^0_3(\mathbf{g})$-presentable.*

**Proof**. An analysis of the definition of *tt*-reducibility. □

**Corollary 2.4.** *Each upper semilattices with least and greatest element that can be realized as initial segments $[\mathbf{0}, \mathbf{g}]$ with $\mathbf{g}^{(2)} \leq \mathbf{y}^{(3)}$ is $\Sigma^0_4(\mathbf{y})$-presentable.*

**Theorem 2.5.** *For any $y$, the upper semilattices with least and greatest element that can be realized as initial segments $[\mathbf{0}, \mathbf{g}]$ with $\mathbf{g}^{(2)} \leq \mathbf{y}^{(3)}$ are exactly the $\Sigma^0_4(\mathbf{y})$-presentable ones.*

**Proof**. By Corollary 2.4 and Theorem 4.4. □

**Theorem 2.6.** *Let $\pi$ be an automorphism of the truth-table degrees with jump and let $\mathbf{x} \geq \mathbf{0}^{(3)}$. Then $\pi(\mathbf{x}) \equiv_T \mathbf{x}$.*

**Proof**. By Theorem 2.2 there is a $\mathbf{y}$ such that $\mathbf{x} = \mathbf{y}^{(3)}$. The initial segments $[\mathbf{0}, \mathbf{y}']$ and $[\mathbf{0}, \pi(\mathbf{y}')]$ are jump-isomorphic via $\pi$, so by Theorem 2.5, the $\Sigma^0_4(\mathbf{y})$- and $\Sigma^0_4(\pi(\mathbf{y}))$-presentable bounded usls coincide. Hence by Proposition 5.8,

$$\pi(\mathbf{y})^{(3)} \equiv_T \mathbf{y}^{(3)}$$

and so

$$\pi(\mathbf{x}) = \pi(\mathbf{y}^{(3)}) = \pi(\mathbf{y})^{(3)} \equiv_T \mathbf{y}^{(3)} = \mathbf{x}.$$

□

**Lemma 2.7.** $\mathbf{a} \equiv_T \mathbf{b} \Rightarrow \mathbf{a}' = \mathbf{b}'$.

**Theorem 2.8.** *Let $\pi$ be an automorphism of the truth-table degrees with jump and let $\mathbf{x} \geq \mathbf{0}^{(4)}$. Then $\pi(\mathbf{x}) = \mathbf{x}$.*

**Proof**. By Theorem 2.2, there is a $\mathbf{y}$ such that $\mathbf{x} = \mathbf{y}^{(4)}$. Let $\mathbf{z} = \mathbf{y}^{(3)}$, so $\mathbf{x} = \mathbf{z}'$ and $\mathbf{z} \geq \mathbf{0}^{(3)}$. By Theorem 2.6, $\pi(\mathbf{z}) \equiv_T \mathbf{z}$ and by Lemma 2.7, $\mathbf{a} \equiv_T \mathbf{b} \Rightarrow \mathbf{a}' = \mathbf{b}'$. Hence

$$\pi(\mathbf{x}) = \pi(\mathbf{z}') = \pi(\mathbf{z})' = \mathbf{z}' = \mathbf{x}.$$

□

# 3 Mal'tsev homogeneous lattice tables

If $(L, \leq)$ is a partial order (transitive, reflexive, antisymmetric relation) such that greatest lower bounds $\alpha \wedge \beta$ of all $\alpha, \beta \in L$ exist then $(L, \leq, \wedge)$ is called a lower semilattice; if least upper bounds $\alpha \vee \beta$ of all pairs $\alpha, \beta \in L$ exist, then $(L, \leq, \vee)$ is called an upper semilattice (usl). If $L$ is both an lower semilattice and an upper semilattice then $L$ is a lattice. $L$ is called bounded if there exist elements $0, 1 \in L$ such that for all $\alpha \in L$, $0 \leq \alpha \leq 1$. In particular every finite lattice is bounded. If $L$ has more than one element (so in the bounded case, $0 \neq 1$) then we say that $L$ is nontrivial. A unary algebra is a collection of functions $f : X \to X$ on a set $X$, closed under composition. The partition lattice $\mathrm{Part}(X)$ on a set $X$ consists of all equivalence relations (considered as sets of ordered pairs) on $X$, ordered by inclusion. We will be interested in the case where $X$ is finite or countably infinite.

A lattice table (see [L]) $\Theta$ is (1) a set $X$ together with (2) a finite set of equivalence relations $\alpha_1, \ldots, \alpha_n$ on $X$, and (3) an order $\leq$ given by $\alpha_i \leq \alpha_j \leftrightarrow \alpha_i \supseteq \alpha_j$ (reverse inclusion of sets of ordered pairs), such that $\{\alpha_1, \ldots, \alpha_n\}$ ordered by *inclusion* is a 0-1 sublattice of Part(X). We write $\widehat{\Theta} = \{\alpha_1, \ldots, \alpha_n\}$. We think of $\Theta$ as equal to $X$, but endowed with additional structure. So $x \in \Theta$ means

$x \in X$, etc. but for emphasis we may write $|\Theta|$ for $X$. Note that $\Theta$ is determined by $\widehat{\Theta}$.

Elements of $|\Theta|$ are denoted by lower-case Roman letters such as $u$, $v$, $w$, $x$, $y$, $z$, and elements of semilattices in general and $\widehat{\Theta}$ in particular by lower-case Greek letters such as $\alpha$, $\beta$, $\gamma$.

If $\alpha \in \widehat{\Theta}$ and $(x, y) \in \alpha$ then we write $x \sim_\alpha y$. If $\Theta$ is a lattice table then an endomorphism of $\Theta$ is a map from $\Theta$ to $\Theta$ preserving all equivalence relations in $\widehat{\Theta}$. That is, $(\forall x, y \in \Theta)(\forall \alpha \in \widehat{\Theta})(x \sim_\alpha y \to f(x) \sim_\alpha f(y))$. End $\Theta$ denotes the unary algebra consisting of all endomorphisms of $\Theta$.

$C_\Theta(x, y)$ denotes the principal equivalence relation in $\Theta$ generated by $(x, y)$, i.e.

$$C_\Theta(x, y) = \cap\{\alpha \in \widehat{\Theta} : (x, y) \in \alpha\}.$$

We define $\mathrm{End}_\Theta(x, y)$ to be the the principal congruence relation in $\Theta$ generated by $(x, y)$, i.e. the equivalence relation generated by all pairs $(f(x), f(y))$ for $f \in \mathrm{End}\ \Theta$.

**Lemma 3.1.** *$End_\Theta(x, y) \subseteq C_\Theta(x, y)$.*

**Proof**. If $(u, v) \in \mathrm{End}_\Theta(x, y)$ then $(c, d)$ is in the transitive closure of

$$\{(f(x), f(y)) \mid f \in \text{ End } \Theta\},$$

so it suffices to show each such $(f(x), f(y)) \in C_\Theta(x, y)$. For this it suffices to show $(f(x), f(y)) \in \alpha$ provided that $(x, y) \in \alpha$ for $\alpha \in \widehat{\Theta}$; this holds since $f \in \mathrm{End}\ \Theta$. $\square$

**Definition 3.2.** *Let $\Theta$ be a lattice table. We say that $\Theta$ is* Mal'tsev homogeneous *if for all $x, y \in \Theta, C_\Theta(x, y) \subseteq End_\Theta(x, y)$ (so by Lemma 3.1, $C_\Theta(x, y) = End_\Theta(x, y)$).*

The following Proposition can readily be proved:

**Proposition 3.3.** *$\Theta$ is Mal'tsev homogeneous iff for all $x, y, u, v \in \Theta$ satisfying*

$$(\forall \alpha \in \widehat{\Theta})(x \sim_\alpha y \to u \sim_\alpha v),$$

*there exist $n \in \omega = \{0, 1, 2, \ldots\}, z_1, \ldots, z_n \in \Theta$ and $f_0, \ldots, f_n \in End\ \Theta$ such that*

$$(\forall i \leq n)(\{f_i(x), f_i(y)\} = \{z_i, z_{i+1}\})$$

*where $z_0 = u$ and $z_{n+1} = v$.*

The $z_i$ are called *homogeneity interpolants*.

This notion of homogeneity is more general (weaker) than those considered in [L].

Note that if $\alpha \wedge \beta = \gamma$ in $\widehat{\Theta}$ then $\alpha$ and $\beta$ generate $\gamma$. That is, if $x \sim_\gamma y$ then there exist *meet interpolants* $z_1, \ldots, z_n$ for $x, y$ such that $x \sim_\alpha z_1 \sim_\beta z_2 \cdots \sim_\alpha z_n \sim_\beta y$.

**Definition 3.4.** *If $\Theta$ is a lattice table and $Y \subseteq |\Theta|$, then for each $\alpha \in \widehat{\Theta}$, $\alpha \restriction Y = \{(x,y) \in Y \times Y : (x,y) \in \alpha\}$. Let $\widehat{\Theta} \restriction Y = \{\alpha \restriction Y : \alpha \in \widehat{\Theta}\}$.*

*If $\Theta_0$ and $\Theta_1$ are lattice tables, then we say that $\Theta_0 \subseteq \Theta_1$ if $|\Theta_0| \subseteq |\Theta_1|$ and $\widehat{\Theta}_1 \restriction |\Theta_0| = \widehat{\Theta}_0$. Note that if $\Theta_0 \subseteq \Theta_1$ then $\widehat{\Theta}_0$ and $\widehat{\Theta}_1$ are isomorphic (nontrivial, finite) lattices.*

*If $\Theta_n, n \in \omega$ are lattice tables such that $\Theta_n \subseteq \Theta_{n+1}$ for each $n$, then $\bigcup_{n\in\omega} \Theta_n$ is the lattice table $\Theta$ such that $|\Theta| = \bigcup_{n\in\omega} |\Theta_n|$ and $\widehat{\Theta} \restriction |\Theta_n| = \widehat{\Theta}_n$ for each $n$. In particular $\Theta_n \subseteq \Theta$ and $\widehat{\Theta}_n$ and $\widehat{\Theta}$ are isomorphic lattices for each $n$.*

**Definition 3.5.** *$\Theta$ is a sequential lattice table if there exist $\Theta_n, n \in \omega$, such that $\Theta = \bigcup_{n\in\omega} \Theta_n$, and*

*(1) each $\Theta_n$ is a $(0,1,\vee)$-substructure of Part $(|\Theta_n|)$ ( $\Theta_n$ is an* usl table*),*

*(2) $\Theta$ is a lattice table, and*

*(3) for each $n$, meet interpolants for elements of $\Theta_n$ exist in $\Theta_{n+1}$.*

*$\Theta$ is a sequential Mal'tsev homogeneous lattice table if in addition*

*(4) $\Theta$ is Mal'tsev homogeneous, with homogeneity interpolants for elements of $\Theta_n$ appearing in $\Theta_{n+1}$ (compare VII.1.1, 1.3 of* [L]*).*

**Definition 3.6** (Direct limit)**.** *Let a sequence $(L^i, \varphi_i)_{i\in\omega}$ be given, where each $L^i$ is a finite lattice, $\varphi_i : L^i \to L^{i+1}$ is a $(0,1,\vee)$ homomorphism, and $L^i \cap L^j = \emptyset$ for $i \neq j$.*

*Let $L' = \bigcup_{i\in\omega} L_i$ as a set. Let $\approx$ be the equivalence relation on $L'$ generated by $a \approx \varphi_i(a)$ for $a \in L^i$. Then $L = L'/\approx$ is an upper semilattice called the direct limit of the sequence $(L^i, \varphi_i)_{i\in\omega}$.*

**Definition 3.7.** *Fix finite lattices $L^0, L^1$ and a $(0,1,\vee)$ homomorphism $\varphi : L^0 \to \varphi(L^0) \subseteq L^1$, and lattice tables $\Theta^0, \Theta^1$. Suppose $\Psi^i : L^i \to \widehat{\Theta}^i, i = 0,1$, are isomorphisms. For $\alpha \in L^i$, we write $\sim_\alpha$ for $\sim_{\Psi^i\alpha}$.*

*We say that $\Theta^1$* embeds in $\Theta^0$ with respect to $\varphi$ and $\Psi_0, \Psi_1$ *if there is a function $\Theta(\varphi) : \Theta^1 \to \Theta^0$ such that for all $x, y \in \Theta^1$, and all $\alpha \in L^0$,*

$$x \sim_{\varphi\alpha} y \Leftrightarrow \Theta(\varphi)(x) \sim_\alpha \Theta(\varphi)(y).$$

In the characterization of intervals $[\mathbf{0}, \mathbf{g}]$ for $\mathbf{g} < \mathbf{0}'$, Definition 3.7 plays a key role which we will now describe.

Suppose a bounded countable upper semilattice $L$ is given such that the ordering $\leq$ of $L$ is computably enumerable but not necessarily computable. That is, there is a computable sequence that consists of all pairs $(\alpha, \beta)$ such that $\alpha \leq \beta$, but if a given pair $(\alpha, \beta)$ does not appear anywhere in the list then this cannot be determined effectively.

For reasons whose explanation would take us too far afield (but see [KH1]), we need a computable sequence of lattice tables $\Theta^0, \Theta^1, \ldots$ such that $\widehat{\Theta}^s$ is isomorphic to our approximation to $L$ at stage $s$. (We will start with a sequence $(L^i, \varphi_i)_{i\in\omega}$ having $L$ as direct limit, and our approximation to $L$ at stage $s$ will be $L^s$.) Suppose we discover at stage $s+1$ that $\alpha \leq \beta$, whereas at stage $s$ we knew that $\beta \leq \alpha$ but thought that $\alpha \not\leq \beta$. Further suppose that we cannot ignore what was done using $\Theta^s$ at stage $s$, but we can let $\Theta^{s+1}$ be a subset of $\Theta^s$. If $\Theta^{s+1}$ embeds into $\Theta^s$ with respect to $\varphi$ (the homomorphism mapping our approximation to $L$ at stage $s$ to our approximation to $L$ at stage $s+1$) then by thinning $\Theta^0$ to $\Theta(\varphi)\Theta^1$, we eliminate all elements $x, y$ that are witnesses to the fact that $\alpha \neq \beta$. This allows us to identify $\alpha$ and $\beta$, even though so far we have been working under the assumption that $\alpha \neq \beta$.

We mention for the reader who is a computability theorist that in the characterization of lattices isomorphic to $[\mathbf{0}, \mathbf{g}]$ for $\mathbf{g} < \mathbf{0}'$, the ordering of $L$ is computably enumerable only relative to the Turing degree $\mathbf{0}''$, and the "stages" above are really levels of a priority tree, the true path of which it requires $\mathbf{0}''$ to identify.

The full result needed for the application to the Turing degrees is contained in Theorem 3.8 and Proposition 3.9.

**Theorem 3.8.** *Let $L$ be a bounded countable nontrivial usl and let $(L^i, \varphi_i)_{i\in\omega}$ be any system of nontrivial finite lattices having $L$ as direct limit in the sense of Definition 3.6. Then there exists*

1. *a function $h : \omega \to \omega$,*
2. *a double sequence of finite lattice tables $(\Theta^i_j)_{i\in\omega, j\geq h(i)}$ with $\Theta^i_j \subseteq \Theta^i_{j+1}$ for each $i \in \omega, j \geq h(i)$, and*
3. *for each $i \in \omega$ an increasing function $m_i : \omega \to \omega$ with $m_i(0) = h(i)$, such that*
    1. *letting $\Theta^i = \bigcup_{j\in\omega} \Theta^i_j$, we have $|\Theta^i| \supseteq |\Theta^{i+1}|$ for each $i \in \omega$,*
    2. *for each $i, j \geq h(i)$ and $k$ such that $m_i(j) \leq k < m_i(j+1)$, we have*
    $$\Theta^i_k = \Theta^i_{m_i(j)},$$
    3. *for each $i \in \omega, (\Theta^i_{m_i(j)})_{j\in\omega}$ is a sequential Mal'tsev homogeneous lattice table,*
    4. *for each $i \in \omega, \widehat{\Theta}_i$ is isomorphic to $L^i$, and*
    5. *there exist isomorphisms $\Psi_i : L^i \to \widehat{\Theta}_i$ such that $\Theta^{i+1}$ embeds in $\Theta^i$ with respect to $\varphi_i$ and $\Psi_i, \Psi_{i+1}$, and the embedding is the identity map. In other words, for all $x, y \in \Theta^{i+1}$ and $\alpha \in L^i$, we have*
    $$x \sim_{\Psi_i\alpha} y \leftrightarrow x \sim_{\Psi_{i+1}\varphi_i\alpha} y.$$

The essential property in Theorem 3.8, and the one that goes beyond those of [LL], is (5). The following Proposition can be proved by inspecting the proof of Theorem 3.8.

**Proposition 3.9** (Computability-theoretic properties). *Let* $\mathbf{a}$ *be a Turing degree and let* $L$ *be a* $\Sigma^0_1(\mathbf{a})$*-presentable usl, as in* [KH1]. *Then in Theorem 3.8, we may assume that* $h$ *is* $\mathbf{a}$*-computable; the array* $\{\Theta^i_j \mid j \geq h(i)\}$ *is* $\mathbf{a}$*-computable; each* $m_i$ *is computable; for each* $i < \omega, (\Theta^i_{m_i(j)})_{j\in\omega}$ *is a computable sequence; each* $\Theta^i$ *is computable; and there is a computable function taking* $L^0, \ldots, L^i$ *to* $\Theta^i$.

We now begin the development that will lead to a proof of Theorem 3.8.

If $A$ is a unary algebra then Con $A$ denotes the congruence lattice of $A$, i.e. the lattice of all equivalence relations $E$ on $X$ preserved by all $f \in A$, ordered by inclusion.

The following observation can be traced back to Mal'tsev [G1, M1, M2].

**Proposition 3.10.** *For any unary algebra* $A$*, the dual of Con* $A$ *is a Mal'tsev homogeneous lattice table.*

**Proof**. Suppose $A$ is a unary algebra on a set $X$. Let $\Theta$ be the lattice table such that $\widehat{\Theta}$ is the dual of Con $A$. Since Con $A$ is a 0-1 sublattice of Part$(A)$, $\Theta$ is a lattice table.

If $f$ is an operation in $A$ and $\alpha \in \widehat{\Theta}$ then $\alpha$ is a congruence relation on $A$ and hence $\forall x, y(x \sim_\alpha y \to f(x) \sim_\alpha f(y))$, which means that $f \in \text{End } \Theta$. So

$$A \subseteq \text{End } \Theta.$$

Clearly for any unary algebras $A, B$ on the same underlying set, we have $A \subseteq B \Rightarrow \text{Con } A \supseteq \text{Con } B$. Hence Con End $\Theta \subseteq \text{Con } A = \widehat{\Theta}$.

If $u$, $v$, $x$, $y \in X$, $f \in \text{End } \Theta$ and $(x, y) \in \text{End}_\Theta(u, v)$ then there exist $z_1, \ldots, z_k$ such that $(z_i, z_{i+1}) = (g_i(u), g_i(v))$ for $g_i \in \text{End } \Theta$ with $z_1 = x, z_k = y$, hence letting $w_i = f(z_i)$ and $h_i = f \circ g_i$ we have $(w_i, w_{i+1}) = (h_i(u), h_i(v)) \in \text{End}_\Theta(u, v)$, $w_1 = f(x), w_k = f(y)$, and so $(f(x), f(y)) \in \text{End}_\Theta(u, v)$. Hence we have shown $\text{End}_\Theta(u, v) \in \text{Con End } \Theta \subseteq \Theta$. Since End $\Theta$ contains the identity map, $\text{End}_\Theta(u, v)$ contains $(u, v)$. Hence $\text{End}_\Theta(u, v)$ is in $\Theta$ and contains $(u, v)$, so it contains $C_\Theta(u, v)$. So $\Theta$ is Mal'tsev homogeneous. □

We recall the construction of [P].

**Definition 3.11.** *Let* $L$ *be a nontrivial lattice.* $\mathcal{A} = (A, r, h)$ *is called an* $L-\{1\}$-colored *graph if* $A$ *is a set,* $r$ *is a set of size-two subsets of* $A$*, i.e.* $(A, r)$ *is an undirected graph without loops, and* $h : r \to L - \{1\}$ *is a mapping of the set* $r$ *of the edges of the graph into* $L - \{1\}$.

*The map* $e : L \to Part(A)$ *is defined by: for* $\alpha \in L, e(\alpha)$ *is the equivalence relation on* $A$ *generated by identifying points* $x, y$ *if there is a path from* $x$ *to* $y$

*in the graph consisting of edges all of which have color* $\geq \alpha$. *In this case we say that* $x, y$ *are connected with color* $\geq \alpha$.

**Definition 3.12** ($\alpha$-cells)**.** *Let* $L$ *be a nontrivial lattice and let* $\alpha \in L - \{1\}$. *An* $\alpha$*-cell* $\mathcal{B}_\alpha = (B_\alpha, s_\alpha, k_\alpha)$ *is an* $L - \{1\}$*-colored graph consisting of (1) a base edge* $\{x, y\}$ *colored* $\alpha$*, and (2) for each pair* $(\alpha_1, \alpha_2)$ *of elements of* $L$ *such that* $\alpha_1 \wedge \alpha_2 \leq \alpha$*, a chain of edges* $\{x, u_1\}, \{u_1, u_2\}, \{u_2, u_3\}, \{u_3, y\}$*, colored* $\alpha_1, \alpha_2, \alpha_1, \alpha_2$*, respectively. Here* $x, y, u_1, u_2, u_3$ *are distinct elements of* $B_\alpha$. *The base edge and chain of edges corresponding to a particular inequality* $\alpha_1 \wedge \alpha_2 \leq \alpha$ *is referred to as a pentagon. So an* $\alpha$*-cell consists of several pentagons, intersecting only in a common base edge.*

**Definition 3.13** (Pudlák graphs)**.** *Let* $L$ *be a nontrivial lattice. The Pudlák graph* [P] *of* $L$ *is an* $L - \{1\}$ *colored graph* $\mathcal{A}^P$*, defined as follows.*

1. $\mathcal{A}_0^P$ *consists of a single edge colored by* $0 \in L$. *(In fact, how we choose to color this one edge has no impact on later proofs.)*
2. $\mathcal{A}_{n+1}^P$ *contains* $\mathcal{A}_n^P$ *as a subgraph and is obtained by attaching to each edge of* $\mathcal{A}_n^P$ *of any color* $\alpha$ *an* $\alpha$*-cell.*
3. $\mathcal{A}^P = \bigcup_{n \in \omega} \mathcal{A}_n^P$.

*We will use the following modification, which contains infinitely many copies of each edge in Pudlák's graph.*

1. $\mathcal{A}_0^{(i)} = \mathcal{A}_0^P$*, for each* $i \in \omega$.
2. $\mathcal{A}_j^{(i)}$ *is obtained by attaching to each edge of* $\mathcal{A}_{j-1}^{(i)}$ *of any color* $\alpha, i$ *many* $\alpha$*-cells.*
3. $\mathcal{A}_j = \mathcal{A}_j^{(j)}$.
4. $\mathcal{A} = \bigcup_{n \in \omega} \mathcal{A}_n = \mathcal{A}(L)$ *is called the homogenized Pudlák graph of* $L$.

*The underlying set of* $\mathcal{A}_n$ *is denoted by* $A_n$.

*Let* $\Theta = \Theta(L)$ *be the lattice table with* $|\Theta| = A$*, and* $\widehat{\Theta} = \{e(\alpha) : \alpha \in L\}$.

*Note that by definition of* $\Theta$ *being a lattice table,* $\widehat{\Theta}$ *is ordered by reverse inclusion. Similarly let* $\Theta_n$ *be the lattice table with* $|\Theta_n| = A_n$,

$$\widehat{\Theta}_n = \{e(\alpha) \upharpoonright \Theta_n \mid \alpha \in L\}.$$

**Lemma 3.14.** *Let* $B_0 \subseteq B_1 \subseteq A$*, where* $A$ *is the underlying set of* $\Theta(L)$.

*For* $i = 0, 1$*, let* $\Xi_i$ *be the usl table whose underlying set is* $B_i$*, and whose equivalence relations are computed using graph points belonging to* $B_i$ *only.*

*Then* $\Xi_0 \subseteq \Xi_1$ *in the sense of Definition 3.4.*

**Proof**. We have to show that if $x \sim_\alpha y$ holds in $\Xi_1$ then $x \sim_\alpha y$ holds in $\Xi_0$. The only way this could fail is if there is a path of edges between $x$ and $y$ leading out of $\Xi_0$ and then back in. We may assume that the path does not leave and re-enter $\Xi_0$ via the same node. So it suffices to show that any path that goes around a pentagon not contained in $\Xi_0$ but whose base is in $\Xi_0$ can be shortened to one contained in $\Xi_0$ with no loss of equivalence. Since the pentagons represent inequalities $\alpha \wedge \beta \leq \gamma$, any path $x, u_1, u_2, u_3, y$ going around the pentagon in $\Xi_1 - \Xi_0$ may be replaced by the edge $x, y$ cutting across which has equal or greater color, i.e. with no loss of equivalence. □

**Lemma 3.15.** $\Theta_n \subseteq \Theta_{n+1}$ *for each* $n$*, so* $\Theta = \bigcup_{n\in\omega} \Theta_n$*.*

**Proof**. Let $\Xi_i = \Theta_{n+i}$ for $i = 0, 1$ and apply Lemma 3.14. □

**Theorem 3.16.** *Let $L$ be a nontrivial finite lattice. $L$ is dual isomorphic to the congruence lattice of End $\Theta(L)$. In fact, $e : L \to \widehat{\Theta}(L)$ is an isomorphism, and $\Theta(L) =$ Con End $\Theta(L)$.*

**Proof**. Pudlák [P] assumes that $L$ is an algebraic lattice [G2], defines a certain algebra $S \subseteq$ End $\Theta^P(L)$, and shows that $e : L \to \widehat{\Theta}^P(L)$ is an isomorphism, and $\widehat{\Theta}^P(L) =$ Con $S$. Now trivially $\widehat{\Theta}^P(L) \subseteq$ Con End $\Theta^P(L)$ holds, and $S \subseteq$ End $\Theta^P(L)$ implies Con End $\Theta^P(L) \subseteq$ Con $S$. So we have $\widehat{\Theta}^P(L) =$ Con End $\Theta^P(L)$.

In fact Pudlák's proof works for our graph $\Theta$ as well, i.e. it shows that $e : L \to \widehat{\Theta}(L)$ is an isomorphism, and $\widehat{\Theta}(L) =$ Con End $\Theta(L)$.

Now let $L$ be a finite lattice. Since every finite lattice is algebraic, $L$ is an algebraic lattice. Hence $e : L \to \widehat{\Theta}(L)$ is an isomorphism and $\widehat{\Theta}(L) =$ Con End $\Theta(L)$. □

**Lemma 3.17.** *The sequence $\Theta_n(L), n \in \omega$ has a subsequence which is a computable Mal'tsev homogeneous sequential lattice table.*

**Proof**. Since $\Theta$ is a congruence lattice, $\Theta$ is a Mal'tsev homogeneous lattice table. Hence as $\Theta = \bigcup_{n\in\omega} \Theta_n$, a subsequence of $\Theta_n, n \in \omega$ will be a sequential Mal'tsev homogeneous lattice table. The sequence is computable since to compute an equivalence relation on elements of $\Theta_n$, it is sufficient to consider paths in $\Theta_n$, since $\Theta_n \subseteq \Theta_{n+1}$ by Lemma 3.15. □

From now on we will assume that in fact $\Theta_n, n \in \omega$ is itself the subsequence from Lemma 3.17. Fix nontrivial finite lattices $L^0, L^1$ and a $(0, 1, \vee)$-isomorphisms $\varphi : L^0 \to \varphi(L^0) \subseteq L^1$. Let the $\wedge$-isomorphism $\varphi^* : L^1 \to L^0$ be defined by $\varphi^*\beta = \bigvee\{\alpha \in L^0 \mid \varphi(\alpha) \leq \beta\}$. This $\varphi^*$ is known as the Galois adjoint of $\varphi$ [GHK$^+$].

The map $\varphi^*$ has many nice properties; we list the ones we need in the following lemma.

**Lemma 3.18.** *1. $\varphi^*$ is a $(\wedge, 1)$-homomorphism.*

*2. If $\beta < 1$ then $\varphi^*\beta < 1$.*

*3. $\varphi^*$ is injective on $\varphi L^0$.*

*4. $\alpha \leq \varphi^*\beta \leftrightarrow \varphi^*\varphi\alpha \leq \varphi^*\beta$.*

**Proof**. These all follow easily from the definition of $\varphi^*$ and the fact that $\{\alpha \in L^0 \mid \varphi(\alpha) \leq \beta\}$ is the principal ideal generated by $\varphi^*(\beta)$, i.e. $\{\alpha \in L^0 \mid \alpha \leq \varphi^*(\beta)\}$. □

**Lemma 3.19.** *Let $\mathfrak{C}(\varphi)\mathcal{A}L^1$ be the graph obtained from $\mathcal{A}L^1$ by replacing each color $\beta$ by $\varphi^*\beta$. Then $\mathfrak{C}(\varphi)\mathcal{A}L^1$ is isomorphic to a subgraph of $\mathcal{A}L^0$.*

**Proof**. Each pentagon of $\mathfrak{C}(\varphi)\mathcal{A}L^1$ represents an inequality of the form

$$\varphi^*\beta_1 \wedge \varphi^*\beta_2 \leq \varphi^*\beta,$$

for $\beta_1, \beta_2, \beta \in L^1$ satisfying $\beta_1 \wedge \beta_2 \leq \beta$. Then $\varphi^*\beta_1 \wedge \varphi^*\beta_2 = \varphi^*(\beta_1 \wedge \beta_2) \leq \varphi^*\beta$, so the represented inequality $\varphi^*\beta_1 \wedge \varphi^*\beta_2 \leq \varphi^*\beta$ holds in $L^0$.

Hence we can obtain an isomorphic copy of $\mathfrak{C}(\varphi)\mathcal{A}L^1$ within $\mathcal{A}L^0$ by running through the construction of $\mathcal{A}L^0$, omitting every pentagon that represents an inequality involving members of $L^0 - \varphi^*L^1$, and omitting pentagons for inequalities that are true in $L^0$ but not in $L^1$. If an edge becomes disconnected from $\mathcal{A}_0$ by such omissions then it too is omitted. Since $L^1$ may have many more elements than $L^0$, we make use of the fact that $\mathcal{A}$ contains infinitely many copies of each edge from Pudlák's original graph $\mathcal{A}^P$. Since $\varphi^*(\beta) = 1 \rightarrow \beta = 1$ by Lemma 3.18, recoloring of points is never identification of points. □

**Lemma 3.20.** *Let $\Theta(\varphi)$ be the isomorphism from Lemma 3.19, sending $\mathcal{A}L^1$ to a subgraph of $\mathcal{A}L^0$ isomorphic to $\mathfrak{C}(\varphi)\mathcal{A}L^1$.*

*Then $\Theta(\varphi)\Theta L^1 \subseteq \Theta L^0$ in the sense of Definition 3.4.*

**Proof**. Let $\Xi_0 = \Theta(\varphi)\Theta L^1$ and $\Xi_1 = \Theta L^0$ and apply Lemma 3.14. □

**Lemma 3.21.** *Let $\Psi_i$ be the map $e$ of Definition 3.11 for $L = L^i, i = 0, 1$.*

*Then $\Theta(L^1)$ embeds in $\Theta(L^0)$ with respect to $\varphi$ and $\Psi_0, \Psi_1$.*

**Proof**. Let $x, y$ be points in $\Theta L^1$, i.e. in $\mathcal{A}L^1$, and let $\alpha \in L^0$. Then obviously $x \sim_{\varphi\alpha} y \rightarrow \Theta(\varphi)x \sim_{\varphi^*\varphi\alpha} \Theta(\varphi)y$. Now suppose $\Theta(\varphi)x \sim_{\varphi^*\varphi\alpha} \Theta(\varphi)y$. Then there is a path witnessing this, which by Lemma 3.19 we may assume lies within $\Theta(\varphi)\mathcal{A}L^1$. Hence the path has an inverse image path under $\Theta(\varphi)^{-1}$. This is then a path from $x$ to $y$ with colors $\beta$ for all of which $\varphi^*\beta \geq \varphi^*\varphi\alpha$. But then $\alpha \leq \varphi^*\beta$ by Lemma 3.18(4), and so $\varphi\alpha \leq \beta$, so $x \sim_{\varphi\alpha} y$. So in fact $x \sim_{\varphi\alpha} y \leftrightarrow \Theta(\varphi)x \sim_{\varphi^*\varphi\alpha} \Theta(\varphi)y$. Colors $\gamma$ of edges in $\Theta(\varphi)\mathcal{A}L^1$ are all of the form $\varphi^*(\beta)$ for some $\beta$. So $\Theta(\varphi)x \sim_{\varphi^*\varphi\alpha} \Theta(\varphi)y$ iff there is a path from $\Theta(\varphi)x$ to $\Theta(\varphi)y$, all edges of which are colored $\gamma \geq \varphi^* \varphi\alpha$, or equivalently by Lemma 3.18(4) (using $\gamma = \varphi^* \beta$), colored $\gamma \geq \alpha$. Hence equivalently $\Theta(\varphi)x \sim_\alpha \Theta(\varphi)y$. $\square$

*Proof of Theorem 3.8.* Let $m_i(n)$ be the least $m$ such that $\Theta(\varphi_i)\Theta_n^{i+1} \subseteq \Theta_m^i$. Let $h(i) = m_i(0)$, for $i \in \omega$. Let $\Theta^0 = \Theta(L^0)$ and for $i \geq 1$, denoting composition by juxtaposition,

$$\Theta^i = \Theta(\varphi_0) \cdots \Theta(\varphi_{i-1})\Theta(L^i).$$

Let $\Theta_k^i = \Theta(\varphi_0) \cdots \Theta(\varphi_{i-1})\Theta_j(L^i)$ if $k = m_0 m_1 \cdots m_{i-1}(j)$ for some $j$; otherwise, let $\Theta_k^i = \Theta_{k-1}^i$. The Theorem now follows easily. $\square$

# 4 Initial segments of the $tt$-degrees

**Lemma 4.1.** *Suppose for each $e$, $g$ lies on a tree $T_e$ which is $e$-splitting for some $c$ for some tables with the properties of Proposition 3.9, in the sense of* [LL]*. Then* $\mathbf{g}$ *is hyperimmune-free.*

**Proof**. For each $e \in \omega$ there exists $e^* \in \omega$ such that for all stages $s$ and all oracles $g$, if $\{e^*\}_s^g(x) \downarrow$ then $\{e^*\}^g(x) = \{e\}^g(x)$ and $\{e\}_s^g(y) \downarrow$ for all $y \leq x$. If $g$ lies on $T_{e^*}$ then it follows that $\{e\}^g$ is total and $\{e^*\}^{T(\sigma)}(x) \downarrow$ for each $\sigma$ of length $x + 1$. Hence $\{e\}^g = \{e^*\}^g$ is dominated by the recursive function $f(x) = \max\{\{e\}^{T(\sigma)}(x) : |\sigma| = x + 1\}$. $\square$

**Proposition 4.2.** *Let $L$ be a $\Sigma_4^0(\mathbf{y})$-presentable upper semilattice with least and greatest element. Then there exist $t$, $i$, $g$ such that*

1. *$t : \omega \to 2$ is $0''$-computable,*
2. *$i$ is the characteristic function of a set $I$ such that $I \leq_m y^{(3)}$,*
3. *$g^{(2)}(e) = t(i(0), \ldots, i(e))$ for all $e \in \omega$,*
4. *$[\mathbf{0}, \mathbf{g}]$ is isomorphic to $L$, and*
5. *$\mathbf{g}$ is hyperimmune-free*

**Proof**. The proof in [LL] must be modified to employ the lattice tables of Proposition 3.9.

By Proposition 3.9, for all $x$, $y \in \Theta^{k+1}$ and $\alpha \in L^k$, we have [identifying the isomorphism between $L^i$ and $\widehat{\Theta}^i$ with the identity]

$$x \sim_{\varphi_k \alpha} y \leftrightarrow x \sim_\alpha y.$$

Lemma 4.1 of [LL] is modified so that $\psi_{T,c}$ is $\psi_{T,\varphi_k c}$. The equivalence

$$uF_{m(i)}(c)v \leftrightarrow uG_i(c)v$$

now becomes

$$uF_{m(i)}(c)v \leftrightarrow uG_i(\varphi_k c)v$$

Just as in Lemma 3.1 it is shown that $\psi_{T,c}$ is Turing equivalent to $\psi_{T_0,c}$, it now follows that $\psi_{T,\varphi_k c}$ is Turing equivalent to $\psi_{T_0,c}$, which is what we want.

$i(e) = 1$ iff the answers to the $\Pi^0_1(y^{(2)})$-question about $L^e$ is yes.

By Lemma 4.1, $\mathbf{g}$ is hyperimmune-free. $\square$

**Lemma 4.3.** *If $t$, $i$, $A$, $q$ satisfy*

1. *$t : \omega \to 2$ is $q$-computable,*
2. *$i$ is the characteristic function of a set $I$ such that $I \leq_m q'$,*
3. *$A(e) = t(i(0), \ldots, i(e))$ for all $e \in \omega$,*

*then $A \leq_{tt} q'$.*

**Proof**. The value of $A(e)$ is determined by the following $e + 2$ many yes-or-no questions to $q'$:

Is $i(0) = 0$? $\cdots$ Is $i(e) = 0$? and, using the answers to the first $e + 1$ many questions: Is $t(i(0), \ldots, i(e)) = 0$? $\square$

**Theorem 4.4.** *Each $\Sigma^0_4(\mathbf{y})$-presentable upper semilattice with least and greatest element can be realized as an initial segment $[\mathbf{0}, \mathbf{g}]$ with $\mathbf{g}^{(2)} \leq \mathbf{y}^{(3)}$.*

**Proof**. Let $g$ be as in Proposition 4.2. By Lemma 4.3 with $q = y^{(2)}$ and $A = g^{(2)}$, we have $g^{(2)} \leq_{tt} y^{(3)}$.

By Proposition 4.2, $L$ is isomorphic to $[\mathbf{0}, \mathbf{g}]_T$. Since $\mathbf{g}$ is hyperimmune-free, $[\mathbf{0}, \mathbf{g}]_T = [\mathbf{0}, \mathbf{g}]_{tt}$. $\square$

# 5 Coding a set into a lattice

**Definition 5.1.** *Let $L$ be an upper semilattice and suppose $G = \{g_i \mid i < \omega\} \subseteq L$. If there exist $p$, $q \in L$ such that*

$$\{g_i \mid i \in \omega\} \subseteq \{x \mid x \vee p \geq q \ \ \& \ \ (\forall y < x)(y \vee p \ngeq q)\}$$

*then $G$ is called a Slaman-Woodin set (SW-set) for $p$, $q$ in $L$.*

*If there exist $e_0$, $e_1$, $f_0$, $f_1 \in L$ such that for each $i < \omega$,*

$$g_{2i+1} = (g_{2i} \vee e_1) \wedge f_1 \ \ \& \ \ g_{2i+2} = (g_{2i+1} \vee e_0) \wedge f_0,$$

*then the function $i \mapsto g_i$ is called a Shore sequence for $e_0$, $e_1$, $f_0$, $f_1$ in $L$.*

**Lemma 5.2.** *Let* $\mathbf{a}$ *be a Turing degree. Let* $L$ *be a* $\Sigma^0_1(\mathbf{a})$*-presented upper semilattice containing elements* $p$, $q$, $e_0$, $e_1$, $f_0$, $f_1$, *and atoms* $g_i$ *for* $i \in \omega$, *such that* $G = \{g_i \mid i < \omega\}$ *is a Slaman-Woodin set for* $p$, $q$ *and* $i \mapsto g_i$ *is a Shore sequence for* $e_0$, $e_1$, $f_0$, $f_1$. *Then* $\{\langle y, i\rangle \mid y = g_i\} \leq_T \mathbf{a}$.

**Proof**.

$$y = g_{2i+1} \Leftrightarrow \exists x(x = g_{2i} \ \ \& \ \ y \leq x \vee e_1 \ \ \& \ \ y \leq f_1 \ \ \& \ \ y \vee p \geq q)$$

$$y = g_{2i+2} \Leftrightarrow \exists x(x = g_{2i+1} \ \ \& \ \ y \leq x \vee e_0 \ \ \& \ \ y \leq f_0 \ \ \& \ \ y \vee p \geq q)$$

Note that the matrices of the formulas on the right hand side are positive formulas in the language with $\vee$ and $\leq$. The function $\vee$ is $\mathbf{a}$-recursive and the relation $\leq$ is $\Sigma^0_1(\mathbf{a})$. Hence the entire right hand sides are $\Sigma^0_1(\mathbf{a})$. So starting with $g_0$ we can find $g_n$, $\mathbf{a}$-recursively. $\square$

**Definition 5.3.** *Let* $a \in \mathcal{D}$. *An usl* $L$ *is said to be of degree* $\mathbf{a}$ *if (1)* $L$ *is* $\mathbf{a}$*-presentable, and (2) if* $\mathbf{b} \in \mathcal{D}$ *and* $L$ *is* $\mathbf{b}$*-presentable then* $\mathbf{a} \leq \mathbf{b}$.

**Definition 5.4.** *Given* $U \subseteq \omega$ *we define a lattice* $L(U)$.

*It consists of* $0, 1$, *atoms* $\{g_i : i \in \omega\}$, *more atoms* $e_0$, $e_1$, $p$, $s$ *and non-atoms* $f_0$, $f_1 < 1$ *with the properties of Lemma 5.2 (taking* $q = 1$*) and an additional element* $s$ *with the following property for each* $n \in \omega$*:*

$$n \in U \Leftrightarrow g_n \vee s = 1.$$

**Remark 5.5.** *Historically, the technique of enumerating the* $g_n$ $\mathbf{a}'$*-recursively was first done in* [S2]. *The idea of the improvement can be seen in* [S1] *Lemma 1.11. The Slaman-Woodin conditions used to combine these ideas to get the above lemma were presented in* [NSS1] *with a proof appearing in* [NSS2] *Lemma 2.13(i). The construction of* $L(U)$ *was presented to the author by Slaman; see also Theorem 3.7 of* [NSS1].

**Remark 5.6.** *Here are some details for the proof that such a lattice* $L(U)$ *exists (thanks to assistance from participants in a 2008 seminar at the University of Hawai'i). We make* $L(U)$ *a height-three lattice, i.e., every element is either 0, 1, an atom or a co-atom. The atoms are* $e_0$, $e_1$, $s$, *and the* $g_i$. *The element* $p$ *may be either an atom or a co-atom, and is incomparable with all other elements except that* $0 \leq p \leq 1$. *The co-atoms are* $e_0 \vee g_{2n+1}$, *and* $e_1 \vee g_{2n}$, $f_0$, $f_1$, *and* $g_i \vee s$ *whenever* $i \notin U$. *These elements are incomparable except as forced by the above conditions. The point of including* $p$ *and* $q$ *is that* $y \vee p \geq q$ *is a positive statement that implies* $y \not\leq 0$.

The following lemma will have many applications:

**Lemma 5.7.** *Let* $U \subseteq \omega$.

1. $L(U)$ *has degree* $\mathbf{u}$.

2. *If $L(U)$ is $\Sigma^0_1(\mathbf{b})$-presentable, then $U \in \Sigma^0_1(\mathbf{b})$.*

3. *If $U \in \Sigma^0_1(\mathbf{b})$ then $L(U)$ is $\Sigma^0_1(\mathbf{b})$-presentable.*

**Proof**. 1. The definition of $L(U)$ appeals to an oracle of degree $\mathbf{u}$ only and so $L(U)$ is $\mathbf{u}$-presentable. Suppose $L(U)$ is presented with degree $\mathbf{v}$. By Lemma 5.2, the relation $y = g_i$ is recursive in $\mathbf{v}$. Now $i \in U \leftrightarrow g_i \vee s \geq 1$, so since $\vee$ and $\geq$ are recursive in $\mathbf{v}, \mathbf{u} \leq \mathbf{v}$.

2. We have

$$n \in U \Leftrightarrow \exists x(x = g_n \;\;\&\;\; x \vee s \geq 1) \Leftrightarrow \forall x(x = g_n \rightarrow x \vee s \geq 1).$$

By Lemma 5.2, $U$ is of the form $\exists x(\triangle^0_1(\mathbf{b}) \;\;\&\;\; \Sigma^0_1(\mathbf{b}))$ $U \in \Sigma^0_1(\mathbf{b})$.

3. Immediate from the fact that all clauses of the definition of $L(U)$ except "$n \in U \leftrightarrow g_n \vee s \geq 1$" are recursive. □

**Proposition 5.8.** *If each $\Sigma^0_4(\mathbf{x})$-presentable bounded usl is $\Sigma^0_4(\mathbf{y})$-presentable, then $\mathbf{x}^{(3)} \leq_T \mathbf{y}^{(3)}$.*

**Proof**. Let $\mathbf{b} = \mathbf{x}^{(3)}$. Since $L(B \oplus \overline{B})$ is $\Sigma^0_1(B)$-presentable, it is $\Sigma^0_1(\mathbf{y}^{(3)})$-presentable. Thus by Lemma 5.7(2), $B \oplus \overline{B}$ is $\Sigma^0_1(\mathbf{y}^{(3)}$ and hence $B \leq_T \mathbf{y}^{(3)}$.

□

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