TOPOLOGICAL DRAWINGS OF COMPLETE BIPARTITE GRAPHS

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ABSTRACT. Topological drawings are natural representations of graphs in the plane, where vertices are represented by points, and edges by curves connecting the points. Topological drawings of complete graphs and of complete bipartite graphs have been studied extensively in the context of crossing number problems. We consider a natural class of simple topological drawings of *complete bipartite* graphs, in which we require that one side of the vertex set bipartition lies on the outer boundary of the drawing.

We investigate the combinatorics of such drawings. For this purpose, we define combinatorial encodings of the drawings by enumerating the distinct drawings of subgraphs isomorphic to $K_{2,2}$ and $K_{3,2}$, and investigate the constraints they must satisfy. We prove that a drawing of $K_{k,n}$ exists if and only if some simple local conditions are satisfied by the encodings. This directly yields a polynomial-time algorithm for deciding the existence of such a drawing given the encoding. We show the encoding is equivalent to specifying which pairs of edges cross, yielding a similar polynomial-time algorithm for the realizability of abstract topological graphs.

We also completely characterize and enumerate such drawings of $K_{k,n}$ in which the order of the edges around each vertex is the same for vertices on the same side of the bipartition. Finally, we investigate drawings of $K_{k,n}$ using straight lines and pseudolines, and consider the complexity of the corresponding realizability problems.

1 Introduction

We consider *topological graph drawings*, which are drawings of simple undirected graphs where vertices are represented by points in the plane, and edges are represented by simple curves that connect the corresponding points. We typically restrict those drawings to satisfy some natural nondegeneracy conditions. In particular, we consider *simple* drawings, in which every pair of edges intersect at most once. A common vertex counts as an intersection.

While being perhaps the most natural and the most used representations of graphs, simple drawings are far from being understood from the combinatorial point of view. A prominent illustration is the problem of identifying the minimum number of edge crossings in a simple topological drawing of K_n [10, 3, 1] or of $K_{k,n}$ [22, 4], for which there are long standing conjectures.

In order to cope with the inherent complexity of the drawings, it is useful to consider combinatorial abstractions. Those abstractions are discrete structures encoding some fea-

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tures of a drawing. One such abstraction, introduced by Kratochvíl, Lubiw, and Nešetřil, is called *abstract topological graphs* (AT-graph) [12]. An AT-graph consists of a graph (V, E) together with a set $\mathcal{X} \subseteq \binom{E}{2}$. A topological drawing is said to *realize* an AT-graph if the pairs of edges that cross are exactly those in $\mathcal{X}$. Another abstraction of a topological drawing is called the *rotation system*. The rotation system associates a circular permutation with every vertex v, which in a realization must correspond to the order in which the neighbors of v are connected to v. Natural realizability problems are: given an AT-graph or a rotation system, is it realizable as a topological drawing? The realizability problem for AT-graphs is known to be NP-complete [13].

For simple topological drawings of complete graphs, the two abstractions are actually equivalent [19]. It is possible to reconstruct the set of crossing pairs of edges by looking at the rotation system, and vice-versa (up to reversal of all permutations). Kynčl recently proved the remarkable result that a complete AT-graph (an AT-graph for which the underlying graph is complete) can be realized as a simple topological drawing of K_n if and only if all the AT-subgraphs on at most 6 vertices are realizable [14, 16]. This directly yields a polynomial-time algorithm for the realizability problem. While this provides a key insight on topological drawings of complete graphs, similar realizability problems already appear much more difficult when they involve complete *bipartite* graphs. In that case, knowing the rotation system is not sufficient for reconstructing the intersecting pairs of edges.

We propose a fine-grained analysis of simple topological drawings of complete bipartite graphs. In order to make the analysis more tractable, we introduce a natural restriction on the drawings, by requiring that one side of the vertex set bipartition lies on a circle at infinity. This gives rise to meaningful, yet complex enough, combinatorial structures.

Definitions. We wish to draw the complete bipartite graph $K_{k,n}$ in the plane in such a way that:

1. vertices are represented by points,
2. edges are continuous curves that connect those points, and do not contain any other vertices than their two endpoints
3. no more than two edges intersect in one point,
4. edges pairwise intersect at most once; in particular, edges incident to the same vertex intersect only at this vertex,
5. the k vertices of one side of the bipartition lie on the outer boundary of the drawing.

Properties 1–4 are the usual requirements for *simple topological drawings*, also known as *good drawings*. As we will see, Property 5 leads to drawings with interesting combinatorial structures. We will refer to drawings satisfying Properties 1–5 as *outer drawings*. Since this is the only type of drawings we consider, we will use the single term *drawing* instead when the context is clear.

The set of vertices of a bipartite graph $K_{k,n}$ will be denoted by $P \cup V$, where P and V are the two sides of the bipartition, with $|P| = k$ and $|V| = n$. When we consider a given drawing, we will use the word "vertex" and "edge" to denote both the vertex or edge

of the graph, and their representation as points and curves. Without loss of generality, we can assume that the k outer vertices $p_1, \ldots, p_k$ lie in clockwise order on the boundary of a disk that contains all the edges, or on the line at infinity. The vertices of V are labeled $1, \ldots, n$. An example of such a drawing is given in Figure 1.
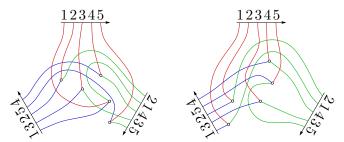

Figure 1: Two outer drawings of $K_{3,5}$. In each of the two drawings the rotation system is $(12345, 21435, 13254)$.

The *rotation system* of the drawing is a sequence of k permutations on n elements associated with the vertices of P in clockwise order. For each vertex of P, its permutation encodes the (say) counterclockwise order in which the n vertices of V are connected to it. Due to our last constraint on the drawings, the rotations of the k vertices of P around each vertex of V are fixed and identical, they reflect the clockwise order of $p_1, \ldots, p_k$ on the boundary.

Unlike for complete graphs, the rotation system of an outer drawing of a complete bipartite graph does not completely determine which pairs of edges are intersecting. This is exemplified with the two drawings in Figure 1.

Results.   The paper is organized as follows. In Section 2, we consider outer drawings with a *uniform* rotation system, in which the k permutations of the vertices of P are all equal to the identity. In this case, we can state a general structure theorem that allows us to completely characterize and count outer drawings of arbitrary bipartite graphs $K_{k,n}$.

In Section 3, we consider outer drawings of $K_{2,n}$ with arbitrary rotation systems. We consider a natural combinatorial encoding of such drawings, and state two necessary consistency conditions involving triples and quadruples of points in V. We show that these conditions are also sufficient, yielding a polynomial-time algorithm for checking consistency of a drawing.

We also observe that our encoding is equivalent to specifying which pairs of edges must intersect in the drawing, hence exactly encodes the corresponding AT-graph. Therefore we show as a corollary that we can decide the realizability of a given AT-graph with underlying graph isomorphic to $K_{2,n}$ in polynomial time.

In Section 4 and 5, we extend these results, first to outer drawings of $K_{3,n}$, then to outer drawings of $K_{k,n}$. We prove that simple consistency conditions on triples and quadruples are sufficient for drawings of $K_{k,n}$, yielding again a polynomial-time algorithm for consistency checking.

In Section 6, we consider outer drawings with the additional property that the edges

can be extended into a pseudoline arrangement, which we refer to as *extendable* drawings. We give a necessary and sufficient condition for the existence of an extendable outer drawing of a complete bipartite graph given the rotation system. We also touch upon the even more restricted problem of finding straight-line outer drawings with prescribed rotation systems.

2   Outer Drawings with uniform rotation system

We first consider the case where k is arbitrary but the rotation system is uniform, that is, the permutation around each of the k vertices p_i is the same. Without loss of generality we assume that this permutation is the identity permutation on $[n]$.

In a given outer drawing, each of the n vertices of V splits the plane into k regions $Q_1, Q_2, \ldots, Q_k$, where each Q_i is bounded by the edges from v to p_i and p_{i+1}, with the understanding that $p_{k+1} = p_1$. We denote by $Q_i(v)$ the ith region defined by vertex v and further on call these regions *quadrants*. We let type$(a, b) = i$, for $a, b \in V$ and $i \in [k]$, whenever $a \in Q_i(b)$. This implies that $b \in Q_i(a)$, see Figure 2. Indeed if $a < b$ and $j \neq i+1$, then edge $p_{i+1}b$ has to intersect all the edges $p_j a$, while edge $p_j b$ has to avoid $p_{i+1}b$ until they meet in b. It follows that none of the edges $p_j b$ can intersect $p_{i+1}a$. This shows that $a \in Q_i(b)$.
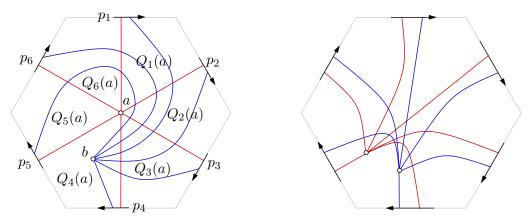

Figure 2: Having placed b in $Q_4(a)$ the crossing pairs of edges and the order of crossings on each edge is prescribed. In particular $a \in Q_4(b)$. On the right a symmetric outer drawing of the pair.

Observation 1 (Symmetry). *For all $a, b \in V$, we have* type$(a, b) =$ type(b, a).

For the case $k = 2$, we have exactly two types of pairs, that we will denote by A and B. The two types are illustrated on Figure 3. Note that the two types can be distinguished by specifying which are the pairs of intersecting edges. The outer drawings of $K_{2,n}$ with uniform rotations can be viewed as *colored pseudoline arrangements*, where:

- each pseudoline is split into two segments of distinct colors,

- no crossing is monochromatic.

Figure 3: The two types of pairs for outer drawings of $K_{2,n}$ with uniform rotation systems.

This is illustrated on Figure 4. The pseudoline of a vertex $v \in V$ is denoted by $\ell(v)$. The left (red) and right (blue) parts of this pseudoline are denote by $\ell_L(v)$ and $\ell_R(v)$. Now having $\text{type}(a, b) = \text{type}(b, a) = A$ means that b lies **above** $\ell(a)$ and a lies **above** $\ell(b)$. While having $\text{type}(a, b) = \text{type}(b, a) = B$ means that b lies **below** $\ell(a)$ and a lies **below** $\ell(b)$.
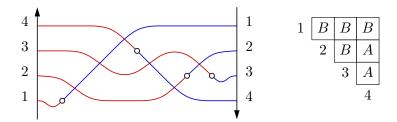

Figure 4: Drawing $K_{2,4}$ as a colored pseudoline arrangement. The type of each pair is given in the table on the right.
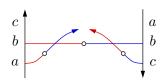
2.1   The triple rule

In what follows, we will say that $a < b$ for a pair $a, b \in V$ whenever a comes before b in the permutation around the vertices p_i.

Lemma 2 (Triple rule). *Consider the complete bipartite graph G with vertex bipartition (P, V) such that $|P| = k$ and $|V| = n$, and an outer drawing of G with a uniform rotation system. Then for all $a, b, c \in V$ such that $a < b < c$, we have*

$$\text{type}(a, c) \in \{\text{type}(a, b), \text{type}(b, c)\}.$$

Proof. **Case $k = 2$.** If $\text{type}(a, b) \neq \text{type}(b, c)$ there is nothing to show since there are only two types. Without loss of generality, suppose that $\text{type}(a, b) = \text{type}(b, c) = B$. This situation is illustrated in the left part of Figure 5. The pseudoline $\ell(c)$ must cross $\ell(b)$ on $\ell_R(b)$, otherwise we would have $\text{type}(b, c) = A$. Hence the point c is on the right of this intersection. Pseudoline $\ell(a)$ must cross $\ell(b)$ on $\ell_L(b)$, and a is left of this intersection. It follows that $\ell(a)$ and $\ell(c)$ cross on $\ell_R(a)$ and $\ell_L(c)$, i.e., $\text{type}(a, c) = B$.

Case $k > 2$. For the general case assume that $\text{type}(a, b) = i$ and $\text{type}(a, c) = j$. If $i = j$ there is nothing to show. Now suppose $i \neq j$. From $c \in Q_j(a)$ it follows that $p_{j+1}a$ and $p_j c$ are disjoint. Edges $p_j b$ and $p_j c$ only share the endpoint p_j, hence c has to be in the region delimited by $p_j b$ and $p_{j+1}a$, see the right part of Figure 5. This region is contained in $Q_j(b)$, whence $\text{type}(b, c) = j$.   □
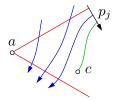
Figure 5: Illustrations for Lemma 2. The $k = 2$ (left), and the $k > 2$ case of (right).

We like to display triples in little tables, e.g., the triple $\mathrm{type}(a,b) = X$, $\mathrm{type}(a,c) = Y$, and $\mathrm{type}(b,c) = Z$ is represented as

$$
\begin{array}{c|c|c}
a & X & Y \\
\cline{1-3}
& b & Z
\end{array}
$$

The triple rule can then be depicted as follows:

$$
\begin{array}{c|c|c}
a & X & Y \\
& b & X
\end{array} \;\Rightarrow\; Y = X.
$$

2.2 The quadruple rule

Lemma 3 (Quadruple rule).

Consider the complete bipartite graph G with vertex bipartition (P,V) such that $|P| = k$ and $|V| = n$, and an outer drawing of G with a uniform rotation system. Then for four vertices $a, b, c, d \in V$ with $a < b < c < d$, if $\mathrm{type}(a,c) = \mathrm{type}(b,c) = \mathrm{type}(b,d) = X$ then $\mathrm{type}(a,d) = X$, which can be depicted as follows:

$$
\begin{array}{c|c|c}
a & & X & Y \\
& b & X & X \\
& c & &
\end{array} \;\Rightarrow\; Y = X.
$$

Proof. **Case $k = 2$.** Suppose, without loss of generality, that $X = B$. Consider the pseudolines representing b and c with their crossing at $\ell_R(b) \cup \ell_L(c)$. Coming from the left the edge $\ell_L(d)$ has to avoid $\ell_L(c)$ and therefore intersects $\ell_R(b)$. On $\ell_R(b)$ the crossing with $\ell_L(c)$ is left of the crossing with $\ell_L(d)$, see Figure 6. Symmetrically from the right the edge $\ell_R(a)$ has to intersect $\ell_L(c)$ and this intersection is left of $\ell_R(b) \cup \ell_L(c)$. To reach the crossings with $\ell_L(c)$ and $\ell_R(b)$ edges $\ell_R(a)$ and $\ell_L(d)$ have to intersect, hence, $\mathrm{type}(a,d) = B$.
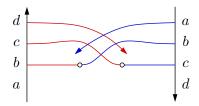
Figure 6: Illustration for the $k = 2$ case of Lemma 3.

Case $k > 2$. In the general case, we let $X = i$, and consider the pseudoline arrangement defined by the two successive vertices p_i and p_{i+1} of P defining the quadrants Q_i. Proving that $\mathrm{type}(a,d) = i$, that is, that $a \in Q_i(d)$, can be done as above for $k = 2$ on the drawing of $K_{2,n}$ induced by $\{p_i, p_{i+1}\}$ and V. $\qquad\square$

2.3   Decomposability and Counting

We can now state a general structure theorem for all outer drawings of $K_{k,n}$ with uniform rotation systems.

Theorem 4. *Consider the complete bipartite graph G with vertex bipartition (P, V) such that $|P| = k$ and $|V| = n$. Given a type in $[k]$ for each pair of vertices in V, there exists an outer drawing of G realizing those types with a uniform rotation system if and only if:*

1. *there exists $s \in \{2, \ldots, n\}$ and $X \in [k]$ such that $\mathrm{type}(a, b) = X$ for all pairs a, b with $a < s$ and $b \geq s$, (in the table this corresponds to maximal rectangle whose cells have all the same entry)*

2. *the same holds recursively when the interval $[1, n]$ is replaced by any of the two intervals $[1, s - 1]$ and $[s, n]$.*

Proof. ($\Rightarrow$) Let us first show that if there exists a drawing, then the types must satisfy the above structure. We proceed by induction on n. Pick the smallest $s \in \{2, \ldots, n\}$ such that $\mathrm{type}(1, b) = \mathrm{type}(1, s)$ for all $b \geq s$. Set $X := \mathrm{type}(1, s)$. We claim that $\mathrm{type}(a, b) = X$ for all a, b such that $1 \leq a < s \leq b \leq n$. For $a = 1$ this is just the condition on s. Now let $1 < a$.

First suppose that $\mathrm{type}(1, a) \neq X$. We can apply the triple rule on the indices $1, a, b$. Since $\mathrm{type}(1, b) \in \{\mathrm{type}(1, a), \mathrm{type}(a, b)\}$, we must have that $\mathrm{type}(a, b) = X$.

Now suppose that $\mathrm{type}(1, a) = X$. We have $\mathrm{type}(1, s - 1) = Y \neq X$ by definition. As in the previous case we obtain $\mathrm{type}(s - 1, b) = X$ from the triple rule for $1, s - 1, b$. Applying the triple rule on $1, a, s - 1$ yields that $\mathrm{type}(a, s - 1) = Y$.

Now apply the quadruple rule on $1, a, s - 1, b$. We already know $\mathrm{type}(1, s - 1) = \mathrm{type}(a, s - 1) = Y$, and by definition $\mathrm{type}(1, b) = X$. Hence we must have $\mathrm{type}(a, b) \neq Y$.

Finally, apply the triple rule on $a, s - 1, b$. We know that $\mathrm{type}(a, s - 1) = Y$, $\mathrm{type}(s - 1, b) = X$. Since $\mathrm{type}(a, b) \neq Y$, we must have $\mathrm{type}(a, b) = X$. This yields the claim.

($\Leftarrow$) Given the recursive structure, one can construct a drawing using a recursive procedure. First, consider the two subintervals as two single vertices, in respective positions prescribed by their type X. Then proceed by recursively blowing up these two vertices. (See Figure 7 for an illustration). $\qquad\square$

The recursive structure yields a corollary on the number of distinct drawings.

Corollary 5 (Counting outer drawings with uniform rotation systems)**.** *For every pair of integers $k, n > 0$ denote by $T(k, n)$ the number of outer drawings of the complete bipartite graph isomorphic to $K_{k,n}$ with uniform rotation systems. Then*

$$T(k + 1, n + 1) = \sum_{j=0}^{n} \binom{n+j}{2j} C_j \, k^j$$

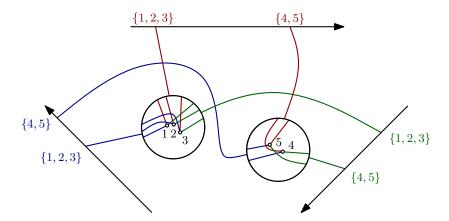*where C_j is the jth Catalan number.*

Figure 7: Illustration of the recursive structure of the outer drawings in the uniform case.

Proof. The recursive structure can be modeled in a labeled binary tree. The root corresponds to $[1, n]$, the subtrees correspond to the intervals $[1, s - 1]$ and $[s, n]$, and the label of the root is type(a, b) for $a < s \leq b$. The definition implies that the label of the left child of a node is different from the label of the node. Leaves have no label.

For the number $T(k, n)$ of labeled binary trees we therefore get a Catalan-like recursion $T(k, n) = k \sum_{i=1}^{n-1} \left(\frac{k-1}{k} T(k, i) \right) \cdot T(k, n - i) + T(k, n - 1)$. The factor k preceeding the sum accounts for the choice of the label for the root. Using symmetry on the labels we find that a $\frac{k-1}{k}$ fraction of the candidates for the left subtree comply with the condition on the labels. The case where the left subtree only consists of a single leaf node is exceptional, in this case there is no label and we have one choice for this subtree, not just $(1 - 1/k)$. This explains the additional summand. The recursion

$$T(k, n) = T(k, n - 1) + (k - 1) \sum_{i=1}^{n-1} T(k, i) \cdot T(k, n - i)$$

together with the initial condition $T(k, 1) = 1$ yields an array of numbers which is is listed as entry A103209 in the encyclopedia of integer sequences[1] (OEIS). The stated explicit expression for $T(k, n)$ can be found there. It can be verified by induction. $\square$

Note that in the case $k = 2$, Corollary 5 provides a bijection between outer drawings with uniform rotation systems and combinatorial structures counted by Schröder numbers, such as separable permutations and guillotine partitions. Separable permutations are permutations that avoid the patterns $\{2413, 3142\}$. To make the bijection more explicit, we can consider the type matrices as matrices of inversions of a permutation, where the type of a pair of elements is A or B depending on whether the elements are reversed in the permutation. The triple rule then enforces that the matrix indeed corresponds to a permutation, while the quadruple rule excludes the two forbidden patterns.

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[1] www.oeis.org

3   Outer Drawings with $k = 2$

In this section we deal with outer drawings with $k = 2$ and arbitrary rotation system. We now have three types of pairs, that we call N, A, and B, as illustrated on Figure 8. The type N (for noncrossing) is new, and is forced whenever the pair is an inversion in exactly one of the two permutations. Note again that the three types exactly encode which are the pairs of crossing edges.
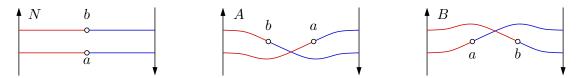

Figure 8: The three types of pairs for outer drawings of $K_{2,n}$ with arbitrary rotation systems.

Recall that an outer drawing of $K_{2,n}$, in which no pair is of type N, can be seen as a colored pseudoline arrangement as defined previously. Similarly, an outer drawing of $K_{2,n}$ in which some pairs are of type N can be seen as an arrangement of colored monotone curves crossing pairwise *at most* once. We will refer to arrangement of monotone curves that cross *at most* once as *quasi-pseudoline arrangements*. The pairs of type N correspond to parallel pseudolines. Without loss of generality, we can suppose that the first permutation in the rotation system, that is, the order of the pseudolines on the left side, is the identity. We denote by π the permutation on the right side.

The first question is whether every permutation π is feasible in the sense that there is a drawing of $K_{2,n}$ such that the rotations are (id, π). The answer is yes, two easy constructions are exemplified in Figure 9
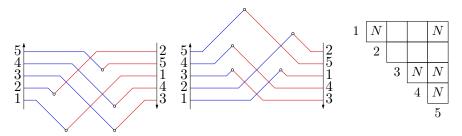

Figure 9: Two outer drawings with rotations $(\mathrm{id}_5, [2, 5, 1, 4, 3])$. On the left, all non-$N$-types are B, on the right they are A.

3.1   Triples

For $a, b, c \in V$, with $a < b < c$, we are interested in the triples of types $(\mathrm{type}(a, b), \mathrm{type}(a, c), \mathrm{type}(b, c))$ that are possible in an outer drawing of $K_{2,n}$, such triples are called *legal*.

Lemma 6 (decomposable triples).
For $a, b, c \in V$, with $a < b < c$, let $X = \mathrm{type}(a, b)$, $Y = \mathrm{type}(a, c)$, and $Z = \mathrm{type}(b, c)$). If $Y \in \{X, Z\}$, then the triple is legal. There are 15 triples of this kind.

Proof. If $Y = X$ take a drawing of $K_{2,2}$ of type X with vertices a, v and $a < v$. In this drawing double the pseudoline corresponding to v and cover vertex v by a small circle. Then plug a drawing of $K_{2,2}$ of type Z with vertices b, c in this circle. This results in a drawing of $K_{2,3}$ with the prescribed types. The construction is very much as in Figure 7.

There are 3 triples
$$\begin{array}{c|c|c} a & X & X \\ \hline b & X \\ \cline{1-2} c \end{array}$$
and in addition for each of the 6 pairs (X, Z) with

$X \neq Z$ a triple of each the 2 types
$$\begin{array}{c|c|c} a & X & X \\ \hline b & Z \\ \cline{1-2} c \end{array} \quad \text{and} \quad \begin{array}{c|c|c} a & X & Z \\ \hline b & Z \\ \cline{1-2} c \end{array}.$$
$\qquad\square$

Note that the triples of the latter lemma are decomposable in the sense of Theorem 4.

Lemma 7.

There are exactly two non-decomposable legal triples:
$$\begin{array}{c|c|c} a & N & A \\ \hline b & B \\ \cline{1-2} c \end{array} \quad \text{and} \quad \begin{array}{c|c|c} a & A & B \\ \hline b & N \\ \cline{1-2} c \end{array}.$$

Proof. From Lemma 2 we know that triples where all entries are A or B are decomposable. If $\text{type}(a, b) = N$, then (a, b) is an inversion of π while pairs (a, b) with $\text{type}(a, b) \in \{A, B\}$ are non-inversions of π. Both, the set of inversion pairs and the set of non-inversion pairs are transitive. Hence, triples of type
$$\begin{array}{c|c} & N \\ \hline & \\ \end{array} \quad \text{and} \quad \begin{array}{c|c} N & \\ \hline & N \\ \end{array}$$
where empty cells represent non-inversion pairs are impossible. It remains to consider the cases where exactly one of X and Z is N and the other two symbols in the triple are A and B. Only the two triples shown in the statement of the lemma remain. $\qquad\square$

With the two lemmas we have classified all 17 legal triples, i.e., all outer drawings of $K_{2,3}$.

Observation 8 (Triple rule)**.**
Any three vertices of V must induce one of the 17 legal triples of types.

3.2 Quadruples

We aim at a characterization of collections of types that correspond to outer drawings. Already in the case of uniform rotations we had to add Lemma 3, a condition for quadruples. In the general case the situation is more complex than in the uniform case, see Figure 10.

Reviewing the proof of Lemma 3 we see that in the case discussed there, where given B types are intended to enforce $\text{type}(a, d) = B$, we need that in π element a is before b. This is equivalent to $\text{type}(a, b) \neq N$. Symmetrically, three A types enforce $\text{type}(a, d) = A$ when d is the last in π, i.e., if $\text{type}(c, d) \neq N$.

Lemma 9. *Consider an outer drawing of the complete bipartite graph G with vertex bipartition (P, V) such that $|P| = 2$ and $|V| = n$, and four vertices $a, b, c, d \in V$ such that*
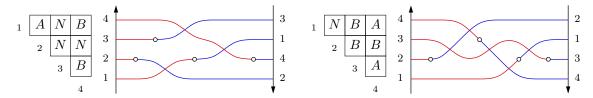
Figure 10: The quadruple rule from Lemma 3 does not hold in the presence of N types.

$a < b < c < d$. If $\text{type}(a,b) \neq N$ and $\text{type}(a,c) = \text{type}(b,c) = \text{type}(b,d) = B$ then $\text{type}(a,d) = B$. If $\text{type}(c,d) \neq N$ and $\text{type}(a,c) = \text{type}(b,c) = \text{type}(b,d) = A$ then $\text{type}(a,d) = A$.

This can be summarized as follows:

3.3 Consistency

With the next theorem we show that consistency on triples and quadruples is sufficient to grant the existence of an outer drawing.

Theorem 10 (Consistency of outer drawings for $k = 2$). *Consider the complete bipartite graph G with vertex bipartition (P, V) such that $|P| = 2$ and $|V| = n$. Given a type in $\{A, B, N\}$ for each pair of vertices in V, there exists an outer drawing of G realizing those types if and only if all triples are legal and the quadruple rule (Lemma 9) is satisfied.*

The proof of this result uses the following known result on *local sequences* in pseudoline arrangements. Given an arrangement of n pseudolines, the local sequences are the permutations α_i of $[n] \setminus \{i\}$, $i \in [n]$, representing the order in which the ith pseudoline intersects the $n - 1$ others.

Lemma 11 (Thm. 6.17 in [5]). *The set $\{\alpha_i\}_{i \in [n]}$ is the set of local sequences of an arrangement of n pseudolines if and only if*

$$ij \in \text{inv}(\alpha_k) \Leftrightarrow ik \in \text{inv}(\alpha_j) \Leftrightarrow jk \in \text{inv}(\alpha_i),$$

for all triples i, j, k, where $\text{inv}(\alpha)$ is the set of inversions of the permutation α.

Proof of Theorem 10. The necessity of the condition was already stated in Observation 8

We proceed by giving an algorithm for constructing an appropriate drawing. First recall from the proof of Lemma 7 that having legal triples implies that the sets of inversion pairs and its complement, the set of non-inversion pairs, are both transitive. Hence, there is a well defined permutation π representing the rotation at p_2.

We aim at defining the local sequences α_i that allow an application of Lemma 11. This will yield a pseudoline arrangement. A drawing of $K_{2,n}$, however, will only correspond to a quasi-pseudoline arrangement. Therefore, we first construct a quasi-pseudoline arrangement T for the pair $(\overline{\pi}, \mathrm{id})$, i.e., only the quasi-pseudolines corresponding to i and j with $\mathrm{type}(i,j) = N$ cross in T. The idea is that appending T on the right side of the quasi-pseudoline arrangement of the drawing yields a full pseudoline arrangement.

Now fix $i \in [n]$. Depending on i we partition the set $[n] \backslash i$ into five parts. For a type X let $X_<(i) = \{j : j < i \text{ and } \mathrm{type}(j,i) = X\}$ and $X_>(i) = \{j : j > i \text{ and } \mathrm{type}(i,j) = X\}$, the five relevant parts are $A_<(i)$, $A_>(i)$, $B_<(i)$, $B_>(i)$, and $N(i) = N_<(i) \cup N_>(i)$. The pseudoline ℓ_i has three parts. The edge incident to p_1 (the red edge) is crossed by pseudolines ℓ_j with $j \in A_>(i) \cup B_<(i)$. The edge incident to p_2 (the blue edge) is crossed by pseudolines ℓ_j with $j \in A_<(i) \cup B_>(i)$. The part of ℓ_i belonging to T is crossed by pseudolines ℓ_j with $j \in N(i)$. The order of the crossings in the third part, i.e., the order of crossings with pseudolines ℓ_j with $j \in N(i)$, is prescribed by T.

Regarding the order of the crossings on the second part we know that the lines for $j \in A_<(i)$ have to cross ℓ_i from left to right in order of decreasing indices and the lines for $j \in B_>(i)$ have to cross ℓ_i from left to right in order of increasing indices, see Figure 11. If $j \in A_<(i)$ and $j' \in B_>(i)$, then consistency of triples implies that $\mathrm{type}(j,j') \in \{A, B\}$. If $\mathrm{type}(j,j') = A$, then on ℓ_i the crossing of j' has to be left of the crossing of j. If $\mathrm{type}(j,j') = B$, then on ℓ_i the crossing of j has to be left of the crossing of j'.
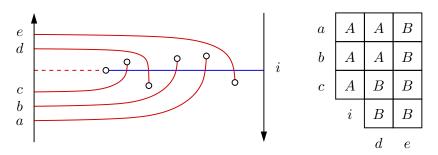

	A	A	B
a	A	A	B
b	A	A	B
c	A	B	B
i	B	B	
	d	e	

Figure 11: Crossings on the edge $i\,p_2$.

The described conditions yield a *"left–to–right"* relation $\to_i$ such that for all $x, y \in A_<(i) \cup B_>(i)$ one of $x \to_i y$ and $y \to_i x$ holds. We have to show that $\to_i$ is acyclic. Since $\to_i$ is a tournament it is enough to show that $\to_i$ is transitive.

Suppose there is a cycle $x \to_i y \to_i z \to_i x$. If $x, y < i$ and $z > i$, then $\mathrm{type}(x,i) = \mathrm{type}(y,i) = A$, moreover, from $x \to_i y$ we get $y < x$ and from $y \to_i z \to_i x$ we get $\mathrm{type}(x,z) = A$, and $\mathrm{type}(y,z) = B$. Since $\mathrm{type}(i,z) = B \neq N$ this is a violation of the second quadruple rule of Lemma 9.

If $x < i$ and $y, z > i$, then we have $\mathrm{type}(i,y) = \mathrm{type}(i,z) = B$. From this together with $y \to_i z$ we obtain $y < z$, and $z \to_i x \to_i y$ yields $\mathrm{type}(x,y) = B$, and $\mathrm{type}(x,z) = A$. This is a violation of the first quadruple rule of Lemma 9.

Adding the corresponding arguments for the order of crossings on the first part of line ℓ_i we conclude that the permutation α_i is uniquely determined by the given types and

the choice of T.

The consistency condition on triples of local sequences needed for the application of Lemma 11 is trivially satisfied because legal triples of types correspond to drawings of $K_{2,3}$ and each such drawing together with T consists of three pairwise crossing pseudolines. □

Since the condition only involves triples and quadruples of vertices in V, this directly yields a polynomial-time algorithm for consistency checking. By observing that the information given by the types is equivalent to specifying which pairs of edges cross, we directly get the analogue statement for AT-graphs.

Corollary 12 (AT-graph realizability). *There exists an $O(n^4)$ algorithm for deciding the existence of an outer drawing of an AT-graph whose underlying graph is of the form $K_{2,n}$.*

Proof. We can check that the three types of pairs in Figure 8 exactly prescribe which pairs of edges cross. Furthermore, given the set of crossing pairs, we can reconstruct the type assignment. We can then check that every triple is legal and that the quadruple rule is satisfied in time proportional to the number of triples and quadruples, hence $O(n^4)$. □

4   Outer Drawings with $k = 3$

At the beginning of the previous section we have seen that any pair of rotations is feasible for outer drawings of $K_{2,n}$. This is not true in the case of $k > 2$. For $k = 4$ the system of rotations $([1,2],[2,1],[1,2],[2,1])$ is easily seen to be infeasible. In the case $k = 3$ it is less obvious that infeasible systems of rotations exist. For $k = 2$ we also had an efficient characterization of consistent assignments of types. We generalize this to $k = 3$.

Again we start by looking at the types for pairs, i.e., at all possible outer drawings of $K_{3,2}$. We already know that if the rotation system is uniform $(\mathrm{id}_2, \mathrm{id}_2, \mathrm{id}_2)$, then there are three types of drawings. The other three options $(\mathrm{id}_2, \overline{\mathrm{id}_2}, \overline{\mathrm{id}_2})$, $(\mathrm{id}_2, \overline{\mathrm{id}_2}, \mathrm{id}_2)$, and $(\mathrm{id}_2, \mathrm{id}_2, \overline{\mathrm{id}_2})$, each have a unique drawing. Figure 12 shows the six possible types and associates them to the symbols B_α, and W_α, for $\alpha = 1, 2, 3$.

Let T_1 be the assignment of types in $\{A, B, N\}$ corresponding to the outer drawing of $K_{2,n}$ with outer vertices (p_3, p_2). Similarly, let T_2 be the table corresponding to (p_1, p_3) and T_3 the table corresponding to (p_2, p_1). Note that these assignments determine the rotation system (π_1, π_2, π_3). An assignment T of types in $\{B_1, B_2, B_3, W_1, W_2, W_3\}$ to every pair of vertices in V translates to the assignments for T_1, T_2, T_3 shown in Table 1:

	B_1	B_2	B_3	W_1	W_2	W_3
T_1	B	A	A	A	N	N
T_2	A	B	A	N	A	N
T_3	A	A	B	N	N	A

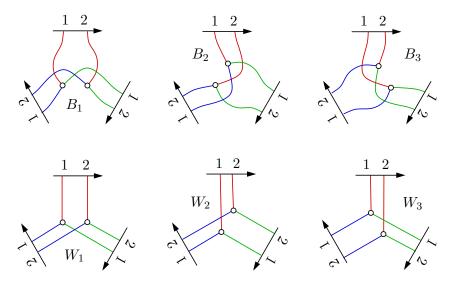Table 1: The projections of the six types of outer drawings of $K_{3,2}$.

Figure 12: The six types of outer drawings of $K_{3,2}$.

This table reveals a dependency between the types in $\{A, B, N\}$ of every pair of vertices in V for different pairs of vertices in P. This dependency is instrumental in the upcoming consistency theorems. For an assignment T of types in $\{B_1, B_2, B_3, W_1, W_2, W_3\}$ to pairs of vertices in V, we will refer to the induced assignments T_1, T_2, T_3 as the *projections* of T.

This allows us to analyze the following example.

Proposition 13. *The system* $(\mathrm{id}_4, [4, 2, 1, 3], [2, 4, 3, 1])$ *is an infeasible set of rotations.*

Proof. The table of types for the given permutations must be of the following form:

$$
\begin{array}{r|c|c|c|}
1 & W_1 & W_3 & W_1 \\ \hline
2 & B_\alpha & W_2 \\ \cline{2-3}
3 & W_1 \\ \cline{2-2}
& 4
\end{array}
$$

From the entries of type W_i we get the following information about the projection tables:

$$
T_1 = \quad
\begin{array}{r|c|c|c|}
2 & N & & A \\ \hline
4 & A & A \\ \cline{2-3}
3 & N \\ \cline{2-2}
& 1
\end{array}
\qquad
T_2 = \quad
\begin{array}{r|c|c|c|}
1 & N & N & N \\ \hline
2 & & A \\ \cline{2-3}
3 & N \\ \cline{2-2}
& 4
\end{array}
\qquad
T_3 = \quad
\begin{array}{r|c|c|c|}
4 & N & N & N \\ \hline
2 & N \\ \cline{2-2}
1 & A \\ \cline{2-2}
& 3
\end{array}
$$

Now we use the triple rule (Observation 8) to deduce the missing entries $T_i(2, 3)$. In T_1 we use the subtable of $\{1, 2, 3\}$ to deduce that $T_1(2, 3) = A$. In T_2 we use the subtable of $\{2, 3, 4\}$ to deduce that $T_2(2, 3) = A$. In T_3 we use the subtable of $\{1, 2, 3\}$ to deduce that $T_3(2, 3) = A$. However, none of the six types of drawings of $K_{3,2}$ has a projection (A, A, A). This proves that there is no drawing for this set of rotations. $\qquad\square$

4.1 The consistency theorem for $k = 3$

Theorem 14 (Consistency of outer drawings for $k = 3$). *Consider the complete bipartite graph G with vertex bipartition (P, V) such that $|P| = 3$ and $|V| = n$. Given the assignments T_1, T_2, T_3 of types in $\{A, B, N\}$ for the pairs (p_3, p_2), (p_1, p_3), and (p_2, p_1) of vertices in P, respectively, there exists an outer drawing of G realizing those types if and only if*

- *all three assignments obey the triple and quadruple consistency rules in the sense of Theorem 10,*

- *for any pair of vertices in V, the assignments correspond to an outer drawing of $K_{3,2}$ in the sense of Table 1.*

This directly yields the following corollary.

Corollary 15 (AT-graph realizability). *There exists an $O(n^4)$ algorithm for deciding the existence of an outer drawing of an AT-graph whose underlying graph is of the form $K_{3,n}$.*

Proof. Again, one can check that the three types of pairs in Figure 12 exactly prescribe which pairs of edges cross, and given the set of crossing pairs, we can reconstruct the type assignment. We can then check that every triple is legal and that the quadruple rule is satisfied in time proportional to the number of triples and quadruples, hence $O(n^4)$. □

Proof of Theorem 14. Let us first note that one direction of the Theorem is easy: if there exists an outer drawing, then the assignments must be consistent.

We now show that consistency of the type assignments is sufficient for the existence of an outer drawing. Let T be the assignment of types in $\{B_1, B_2, B_3, W_1, W_2, W_3\}$ to the pairs of V given by Table 1, and let (π_1, π_2, π_3) be the corresponding rotations. The consistency of the tables T_2 and T_3 in the sense of Theorem 10 implies that there are drawings D_2 and D_3 of $K_{2,n}$ realizing the type assignments T_2 and T_3. The vertex p_1 (the outer vertex with rotation π_1) and its edges form a non-crossing star in both drawings.

Let the drawing D_2 live on plane Z_2 and D_3 live on plane Z_3 and consider a fixed homeomorphism ϕ between the planes. There is a homeomorphism $\psi : Z_2 \to Z_2$ such that mapping D_2 via $\phi \circ \psi$ to Z_3 yields a superposition of the two drawings with the following properties.

- Corresponding vertices are mapped onto each other.

- The stars of p_1 are mapped onto each other, i.e., the edges at p_1 of the two drawings are represented by the same curves.

- At each vertex $v \in V$ the rotation is correct, i.e., we see the edges to p_1, p_2, p_3 in clockwise order.

- The drawing has no touching edges, i.e., when two edges meet they properly cross in a single point.

The drawing D obtained by superposing D_2 and D_3 is an outer drawing of $K_{3,n}$. We color the edges of D as in our figures, for example the edges incident to p_2 are the green edges. In D each color class of edges is a non-crossing star. For the blue and the green this is true because the edges come from only one of D_2 and D_3. For the red star it is true due to construction. Moreover, all the red-blue and red-green crossings are as prescribed by the original table T. The problem we face is that there is little control on blue-green crossings. Let $e_{red}(v), e_{green}(v), e_{blue}(v)$ be the red, green, and blue edge of v.

Claim: For all v, w the parity of the number of crossings between $e_{green}(v)$ and $e_{blue}(w)$ in D is prescribed by T, i.e., if T requires a crossing between two edges, then they have an odd number of crossings and an even number of crossings otherwise.

Consider the curves $e_{green}(v) \cup e_{red}(v)$ and $e_{blue}(w) \cup e_{red}(w)$. The rotations prescribe whether the number of intersections of the two curves is odd or even. Hence, the parity is respected by D. The crossings of the pairs $(e_{green}(v), e_{red}(w))$, $(e_{red}(v), e_{blue}(w))$ in D are as prescribed by $T(v, w)$. Hence, the parity of the number of crossings of $e_{green}(v)$ and $e_{blue}(w)$ in D is also prescribed. This proves the claim.

Because the rotation at v in D is correct we also note: If $e_{green}(v)$ and $e_{blue}(v)$ cross, then the number of crossings is even. Hence, if D has no pair of a green and a blue edge crossing more than once, then D is an outer drawing realizing the types given by T.

Now consider a pair of edges $e_{green}(v)$ and $e_{blue}(w)$ crossing more than once, $v = w$ is allowed. Use a homeomorphism of the plane to make $e_{green}(v)$ a horizontal straight line segment, see Figure 13. (In the literature intersection patterns of two simple curves in the plane are often called *meanders*. They are of interest in enumerative and algebraic combinatorics.)

The intersections with $e_{green}(v)$ subdivide $e_{blue}(w)$ into a family of *arcs* and two extremal pieces. A blue arc defines an interval on the green edge, this is the *interval of the arc*. An arc together with its interval enclose a bounded region, this is the *region of the arc*.

Fact. Apart from intersecting intervals any two regions of arcs over a fixed green edge $e_{green}(v)$ are either disjoint or nested. Because blue edges are pairwise disjoint this also holds if the arcs are defined by different blue edges.
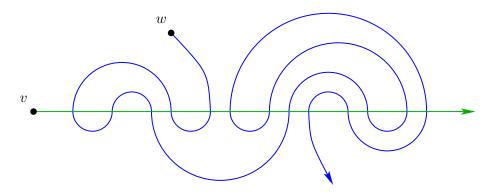

Figure 13: A meander with 7 empty lenses.

An inclusionwise minimal region of an arc is a *lens*. Since D has a finite number of crossings and hence a finite number of regions we have:

Fact. Every region of an arc contains a lens.

Consider a lens L formed by pair of a green and a blue edge in D. Suppose that L is an *empty lens*, i.e., there is no vertex $u \in V$ inside of L. It follows that the boundary of L is only intersected by red edges, moreover, if a red edge e intersects the boundary of L on the green side, then e also intersects L on the blue side. Therefore, we can make $e_{green}(v)$ and $e_{blue}(w)$ switch sides at L and with small deformations at the two crossings get rid of them. It is important to note that the *switch at a lens* does not change the types. In particular after the switch the drawing still represents assignment T.

Apply switching operations until the drawing D' obtained by switching has the property that every lens in D' contains a vertex.

In the following we show that D' has no lens. For the proof we use that the table T_1 corresponding to (π_3, π_2) also has to be consistent. Since D' has no lens we conclude that it is an outer drawing that realizes the types given by T. The existence of such a drawing was the statement of the theorem. $\qquad\square$

Let D' be a drawing with the property that every lens contains a vertex. Before going into the proof that there is no lens in D' we fix some additional notation. For a given green edge $e_{green}(v)$ the regions defined by blue arcs over $e_{green}(v)$ can be classified as *above*, *below*, and *wrapping*. A wrapping region is a region with one contact between $e_{green}(v)$ and $e_{blue}(w)$ from above and one contact from below.

Lemma 16. *There is no wrapping region.*

Proof. Let R be the wrapping region formed by a blue edge wrapping around $e_{green}(v)$ and note that $v \in R$. If the wrapping blue edge is $e_{blue}(v)$, then $e_{red}(v)$ has to intersect one of $e_{green}(v)$ and $e_{blue}(v)$ to leave the region. This is not allowed. If the wrapping blue edge is $e_{blue}(u)$ with $u \neq v$, then $e_{blue}(v)$ has to intersect one of $e_{green}(v)$ and $e_{blue}(u)$ to leave the region. Again, this is not allowed. $\qquad\square$

With a similar proof we get:

Lemma 17. *Vertex w is not contained in the region defined by an arc of $e_{blue}(w)$ over $e_{green}(v)$.*

Proof. The green edge of w would be trapped in such a region. $\qquad\square$

For a region defined by an arc on $e_{blue}(w)$ we speak of a *forward* or *backward* region depending on the direction of the arc above $e_{green}(v)$. Formally: label the t crossings of $e_{green}(v)$ and $e_{blue}(w)$ as $1, .., t$ according to the order on $e_{blue}(w)$. Arcs correspond to consecutive crossings. An arc $[i, i+1]$ is *forward* if crossing i is to the left of crossing $i+1$ on $e_{green}(v)$, otherwise it is *backward*. The permutation of $1, .., t$ obtained by reading the crossings from v to p_2 along $e_{green}(v)$ is called the *meander permutation* and denoted $\sigma_{green}(v)$.

A region is a *relative lens* for $e_{green}(v)$ and $e_{blue}(w)$ if it is above or below $e_{green}(v)$ and minimal in the nesting order of regions defined by $e_{green}(v)$ and $e_{blue}(w)$. In the sequel we sometimes abuse notation by talking of lenses when we mean relative lenses.

Proposition 18. *For all $v \in V$, the green and blue edges of v do not cross in D'.*

Proof. If $e_{green}(v)$ and $e_{blue}(v)$ cross, then there is at least one blue arcs on $e_{blue}(w)$ over $e_{green}(v)$ and, hence, there are regions.

From the order of the three outer vertices and the fact that $e_{red}(v)$ has no crossing with the two other edges of v we conclude that at the last crossing of $e_{blue}(v)$ and $e_{green}(v)$ the blue edge is crossing $e_{green}(v)$ downwards.

Since at the first crossing the blue edge is crossing upwards there is an arc and consequently also a lens above $e_{green}(v)$.

Suppose there is a forward lens above $e_{green}(v)$. Let u be a vertex in the lens. Vertex u is below the line of v in the green-red arrangement. Hence, either $T_3(u, v) = B$ or $T_3(u, v) = N$ and $v <_1 u$. Vertex u is below the forward arc of $e_{blue}(v)$ forming the lens. Hence, it is below the line of v in the red-blue arrangement, and either $T_2(u, v) = B$ or $T_2(u, v) = N$ and $u <_1 v$. From Table 1 we infer that the only legal assignment is $T(u, v) = W_1$ and the projections are $T_2(u, v) = N$ and $T_3(u, v) = N$. However, there is no consistent choice for the order of u and v in $<_1$. This shows that there is no forward lens above $e_{green}(v)$.

Now let $[i, i + 1]$ be a backward lens above $e_{green}(v)$. Let u be a vertex in the lens. We distinguish whether the last crossing on the blue edge is to the right/left of the lens on $e_{green}(v)$, see Figure 14.
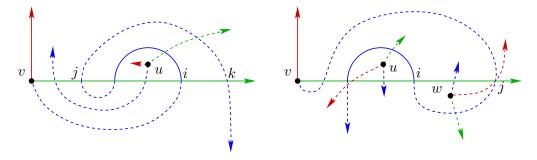

Figure 14: The two cases for a backward lens above $e_{green}(v)$. Dashed curves indicate the order of some crossings along the corresponding edges.

Suppose the last crossing k on the blue edge is to the right of the lens, Figure 14 (left). To get from the arc $[i, i + 1]$ to k the edge $e_{blue}(v)$ has to cross $e_{green}(v)$ upwards at some crossing j left of $i + 1$. The edge $e_{blue}(u)$ has to stay disjoint from $e_{blue}(v)$, therefore, after crossing $e_{green}(v)$ to leave the lens it has a second crossing left of j. The edge $e_{green}(u)$ has a crossing with $e_{blue}(v)$ in the arc $[i, i+1]$ and another one between j and k. Now consider the edge $e_{red}(u)$. If it leaves the lens through $e_{green}(v)$, then it has entered a region whose boundary consists of a piece of $e_{blue}(u)$ and a piece of $e_{green}(v)$. Edge $e_{red}(u)$ is disjoint from $e_{blue}(u)$ and it has already used its unique crossing with $e_{green}(v)$. Hence, $e_{red}(u)$ has

to leave the lens through the blue arc on $e_{blue}(v)$. In this case, however, $e_{red}(u)$ enters a region whose boundary consists of a piece of $e_{blue}(v)$ and a piece of $e_{green}(u)$. Again $e_{red}(u)$ is trapped. Hence, this configuration is impossible.

Suppose the last crossing on the blue edge is to the left of the lens, Figure 14 (right). To get from v to the backward arc $[i, i+1]$ edge $e_{blue}(v)$ has to cross $e_{green}(v)$ downwards at some crossing j right of i. Let w be a vertex in the region of some blue arc $[k, k+1]$ below $e_{green}(v)$ with the property that $j \leq k < i$. The edges $e_{red}(u)$ and $e_{red}(w)$ both need at least two crossings with the union of $e_{green}(v)$ and $e_{blue}(v)$. Hence, they both cross $e_{green}(v)$ and $e_{blue}(v)$. The order of the red edges at p_1 implies $u <_1 v <_1 w$. Matching these data with the drawings of Figure 12 we find that $T(u,v) = B_3$ and $T(v,w) = B_2$. These types together with the already known permutation π_1 imply that π_2 and π_3 also equal π_1. Hence, the triple is a uniform system, and we must have $T(u,w) \in \{B_2, B_3\}$. Since edge $e_{blue}(w)$ has to follow the 'tunnel' prescribed by $e_{blue}(v)$ there is a crossing of edges $e_{green}(v)$ and $e_{blue}(w)$. There is also a crossing of $e_{green}(v)$ and $e_{red}(w)$. However, neither in B_2 nor in B_3 we see crossings of $e_{green}(1)$ with $e_{blue}(2)$ and $e_{red}(2)$. Hence, again the configuration is impossible.   □

We now come to the discussion of the general case.

Proposition 19. *For all $v, w \in V$, there is no lens formed by $e_{green}(v)$ and $e_{blue}(w)$ in D'.*

Our proof of this proposition unfortunately depends on a lengthy case analysis, an outline of which is given in the following subsection.

4.2   Outline of the Proof of Proposition 19

Suppose that there is a lens L formed by $e_{green}(v)$ and $e_{blue}(w)$ in D'. From Proposition 18 we know that $v \neq w$. In fact the proposition implies that the star of every vertex is non-intersecting. For emphasis we collect this and the other restrictions on crossings in D' in a list.

1. Edges of the same color do not cross.

2. Edges of different color that belong to the same vertex do not cross.

3. Red edges have at most one intersection with any other edge.

These properties will be crucial throughout the argument. We also know that there are no wrapping regions (Lemma 16) and the vertex in a region or lens is always different from the vertices of the edges defining the region (Lemma 17).

In section 4.3.1, we discuss eight configurations that may appear in a meander of $e_{blue}(w)$ over $e_{green}(v)$. The eight configurations shown in Figure 15 correspond to the simplest meanders for the following three binary decisions:

- Vertex w is *above/below* edge $e_{green}(v)$.

- The last crossing is to the *left/right* of the first crossing.

- At the last crossing $e_{blue}(w)$ is cutting $e_{green}(v)$ *upward/downward*.

For example the meander labeled VII corresponds to below/left/up. In the case discussion we impose additional conditions on vertices u that are contained in the regions defined by arcs of $e_{blue}(w)$ over $e_{green}(v)$. These conditions either ask for $u \in S_1(w)$ or for $u \notin S_1(w)$. The conditions are so that they are satisfied for free if the respective regions are relative lenses of $e_{blue}(w)$ over $e_{green}(v)$. In each of the cases we show that the T_1 projections yield an illegal table. Hence the configurations do not appear in the drawing D'.

Figure 15: The eight configurations for the first phase of the case analysis.

In the second part we use the results from the first part to show that meanders of the drawing D' have no *inflections*, i.e., in the meander permutation we never see $i-1$ and $i+1$ on the same side of i. This is split in the following two lemmas. The first, easy one, is the following.

Lemma 20. *If a meander permutation $\sigma_{green}(v)$ has an inflection, then it has one of the four turn-back patterns shown in Figure 16.*

Proof. Suppose that a meander permutation $\sigma = \sigma_{green}(v)$ has an inflection at i. We discuss the case where $i+1 <_\sigma i-1 <_\sigma i$ and the blue arc $[i+1, i]$ of $e_{blue}(w)$ is above $e_{green}(v)$. All the other cases can be treated with symmetrical arguments. First note that $i-1 > 1$, otherwise, w is in the region defined by the arc $[i+1, i]$. This is impossible (Lemma 17).

Now suppose that $i-1 <_\sigma i-2 <_\sigma i$. Let x be a vertex in the region defined by the arc $[i-1, i-2]$. Confined by $e_{blue}(w)$ the edge $e_{blue}(x)$ has to cross $e_{green}(v)$ at least three times and it contains an arc whose region contains x. This is impossible (Lemma 17). Hence, $i+1 <_\sigma i-2 <_\sigma i-1 <_\sigma i$ and the meander contains the second of the turn-back patterns shown in Figure 16. Other cases of inflections are related to the other cases of turn-back patterns. □

The proof of the second lemma involves a delicate case analysis and is deferred to Subsection 4.3.2.

Lemma 21. *A meander permutation has no turn-back.*

The consequence of Lemma 21 is that the meanders in D' are very simple, their meander permutations are either the identity permutation or the reverse of the identity. In particular every arc defines a relative lens. It then follows that each meander in D'

Figure 16: The four turn-back pattern.

can be classified according to the three binary decisions mentioned above. Hence, it falls into one of the configurations that have been discussed in Subsection 4.3.1. The additional conditions are satisfied because the arcs all define relative lenses. Hence, there are no nontrivial meanders, i.e., every pair of a green and a blue edge crosses at most once in D'. This concludes the proof of Proposition 19.

4.3 Case analysis for the proof of Proposition 19

4.3.1 The eight basic configurations

We now deal with special instances of the configurations from Figure 15.

Case I. *The first intersection along $e_{blue}(w)$ is downwards and the last intersection is upwards and to the right of the first intersection along $e_{green}(v)$.*

Figure 17: Case I.

Its last intersection with $e_{green}(v)$ being upward forces $e_{blue}(w)$ to intersect $e_{red}(v)$ on its final piece. Therefore, edge $e_{green}(w)$ has to turn around v as shown in Figure 17. Now consider $e_{red}(w)$, its first crossing (among those relevant to the argument) has to be with $e_{green}(v)$. With this crossing, however, $e_{red}(w)$ gets separated from its destination p_1 by the union of $e_{blue}(w)$ and $e_{green}(v)$. Therefore, this case is impossible.
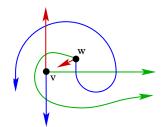
Case II. *The first intersection along $e_{blue}(w)$ is upwards and the last intersection is downwards and to the right of the first along $e_{green}(v)$ and there is a forward arc of $e_{blue}(w)$ above $e_{green}(v)$ whose region contains a vertex $u \notin S_1(w)$.*

Figure 18: Case II.

w	A	B
u	A	

v

Edge $e_{green}(w)$ has to turn around v and also $e_{red}(w)$ has to cross $e_{blue}(v)$, see Figure 19. Now consider a vertex u in a forward region of $e_{blue}(w)$ above $e_{green}(v)$. Vertex u is not under an arc of $e_{blue}(u)$ (Lemma 17) and there is at most one crossing of $e_{blue}(u)$ and $e_{red}(w)$. Therefore, $e_{blue}(u)$ behaves as shown in the sketch, i.e., $w <_3 u <_3 v$. Also $w <_2 u <_2 v$. From the intersections of green and blue edges we conclude that the projections T_1 are as given by the table on the right. This table is not legal. Hence, this case is impossible.
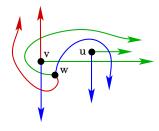
Case III. *The first intersection along $e_{blue}(w)$ is upwards and the last intersection is downwards and to the left of the first along $e_{green}(v)$ and there is a backward arc of $e_{blue}(w)$ above $e_{green}(v)$ whose region contains a vertex $u \in S_1(w)$.*

Edge $e_{green}(w)$ has to stay below $e_{green}(v)$. Now consider a vertex u in a backward region of $e_{blue}(w)$ above $e_{green}(v)$. Vertex u is not under an arc of $e_{blue}(u)$ and we exclude the configuration of Case I. Therefore, $e_{blue}(u)$ behaves as shown in Figure 18, i.e., $u <_3 w <_3 v$. It follows that $e_{green}(u)$ has to stay above $e_{green}(v)$ and $u <_2 v <_2 w$. From the intersections of green and blue edges we conclude that the projections T_1 are as given by the table on the right. This table is not legal. Hence, this case is impossible.

Figure 19:   Case III.

u	B	A
w	N	
v		

Case IV. *The first intersection along $e_{blue}(w)$ is downwards and the last intersection is upwards and to the left of the first along $e_{green}(v)$ and there is a backward arc of $e_{blue}(w)$ below $e_{green}(v)$ whose region contains a vertex $u \notin S_1(w)$.*

Edge $e_{blue}(w)$ has to cross $e_{red}(v)$ and the other edges of w have no intersections with edges of v. Edge $e_{blue}(u)$ either turn around w or it crosses $e_{red}(v)$. In the first case $e_{green}(u)$ has to turn around v. Now, $e_{red}(u)$ has to cross $e_{blue}(w)$ and gets separated from its destination p_1 by the union of $e_{blue}(w)$ and $e_{green}(u)$. Hence, there is no possible routing for edge $e_{red}(u)$ respecting the restrictions on crossings in D'. In the second case $e_{blue}(u)$ crosses $e_{red}(v)$ and $v <_3 w <_3 u$. From the green edges we obtain $w <_2 v <_2 u$. The intersections of green and blue edges, see Figure 20 imply that the projections T_1 are as given by the table below the figure. This table is not legal. Hence, this case is impossible.

Figure 20:   Case IV.

v	N	B
w	A	
u		

Case V. *The first intersection along $e_{blue}(w)$ is downwards and the last intersection is downward and to the right of the first. Moreover, along $e_{green}(v)$ there is a forward arc of $e_{blue}(w)$ below $e_{green}(v)$ whose region contains a vertex $u \in S_1(w)$ and further to the right a forward arc of $e_{blue}(w)$ above $e_{green}(v)$ containing a vertex $x \notin S_1(w)$.*

If $u <_3 w$, then edge $e_{blue}(w)$ makes sure that $x \notin S_1(u)$. Therefore, we have Case II with v, u, and x. If $w <_3 u <_3 v$, then $w \in S_1(u)$. Therefore, we have a Case III.

Now let $w <_3 v <_3 u$. Since $e_{blue}(x)$ is crossing $e_{green}(v)$ downwards we have $e_{green}(x)$ above $e_{green}(v)$ and
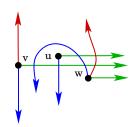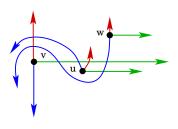
Figure 21:   Case V.

w	A	B
x	A	
u			.

a crossing of $e_{blue}(w)$ with $e_{green}(x)$. Therefore $w <_3 x$.

If $w <_3 x <_3 v$, then the intersections of green and blue edges areas shown in Figure 21. The projections T_1 are given by the table below the figure. This table is not legal. Hence, this case is impossible. If $v <_3 x <_3 u$, then we have Case IV with x and u. Finally, if $u <_3 x$, then the green and blue edges behave as shown in Figure 22. Now, $e_{red}(w)$ would have to cross one of $e_{blue}(x)$ and $e_{green}(v)$ twice to get to its destination p_1.

Figure 22: Case V, 2nd illustration.

Case VI. *The first intersection along $e_{blue}(w)$ is downwards and the last intersection is downward and to the left of the first. Moreover, along $e_{green}(v)$ there is a backward arc of $e_{blue}(w)$ above $e_{green}(v)$ whose region contains a vertex $u \in S_1(w)$ and further to the right a backward arc of $e_{blue}(w)$ below $e_{green}(v)$ containing a vertex $x \notin S_1(w)$.*

If $x <_3 v$, then edge $e_{blue}(x)$ either yields a Case II with $w \notin S_1(x)$ or a Case III with $u \in S_1(x)$. Therefore, $v <_3 x$.

If $w <_3 u$, then edge $e_{blue}(u)$ has an arc above u or it forms a Case I. Therefore, $u <_3 w$ and the complete order is $u <_3 w <_3 v <_3 x$.

Figure 23: Case VI.

u	A	B
w	A	A
v		B
		x

From green edges we get $w <_2 v <_2 x$ and $u <_2 v$, see Figure 23. Independent of the entry $T_1(u, w)$ the resulting table as shown below Figure 23 violates the quadruple rule, Lemma 9. Hence, this case is impossible.

Case VII. *The first intersection along $e_{blue}(w)$ is upwards and the last intersection is upward and to the left of the first. Along $e_{green}(v)$ there is a backward arc of $e_{blue}(w)$ below $e_{green}(v)$ whose region contains a vertex $u \notin S_1(w)$ and further to the right a backward arc of $e_{blue}(w)$ above $e_{green}(v)$ containing a vertex $x \in S_1(w)$.*

If $v <_3 x$, then edge $e_{blue}(x)$ either yields a Case I or a Case IV with $u \notin S_1(x)$. Therefore, $x <_3 v$.

If $u <_3 x$, then vertices v, u and x are in the configuration of Case II. If $x <_3 u <_3 w$, then $u <_3 v$ since otherwise u would be in a region of $e_{blue}(u)$, a violation of Lemma 17. Now $u <_3 v <_3 w$ and $T_1(u, v) = B$, $T_1(u, w) = A$ and $T_1(v, w) = B$ is an illegal table.

Figure 24: Case VII.

x	A	B	A
v		B	B
		w	u

If $v <_3 u$, then $x <_3 v <_3 w <_3 u$ and the situation is as sketched in Figure 24. Irrespective of $u <_2 w$ or $w <_2 u$ the T_1 projections of the remaining 5 pairs yield the partial table shown below the figure. This violates the quadruple rule, Lemma 9. Hence, this case is impossible.
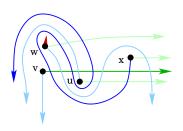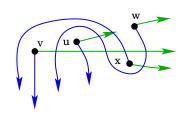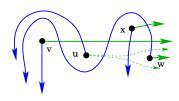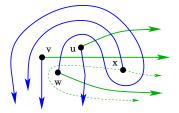
Case VIII. *The first intersection along $e_{blue}(w)$ is upwards and the last intersection is upward and to the right of the first. Along $e_{green}(v)$ there is a forward arc of $e_{blue}(w)$ above $e_{green}(v)$ whose region contains a vertex $u \notin S_1(w)$ and further to the right a forward arc of $e_{blue}(w)$ below $e_{green}(v)$ containing a vertex $x \in S_1(w)$.*

Figure 25: Case VIII.

If $v <_3 u$, then edge $e_{blue}(u)$ either yields a Case I or a Case IV with $w \notin S_1(x)$. Therefore, $u <_3 v$.

If $u <_3 x <_3 v$, then edge $e_{blue}(x)$ yields a Case III with $u \in S_1(x)$. If $x <_3 u$, then $x <_3 u <_3 w$ and $T_1(x, u) = N$, $T_1(x, w) = B$ and $T_1(u, w) = A$ is an illegal table.

u	A	N	B	A
v	B		B	

x w

Therefore, $v <_3 x$ and in total $u <_3 v <_3 x <_3 w$, see Figure 25. Also $u <_2 v <_2 w$ and $v <_2 x$ only the order of x and w in $<_2$ is not determined. If $w <_2 x$ then $T_1(x, w) = N$ otherwise $T_1(x, w) = B$. The resulting table is shown below Figure 25. In the first case the triple u, v, x yields an illegal subtable. In the second case there is a violation of the quadruple rule, Lemma 9. Hence, this case is impossible.

4.3.2   Proof of Lemma 21

Given v suppose that some $e_{blue}(w)$ has a meander over $e_{green}(v)$ with a turn-back. Among all the turn-backs over $e_{green}(v)$ choose the one with the turn-back point i as far as possible, where i is the crossing of the turn-back whose two adjacent crossings are on the same side.

It will be seen in the proof that this specific choice of turn-back substantially simplifies the analysis of two of the cases.

Case TB 1. *The turn back corresponds to a substring $i, i-1, i-2, i+1$ or to a substring $i, i+1, i+2, i-1$ in $\sigma_{green}(v)$ and the turn-back arc is above $e_{green}(v)$.*

Let us assume that the substring is $i, i-1, i-2, i+1$ so that the turn-back arc is $[i, i-1]$. Let u be a vertex in the region of the turn-back arc and let x be a vertex in the region of the arc $[i-1, i-2]$.

The edge $e_{blue}(u)$ is confined by $e_{blue}(w)$ and has a forward arc below x. If $e_{blue}(u)$ has a crossing with $e_{red}(v)$ then we have a Case I. From Lemma 17 it follows that the last crossing of $e_{blue}(u)$ and $e_{green}(v)$ is downwards and to the right of $i+1$.

If $x \notin S_1(u)$, then $e_{blue}(u)$ has a backward arc a in the interval $[i-1, i-2]$ such that x is in the region of arc a. The left part of Figure 26 shows a schematic view of the situation. Let y be a vertex in the region of an arc of $e_{blue}(u)$ above $e_{green}(v)$ and preceeding a along $e_{blue}(u)$ such that $y \notin S_1(x)$. Edges $e_{blue}(u)$ and Lemma 17 force $e_{blue}(x)$ to have its last intersection with $e_{green}(v)$ downwards and to the right of $i+1$. Hence $e_{blue}(x)$ and y are in the configuration of Case II.

We know that $x \in S_1(u)$. Let y be a vertex below the last forward arc of $e_{blue}(u)$ above $e_{green}(v)$. If $y \notin S_1(u)$, then $e_{blue}(u)$ with x and y are in the configuration of Case V.

Otherwise, if $y \in S_1(u)$, then there is a backward arc a of $e_{blue}(u)$ above y. Choose a vertex z from a backward arc below $e_{green}(v)$ and to the right of a. The right part of Figure 26 shows a schematic view of the situation. Edge $e_{blue}(u)$ is forcing $e_{blue}(z)$ to have an arc over y.
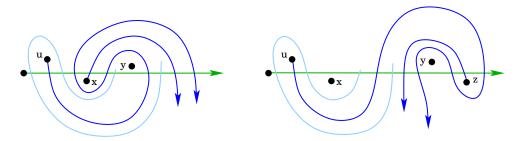

Figure 26: Illustrations for case TB 1.

If the last intersection of $e_{blue}(z)$ with $e_{green}(v)$ is downwards and left of the last intersection of $e_{blue}(u)$ and $e_{green}(v)$, then v with z and y are in the configuration of Case III.

If the last intersection of $e_{blue}(z)$ with $e_{green}(v)$ is downwards and right of the last intersection of $e_{blue}(u)$ with $e_{green}(v)$, then $e_{red}(z)$ has to cross $e_{blue}(u)$ twice to reach its destination p_1 without intersecting $e_{blue}(z)$ and $e_{green}(z)$. The left part of Figure 27 shows a schematic view of the situation, the light-blue arc is spanned by two arcs of $e_{blue}(z)$.

If the last intersection of $e_{blue}(z)$ with $e_{green}(v)$ is upwards and $e_{blue}(z)$ crosses $e_{red}(v)$, then there are two options. The first is that $x \notin S_i(z)$ and there is a configuration of Case VII with v, z, x and y. The second is $x \in S_i(z)$, this requires that $e_{blue}(z)$ has a turn-back interior of the original turn-back of $e_{blue}(w)$ before crossing $e_{red}(v)$, in this situation $e_{blue}(x)$ shows a Case I or together with u a Case III. The right part of Figure 27 illustrates this case.
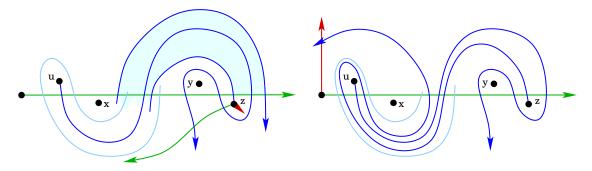

Figure 27: Case TB 1, continued.

Case TB 2. *The turn back corresponds to a substring $i+1, i-2, i-1, i$ or to a substring $i-1, i+2, i+1, i$ and the turn-back arc is below $e_{green}(v)$.*

Let us assume that the substring is $i+1, i-2, i-1, i$ so that the turn-back is the arc $[i, i-1]$. Let u be a vertex in the region of the turn-back arc and let x be a vertex in the region of the arc $[i-2, i-1]$.

The first intersection of $e_{blue}(u)$ with $e_{green}(v)$ is left of the turn back point i of the meander $e_{blue}(w)$. The choice of the turn-back as the leftmost implies that $e_{blue}(u)$ has no turn-back. Hence, $\sigma_{green}(u)$ is the reverse of the identity.

If the last intersection of $e_{blue}(u)$ with $e_{green}(v)$ is downwards, then this is a Case III with x. Otherwise the last intersection is upwards and $e_{blue}(u)$ intersects $e_{red}(v)$. Then there is a backward arc of $e_{blue}(u)$ below $e_{green}(v)$ and a vertex y from such an arc together with x shows that the situation of Case VII. See Figure 28.
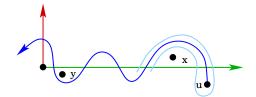

Figure 28: Illustrations for case TB 2.

Case TB 3. *The turn back corresponds to a substring $i+1, i-2, i-1, i$ or to a substring $i-1, i+2, i+1, i$ and the turn-back arc is above $e_{green}(v)$.*

Let us assume that the substring is $i+1, i-2, i-1, i$ so that the turn-back is the arc $[i, i-1]$. Let u be a vertex in the region of the turn-back arc and let x be a vertex in the region of the arc $[i-2, i-1]$.

The first intersection of $e_{blue}(u)$ with $e_{green}(v)$ is left of the turn back point i of the meander $e_{blue}(w)$. The choice of the turn-back as the leftmost implies that $e_{blue}(u)$ has no turn-back. Hence, $\sigma_{green}(u)$ is the reverse of the identity.

If $e_{blue}(u)$ intersects $e_{red}(v)$, then with x this yields a Case IV. Otherwise there is a backward arc of $e_{blue}(u)$ above $e_{green}(v)$. Let y be a vertex from such an arc. Now $e_{blue}(u)$ together with x and y are a Case VI.

Case TB 4. *The turn back corresponds to a substring $i, i-1, i-2, i+1$ or to a substring $i, i+1, i+2, i-1$ and the turn-back arc is below $e_{green}(v)$.*

Let us assume that the substring is $i, i-1, i-2, i+1$ so that the turn-back is the arc $[i, i-1]$. Let u be a vertex in the region of the turn-back arc and let x be a vertex in the region of the arc $[i-2, i-1]$.

If $e_{blue}(x)$ intersects $e_{red}(v)$ this is a Case I. It follows that that $x <_2 v$. If $u <_3 x$ then u and x are in the configuration of Case II.

If $x <_3 u$, then there is a forward arc of $e_{blue}(u)$ below $e_{green}(v)$ and to the right of $i-2$. Let y be a vertex from the region of such an arc.

We first consider the case $y \in S_1(u)$. If $e_{blue}(u)$ intersects $e_{red}(v)$ then u with y yields a Case VIII. If $x <_3 y$, then it follows with Lemma 17 that $x \in S_1(y)$ so that they form a Case III. For $y <_3 x$ edge $e_{blue}(y)$ has to turn back as shown in the left part of Figure 29. In this configuration $e_{red}(u)$ has to cross one of $e_{green}(v)$ and $e_{blue}(y)$ twice to get to its destination p_1.

Now we are in the case $x <_3 u$ and $y \in S_1(u)$. This requires a backward arc of $e_{blue}(u)$ whose region contains y. To the right of this arc there has to be a backward arc above $e_{green}(v)$. Let z be a vertex from the region of this arc. The situation is shown in the right part of Figure 29. If $y <_3 x$, then $z \notin S_1(y)$ so that y and z form a Case II. If $x <_3 y$, then either y is a Case I or it forms a Case III with x.
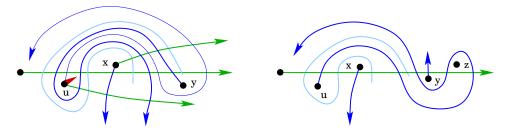

Figure 29: Illustrations for case TB 4.

5 Outer Drawings with arbitrary k

We now generalize our result on consistency of outer drawings for $k = 2, 3$ to any $k \geq 2$.

In the previous section, we defined types for each pair of vertices in V in an outer drawing of $K_{3,n}$. Fortunately, we do not need to go beyond the enumeration of the outer drawings of $K_{3,2}$ to be able to generalize our result to arbitrary values of k.

Theorem 22 (Consistency of outer drawings for $k \geq 2$). *Consider the complete bipartite graph G with vertex bipartition (P, V) such that $|P| = k$ and $|V| = n$. Given assignments T_{ij} of types in $\{A, B, N\}$ for each pair of vertices (p_i, p_j) in P and each pair of vertices in V, there exists an outer drawing of G realizing those types if and only if*

- *all assignments T_{ij} for $1 \leq i < j \leq k$ obey the triple and quadruple consistency rules in the sense of Theorem 10,*

- *for any triple of vertices from P and and each pair of vertices in V the assignments correspond to an outer drawing of $K_{3,2}$ in the sense of Table 1.*

Proof. We proceed by induction, suppose that the result holds for some k and prove that the result holds for $k + 1$. The base cases for $k = 2, 3$ were proved in the previous sections.

From the induction hypothesis, there exists a drawing D_{k-1} of $K_{k-1,n}$ realizing the types involving vertices $p_1, p_2, \ldots, p_{k-1}$ in P. Similarly, there exists a drawing D_k of $K_{2,n}$ realizing the types involving the two vertices p_1 and p_k.

Similarly to what we did in the proof of Theorem 14, we can superpose the two drawings in such a way that the drawings of the two stars $K_{1,n}$ corresponding to p_1 match. More precisely, let the drawing D_{k-1} live on plane Z_{k-1} and D_k live on plane Z_k. Consider a fixed homeomorphism ϕ between the planes. There is a homeomorphism $\psi : Z_{k-1} \to Z_{k-1}$ such that mapping D_{k-1} via $\phi \circ \psi$ to Z_k yields a superposition of the two drawings with the following properties:

- Corresponding vertices are mapped onto each other.

- The stars of p_1 are mapped onto each other, i.e., the edges at p_1 of the two drawings are represented by the same curves.

- At each vertex $v \in V$ the rotation is correct, i.e., we see the edges to $p_1, p_2, \ldots, p_k$ in clockwise order.

- The drawing has no touching edges, i.e., when two edges meet they properly cross in a single point.

The drawing D obtained by superposing D_{k-1} and D_k is a drawing of $K_{k,n}$, possibly with some multiple crossings. We color the edges of D according to the vertex $p_i \in P$ it is incident to, using colors $1, 2, \ldots, k$. Each color class of edges is a non-crossing star, and the parity of the number of crossings between two edges of distinct colors is prescribed by the type assignments. Furthermore, since D_{k-1} and D_k are proper outer drawings, all pairs of edges crossing more than once have respective colors i and k for some $i \in \{2, 3, \ldots, k-1\}$.

Consider all lenses formed by an edge of color k and another edge of color $i \in \{2, 3, \ldots, k-1\}$. Such a lens is said to be empty whenever it does not contain any vertex from V. It is inclusionwise minimal whenever it does not fully contain any other lens. Consider an empty inclusionwise minimal lens formed by an edge $e_k(v)$ of color k and another edge $e_i(w)$ of color i.

If an edge of color $i' \in \{1, 2, 3, \ldots, k-1\} \setminus \{i\}$ intersects the lens, it can only cross $e_i(w)$ once. Similarly, it cannot cross the boundary of the lens on $e_k(v)$ more than once, either because $i' = 1$, or because otherwise the lens would not be minimal. Another edge of color k cannot intersect the lens, because it could only do so by intersecting $e_i(w)$ at least twice, contradicting the minimality of the lens.

Hence all edges intersecting the lens intersect its boundary exactly once on each edge. Therefore we can safely get rid of the two crossings by making $e_k(v)$ and $e_i(w)$ switch sides. The *switch at a lens* does not change the type assignment. We can perform this iteratively until no empty lens is remaining. We call the final drawing D'. This drawing has the property that every lens contains a vertex.

We now wish to show that D' has no lens. For the sake of contradiction, suppose that there is a lens formed by $e_i(v)$ and $e_k(w)$ in D'. We can apply our previous analysis for the case $k = 3$ on the restriction D'' of D' formed by the edges of color $1, i$, and k, letting red$= 1$, green$= i$, and blue$= k$. In particular Proposition 19 holds, and D'' has no lens. Therefore, neither does D', which is a suitable outer drawing of $K_{k,n}$. □

This yields the following corollary for AT-graphs.

Corollary 23 (AT-graph realizability)**.** *There exists an $O(k^3 n^2 + k^2 n^4)$ algorithm for deciding the existence of an outer drawing of an AT-graph whose underlying graph is of the form $K_{k,n}$.*

Proof. We know that the three types of pairs A, B, N (see Figure 12) exactly prescribe which pairs of edges cross, hence given the set of crossing pairs, we can reconstruct the

type assignments T_{ij}. From these data, we can consider every 5-tuple (p_a, p_b, p_c, u, v), with $p_a, p_b, p_c \in P$ and $u, v \in V$, and check that the corresponding drawing of the induced $K_{3,2}$ graph is one of the drawings given in Table 1. This can be done in time $O(k^3 n^2)$. If this is correct, we can then check that every triple is legal and that the quadruple rule is satisfied for each T_{ij}. This can be done in time $O(k^2 n^4)$, hence the result. $\square$

In fact, this implies that checking consistency on all 6-tuples of vertices is sufficient. This matches the result of Kynčl for complete graphs [15].

6 Extendable and Straight-line Outer Drawings

A simple topological drawing of a graph is called *extendable* if its edges can be extended into a pseudoline arrangement. We consider the problem of the existence of an extendable outer drawing for a given rotation system. We also further restrict to straight-line drawings. We exploit a connection between topological drawings and generalized configurations of points which was also used by Kynčl [15]. The main result uses the notion of *suballowable sequence*, as defined by Asinowski [2].

6.1 Extendable Outer Drawings

We will use the notion of *allowable sequence*. These sequences were introduced by Goodman and Pollack in a series of papers [7, 8, 9] as an abstraction of the sequences of successive permutations obtained by projecting a set of n points in $\mathbb{R}^2$ on a rotating line. A good summary of this and related notions can be found in article *Pseudoline Arrangements* in the Handbook of Discrete and Computational Geometry [6]. We restrict to simple allowable sequences, corresponding to point sets with no three points on a line and such that no two pair of points determine the same slope.

A sequence $\pi_1, \pi_2, \ldots, \pi_{m+1}, \ldots, \pi_{2m}$ of permutations of an n-element set, with $m = \binom{n}{2}$, is an *allowable sequence* if for all $i \in [m]$, π_i and π_{m+i} are reverse of each other, each consecutive pair of permutations differ only by a single adjacent transposition, and every pair of element is reversed exactly twice in the sequence.

We will also need the following additional definition: a *generalized configuration* is a pair (C, A) such that C is a collection of n points in the projective plane $\mathbb{P}^2$ and A is an arrangement of pseudolines, one of which is the *pseudoline at infinity* L_∞, satisfying the following conditions:

- every pair of points of C lie on a pseudoline of A,

- each pseudoline in A contains exactly two points of C, except L_∞,

- L_∞ does not contain any point of C.

We say that an allowable sequence on an n-element set is *realized* by a generalized configuration on n points when the successive permutations of the sequence correspond to

successive segments of the pseudoline L_∞, and the transposition between a pair of permutations corresponds to the intersection of L_∞ with the pseudoline containing the two corresponding points. In particular, we observe that if a permutation π is realized at some point $p \in L_\infty$, then p can be connected by additional pseudolines to the n other points in the radial order prescribed by π.

The following result is known.

Theorem 24 (Goodman and Pollack [8], Thm. 4.4). *Every allowable sequence can be realized by a generalized configuration.*

This directly yields that allowable sequences exactly encode which are the achievable rotation systems for our drawings. We first define *suballowable sequences*, see Asinowski [2]: A sequence $\pi_1, \pi_2, \ldots, \pi_k$ of permutations of an n-element set is said to be a *suballowable sequence* if there exists an allowable sequence admitting $\pi_1, \pi_2, \ldots, \pi_k$ as a subsequence.

Theorem 25. *Given a sequence $\pi_1, \pi_2, \ldots, \pi_k$ of permutations of n elements, there exists an extendable outer drawing of $K_{k,n}$ with this rotation system if and only if $\pi_1, \pi_2, \ldots, \pi_k$ is a suballowabe sequence.*

Proof. First, suppose that the rotation system is a suballowable sequence. Then by definition there exists an allowable sequence admitting it as a subsequence. Then from Theorem 24 there exists a generalized configuration on n points realizing this sequence. The n points form the set V. For each permutation π_i, there exists a point on L_∞ that can be connected by additional pseudolines to the n points of V in the order prescribed by π_i. These k points form the set P. By keeping only the inner segments of the pseudolines connecting every point of P with all points of V, we obtain an extendable outer drawing satisfying our conditions.

Now suppose there exists an extendable outer drawing. Consider the generalized configuration obtained as follows. First, we extend the edges between the points of P and V into pseudolines, which is possible from the extendability of the drawing. Then we connect the points of P by the pseudoline at infinity. Finally, by iteratively applying Levi's enlargement Lemma, we can also connect every pair of points in V by a new pseudoline. The order in which each point of P is connected to all points of V realizes one permutation in the allowable sequence defined by the generalized configuration. The collection of such permutations therefore forms a suballowable sequence. $\square$

Note that since suballowable sequences can be recognized efficiently (see [2]), this directly yields a polynomial-time algorithm for deciding whether there exists an extendable outer drawing realizing a given rotation system.

6.2 Straight-line Outer Drawings

We have a direct equivalent statement for straight-line outer drawings.

Theorem 26. *Given a sequence $\pi_1, \pi_2, \ldots, \pi_k$ of permutations of n elements, there exists a straight-line outer drawing of $K_{k,n}$ with this rotation system if and only if $\pi_1, \pi_2, \ldots, \pi_k$ is a suballowable sequence that can be realized by an actual point arrangement.*

This does not yield an efficient algorithm, since deciding whether a given allowable sequence can be realized by an arrangement of straight lines is a hard problem [21, 18, 11]. The problem, however, is tractable in the special case where $k = 3$. This is due to the fact that we can fix the orientation of the three sets of edges. There are two cases to consider. If the suballowable sequence π_1, π_2, π_3 can be extended into an allowable sequence such that the three permutations π_i appear in the first m permutations of the whole sequence, then there exists a straight-line outer drawing with $p_1 = (-\infty, \infty), p_2 = (0, \infty)$, and $p_3 = (\infty, \infty)$. This corresponds to the case where the three projection directions span less than a half-circle. Otherwise there exists a drawing with $p_1 = (\infty, \infty)$, $p_2 = (0, -\infty)$ and $p_3 = (-\infty, 0)$.

We first give an example showing that there are rotation systems for $k = 3$ that are suballowable sequences but do not admit a straight line outer drawing. Up to renaming, this example is the same as Asinowski's example of nonrealizable suballowable sequence ([2], Proposition 8). We give the proof for completeness. Furthermore, our proof directly indicates how to solve the straight-line realizability problem in the case $k = 3$.

Theorem 27. *The triple of rotations given by:*

$$\begin{aligned}
\pi_1 &= (q_1, q_6, q_5, q_2, q_3, q_4) \\
\pi_2 &= (q_3, q_2, q_1, q_4, q_5, q_6) \\
\pi_3 &= (q_5, q_4, q_3, q_6, q_1, q_2)
\end{aligned}$$

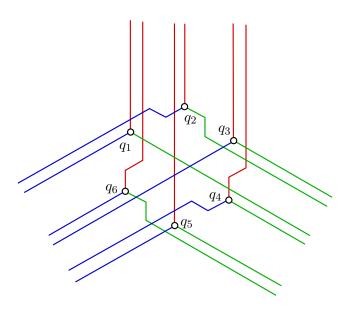*is a suballowable sequence, but is not realizable with straight lines.*

Figure 30: An outer drawing of the system from Theorem 27

Proof. Figure 30 shows an outer drawing of the system. Suppose that the system is stretchable, then there is a drawing with $p_1 = (\infty, \infty)$, $p_2 = (0, -\infty)$ and $p_3 = (-\infty, 0)$, i.e., with horizontal rays, vertical rays, and rays of slope 1. The coordinates of a vertex v are denoted $(x(v), y(v))$.

From π_1 we deduce the inequality $y(q_6) - x(q_6) < y(q_1) - x(q_1)$. Similarly $y(q_2) - x(q_2) < y(q_5) - x(q_5)$ and $y(q_4) - x(q_4) < y(q_3) - x(q_3)$. We will refer to these inequalities as I1, I2, and I3 in this order.

From π_2 we get $x(q_6) < x(q_5)$, $x(q_4) < x(q_1)$, and $x(q_2) < x(q_3)$. Finally, from π_3 we get $y(q_1) < y(q_2)$, $y(q_3) < y(q_6)$, and $y(q_5) < y(q_4)$. These six inequalities are the *helper inequalities*.

Adding I1 and I2 and rearranging terms we obtain

$$y(q_6) + y(q_4) + x(q_1) + x(q_3) < y(q_1) + y(q_3) + x(q_6) + x(q_4)$$

Replacing $y(q_6)$, $y(q_4)$, and $x(q_3)$ on the left side and $y(q_3)$, $x(q_6)$, and $x(q_4)$ on the right side using the six helper inequalities we obtain $y(q_5) + x(q_2) < y(q_2) + x(q_5)$. This contradicts I3. □

A straight line outer drawing of such a rotation system can be regarded as a *tropical arrangement of lines*. We refer to [17] for an introduction to tropical geometry. Such outer drawings can consequently be regarded as *tropical arrangements of pseudolines*. Restated as a result in tropical geometry Theorem 27 tells us that there are non-stretchable tropical arrangements of pseudolines.

The classical construction of a simple non-stretchable example of pseudolines due to Ringel is based on the 9 lines of a Pappus configuration. In [20] tropical versions of Pappus are discussed. They note that a configuration on 9 tropical lines which can be obtained from the configuration of Theorem 27 by replacing points q_ℓ by triple incidences of lines a_i, b_j, c_k with $\{i, j, k\} = \{1, 2, 3\}$ is a valid tropical version of Pappus' Theorem.

The stretchability problem for a tropical arrangements of pseudolines is polynomial-time solvable, as it boils down to the feasibility of a linear program. The proof of Theorem 27 indicates how to set up such a linear program.

7   Open problems

A problem that remains for $k = 2$ is to find a good description of all outer drawings for a given pair (π_1, π_2) of rotations. We state the same problem differently:

Problem 1. *For a given pair (π_1, π_2) of permutations characterize all consistent assignments of types with the property that if $a < b$ in both permutations, then $\mathrm{type}(a, b) \in \{A, B\}$, and if $a <_1 b$ and $b <_2 a$, then $\mathrm{type}(a, b) = N$.*

The simpler case where $\pi_1 = \pi_2$ (hence all types are in $\{A, B\}$) is covered by Corollary 5, which provides a bijection with combinatorial structures counted by Schröder numbers. The following natural question also remains unsolved:

Problem 2. *For a given triple (π_1, π_2, π_3) of rotations, can we decide in polynomial time whether there exists a consistent assignment of types?*

The problem could be generalized to arbitrary k: how hard is it to decide whether there exists an outer drawing of $K_{k,n}$ for a given rotation system? We answered this question partially, by showing that it is polynomial-time decidable when the drawing is further required to be extendable.

We also note that outer drawings are not really enlightening with respect to crossing numbers, as minimally crossing outer drawings of $K_{k,n}$ are easy to describe. It can be checked that by drawing the n vertices of V on a line so that $\lfloor k/2 \rfloor$ vertices of P see them in some order, and the other $\lceil k/2 \rceil$ vertices see them in the reversed order, we achieve the minimum number of crossings. This minimum has value exactly $\binom{n}{2}\left(\binom{k}{2} - \lfloor k/2 \rfloor \lceil k/2 \rceil\right)$. Further insight into crossing numbers could be gained by considering the obvious remaining problem of generalizing our analysis to general simple topological drawings, dropping the outer property.

Acknowledgments

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